$\mathcal{L}_2$ State Estimation with Guaranteed Convergence Speed in the Presence of Sporadic Measurements

Francesco Ferrante, Frédéric Gouaisbaut, Ricardo G. Sanfelice and Sophie Tarbouriech

***Abstract*—This paper deals with the problem of estimating the state of a nonlinear time-invariant system in the presence of sporadically available measurements and external perturbations. An observer with a continuous intersample injection term is proposed. Such an intersample injection is provided by a linear dynamical system, whose state is reset to the measured output estimation error whenever a new measurement is available. The resulting system is augmented with a timer triggering the arrival of a new measurement and analyzed in a hybrid system framework. The design of the observer is performed to achieve exponential convergence with a given decay rate of the estimation error. Robustness with respect to external perturbations and $\mathcal{L}_2$-external stability from plant perturbations to a given performance output are considered. Computationally efficient algorithms based on the solution to linear matrix inequalities are proposed to design the observer. Finally, the effectiveness of the proposed methodology is shown in an example.**

I. INTRODUCTION

A. Background

In most real-world control engineering applications, measurements of the output of a continuous-time plant are only available to the algorithms at isolated times. Due to the use of digital systems in the implementation of the controllers, such a constraint is almost unavoidable and has lead researchers to propose algorithms that can cope with information not being available continuously. In what pertains to state estimation, such a practical need has brought to life a new research area aimed at developing observer schemes accounting for the discrete nature of the available measurements. When the information is available at periodic time instances, there are numerous design approaches in the literature that consist of designing a discrete-time observer for a discretized version of the process; see, e.g., [2] where the proposed approach relies on the results in [18]. Unfortunately, such an approach focuses on periodic sampling and leads in general only to semiglobal practical stability properties (extending such an approach to aperiodic sampling should be possible via the results in [27]). Furthermore, with such an approach no mismatch between the actual sampling time and the one used to discretize the plant is allowed in the analysis or in the discrete-time model used to solve the estimation problem. Very importantly, in many modern applications, such as networked control systems; see [14], [3] and the references therein, the output of the plant is often accessible only sporadically, making the fundamental assumption of measuring it periodically unrealistic.

To overcome the issues mentioned above, several state estimation strategies that accommodate information being available sporadically, at isolated times, have been proposed in the literature. Such strategies essentially belong to two main families. The first family pertains to observers whose state is entirely reset whenever a new measurement is available and that run open loop in between such events – these are typically called *continuous-discrete observers*. The design of such observers is pursued, e.g., in [8], [17]. In particular, in [8] the authors propose a hybrid systems approach to model and design, via *Linear Matrix Inequalities* (LMIs), a continuous-discrete observer ensuring exponential convergence of the estimation error and input-to-state stability with respect to measurement noise. In [17], a new design for continuous-discrete observers based on cooperative systems is proposed for the class of Lipschitz nonlinear systems.

The second family of strategies pertains to continuous-time observers whose output injection error between consecutive measurement events is estimated via a continuous-time update of the latest output measurement. This approach is pursued in [6], [15], [22], [23], [24]. Specifically, the results in [15], [6] show that if a system admits a continuous-time observer and the observer has suitable robustness properties, then, one can build an observer guaranteeing asymptotic state reconstruction in the presence of intermittent measurements, provided that the time in between measurements is small enough. Later, the general approach in [15] has been also extended by [22] to the more general context on networked systems, in which communication protocols are considered. A different approach is pursued in [24]. In particular, in this work, the authors, building on a sampled-data systems approach, propose sufficient conditions in the form of LMIs to design a sampled-and-hold observer to estimate the state of a Lipschitz nonlinear system in the presence of sporadic measurements.

B. Contribution

In this paper, we consider the problem of exponentially estimating the state of continuous-time Lipschitz nonlinear systems subject to external disturbances and in the presence of sporadic measurements, i.e., we assume the plant output to be sampled with a bounded nonuniform sampling period, possibly very large. To address this problem, we propose an observer

Francesco Ferrante is with Univ. Grenoble Alpes, CNRS, GIPSA-lab, F-38000 Grenoble, France. Email: francesco.ferrante@gipsa-lab.fr

Ricardo G. Sanfelice is with Electrical and Computer Engineering department, University of California, Santa Cruz, CA 95064. Email: ricardo@ucsc.edu

Frédéric Gouaisbaut, and Sophie Tarbouriech are with LAAS-CNRS, Université de Toulouse, UPS, CNRS, Toulouse, France. Email:{fgouaisb, tarbour}@laas.fr

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with a continuous intersample injection and state resets. Such an intersample injection is provided by a linear time-invariant system, whose state is reset to the measured output estimation error at each sampling time.

Our contributions in the solution to this problem are as follows. Building on a hybrid system model of the proposed observer and of its interconnection with the plant, we propose results for the simultaneous design (co-design) of the observer and the intersample injection dynamics for the considered class of nonlinear systems. The approach we pursue relies on Lyapunov theory for hybrid systems in the framework in [13]; similar Lyapunov-based analyses for observers are also available in [23, Section VIII], [28], [1]. The use of the hybrid systems framework [13] can be seen as an alternative approach to the impulsive approach pursued, e.g., in [6]. The design we propose ensures exponential convergence of the estimation error with guaranteed convergence speed and robustness with respect to measurement noise and plant perturbations. More precisely, the decay rate of the estimation error can be specified as a design requirement cf. [10]. In addition, for a given performance output, we propose conditions to guarantee a particular $\mathcal{L}_2$-gain between the disturbances entering the plant and the desired performance output. The conditions in these results are turned into matrix inequalities, which are used to derive efficient design procedures of the proposed observer. The methodology we propose gives rise to novel observer designs and allows one to recover as special cases the schemes in [15], [24].

The remainder of the paper is organized as follows. Section II presents the system under consideration and the state estimation problem we solve. Section III illustrates the proposed observer and the resulting hybrid model. Section IV is dedicated to the design of the proposed observer and to some optimization aspects. Finally, in an example, Section V shows the effectiveness of the results presented.

Notation: The set $\mathbb{N}$ is the set of positive integers including zero, the set $\mathbb{N}_{>0}$ is the set of strictly positive integers, $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) represents the set of nonnegative (positive) reals, $\mathbb{R}^{n\times m}$ represents the set of the $n \times m$ real matrices, and $\mathcal{S}_+^n$ is the set of $n \times n$ symmetric positive definite matrices. The identity matrix is denoted by I, whereas the null matrix is denoted by 0. For a matrix $A \in \mathbb{R}^{n\times m}$, A^T denotes the transpose of A, $\mathrm{He}(A) = A+A^\mathsf{T}$, and, when A is nonsingular, $A^{-\mathsf{T}} = (A^\mathsf{T})^{-1}$. For a symmetric matrix A, $A > 0$ and $A \geq 0$ ($A < 0$ and $A \leq 0$) mean that A ($-A$) is, respectively, positive definite and positive semidefinite. In partitioned symmetric matrices, the symbol $\bullet$ stands for symmetric blocks. Given matrices A and B, the matrix $A \oplus B$ is the block-diagonal matrix having A and B as diagonal blocks. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. Given two vectors x, y, we denote $(x, y) = [x^\mathsf{T}\ y^\mathsf{T}]^\mathsf{T}$. Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A}$, the distance of x to $\mathcal{A}$ is defined as $|x|_\mathcal{A} = \inf_{y\in\mathcal{A}} |x - y|$. For any function $z : \mathbb{R} \to \mathbb{R}^n$, we denote $z(t^+) := \lim_{s\to t^+} z(s)$ when it exists. Given a hybrid signal u, $\mathrm{dom}_t\, u := \{t \in \mathbb{R}_{\geq 0} \colon \exists j \in \mathbb{N}_0 \text{ s.t. } (t, j) \in \mathrm{dom}\, u\}$ and $\mathrm{dom}_j\, u := \{j \in \mathbb{N}_0 \colon \exists t \in \mathbb{R}_{\geq 0} \text{ s.t. } (t, j) \in \mathrm{dom}\, u\}$, and for any $(s, i) \in \mathrm{dom}\, u$, $j(s) = \min\{j \in \mathbb{N}_0 \colon\ (s, j) \in \mathrm{dom}\, u\}$ and $t(i) = \min\{t \in \mathbb{R}_{\geq 0} \colon\ (t, i) \in \mathrm{dom}\, u\}$; see [5] for formal definitions of hybrid arcs and hybrid signals.

II. Problem Statement and Outline of Proposed Observer

A. System Description

We consider continuous-time nonlinear time-invariant systems with disturbances of the form

$$\dot{z} = Az + B\psi(Sz) + Nw, \quad y = Cz + \eta \tag{1}$$

where $z \in \mathbb{R}^{n_z}, y \in \mathbb{R}^{n_y}$, $w \in \mathbb{R}^{n_w}$, and $\eta \in \mathbb{R}^{n_y}$ are, respectively, the state, the measured output of the system, a nonmeasurable exogenous input, and the measurement noise affecting the output y, while $\psi \colon \mathbb{R}^{n_q} \to \mathbb{R}^{n_s}$ is a Lipschitz function with Lipschitz constant $\ell > 0$, i.e., for all $v_1, v_2 \in \mathbb{R}^{n_q}$

$$|\psi(v_1) - \psi(v_2)| \leq \ell |v_1 - v_2| \tag{2}$$

The matrices A, C, B, S, and N are constant and of appropriate dimensions and such that the pair (A, C) is detectable. The output y is available only at some time instances t_k, $k \in \mathbb{N}_{>0}$, not known *a priori*. We assume that the sequence $\{t_k\}_{k=1}^\infty$ is strictly increasing and unbounded, and that (uniformly over such sequences) there exist two positive real scalars $T_1 \leq T_2$ such that

$$0 \leq t_1 \leq T_2, \qquad T_1 \leq t_{k+1} - t_k \leq T_2 \quad \forall k \in \mathbb{N}_{>0} \tag{3}$$

The lower bound in condition (3) prevents the existence of accumulation points in the sequence $\{t_k\}_{k=1}^\infty$, and, hence, avoids the existence of Zeno behaviors, which are typically undesired in practice. In fact, T_1 defines a strictly positive minimum time in between consecutive measurements. Furthermore, T_2 defines the *Maximum Allowable Transfer Time (MATI)* [22].

Given a performance output $y_p := C_p(z - \hat{z})$, where $\hat{z}$ is the estimate of z to be generated, the problem to solve is as follows:

Problem 1. *Design an observer providing an estimate $\hat{z}$ of z, such that the following three properties are fulfilled:*

($P1$) *The set of points where the plant state z and its estimate $\hat{z}$ coincide (and any other state variables[1] are bounded) is globally exponentially stable with a prescribed convergence rate for the plant* (1) *interconnected with the observer whenever the input w and η are identically zero;*

($P2$) *The estimation error is bounded when the disturbances w and η are bounded;*

($P3$) *$\mathcal{L}_2$-external stability from the input w to the performance output y_p is ensured with a prescribed $\mathcal{L}_2$-gain when $\eta \equiv 0$.*

B. Outline of the Proposed Solution

Since measurements of the output y are available in an impulsive fashion, assuming that the arrival of a new measurement can be instantaneously detected, inspired by [15], [22],

[1] The observer may have extra state variables that are used for estimation. In our setting, the sporadic nature of the available measurements of y will be captured by a timer with resets.

[24] to solve Problem 1, we propose the following observer with jumps

$$\left.\begin{array}{rcl}\dot{\hat{z}}(t) &=& A\hat{z}(t)+B\psi(S\hat{z}(t))+L\theta(t)\\ \dot{\theta}(t) &=& H\theta(t)\end{array}\right\}\forall t\neq t_k, k\in\mathbb{N}_{>0}$$
$$\left.\begin{array}{rcl}\hat{z}(t^+) &=& \hat{z}(t)\\ \theta(t^+) &=& y(t)-C\hat{z}(t)\end{array}\right\}\qquad \forall t=t_k, k\in\mathbb{N}_{>0} \tag{4}$$

where L and H are real matrices of appropriate dimensions to be designed and $\hat{z}$ represents the estimate of z provided by the observer. The operating principle of the observer in (4) is as follows. The arrival of a new measurement triggers an instantaneous jump in the observer state. Specifically, at each jump, the measured output estimation error, *i.e.*, $e_y := y - M\hat{z}$, is instantaneously stored in θ. Then, in between consecutive measurements, θ is continuously updated according to continuous-time dynamics, and its value is continuously used as an intersample correction to feed a continuous-time observer. At this stage, we introduce the following change of variables $\varepsilon := z-\hat{z}, \tilde{\theta} := C(z-\hat{z})-\theta$, which defines, respectively, the estimation error and the difference between the output estimation error and θ. Moreover, by defining as a performance output $y_p = C_p\varepsilon$, where $C_p \in \mathbb{R}^{n_{y_p}\times n_z}$, we consider the following dynamical system with jumps:

$$\begin{cases}\dot{z}(t) &= Az(t)+G\psi(Sz(t))+Nw(t)\\ \begin{bmatrix}\dot{\varepsilon}(t)\\ \dot{\tilde{\theta}}(t)\end{bmatrix} &= \mathcal{F}\begin{bmatrix}\varepsilon(t)\\ \tilde{\theta}(t)\end{bmatrix}+\mathcal{Q}\zeta(z(t),\varepsilon(t))+\mathcal{T}w(t)\end{cases}\forall t\neq t_k$$
$$\begin{cases}z(t^+) &= z(t)\\ \begin{bmatrix}\varepsilon(t^+)\\ \tilde{\theta}(t^+)\end{bmatrix} &= \mathcal{G}\begin{bmatrix}\varepsilon(t)\\ \tilde{\theta}(t)\end{bmatrix}+\mathcal{N}\eta(t)\end{cases}\qquad \forall t=t_k$$
$$y_p(t) = C_p\varepsilon(t) \tag{5}$$

where for each $v_1, v_2\in\mathbb{R}^{n_z}$, $\zeta(v_1,v_2) := \psi(Sv_1)-\psi(S(v_1-v_2))$ and

$$\begin{array}{l}\mathcal{F} := \begin{bmatrix}A-LC & L\\ CA-CLC-HC & CL+H\end{bmatrix}, \mathcal{T} := \begin{bmatrix}N\\ CN\end{bmatrix}\\ \mathcal{Q} := \begin{bmatrix}B\\ CB\end{bmatrix},\quad \mathcal{G} := \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix},\qquad \mathcal{N} := \begin{bmatrix}0\\ -I\end{bmatrix}\end{array} \tag{6}$$

Our approach consists of recasting (5) and the events at instants t_k satisfying (3) as a hybrid system with nonunique solutions and then applying hybrid systems theory to guarantee that (5) solves Problem 1.

III. Construction of the Observer and First Results

A. Hybrid Modeling

The fact that the observer experiences jumps when a new measurement is available and evolves according to a differential equation in between updates suggests that the updating process of the error dynamics can be described via a hybrid system. Due to this, we represent the whole system composed by the plant (1), the observer (4), and the logic triggering jumps as a hybrid system. The proposed hybrid systems approach also models the hidden time-driven mechanism triggering the jumps of the observer.

To this end, in this work, and as in [8], we augment the state of the system with an auxiliary timer variable τ that keeps track of the duration of flows and triggers a jump whenever a certain condition is verified. This additional state allows to describe the time-driven triggering mechanism as a state-driven triggering mechanism, which leads to a model that can be efficiently represented by relying on the framework for hybrid systems in [13]. More precisely, we make τ decrease as ordinary time t increases and, whenever $\tau=0$, reset it to any point in $[T_1,T_2]$, so as to enforce (3). After each jump, we allow the system to flow again. The whole system composed by the states z, ε and $\tilde{\theta}$, and the timer variable τ can be represented by the following hybrid system, which we denote by $\mathcal{H}_e$, with state $x = (z,\varepsilon,\tilde{\theta},\tau)\in\mathbb{R}^{n_x}$ where $n_x := 2n_z+n_y+1$, input $u=(w,\eta)\in\mathbb{R}^{n_u}$, $n_u := n_w+n_y$, and output y_p:

$$\begin{cases}\dot{x} &= f(x,w) \quad x\in\mathcal{C}, w\in\mathbb{R}^{n_w}\\ x^+ &\in G(x,\eta)\quad x\in\mathcal{D},\eta\in\mathbb{R}^{n_y}\end{cases}\\ y_p = C_p\varepsilon \tag{7a}$$

where

$$f(x,w) = \begin{bmatrix}Az+B\psi(Sz)+Nw\\ \mathcal{F}\begin{bmatrix}\varepsilon\\ \tilde{\theta}\end{bmatrix}+\mathcal{Q}\zeta(z,\varepsilon)+\mathcal{T}w\\ -1\end{bmatrix}\qquad \forall x\in\mathcal{C}, w\in\mathbb{R}^{n_w} \tag{7b}$$

$$G(x,\eta) = \begin{bmatrix}z\\ \mathcal{G}\begin{bmatrix}\varepsilon\\ \tilde{\theta}\end{bmatrix}+\mathcal{N}\eta\\ [T_1,T_2]\end{bmatrix}\qquad \forall x\in\mathcal{D},\eta\in\mathbb{R}^{n_y} \tag{7c}$$

and the flow set $\mathcal{C}$ and the jump set $\mathcal{D}$ are defined as follows

$$\mathcal{C} = \mathbb{R}^{2n_z+n_y}\times[0,T_2],\quad \mathcal{D} = \mathbb{R}^{2n_z+n_y}\times\{0\} \tag{7d}$$

The set-valued jump map allows to capture all possible sampling events occurring within T_1 or T_2 units of time from each other. Specifically, the hybrid model in (7) is able to characterize not only the behavior of the analyzed system for a given sequence $\{t_k\}_{k=1}^{\infty}$, but for any sequence satisfying (3).

Concerning the nature of solution pairs to[2] (7), observe that given any maximal solution pair (ϕ,u) to (7), the definition of the sets $\mathcal{C}$ and $\mathcal{D}$ ensures that $\operatorname{dom}\phi = \operatorname{dom}u = \bigcup_{j\in\mathbb{N}}([t_j,t_{j+1}])\times\{j\}$ with $t_0=0$, $0\leq t_1\leq T_2$, and for all $j\in\mathbb{N}_{>0}$, $T_1\leq t_{j+1}-t_j\leq T_2$. In addition, notice that if (ϕ,u) is maximal then it is also complete; see [9] for more details.

To solve Problem 1 our approach is to design the matrices L and H in the proposed observer in (7) such that without disturbances, i.e., $w\equiv 0, \eta\equiv 0$, the following set[3]

$$\mathcal{A} = \mathbb{R}^{n_z}\times\{0\}\times\{0\}\times[0,T_2] \tag{8}$$

is globally exponentially stable and, when the disturbances are nonzero, the system $\mathcal{H}_e$ is input-to-state stable with respect to $\mathcal{A}$. These properties are captured by the notions defined below:

[2] A pair (ϕ,u), where ϕ is a hybrid arc and u is a hybrid signal, is a solution pair to $\mathcal{H}_e$ if $\operatorname{dom}\phi = \operatorname{dom}u$ and it satisfies its dynamics; see [5] for more details.

[3] By the definition of the system $\mathcal{H}_e$ and of the set $\mathcal{A}$, for every $x\in C\cup D\cup G(D)$, $|x|_{\mathcal{A}} = |(\varepsilon,\tilde{\theta})|$.

Definition 1. *($\mathcal{L}_\infty$ norm) Let u be a hybrid signal and $T \in \mathbb{R}_{\geq 0}$. The T-truncated $\mathcal{L}_\infty$ norm of u is given by*

$$\|u_{[T]}\|_\infty := \max\left\{\underset{(s,k)\in \operatorname{dom} u\setminus\Gamma(u), s+k\leq T}{\operatorname{ess.sup}} |u(s,k)| \quad , \sup_{(s,k)\in\Gamma(w), s+k\leq T} |u(s,k)|\right\}$$

where $\Gamma(u)$ denotes the set of all $(t,j) \in \operatorname{dom} u$ such that $(t, j+1) \in \operatorname{dom} u$; see [5] for further details. The $\mathcal{L}_\infty$ norm of u, denoted by $\|u\|_\infty$ is given by $\lim_{T\to T^\star} \|u_{[T]}\|_\infty$, where $T^\star = \sup\{t + j : (t,j) \in \operatorname{dom} u\}$. When, in addition, $\|u\|_\infty$ is finite, we say that $u \in \mathcal{L}_\infty$.

Definition 2 (Exponential input-to-state stability)**.** *Let $\mathcal{A} \subset \mathbb{R}^{n_z+n_y+1}$ be closed. The system $\mathcal{H}_e$ is* exponentially input-to-state-stable *(eISS) with respect to $\mathcal{A}$ if there exist $\kappa, \lambda > 0$ and $\rho \in \mathcal{K}$ such that each maximal solution pair (ϕ, u) to $\mathcal{H}_e$ is complete and if $u \in \mathcal{L}_\infty$ it satisfies*

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\{\kappa e^{-\lambda(t+j)}|\phi(0,0)|_{\mathcal{A}}, \rho(\|u\|_\infty)\} \tag{9}$$

for each $(t,j) \in \operatorname{dom}\phi$.

When $u \equiv 0$, the bound (9) yields global exponential stability as defined by [26].

B. Sufficient conditions

In this section we provide sufficient conditions to solve Problem 1. To this end, let us consider the following assumption, which is somehow driven by [12, Example 27] and whose role will be clarified later via Theorem 1.

Assumption 1. *Let $\lambda_t, \gamma \in \mathbb{R}_{>0}$ be given. There exist two continuously differentiable functions $V_1 : \mathbb{R}^{n_z} \to \mathbb{R}$, $V_2 : \mathbb{R}^{n_y+1} \to \mathbb{R}$, positive real numbers $\alpha_1, \alpha_2, \omega_1, \omega_2$ such that*

(A1) $\alpha_1|\varepsilon|^2 \leq V_1(\varepsilon) \leq \alpha_2|\varepsilon|^2 \quad \forall x \in \mathcal{C}$;
(A2) $\omega_1|\tilde\theta|^2 \leq V_2(\tilde\theta,\tau) \leq \omega_2|\tilde\theta|^2 \quad \forall x \in \mathcal{C}$;
(A3) *the function $x \mapsto V(x) := V_1(\varepsilon) + V_2(\tilde\theta, \tau)$ satisfies for each $x \in \mathcal{C}, w \in \mathbb{R}^{n_w}$*

$$\langle\nabla V(x), \begin{bmatrix} Az + B\psi(Sz) + Nw \\ \mathcal{F}\begin{bmatrix}\varepsilon\\ \tilde\theta\end{bmatrix} + \mathcal{Q}\zeta(z,\varepsilon) + \mathcal{T}w \\ -1\end{bmatrix}\rangle \leq -2\lambda_t V(x) - \varepsilon^\mathsf{T} C_p^\mathsf{T} C_p \varepsilon + \gamma^2 w^\mathsf{T} w \tag{10}$$

$\triangle$

The following theorem shows that if there exist matrices $L \in \mathbb{R}^{n_z\times n_y}$ and $H \in \mathbb{R}^{n_y\times n_y}$ such that Assumption 1 holds, then such matrices provide a solution to Problem 1.

Theorem 1. *Let Assumption 1 hold. Then:*

(i) There exists $\vartheta \in \mathbb{R}_{>0}$ such that for each maximal solution to (7) of the form $(\phi, 0)$, one has

$$|\phi(t,j)|_{\mathcal{A}} \leq \vartheta e^{-\lambda_t t}|\phi(0,0)|_{\mathcal{A}} \qquad \forall (t,j) \in \operatorname{dom}\phi$$

(ii) The hybrid system $\mathcal{H}_e$ is eISS with respect to $\mathcal{A}$;

(iii) There exists $\alpha > 0$ such that any solution pair (ϕ, u) to $\mathcal{H}_e$ with $\eta \equiv 0$ satisfies

$$\sqrt{\int_{\mathcal{I}} |y_p(s,j(s))|^2 ds} \leq \alpha|\phi(0,0)|_{\mathcal{A}} + \gamma\sqrt{\int_{\mathcal{I}} |w(s,j(s))|^2 ds}$$

where $\mathcal{I} := [0, \sup_t \operatorname{dom}\phi] \cap \operatorname{dom}_t \phi$.

Proof. Consider the following Lyapunov function candidate for the hybrid system (7), $\mathbb{R}^{2n_z+n_y} \times \mathbb{R}_{\geq 0} \ni x \mapsto V(x) := V_1(\varepsilon) + V_2(\tilde\theta, \tau)$. We prove (i) first. Set $\rho_1 = \min\{\alpha_1, \omega_1\}$ and $\rho_2 = \max\{\alpha_2, \omega_2\}$. Then, in view of the definition of the set $\mathcal{A}$ in (8), one gets

$$\rho_1|x|^2_{\mathcal{A}} \leq V(x) \leq \rho_2|x|^2_{\mathcal{A}} \qquad \forall x \in \mathcal{C}\cup\mathcal{D}\cup G(\mathcal{D}) \tag{11}$$

Moreover, from Assumption 1 item (A3) one has

$$\langle\nabla V(x), f(x,w)\rangle \leq -2\lambda_t V(x) + \gamma^2 w^\mathsf{T} w \qquad \forall x \in \mathcal{C}, w \in \mathbb{R}^{n_w} \tag{12}$$

and for each $g = \left(z, \mathcal{G}\begin{bmatrix}\varepsilon\\ \tilde\theta\end{bmatrix} + \mathcal{N}\eta, v\right) \in G(x,\eta), x \in \mathcal{D}, \eta \in \mathbb{R}_{n_y}$ one has

$$V(g) - V(x) = -V_2(\tilde\theta, 0) + V_2(-\eta, v) \leq \omega_2|\eta|^2 \tag{13}$$

Pick $u = (w,\eta) \in \mathcal{L}_\infty$, let (ϕ, u) be a maximal solution pair to (7), and pick $(t,j) \in \operatorname{dom}\phi$. Furthermore, let $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{j+1} = t$ be such that $\operatorname{dom}\phi \cap ([0,t]\times\{0,1,\ldots,j\}) = \cup_{i=0}^{j}([t_i, t_{i+1}]\times\{i\})$. By integrating $(t,j) \mapsto V(\phi(t,j))$, thanks to (12) and (13), one gets[4]

$$\begin{aligned} V(\phi(t,j)) \leq{}& e^{-2\lambda_t t}V(\phi(0,0)) + \\ & \gamma^2 e^{-2\lambda_t t}\int_{[0,t]\cap\operatorname{dom}_t\phi} e^{2\lambda_t s}|w(s,j(s))|^2 ds \\ & + \omega_2\sum_{i=1}^{j} e^{-2\lambda_t(t-t_i)}|\eta(t_i, i-1)|^2 \quad \forall (t,j) \in \operatorname{dom}\phi \end{aligned} \tag{14}$$

By bounding the integral term in (14), thanks to [9, Lemma 2], one gets for each $(t,j) \in \operatorname{dom}\phi$

$$V(\phi(t,j) \leq e^{-2\lambda_t t}V(\phi(0,0)) + \frac{\gamma^2}{2\lambda_t}\|w\|^2_\infty + \omega_2\frac{e^{4\lambda_t T_1}}{e^{2\lambda_t T_1}-1}\|\eta\|^2_\infty$$

which, thanks to (11), implies that

$$\begin{aligned} |\phi(t,j)|^2_{\mathcal{A}} \leq{}& \frac{\rho_2}{\rho_1}e^{-2\lambda_t t}|\phi(0,0)|^2_{\mathcal{A}} + \frac{\gamma^2}{2\lambda_t\rho_1}\|w\|^2_\infty \\ & + \frac{e^{4\lambda_t T_1}}{(e^{2\lambda_t T_1}-1)\rho_1}\omega_2\|\eta\|^2_\infty \quad \forall (t,j) \in \operatorname{dom}\phi \end{aligned} \tag{15}$$

Hence, for each $(t,j) \in \operatorname{dom}\phi$ one has

$$\begin{aligned} |\phi(t,j)|_{\mathcal{A}} \leq{}& \sqrt{\tfrac{\rho_2}{\rho_1}}e^{-\lambda_t t}|\phi(0,0)|_{\mathcal{A}} + \tfrac{\gamma}{\sqrt{2\lambda_t\rho_1}}\|w\|_\infty \\ & + \sqrt{\omega_2\tfrac{e^{4\lambda_t T_1}}{e^{2\lambda_t T_1}-1}}\|\eta\|_\infty \\ \leq{}& \max\Big\{2\sqrt{\tfrac{\rho_2}{\rho_1}}e^{-\lambda_t t}|\phi(0,0)|_{\mathcal{A}}, 2\max\{\tfrac{\gamma}{\sqrt{2\lambda_t\rho_1}}, \\ & \sqrt{\omega_2\tfrac{e^{4\lambda_t T_1}}{e^{2\lambda_t T_1}-1}}\}\|u\|_\infty\Big\} \end{aligned} \tag{17}$$

which gives (i) with $\vartheta = \sqrt{\frac{\rho_2}{\rho_1}}$.

To show (ii) is suffices to notice that thanks to [9, Lemma 1], (17) gives (9) with $\lambda \in \left(0, \frac{\lambda_t T_1}{1+T_1}\right]$, $\kappa = 2\sqrt{\frac{\rho_2}{\rho_1}}e^{\omega}$, where $\omega \geq \lambda$, and

$$s \mapsto \rho(s) := 2\max\left\{\tfrac{\gamma}{\sqrt{2\lambda_t\rho_1}}, \sqrt{\omega_2\tfrac{e^{4\lambda_t T_1}}{e^{2\lambda_t T_1}-1}}\right\}s$$

[4]Given a sequence $\{a_k\}$, we adopt the convention $\sum_{k=a}^{b} a_k = 0$ if $a > b$.

$$\mathcal{M}(\tau)=\begin{bmatrix}\operatorname{He}(P_1(A-LC))+2\lambda_t P_1+C_p^{\mathsf{T}}C_p+\chi\ell^2S^{\mathsf{T}}S & P_1L+e^{\delta\tau}(CA-CLC-HC)^{\mathsf{T}}P_2 & P_1N & P_1B\\ \bullet & e^{\delta\tau}(\operatorname{He}(P_2(CL+H))+(2\lambda_t-\delta)P_2) & e^{\delta\tau}P_2CN & e^{\delta\tau}P_2CB\\ \bullet & \bullet & -\gamma^2I_{n_w} & 0\\ \bullet & \bullet & \bullet & -\chi I_{n_s}\end{bmatrix} \quad (16)$$

Hence, since every maximal solution to $\mathcal{H}_e$ is complete, (ii) is established.

To establish (iii), we follow a similar approach as in [19]. Pick $u=(w,0)$ and let (ϕ,u) be a maximal solution pair to $\mathcal{H}_e$. Pick $t>0$, from Assumption 1 item (A3), since, as shown in (12), V is nonincreasing at jumps, by integrating $V\circ\phi$ one gets

$$\int_{\mathcal{I}(t)}\varepsilon(s,j(s))^{\mathsf{T}}C_p^{\mathsf{T}}C_p\varepsilon(s,j(s))ds\leq V(\phi(0,0)) + \gamma^2\int_{\mathcal{I}(t)}|w(s,j(s))|^2ds$$

where $\mathcal{I}(t):=[0,t]\cap\operatorname{dom}_t\phi$. By taking the limit for t approaching $\sup_t\operatorname{dom}\phi$, thanks to (11), one gets (iii) with $\alpha=\rho_2$. □

Remark 1. *Notice that since (iii) holds for any solution pair (ϕ,u) with $\eta\equiv 0$ and w any hybrid signal, it holds in particular when the hybrid signal w is obtained from a continuous-time signal of the original plant* (1). *Passing from hybrid signals w and y_p to right continuous signals $\tilde{u},\tilde{y}_p$, respectively, (see [16]), item (iii) leads to $\sqrt{\int_{\mathcal{I}}|y_p(s,j(s))|^2ds}=\|\tilde{y}_p\|_2\leq\alpha|(\varepsilon_0,\tilde{\theta}_0)|+\gamma\|\tilde{w}\|_2$.*

C. Construction of the functions V_1 and V_2 in Assumption 1

A possible construction for the functions V_1 and V_2 is illustrated by the result given next.

Theorem 2. *Let $\lambda_t,\gamma\in\mathbb{R}_{>0}$. If there exist $P_1\in\mathcal{S}_+^{n_z},P_2\in\mathcal{S}_+^{n_y}$, $\delta,\chi\in\mathbb{R}_{>0}$, and two matrices $L\in\mathbb{R}^{n_z\times n_y},H\in\mathbb{R}^{n_y\times n_y}$, such that*

$$\mathcal{M}(0)\leq 0,\quad \mathcal{M}(T_2)\leq 0 \quad (18)$$

where the function $[0,T_2]\ni\tau\mapsto\mathcal{M}(\tau)$ is defined in (16) *(at the top of the page). Then, the functions $\varepsilon\mapsto V_1(\varepsilon):=\varepsilon^{\mathsf{T}}P_1\varepsilon$ and $(\tilde{\theta},\tau)\mapsto V_2(\tilde{\theta},\tau):=e^{\delta\tau}\tilde{\theta}^{\mathsf{T}}P_2\tilde{\theta}$ satisfy Assumption 1.*

Proof. Pick $\alpha_1=\lambda_{\min}(P_1),\omega_1=\lambda_{\min}(P_2)$, $\alpha_2=\lambda_{\max}(P_1)$, and $\omega_2=\lambda_{\max}(P_2)e^{\delta T_2}$. Then, items (A1) and (A2) of Assumption 1 are satisfied. Define for each $x\in\mathcal{C},w\in\mathbb{R}^{n_w}$, $\Omega(x,w):=\langle\nabla(V_1(\varepsilon)+V_2(\tilde{\theta},\tau)),f(x,w)\rangle+\varepsilon^{\mathsf{T}}C_p^{\mathsf{T}}C_p\varepsilon+2\lambda_t(V_1(\varepsilon)+V_2(\tilde{\theta},\tau))$. Then, thanks to (2), for any $\chi\in\mathbb{R}_{>0}$, one has that for each $x\in\mathcal{C},w\in\mathbb{R}^{n_w}$ $\Omega(x,w)\leq\Omega(x,w)-\chi(\zeta(z,\varepsilon)^{\mathsf{T}}\zeta(z,\varepsilon)-\ell^2\varepsilon^{\mathsf{T}}S^{\mathsf{T}}S\varepsilon)=:\Pi(x,w)$. Therefore, by defining $\Psi(x,w)=(\varepsilon,\tilde{\theta},w,\zeta(z,\varepsilon))$, straightforward calculations show that for each $x\in\mathcal{C},w\in\mathbb{R}^{n_w}$ one has $\Pi(x,w)=\Psi(x,w)^{\mathsf{T}}\mathcal{M}(\tau)\Psi(z,w)$, where the symmetric matrix $\mathcal{M}(\tau)$ is defined in (16). Hence, one has $\Omega(x,w)\leq\Psi(x,w)^{\mathsf{T}}\mathcal{M}(\tau)\Psi(z,w)$. To conclude the proof, notice that it is straightforward to show that there exists $\lambda\colon[0,T_2]\to[0,1]$ such that for each $\tau\in[0,T_2]$, $\mathcal{M}(\tau)=\lambda(\tau)\mathcal{M}(0)+(1-\lambda(\tau))\mathcal{M}(T_2)$; see [9]. Therefore, it follows that the satisfaction of (18) implies $\mathcal{M}(\tau)\leq 0$ for each $\tau\in[0,T_2]$, that is item (A3) of Assumption 1 is fulfilled, concluding the proof. □

IV. LMI-based Observer Design

In the previous section, sufficient conditions turning the solution to Problem 1 into the feasibility problem of certain matrix inequalities were provided. However, condition (18) is nonlinear in the variables P_1,P_2,δ,H, and L; so further work is needed to derive a computationally tractable design procedure for the observer. While from a numerical standpoint the nonlinearities involving δ are easily manageable in a numerical scheme, the other nonlinearities present in (18) need to be properly handled. To this end, in the sequel, we provide several sufficient conditions to solve Problem 1 via the solution to some LMIs.

Proposition 1. *Let λ_t,γ be given positive real numbers. If there exist $P_1\in\mathcal{S}_+^{n_z},P_2\in\mathcal{S}_+^{n_y}$, positive real numbers δ,χ, matrices $J\in\mathbb{R}^{n_z\times n_y}$ and $Y\in\mathbb{R}^{n_y\times n_y}$ such that $\widehat{\mathcal{M}}(0)\leq 0$ and $\widehat{\mathcal{M}}(T_2)\leq 0$, where the function $[0,T_2]\ni\tau\mapsto\widehat{\mathcal{M}}(\tau)$ is defined in* (19) *(at the top of the next page). Then, $L=P_1^{-1}J,H=P_2^{-1}Y^{\mathsf{T}}-CL$ is a solution to Problem 1.*

Proof. By setting $H=P_2^{-1}Y^{\mathsf{T}}-CL$ and $L=P_1^{-1}J$ in (18) yields (19), thus by the virtue of Theorem 2, this concludes the proof. □

Remark 2. *By selecting $Y=0$, the above result leads to the predictor-based observer in [15], though written in different coordinates. Indeed, whenever $H=-CL$, up to an invertible change of variables,* (4) *yields the same observer as in [15].*

The main idea behind the above result consists of selecting the design variable H so as to cancel out the terms CLC and the term involving the product of P_2 and L (which is hard to handle in an LMI setting). Next, we present other design procedures, whose derivation is based on an equivalent condition to (18) that is formulated following an approach inspired by [20].

A. Slack Variables-Based Design

Theorem 3. *Let $P_1\in\mathcal{S}_+^{n_z},P_2\in\mathcal{S}_+^{n_y}$, $H\in\mathbb{R}^{n_y\times n_y},L\in\mathbb{R}^{n_z\times n_y}$, and $\lambda_t,\gamma,\delta,\chi\in\mathbb{R}_{>0}$. The following statements are equivalent:*

(i) *The matrix inequalities in* (18) *are satisfied with strict inequalities;*

(ii) *There exist matrices $X_1,Y_1,X_3,Y_3\in\mathbb{R}^{n_z\times n_z}$, $X_2,X_4,Y_2,Y_4\in\mathbb{R}^{n_z\times n_y}$ $X_5,Y_5,X_7,Y_7\in\mathbb{R}^{n_y\times n_z}$, $X_6,X_8,Y_6,Y_8\in\mathbb{R}^{n_y\times n_y}$ such that*

$$\begin{bmatrix}\operatorname{He}(S_1(X)) & S_2(X)+\mathcal{P} & S_3(X) & S_4(X)\\ \bullet & \mathcal{N}+\operatorname{He}(S_5(X)) & S_6(X) & S_7(X)\\ \bullet & \bullet & -\gamma^2I & 0\\ \bullet & \bullet & \bullet & -\chi I\end{bmatrix}<0$$
$$\begin{bmatrix}\operatorname{He}(S_1(Y)) & S_2(Y)+\mathcal{P} & S_3(Y) & S_4(Y)\\ \bullet & \mathcal{N}_{T_2}+\operatorname{He}(S_5(Y)) & S_6(Y) & S_7(Y)\\ \bullet & \bullet & -\gamma^2I & 0\\ \bullet & \bullet & \bullet & -\chi I\end{bmatrix}<0 \quad (20)$$

$$
\widehat{\mathcal{M}}(\tau) = \begin{bmatrix} \mathrm{He}(P_1A - JC) + 2\lambda_t P_1 + C_p^\mathsf{T} C_p + \ell^2\chi S^\mathsf{T} S & J + e^{\delta\tau}(A^\mathsf{T} C^\mathsf{T} P_2 - C^\mathsf{T} Y) & P_1 N & P_1 B \\ \bullet & (\mathrm{He}(Y) + (2\lambda_t - \delta)P_2)e^{\delta\tau} & e^{\delta\tau}P_2CN & e^{\delta\tau}P_2CB \\ \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & -\chi I \end{bmatrix} \tag{19}
$$

where

$$
\begin{aligned}
&\mathcal{P} = P_1 \oplus P_2, \mathcal{P}_{T_2} = P_1 \oplus P_2 e^{\delta T_2} \\
&\mathcal{N} = (\lambda_t P_1 + C_p^\mathsf{T} C_p + \chi\ell^2 S^\mathsf{T} S) \oplus ((-\delta + 2\lambda_t)P_2) \\
&\mathcal{N}_{T_2} = (\lambda_t P_1 + C_p^\mathsf{T} C_p + \chi\ell^2 S^\mathsf{T} S) \oplus ((-\delta + 2\lambda_t)e^{\delta T_2}P_2) \\
&X = \left[\begin{smallmatrix} X_1 & X_2 & X_3 & X_4 \\ X_5 & X_6 & X_7 & X_8 \end{smallmatrix}\right] \quad Y = \left[\begin{smallmatrix} Y_1 & Y_2 & Y_3 & Y_4 \\ Y_5 & Y_6 & Y_7 & Y_8 \end{smallmatrix}\right]
\end{aligned} \tag{21}
$$

and for each $\mathcal{X} = \left[\begin{smallmatrix} \mathcal{X}_1 & \mathcal{X}_2 & \mathcal{X}_3 & \mathcal{X}_4 \\ \mathcal{X}_5 & \mathcal{X}_6 & \mathcal{X}_7 & \mathcal{X}_8 \end{smallmatrix}\right]$, *where the matrices* $\mathcal{X}_i$, *for* $i = 1, 2, \dots, 8$, *have suitable dimensions*

$$
\begin{aligned}
&S_1(\mathcal{X}) = \left[\begin{smallmatrix} -\mathcal{X}_1 + C^\mathsf{T}\mathcal{X}_5 & -\mathcal{X}_2 + C^\mathsf{T}\mathcal{X}_6 \\ -\mathcal{X}_5 & -\mathcal{X}_6 \end{smallmatrix}\right] \\
&S_2(\mathcal{X}) = \left[\begin{smallmatrix} \mathcal{X}_1^\mathsf{T}(A-LC) - \mathcal{X}_5^\mathsf{T} HC - \mathcal{X}_3 + C^\mathsf{T}\mathcal{X}_7 & -\mathcal{X}_4 + C^\mathsf{T}\mathcal{X}_8 + \mathcal{X}_1^\mathsf{T} L + \mathcal{X}_5^\mathsf{T} H \\ \mathcal{X}_2^\mathsf{T}(A-LC) - \mathcal{X}_6^\mathsf{T} HC - \mathcal{X}_7 & -\mathcal{X}_8 + \mathcal{X}_2^\mathsf{T} L + \mathcal{X}_6^\mathsf{T} H \end{smallmatrix}\right] \\
&S_3(\mathcal{X}) = \left[\begin{smallmatrix} \mathcal{X}_1^\mathsf{T} N \\ \mathcal{X}_2^\mathsf{T} N \end{smallmatrix}\right] \quad S_4(\mathcal{X}) = \left[\begin{smallmatrix} \mathcal{X}_1^\mathsf{T} B \\ \mathcal{X}_2^\mathsf{T} B \end{smallmatrix}\right] \\
&S_5(\mathcal{X}) = \left[\begin{smallmatrix} (A-LC)^\mathsf{T}\mathcal{X}_3 - C^\mathsf{T} H^\mathsf{T}\mathcal{X}_7 & (A-LC)^\mathsf{T}\mathcal{X}_4 - C^\mathsf{T} H^\mathsf{T}\mathcal{X}_8 \\ L^\mathsf{T}\mathcal{X}_3 + H^\mathsf{T}\mathcal{X}_7 & L^\mathsf{T}\mathcal{X}_4 + H^\mathsf{T}\mathcal{X}_8 \end{smallmatrix}\right] \\
&S_6(\mathcal{X}) = \left[\begin{smallmatrix} \mathcal{X}_3^\mathsf{T} N \\ \mathcal{X}_4^\mathsf{T} N \end{smallmatrix}\right] \quad S_7(\mathcal{X}) = \left[\begin{smallmatrix} \mathcal{X}_3^\mathsf{T} B \\ \mathcal{X}_4^\mathsf{T} B \end{smallmatrix}\right]
\end{aligned}
$$

Proof. Let us define

$$
\mathcal{B} = \begin{bmatrix} \mathcal{F} & \mathcal{T} & \mathcal{Q} \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \mathcal{Q}_1 := \begin{bmatrix} 0 & \mathcal{P} \\ \bullet & \mathcal{N} \end{bmatrix} \oplus \begin{bmatrix} -\gamma^2 I & 0 \\ \bullet & -\chi I \end{bmatrix}
$$

$$
\mathcal{Q}_2 := \begin{bmatrix} 0 & \mathcal{P}_{T_2} \\ \bullet & \mathcal{N}_{T_2} \end{bmatrix} \oplus \begin{bmatrix} -\gamma^2 I & 0 \\ \bullet & -\chi I \end{bmatrix}
$$

where $\mathcal{F}$ and $\mathcal{T}$ are defined in (6). Then, one has $\mathcal{M}(0) = \mathcal{B}^\mathsf{T}\mathcal{Q}_1\mathcal{B}$ and $\mathcal{M}(T_2) = \mathcal{B}^\mathsf{T}\mathcal{Q}_2\mathcal{B}$. Moreover, by defining $\mathcal{U} = \begin{bmatrix} 0_{2(n_z+n_y)\times(n_w+n_s)} \\ I \end{bmatrix}$, it turns out that item (i) in our statement is equivalent to

$$
\begin{cases} \mathcal{U}^\mathsf{T}\mathcal{Q}_1\mathcal{U} < 0 & \mathcal{B}^\mathsf{T}\mathcal{Q}_1\mathcal{B} < 0 \\ \mathcal{U}^\mathsf{T}\mathcal{Q}_2\mathcal{U} < 0 & \mathcal{B}^\mathsf{T}\mathcal{Q}_2\mathcal{B} < 0 \end{cases} \tag{22}
$$

Moreover, by the projection lemma; (see [11]) (22) holds iff there exist two matrices X, Y such that

$$
\begin{cases} \mathcal{Q}_1 + \mathcal{B}_r^{\perp^\mathsf{T}} X \mathcal{U}_r^\perp + \mathcal{U}_r^{\perp^\mathsf{T}} X^\mathsf{T} \mathcal{B}_r^\perp < 0 \\ \mathcal{Q}_2 + \mathcal{B}_r^{\perp^\mathsf{T}} Y \mathcal{U}_r^\perp + \mathcal{U}_r^{\perp^\mathsf{T}} Y^\mathsf{T} \mathcal{B}_r^\perp < 0 \end{cases} \tag{23}
$$

where $\mathcal{B}_r^\perp$ and $\mathcal{U}_r^\perp$ are some matrices such that $\mathcal{B}_r^\perp\mathcal{B} = 0$ and $\mathcal{U}_r^\perp\mathcal{U} = 0$. Specifically, by noticing that $\mathcal{F} = \underbrace{\left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)}_{\mathcal{F}_l}\underbrace{\left(\begin{smallmatrix} A-LC & L \\ -HC & H \end{smallmatrix}\right)}_{\mathcal{F}_r}$, where $\mathcal{F}_l$ is nonsingular, one can select

$$
\mathcal{B}_r^\perp = \begin{bmatrix} -\mathcal{F}_l^{-1} & \mathcal{F}_r & \mathcal{F}_l^{-1}\mathcal{T} & \mathcal{F}_l^{-1}\mathcal{Q} \end{bmatrix}
$$

while $\mathcal{U}_r^\perp = \begin{bmatrix} I_{2(n_z+n_y)} & 0_{2(n_z+n_y)\times(n_w+n_s)} \end{bmatrix}$. Thus, according to partitioning of X and Y in (21), relation (23) turns into (20), hence $(i) \iff (ii)$, concluding the proof. □

The above result yields an equivalent condition to (18) that can be exploited to derive an efficient design procedure for the proposed observer. To this end, one needs to suitably manipulate (20) to obtain conditions that are linear in the decision variables. Specifically, the two results given next provide some possible approaches to derive sufficient conditions that, when δ is selected, are genuinely LMIs.

Proposition 2. *Let* $\lambda_t, \gamma \in \mathbb{R}_{>0}$. *If there exist* $P_1 \in \mathcal{S}_+^{n_z}, P_2 \in \mathcal{S}_+^{n_y}$, *positive real numbers* δ, χ, *matrices* $X \in \mathbb{R}^{n_z\times n_z}, U, W \in \mathbb{R}^{n_y\times n_y}, J \in \mathbb{R}^{n_z\times n_y}$ *such that*

$$
\begin{aligned}
&\begin{bmatrix} \mathrm{He}(Z_1) & Z_2 + \mathcal{P} & Z_3 & Z_4 \\ \bullet & \mathcal{N} + \mathrm{He}(Z_5) & Z_6 & Z_7 \\ \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & -\chi I \end{bmatrix} < 0 \\
&\begin{bmatrix} \mathrm{He}(Z_1) & Z_2 + \mathcal{P}_{T_2} & Z_3 & Z_4 \\ \bullet & \mathcal{N}_{T_2} + \mathrm{He}(Z_5) & Z_6 & Z_7 \\ \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & -\chi I \end{bmatrix} < 0
\end{aligned} \tag{24}
$$

where $\mathcal{P}, \mathcal{P}_{T_2}, \mathcal{N}, \mathcal{N}_{T_2}$ *are defined in* (21) *and*

$$
\begin{aligned}
&Z_1 = \begin{bmatrix} -X & C^\mathsf{T} U \\ 0 & -U \end{bmatrix}, Z_2 = \begin{bmatrix} -X + X^\mathsf{T} A - JC & J \\ -WC & W \end{bmatrix} \\
&Z_3 = \begin{bmatrix} X^\mathsf{T} N \\ 0 \end{bmatrix}, Z_4 = \begin{bmatrix} X^\mathsf{T} B \\ 0 \end{bmatrix}, Z_5 = \begin{bmatrix} A^\mathsf{T} X - C^\mathsf{T} J^\mathsf{T} & 0 \\ J^\mathsf{T} & 0 \end{bmatrix} \\
&Z_6 = \begin{bmatrix} X^\mathsf{T} N \\ 0 \end{bmatrix}, Z_7 = \begin{bmatrix} X^\mathsf{T} B \\ 0 \end{bmatrix}
\end{aligned}
$$

then $L = X^{-\mathsf{T}} J$ *and* $H = U^{-\mathsf{T}} W$ *solve Problem 1.*

Proof. By selecting in (20) $X_1 = X_3 = Y_1 = Y_3 = X, X_2 = Y_2 = 0, X_4 = Y_4 = 0, X_5 = Y_5 = 0, X_6 = Y_6 = U, X_7 = Y_7 = 0, X_8 = Y_8 = 0, X^\mathsf{T} L = J, U^\mathsf{T} H = W$, one gets (24). Thus, thanks to Theorem 2 and Theorem 3 the result is proven. □

Remark 3. *In Proposition 2, to obtain sufficient conditions in the form of (quasi)-LMIs, the following constraint is enforced* $X_8 = Y_8 = 0$. *Although this allows to obtain numerically tractable conditions, enforcing such a constraint, for a given* λ_t, *restricts the range of values of* δ *for which feasibility is not lost. Indeed, when* $X_8 = Y_8 = 0$, *a necessary condition for* (24) *to be feasible is*[5] $-\delta + 2\lambda_t < 0$.

***Sample-and-hold Implementation**:* Whenever $H = 0$, the general observer scheme presented in this paper reduces to the *zero order holder (ZOH) sample-and-hold* considered, e.g., in [24]. Although such an observer is perfectly captured by our scheme, the implementation of ZOH sample-and-hold observer schemes only requires to store the last measured output estimation error and hold it in between sampling times. Thus, such schemes may be preferable in some applications. For this reason, it appears useful to derive computationally tractable design algorithms in which $H = 0$. This is realized through the following result.

Proposition 3 (Sample-and-hold Implementation)**.** *Let* λ_t, γ *be given positive real numbers. If there exist* $P_1 \in$

[5] A way to overcome this limitation is illustrated in [9].

$\mathcal{S}_+^{n_z}, P_2 \in \mathcal{S}_+^{n_y}$, *positive real numbers* δ, χ, *a nonsingular matrix* $X \in \mathbb{R}^{n_z \times n_z}$, *and matrices* $X_5, Y_5, X_7, Y_7 \in \mathbb{R}^{n_y \times n_z}, X_6, Y_6, X_8, Y_8 \in \mathbb{R}^{n_y \times n_y}, J \in \mathbb{R}^{n_z \times n_y}$ *such that*

$$\begin{aligned}
&\begin{bmatrix} \mathrm{He}(Q_1) & Q_2+\mathcal{P} & Q_3 & Q_4 \\ \bullet & \mathcal{N}+\mathrm{He}(Q_5) & Q_6 & Q_7 \\ \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & -\chi I \end{bmatrix} < 0 \\
&\begin{bmatrix} \mathrm{He}(\widehat{Q}_1) & \widehat{Q}_2+\mathcal{P}_{T_2} & Q_3 & Q_4 \\ \bullet & \mathcal{N}_{T_2}+\mathrm{He}(Q_5) & Q_6 & Q_7 \\ \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & -\chi I \end{bmatrix} < 0
\end{aligned} \tag{25}$$

where $\mathcal{P}, \mathcal{P}_{T_2}, \mathcal{N}, \mathcal{N}_{T_2}$ *are defined in* (21) *and*

$$Q_1 = \begin{bmatrix} -X + C^\mathsf{T} X_5 & C^\mathsf{T} X_6 \\ -X_5 & -X_6 \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} -X + X^\mathsf{T} A - JC + C^\mathsf{T} X_7 & J + C^\mathsf{T} X_8 \\ -X_7 & -X_8 \end{bmatrix}$$
$$Q_3 = \begin{bmatrix} X^\mathsf{T} N \\ 0 \end{bmatrix} \quad Q_4 = \begin{bmatrix} X^\mathsf{T} B \\ 0 \end{bmatrix} \quad Q_5 = \begin{bmatrix} A^\mathsf{T} X - C^\mathsf{T} J^\mathsf{T} & 0 \\ J^\mathsf{T} & 0 \end{bmatrix}$$
$$Q_6 = \begin{bmatrix} X^\mathsf{T} N \\ 0 \end{bmatrix}, Q_7 = \begin{bmatrix} X^\mathsf{T} B \\ 0 \end{bmatrix}, \widehat{Q}_1 = \begin{bmatrix} -X + C^\mathsf{T} Y_5 & C^\mathsf{T} Y_6 \\ -Y_5 & -Y_6 \end{bmatrix}$$
$$\widehat{Q}_2 = \begin{bmatrix} -X + X^\mathsf{T} A - JC + C^\mathsf{T} Y_7 & J + C^\mathsf{T} Y_8 \\ -Y_7 & -Y_8 \end{bmatrix}$$

then $L = X^{-\mathsf{T}} J$ *and* $H = 0$ *are a solution to Problem 1.*

Proof. By selecting in (20) $H = 0, X_1 = X_3 = Y_1 = Y_3 = X, X_2 = Y_2 = 0, X_4 = Y_4 = 0, X^\mathsf{T} L = J$ one gets (25). Thus, thanks to Theorems 2 and 3 the result is proven. □

Remark 4. *The applicability of the above result requires the matrix* X *to be nonsingular and such a constraint cannot be directly imposed in an LMI setting. Nonetheless, if one wants to ensure the nonsingularity of* X*, at the expense of some additional conservatism, then the following constraint can be included* $X^\mathsf{T} + X > 0$.

B. Optimization aspects

So far, we assumed γ to be given. Nonetheless, most of the time one is interested in designing the observer to reduce the effect of the exogenous signal w. This can be realized in our setting by embedding the proposed design conditions into suitable optimization schemes aimed at minimizing γ. In particular, by setting $\gamma^2 = \mu$, the minimization of the $\mathcal{L}_2$ gain from the disturbance w to the performance output y_p can be achieved, for a given $\lambda_t > 0$, by designing the observer via the solution to the following optimization problem:

$$\begin{aligned}
&\underset{P_1, P_2, L, H, \mu, \delta, \chi}{\text{minimize}} \quad \mu \\
&\text{s.t.} \\
&P_1 \in \mathcal{S}_+^{n_z}, P_2 \in \mathcal{S}_+^{n_y}, \mu > 0, \delta > 0, \chi \geq 0 \\
&\mathcal{M}(0) \leq 0, \mathcal{M}(T_2) \leq 0
\end{aligned} \tag{26}$$

Clearly the above optimization problem is hardly tractable from a numerical standpoint due to nonlinear constraints in the decision variables. However, whenever δ is given, the results given in Section IV allows to obtain sufficient conditions in the form of linear matrix inequalities for the satisfaction of (18). Thus, a suboptimal solution to the above optimization problem can be obtained via semidefinite programming (*SDP*) software by performing a grid search for the scalar δ. Analogously, also the maximum transfer time T_2 can be considered as a design parameter within an optimization scheme as the one outlined above. In particular, when one is interested in simultaneously minimizing γ and maximizing T_2, our design conditions can be used to obtain a tradeoff between these two objectives via semidefinite programing tools; see [4, Chapter 4.7].

V. Numerical example

Consider the following model of the flexible one-link manipulator [25]

$$\begin{aligned}
\dot{z} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{bmatrix} \sin(z_3) + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} w \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} z
\end{aligned}$$

where z_1 and z_2 are, respectively, the motor shaft angle and the motor shaft angular speed, while z_3 and z_4 are, respectively, the link angle and the link angular speed. The exogenous input w represents a disturbance torque acting on the motor shaft. Assuming the output y can be measured sporadically, we want to design an observer providing an estimate $\hat{z}$ of z while reducing the effect of the exogenous signal w on the estimate of the unmeasured link variables z_3 and z_4. By setting $B = \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix}^\mathsf{T}$, $S = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$, $\ell = 3.33$, $C_p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the considered plant can be rewritten as (1), so that the methodology proposed in the paper can be applied. Figure 1 shows the tradeoff curves of the two objective γ and T_2 obtained via the the relaxations issued from Proposition 1, Proposition 2, and Proposition 3; in this example $\lambda_t = 0.01$, and δ and T_2 are selected over a grid, respectively, on $[1, 100]$ and on $[0.01, 0.3]$. In [24], sufficient conditions in the form of LMIs are given for the design of a sample-and-hold observer that solves item $(P1)$ of Problem 1. In particular for this example, the conditions given in [24] are feasible for T_2 up to 0.1. Figure 1 shows that our methodology not only ensures robustness with respect to external inputs and $\mathcal{L}_2$-gain performance, but also leads to a larger allowable value for T_2.

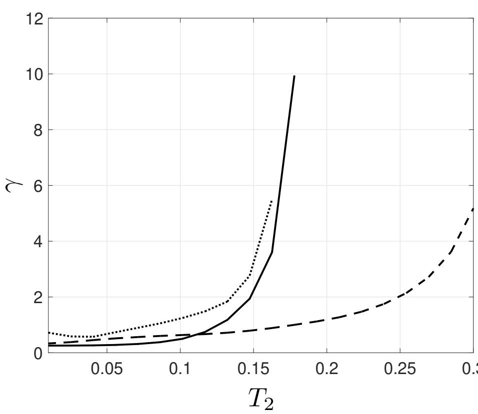

Fig. 1: Tradeoff curves obtained by considering different relaxations: Proposition 1 (dashed line), Proposition 2 (solid line) and, Proposition 3 (dotted line).

Specifically, T_2 can be selected up to 0.3, i.e., an improvement of 200% with respect to [24].

Before concluding, we want to show how our approach compares with other methodologies not relying on LMIs in terms of conservatism in the estimation of the largest allowable value of T_2 for a given design. In particular, we focus on the observer in [15] for which the results in [22] can be used to estimate the largest allowable value T_2 for a given gain L. Such an observer, as pointed out in Remark 2, can be recovered in our setting by selecting $H = -CL$. Specifically, let us consider the following gain from [21, Chapter 6.6.2]

$$L = \begin{bmatrix} 9.328 & 1 \\ -48.78 & 22.11 \\ -0.0524 & 3.199 \\ 19.41 & -0.9032 \end{bmatrix}$$

and set $H = -CL$. An estimate of the largest allowable value T_2 for the given gains can be obtained by determining the largest value of T_2 for which the conditions in (18) are feasible. Notice that when L, H, and δ are given, (18) are LMIs, thus feasibility of those can be checked via semidefinite programming software. By picking $\lambda_t = 0.01$, and by performing a line search on the scalar δ, it turns out that the conditions in (18) are feasible for T_2 up to 0.1016. In [21, Chapter 6.6.2], the authors show that the approach in [22] leads to an estimate of the largest allowable value of T_2 equal to 1.08×10^{-8}. This shows how our approach allows one to get less conservative estimates of the largest allowable value of T_2–the improvement is an increase in T_2 by about 9.5×10^6 times.

VI. Conclusion

This paper proposed a novel methodology to design, via linear matrix inequalities, an observer with intersample injection to exponentially estimate, with a given decay rate, the state of a continuous-time Lipschitz nonlinear system in the presence of sporadically available measurements. Moreover, the observer is robust to measurement noise, plant disturbances, and ensures a given level of performance in terms of $\mathcal{L}_2$-gain between plant exogenous disturbances and a performance output. Several design methodologies to design the observer based on semidefinite programming have been provided. Two of them lead back respectively to the observer proposed in [15] and to the zero order sample-and-hold in [24], while the remaining lead to completely novel schemes. Several suboptimal design algorithms based on SDP programming are presented for the observer. Numerical experiments underlined the significance of the proposed suboptimal design.

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