2019 American Control Conference (ACC)
Philadelphia, PA, USA, July 10-12, 2019

Multiple Barrier Function Certificates for Forward Invariance in Hybrid Inclusions

Mohamed Maghenem and Ricardo G. Sanfelice

***Abstract*— As a continuation of [1] and using multiple barrier functions, this paper studies forward invariance in hybrid systems modeled by hybrid inclusions. After introducing the notion of a multiple barrier function, we propose sufficient conditions to guarantee different forward invariance properties of a closed set for hybrid systems with nonuniqueness of solutions, solutions terminating prematurely, and Zeno solutions. More precisely, we consider forward (pre-)invariance of sets, which guarantees solutions to stay in a set, and (pre-)contractivity, which further requires solutions that stay in the boundary of the set to evolve (continuously or discretely) towards its interior. Our conditions for forward invariance involve infinitesimal conditions in terms of multiple barrier functions while our conditions for pre-contractivity (and contractivity) involve Minkowski functionals. Examples illustrate the results.**

I. INTRODUCTION

The study of forward invariance in dynamical systems can constitute an important step to conclude some stability properties [2]. Also, it offers a powerful tool to guarantee safety in many real-world applications [3]. The main challenge when studying forward invariance notions consists in providing the tightest possible conditions ensuring that such a property holds, while avoiding explicit computation of the systems solutions. A primary, and yet fundamental answer to this problem was given by Nagumo in [4]. In this reference, conditions involving the contingent cone at the boundary of a given closed set are shown to be necessary and sufficient to conclude that, for each point in the set, there exists at least one solution that remains in it. The set, in this case, is invariant in a weak sense. The same result, under the name viablility, is shown to remain valid when the system is represented by a general differential inclusion satisfying certain regularity conditions. The latter result has been, in turn, extended to impulse differential inclusions and hybrid inclusions in [5] and [6], respectively. When all the solutions starting from a given closed set are required to remain in the set, in this case as stressed in [7], the invariance conditions concern the systems behavior outside the set rather than on its boundary. Sufficient conditions are proposed in [8], [7], and [6] for hybrid automata, differential inclusions, and hybrid inclusions, respectively. Moreover, one can strengthen this invariance property by requiring, additionally, that whenever a solution reaches the boundary of the set to render invariant, it instantaneously leaves the boundary towards the interior of the set. In this case the set is contractive [9] or invariant in a strict sense [7]. Characterizations of the latter property are proposed in [9] for differential and difference equations and in [7] for differential inclusions.

For general hybrid inclusions, in our previous work [1], characterizations of the aforementioned invariance notions are proposed using scalar barrier functions. That is, when the considered closed set is defined as the set of points where the dynamics are defined and, at the same time, a scalar function is nonpositive, we say that the set admits a scalar barrier function candidate. In [1], infinitesimal sufficient conditions involving the scalar barrier function candidate are derived to guarantee the aforementioned invariance notions. Our approach in [1] can be seen as an alternative to the previously cited literature using cone conditions [7], [6]. Moreover, it extends the existing results using barrier functions [10], [11] for hybrid inclusions. In many applications, it is often the case the closed set to render invariant corresponds to the region where multiple scalar functions are nonpositive simultaneously. In such cases, it is typically difficult to find a single scalar function that defines the set of interest and, at the same time smooth. Hence, it is natural to want sufficient conditions guaranteeing forward invariance of a closed set in terms of the multiple barrier function for hybrid inclusions.

A hybrid inclusion is defined as a differential inclusion with a constraint, which models the flow or continuous evolution of the system, plus a difference inclusion with a constraint, modeling the jumps or discrete events. For this general framework, sufficient conditions in terms of infinitesimal inequalities involving the multiple barrier candidates are proposed in this paper. More precisely, under mild conditions on the data defining the hybrid inclusion, we first propose sufficient conditions that guarantee forward invariance notions; see Section III. As a second step, we provide conditions to conclude the relatively stronger notion named strict invariance or contractivity. For the latter property, we distinguish two cases. When the closed set is a C-set, necessary and sufficient conditions are provided. When the closed set is a non-C set, only sufficient conditions are provided; see Section IV. The results in this paper cover the existing results in [12], [11], [8], [1] where only scalar barrier functions are used. Also, we provide alternative conditions to those proposed in [6] and [7] in terms of tangent cones. Indeed, our conditions are alternatives to those therein since

M. Maghenem and R. G. Sanfelice are with Department of Electrical and Computer Engineering, University of California, Santa Cruz. Email: mmaghene,ricardo@ucsc.edu.

This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484, Grant no. ECS-1710621, and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, by the Air Force Research Laboratory under Grant no. FA9453-16-1-0053, and by CITRIS and the Banatao Institute at the University of California.

978-1-5386-7926-5/$31.00 ©2019 AACC

they exploit the fact that the set is a zero-sublevel set of a barrier function. Hence, the obtained conditions avoid the computation of tangent cones, which can be numerically expensive. However, our results build upon the well-known tangent-cone based conditions that can be found in [7] and [13]. To the best of our knowledge, this is the first time in the literature where the concept of multiple barrier functions is used for hybrid inclusions.

Due to space limitation, some proofs are omitted and will be published elsewhere.

Notations. For $x, y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the Euclidean norm of x, $|x|_K := \min_{y \in K} |x-y|$ defines the distance between x and the nonempty set K, and $\langle x, y\rangle = x^\top y$ the inner product between x and y. The inequalities $x \leq 0$ and $x < 0$ mean that $x_i \leq 0$ and, respectively, $x_i < 0$ for all $i \in \{1, ..., n\}$. The inequalities $x \nleq 0$ and $x \nless 0$ mean that there exists $i \in \{1, ..., n\}$ such that $x_i > 0$ and, respectively, $x_i \geq 0$. For a closed set $K \subset \mathbb{R}^n$, we use $\text{int}(K)$ to denote its interior, ∂K its boundary, $\text{cl}(K)$ its closure, and $U(K)$ to denote an open neighborhood around K, namely, $\text{cl}(K) \subset U(K)$. For an open set $O \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O. The open unit ball in $\mathbb{R}^n$ centered at the origin is denoted $\mathcal{B}$. For a continuously diferentiable function $B : O \to \mathbb{R}$, $\nabla B(x)$ denotes the gradient of the function B evaluated at x. The set of continuously differentiable functions is denoted by $\mathcal{C}^1$. Finally, $F : O \rightrightarrows \mathbb{R}^n$ denotes a set-valued map associating each element $x \in O$ to a subset $F(x) \subset O$.

II. Preliminaries and Standing Assumptions

A. Hybrid inclusions

We consider general hybrid inclusions of the form

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \tag{1}$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$, the jump set $D \subset \mathbb{R}^n$, the flow and the jump set-valued maps, respectively, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. A solution x to $\mathcal{H}$ is defined on a hybrid time domain denoted $\text{dom}\, x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ where $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. The solution x is parametrized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a discrete jump variable $j \in \mathbb{N}$. Its domain of definition $\text{dom}\, x$ is such that for each $(T, J) \in \text{dom}\, x$, $\text{dom}\, x \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0}^{J} ([t_j, t_{j+1}], j)$ for a sequence $\{t_j\}_{j=0}^{J+1}$, such that $t_{j+1} \geq t_j$ and $t_0 = 0$; see [14].

Definition 1: (Solution to $\mathcal{H}$) A function $x : \text{dom}\, x \to \mathbb{R}^n$ defined on a hybrid time domain $\text{dom}\, x$ and such that, for each $j \in \mathbb{N}$, $t \mapsto x(t, j)$ is locally absolutely continuous is a *solution* to $\mathcal{H}$ if

(S0) $x(0, 0) \in \text{cl}(C) \cup D$;

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t \ : \ (t, j) \in \text{dom}\, x \ \}$ has nonempty interior

$$\begin{aligned} x(t, j) &\in C && \text{for all} \quad t \in \text{int}(I^j), \\ \dot{x}(t, j) &\in F(x(t, j)) && \text{for a.a.} \quad t \in I^j; \end{aligned} \tag{2}$$

(S2) for all $(t, j) \in \text{dom}\, x$ such that $(t, j+1) \in \text{dom}\, x$,

$$x(t, j) \in D, \qquad x(t, j+1) \in G(x(t, j)). \tag{3}$$

•

A solution x to $\mathcal{H}$ is said to be complete if it is defined on an unbounded hybrid time domain; that is, the set $\text{dom}\, x$ is unbounded. Furthermore, it is said to be maximal if there is no solution y to $\mathcal{H}$ such that $x(t, j) = y(t, j)$ for all $(t, j) \in \text{dom}\, x$ with $\text{dom}\, x$ a proper subset of $\text{dom}\, y$. Finally, it is said to be trivial if $\text{dom}\, x$ contains only one element of the form $(0, 0)$.

B. Forward invariance notions for hybrid inclusions

For a set $K \subset C \cup D$, following [6], we distinguish the two following forward invariance notions.

Definition 2 (forward pre-invariance): The set K is said to be forward pre-invariant if for each $x_o \in K$, each maximal solution x starting from x_o satisfies $x(t, j) \in K$ for all $(t, j) \in \text{dom}\, x$. •

Definition 3 (forward invariance): The set K is said to be forward invariant if for each $x_o \in K$, each maximal solution x starting from x_o is complete and satisfies $x(t, j) \in K$ for all $(t, j) \in \text{dom}\, x$. •

C. Cones

Different types of cones have been used in the study of differential inclusions. In the following, we recall from [7] the definition of some of them, for a closed set $K \subset \mathbb{R}^n$, that are used in this paper.

Definition 4: The *contingent* cone of K at x is given by

$$T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{|x + hv|_K}{h} = 0 \right\}. \tag{4}$$

•

Also, we recall the following equivalence [15, Page 122]:

$$\begin{aligned} &v \in T_K(x) \Leftrightarrow \\ &\quad \exists \{h_i\}_{i \in \mathbb{N}} \to 0^+ \text{ and } \{v_i\}_{i \in \mathbb{N}} \to v : x + h_i v_i \in K. \end{aligned} \tag{5}$$

Definition 5: The *Dubovitsky-Miliutin* cone of K at x is given by

$$D_K(x) := \{ v \in \mathbb{R}^n : \exists \epsilon, \alpha > 0 : x + (0, \alpha](v + \epsilon \mathcal{B}) \subset K \}. \tag{6}$$

•

Also, we recall from [7] the following useful property:

$$D_K(x) = \mathbb{R}^n \backslash T_{\mathbb{R}^n \backslash K}(x) = T_K(x) \backslash T_{\mathbb{R}^n \backslash K}(x) \quad \forall x \in \partial K. \tag{7}$$

D. Basic assumptions

Our results are obtained under the following standing assumptions.

Standing Assumptions. The data (C, F, D, G) of the hybrid inclusion $\mathcal{H}$ is such that the flow map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded on C, $F(x)$ is nonempty and convex for all $x \in C$, and $G(x)$ is nonempty for all $x \in D$. •

Before going further, we state the following general fact.

Fact 1: Consider the hybrid inclusion $\mathcal{H} = (C, F, D, G)$ and a closed set $K \subset C \cup D$. Starting from $x_o \in K$, if a solution x leaves the set K, then it has to be under one of the two following scenarios:

(Sc1) The solution x leaves the set K after a jump. It implies the existence of $(t, j) \in \operatorname{dom} x$ such that $x(t, j) \in K \cap D$ and $(t, j+1) \in \operatorname{dom} x$ with $x(t, j+1) \notin K$ and $x(t, j+1) \in G(x(t, j))$.

(Sc2) The solution x leaves the set K by flowing. It implies the existence of $t_2' > t_1' \geq 0$ and $j \in \mathbb{N}$ such that $([t_1', t_2'], j) \subset \operatorname{dom} x$ and $x((t_1', t_2'), j) \subset (U(\partial K) \backslash K) \cap C$, with $x(t_1', j) \in \partial K$ and $x(t_2', j) \notin K$.

•

In fact, when the set K is closed, according to (Sc2), $x(t_1', 0) \in \partial K \cap K$ and since the solution leaves the set K, under Definition 1, $x((t_1', t_2'), 0)$ is a subset of $C \backslash K$ for some $t_2' > t_1'$ and sufficiently close to t_1'. However, when the set K is not closed, the case in (Sc2) is replaced by the following more general scenario:

(Sc3) The solution x leaves the set K by flowing. It implies the existence of $t_2' > t_1' \geq 0$ and $j \in \mathbb{N}$ such that $([t_1', t_2'], j) \subset \operatorname{dom} x$ and either
  a) $x((t_1', t_2'), j) \subset (U(\partial K) \backslash K) \cap C$, with $x(t_1', j) \in \partial K$ and $x(t_2', j) \notin K$, or
  b) $x([t_1', t_2'), j) \subset K$, with $\lim_{t \uparrow t_2'} x(t, j) \in \operatorname{cl}(K) \backslash K$.

For the case in (Sc3)b, the solution dies on the boundary $\partial K \backslash K$.

III. Sufficient Conditions for Forward Invariance Using Multiple Barrier Functions

Given a hybrid system $\mathcal{H} = (C, F, D, G)$, we consider closed sets K defined in $C \cup D$ and collecting points where multiple scalar functions are simultaneously nonpositive.

Definition 6: For a hybrid system $\mathcal{H}$, a (vector) function $B : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a multiple barrier candidate defining the set K if

$$K = \{x \in C \cup D : B(x) \leq 0\}, \tag{8}$$

where $B(x) := [B_1(x) \ B_2(x) \ \ldots \ B_m(x)]^\top$, $B_i : \mathbb{R}^n \to \mathbb{R}$ for all $i \in \{1, \ldots, m\}$, and $B(x) \leq 0$ means that $B_i(x) \leq 0$ for all $i \in \{1, 2, ..., m\}$. •

If B is continuous, the set K is closed relative to $C \cup D$. In addition, when $C \cup D$ is closed, K is automatically closed. Furthermore, we introduce the following sets that we use in some statements and proofs. Given B, $\mathcal{H}$, and the set K such that (8) holds, define, for any $i \in \{1, 2, ..., m\}$,

$$K_e := \{x \in \mathbb{R}^n : B(x) \leq 0\}, \tag{9}$$

$$K_{ei} := \{x \in \mathbb{R}^n : B_i(x) \leq 0\}, \tag{10}$$

$$\partial K_i := \{x \in \partial K : B_i(x) = 0\}. \tag{11}$$

It is useful to notice that $K_e = \cap_{i=1}^m K_{ei}$, $K = K_e \cap (C \cup D)$, and that $\partial K = \cup_{i=1}^m \partial K_i \cup [\partial K \cap \partial(C \cup D)]$. Note that, in general, $\partial K_i \neq \partial K_{ei}$. Also, we notice that it is possible from (8) to construct a scalar barrier candidate defining the closed set K as

$$\bar{B}(x) := \max_{i \in \{1, \ldots, m\}} B_i(x).$$

However, by doing so, the resulting barrier candidate $\bar{B}$ is not guaranteed to be $\mathcal{C}^1$. Indeed, at points x where multiple B_i's are equal, if their gradients are not identical, then, $\bar{B}$ is not differentiable at these elements.

A. Sufficient conditions for forward pre-invariance

In this section, we present extensions of the results for a particular class of hybrid systems in [12], [11], [8] to general hybrid inclusions. For differential inclusions, as stressed in [7], forward invariance of a set is a property that depends on the dynamics of the system outside the considered set. Due to this, the conditions along the flow of $\mathcal{H}$ need to hold on a neighborhood of the boundary of K, relative to its complement.

Theorem 1: Consider a hybrid system $\mathcal{H}$ and a $\mathcal{C}^1$ multiple barrier function candidate B defining the closed set K as in (8). The set K is forward pre-invariant if, for all $i \in \{1, 2, \ldots, m\}$,

$$\langle \nabla B_i(x), \eta \rangle \leq 0 \qquad \forall x \in (U(\partial K_i) \backslash K_{ei}) \cap C \text{ and } \forall \eta \in F(x) \cap T_C(x), \tag{12}$$

$$B(\eta) \leq 0 \qquad \forall \eta \in G(x) \quad \forall x \in D \cap K, \tag{13}$$

$$G(x) \subset C \cup D \quad \forall x \in D \cap K. \tag{14}$$

□

Example 1: Consider the hybrid system $\mathcal{H}$ as in (1) with the data

$$F(x) := \begin{bmatrix} -x_2^2 \\ x_2 x_1 - x_2([2,4] - |x|^2) \end{bmatrix} \quad \forall x \in C,$$
$$C := \left\{x \in \mathbb{R}^2 : x_2 \geq 0, \ x_1 \in [-1, 1]\right\},$$
$$G(x) := [0, 1] \begin{bmatrix} x_2 \\ |x_1| \end{bmatrix} \quad \forall x \in D,$$
$$D := \left\{x \in \mathbb{R}^2 : x_2 \leq 0, \ |x| < 1\right\}.$$

We establish forward pre-invariance for the closed set $K := \left\{x \in C \cup D : |x|^2 - 1 \leq 0, \ x_2 \geq 0\right\}$. To this end, we start noticing that the set K can be written as in (8) with the multiple $\mathcal{C}^1$ barrier function candidate $B(x) := [(|x|^2 - 1) \ \ -x_2]^\top$. Also, it is easy to see that $G(x) \subset C \cup D$ for all $x \in D \cap K = \{x \in D : x_2 = 0\}$; hence, (14) holds. Moreover, a simple computation shows that, for each $x_o \in (K \cap D)$, $B(G(x_o)) \subset [([0,1]|x|^2 - 1) \quad -[0,1]|x_1|]^\top \subset \mathbb{R}_{\leq 0} \times \mathbb{R}_{\leq 0}$;

thus, (13) holds. Furthermore, the set $(U(\partial K_i)\backslash K_{ei}) \cap C$ is empty when $i = 2$ and can be chosen, when $i = 1$, as

$$(U(\partial K_1)\backslash K_{e1}) \cap C = \{x \in C : |x| \in (1, 2)\}.$$

Consequently,

$$\langle \nabla B_1(x), \eta \rangle \in (0, -x_2^2(4 - |x|^2)) \subset \mathbb{R}_{\leq 0}$$

for all $\eta \in F(x)$ and for all $x \in U(\partial K_1)\backslash K_{e1}) \cap C$. Hence, (12) holds and forward pre-invariance of the set K defined by B follows. □

B. Sufficient conditions for forward invariance

According to Definition 3, a forward pre-invariant set $K \subset C \cup D$ is forward invariant if, in addition, all the maximal solutions starting from that set are complete. Hence, one has to exclude the case of non-complete maximal solutions dying on the set $(K \cap \partial C)\backslash D$, as well as the case of maximal solutions escaping in finite time along flows inside the set $K \cap C$.

Proposition 1: For a hybrid system $\mathcal{H}$, a forward pre-invariant set $K \subset C \cup D$ is forward invariant if the solutions starting from K cannot escape in finite time along the flows inside the set $K \cap C$, and from any initial condition in the set $(K \cap \partial C)\backslash D$, a nontrivial flow exists. □

Remark 1: One can guarantee that the solutions do not have a finite escape time along the flows inside the set $K \cap C$ when, for example, the set $K \cap C$ is compact or when the flow map F is globally bounded in $K \cap C$. •

Example 2: Using Proposition 1, we are able to extend the conclusions in Example 1 and conclude forward invariance of the set K. Indeed, after Example 1, the set K is forward pre-invariant; and since it is compact, it follows that there is no possibility of finite time blow-up along the flows inside $K \cap C$. Hence, the forward invariance follows if we show that from every initial condition in the set $(K \cap \partial C)\backslash D = \{[1 \ \ 0]^\top, [-1 \ \ 0]^\top\}$, a nontrivial flow exists. Indeed, $F(x) = \{0\}$ for all $x \in \{[1 \ \ 0]^\top, [-1 \ \ 0]^\top\}$, hence, the system admits nontrivial constant solutions of the form $x(t, 0) = x_o$ for all $t \geq 0$ and $x_o \in \{[1 \ \ 0]^\top, [-1 \ \ 0]^\top\}$. □

IV. Sufficient Conditions for Contractivity Using Barrier Functions

One possible way to guarantee forward pre-invariance while relaxing the flow conditions to hold only on the boundary of the set K is by considering strict inequalities instead of the weak inequalities in (12)-(13). However, as we show, such strict conditions are much stronger than needed, as they induce a contractivity property for the set K. Roughly speaking, a pre-contractive set is forward pre-invariant and whenever a solution reaches its boundary, it moves back towards the interior. A definition of contractivity for particular sets named $C-$sets can be found in [9] in terms of the *Minkowski* functional (also named *gauge* function) for both differential and difference equations separately, see Definitions 3.3 and 3.4 in [9]. In this section, we propose two different definitions of contractivity notions for general hybrid inclusions. The first definition concerns the particular case of $C-$sets and extends what is proposed in [9] using the Minkowski functional. The second definition concerns general closed sets that are not $C-$sets. In the latter case, the definition is based on the system's behavior after reaching the boundary of the considered closed set. Furthermore, necessary and sufficient conditions in terms of barrier candidates defining the closed set are established.

A. The case of $C-$sets

We recall that a set $K \subset C \cup D$ is said to be a $C-$set if it is compact, convex, and includes the origin in its interior. Furthermore, the corresponding Minkowski functional $\Psi_K : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is given by

$$\Psi_K(x) := \inf\{\mu \geq 0 : \ x \in \mu K\}. \tag{15}$$

Definition 7 (pre-contractivity for $C-$sets): For a hybrid system $\mathcal{H}$, a C-set $K \subset C \cup D$ is said to be pre-contractive if

$$\limsup_{h \to 0^+} \frac{\Psi_K(x + \eta h) - 1}{h} < 0 \quad \forall x \in \partial K \cap C \quad \text{and} \quad \forall \eta \in F(x) \cap T_C(x), \tag{16}$$

$$\Psi_K(\eta) < 1 \quad \forall x \in D \cap K, \ \forall \eta \in G(x). \tag{17}$$

•

Definition 8 (contractivity for $C-$sets): For a hybrid system $\mathcal{H}$, a C-set $K \subset C \cup D$ is said to be contractive if it is pre-contractive and, in addition, starting from each element in the set $(\partial(K \cap C) \cap \partial C)\backslash D$, a nontrivial flow of $\mathcal{H}$ exists. •

The following lemma establishes important consequences of the aforementioned contractivity notions on the behavior of the system's solutions.

Lemma 1: For a hybrid system $\mathcal{H}$, if a $C-$set $K \subset C \cup D$ is pre-contractive (respectively, contractive) according to Definition 7 (respectively, Definition 8), then it is forward pre-invariant (respectively, forward invariant) and, for any $x_o \in \partial K$ and any nontrivial solution x starting from x_o, there exists $T > 0$ and $J \in \mathbb{N}^*$ such that $x(t, j) \in \text{int}(K)$ for all $(t, j) \in \text{dom}\, x \cap [([0, T] \times \{0\}) \cup (\{0\} \times \{0, \ldots, J\})]$, $(t, j) \neq (0, 0)$. □

Next, we propose necessary and sufficient conditions for pre-contractivity using barrier functions.

Theorem 2: For a hybrid system $\mathcal{H}$, the $C-$set $K \subset \text{int}(C \cup D)$ is pre-contractive if and only if there exists a Lipschitz continuous barrier candidate B defining the C-set K as in (8) such that

$$\limsup_{h \to 0^+} \frac{B_i(x + \eta h)}{h} < 0 \quad \forall x \in \partial K_i \cap C \quad \forall i \in \{1, ..., m\} \quad \text{and } \forall \eta \in F(x) \cap T_C(x), \tag{18}$$

$$B(\eta) < 0 \qquad \forall x \in K \cap D \quad \forall \eta \in G(x), \tag{19}$$

$$G(x) \subset C \cup D \quad \forall x \in K \cap D. \tag{20}$$

□

Remark 2: The equivalence in the previous statement is shown in the particular case where the $C-$set K satisfies $K \subset \text{int}(C \cup D)$. However, the same result remains valid, using the same proof, when K is a C-set satisfying $K \subset (\text{int}(C) \cup D) \backslash (\partial C \cap \partial D)$ and the following extra jump condition holds:

$$B(\eta) \not< 0 \quad \forall \eta \in G(x) \cap \partial(C \cup D) \quad \forall x \in K \cap D. \quad (21)$$

Indeed, under (19)-(20), having (21) satisfied is important to conclude that the solutions starting from ∂K cannot jump towards $\partial K \cap \text{int}(K_e) \subset \partial(C \cup D)$. Otherwise, it is possible to have $B(x) < 0$ while $x \in \text{int}(K_e) \cap \partial(C \cup D) \subset \partial K$. •

Remark 3: In the general case where $K \not\subset (\text{int}(C) \cup D) \backslash (\partial C \cap \partial D)$, we cannot guarantee for a nontrivial solution flowing from $x_o \in K \cap \partial C$ to satisfy $x([0, \epsilon], x_o) \subset \text{int}(K)$ for some $\epsilon > 0$, since the solution can flow in ∂C while remaining in $\text{int}(K_e)$. In other words, the barrier candidate does not define the set K on any neighborhood of $C \cup D$ as opposed to the Minkowski functional which defines the set K in $\mathbb{R}^n$. Therefore, in order to extend Theorem 2 to the general case where $K \subset C \cup D$, we need to guarantee, additionally, that there is no possibility of flowing in $\partial C \cap \partial K$ while flowing in $\text{int}(K_e)$. The latter fact cannot be characterized in terms of a general barrier candidate defining the set according to (8). •

Example 3: Consider the hybrid system

$$\begin{aligned} F(x) &:= \begin{bmatrix} -[1,2]x_1 + (1/2)x_2 \\ -x_2 - (1/2)x_1 \end{bmatrix} \quad \forall x \in C, \\ C &:= \left\{ x \in \mathbb{R}^2 : x_2 \geq -1 \right\}, \\ G(x) &:= [0, 1/2] \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \quad \forall x \in D, \\ D &:= \left\{ x \in \mathbb{R}^2 : x_2 \leq -1 \right\} \backslash \left\{ [-1 \quad -1]^\top \right\}. \end{aligned}$$

We study the pre-contractivity of the C-set

$$K := \left\{ x \in C \cup D : x_1^2 + x_2^2 \leq 2, \ x_2 \geq -1 \right\},$$

which can be defined according to (8) using the $\mathcal{C}^1$ barrier candidate $B(x) := [|x|^2 - 1 \quad -(x_2+1)]^\top$. That is, we note that the C-set K satisfies $K \subset \text{int}(C \cup D)$; hence, Theorem 2 is applicable. Indeed, since the candidate B is continuously differentiable, it follows that $\limsup_{h \to 0^+} \frac{B_i(x+\eta h)}{h} = \langle \nabla B_i(x), \eta \rangle$ for all $i \in \{1, 2\}$. Furthermore, $\langle \nabla B_1(x), \eta \rangle \in [-x_1^2 + x_2^2, -2x_1^2 + x_2^2] \subset \mathbb{R}_{<0}$ for all $\eta \in F(x)$ and for all $x \in \partial K_1 \cap C = \{x \in \partial K : x_2 \geq -1\}$. Similarly, $\langle \nabla B_2(x), \eta \rangle = x_2 + (1/2)x_1 = -1 + (1/2)x_1 \leq -1/2 < 0$ for all $x \in \partial K_2 \cap C = \left\{ x \in \mathbb{R}^2 : x_2 = -1, \ \ |x_1| \leq 1 \right\}$ and for all $\eta \in F(x)$. Moreover, $G(x) = [0, 1/2][x_1 \quad 1]^\top \subset [-1/2, 1/2] \times [0, 1/2]$ for all $\eta \in G(x)$ and for all $x \in K \cap D = \left\{ x \in \mathbb{R}^2 : x_2 = -1, \ \ |x_1| \leq 1 \right\}$. Hence, $B_1(\eta) < 0$ and $B_2(\eta) < 0$. Hence, the pre-contractivity of the set K follows using Theorem 2. □

The previous sufficient conditions can be complemented in order to conclude contractivity rather than only pre-contractivity. That is, in the following, we propose sufficient qualitative conditions allowing the existence of nontrivial flows starting from any element in the set $(\partial(K \cap C) \cap \partial C) \backslash D$ as required in Definition 8.

Proposition 2: For a hybrid system $\mathcal{H}$, a C-set $K \subset C \cup D$ is contractive if it is pre-contractive and

$$\begin{aligned} F(x) \cap T_C(x) \neq \emptyset \quad & \forall x \in U(x_o) \cap \partial(K \cap C) \cap \partial C, \\ & \forall x_o \in (\partial(K \cap C) \cap \partial C) \backslash D. \quad (22) \end{aligned}$$

□

Example 4: We propose to build upon the pre-contractivity conclusions in Example 3 in order to conclude contractivity, using Proposition 2. To do so, it is enough to show that the set $(\partial(K \cap C) \cap \partial C) \backslash D$ satisfies (22). Indeed, the set $(\partial(K \cap C) \cap \partial C) \backslash D$ reduces to the singleton $\left\{ [-1 \quad -1]^\top \right\}$ and the neighborhood $U(x) \cap \partial(K \cap C) \cap \partial C$ can be chosen as $U(x) \cap \partial(K \cap C) \cap \partial C = \left\{ x \in \mathbb{R}^2 : x_2 = -1, \ x_1 \in [-1, -1/2) \right\}$ on which $F(x) = [-[1,2]x_1 - 1/2 \quad 1 - (1/2)x_1]^\top$; hence, (22) follows since $1 - (1/2)x_1 > 0$. □

B. The case of non-C closed sets

For general closed sets, we cannot use the Minkowski functional in order to define contractivity notions. Consequently, in the case of closed sets that are not C-sets, the following alternative definitions are proposed in order to preserve the properties in Lemma 1.

Definition 9 (Contractivity for non-C closed sets): For a hybrid system $\mathcal{H}$, a closed set $K \subset C \cup D$ is said to be pre-contractive (respectively, contractive) if it is forward pre-invariant (respectively, invariant) and for each $x_o \in \partial K$ and a nontrivial solution x starting from x_o, there exist $T > 0$ and $J \in \mathbb{N}^*$ such that $x(t,j) \in \text{int}(K)$ for all $(t,j) \in \text{dom}\, x \cap [([0,T] \times \{0\}) \cup (\{0\} \times \{0, \ldots, J\})]$ and $(t,j) \neq (0,0)$. •

Remark 4: It is useful to notice that, in the particular case of differential inclusions, the pre-contractivity of a closed set $K \subset O$ reduces to the nonexistence of any solution x starting from any $x_o \in \partial K$ such that $x([0,T], x_o) \subset \mathbb{R}^n \backslash \text{int}(K)$ for some $T > 0$. Pre-contractivity is also named *strict invariance* in [7]. •

Next, we propose to characterize contractivity notions using barrier functions. Our approach is mainly based on the characterization of the tangent cone $D_{\text{int}(K)}$ on the boundary of the considered closed set using the barrier candidate defining the set. Furthermore, the latter fact is combined with [7, Theorem 4.3.4] to certify the system's behavior required in contractivity notions.

Theorem 3: Consider a hybrid system $\mathcal{H}$ and a $\mathcal{C}^1$ barrier function candidate B defining the closed set K as in (8). The set K is pre-contractive if

$$\begin{aligned} \langle \nabla B_i(x), \eta \rangle < 0 \ \ \forall x \in \partial K_i \cap C \ \ \forall i = 1, 2, ..., m \\ \forall \eta \in F(x) \cap T_C(x), \quad (23) \end{aligned}$$

$$F(x) \cap T_{\partial C \cap \partial K}(x) = \emptyset \ \ \forall x \in \partial K \cap \partial C, \quad (24)$$

$$B(\eta) < 0 \qquad \forall x \in K \cap D \quad \forall \eta \in G(x), \tag{25}$$
$$G(x) \subset C \cup D \qquad \forall x \in K \cap D, \tag{26}$$
$$G(x) \subset \text{int}(C \cup D) \qquad \forall x \in \partial K \cap D. \tag{27}$$

□

Example 5: Consider the hybrid system

$$F(x) := \begin{bmatrix} -(x_2+1) \\ -2(x_2+1)+x_1 \end{bmatrix} \quad \forall x \in C,$$
$$C := \left\{ x \in \mathbb{R}^2 : x_2 \in [0,1],\ x_1 \in \sqrt{3}[-1,1] \right\},$$
$$G(x) := \frac{1}{\sqrt{3}} \begin{bmatrix} x_1 \\ \frac{\sqrt{3}}{2} \end{bmatrix} \quad \forall x \in D,$$
$$D := \left\{ x \in \mathbb{R}^2 : x_2 \geq 0,\ x_1 \in \sqrt{3}[-1,1] \right\}.$$

We study the pre-contractivity of the set $K := \left\{ x \in \mathbb{R}^2 : x_1^2 + (x_2+1)^2 \leq 4, \quad x_2 \geq 0 \right\}$, which can be defined using the $\mathcal{C}^1$ barrier function candidate $B(x) := [x_1^2 + (x_2+1)^2 - 4 \quad -x_2]^\top$. Indeed, we start noticing that the set K is not a $C-$set and does not satisfy $K \subset \text{int}(C \cup D)$; hence, we will use Theorem 3 in order to analyze its pre-contractivity. Indeed, in order to satisfy the jump conditions (25)-(27), we notice that, $G(x) = [\frac{x_1}{\sqrt{3}} \quad \frac{1}{2}]^\top \subset \text{int}(C \cup D)$ for all $x \in K \cap D = \left\{ x \in \mathbb{R}^2 : x_2 = 0, \quad |x_1| \leq \sqrt{3} \right\}$ since $(1/\sqrt{3})x_1 \leq 1$; and hence, (26)-(27) hold. Moreover, it is clear to see that $B(G(x)) = [x_1^2/3 - 7/4 \quad -1/2]^\top < 0$ for all $x \in K \cap D$; since $|x_1| \leq \sqrt{3}$; and hence, (25) also follows. In order to conclude the pre-contractivity, it remains to show that (24) and (23) are both satisfied. Indeed, we have $\langle \nabla B_1(x), F(x) \rangle = -(x_2+1)^2 < 0$ for all $x \in \partial C \cap \partial K_1 = \left\{ x \in \mathbb{R}^2 : x_2 \geq 0,\ |[x_1 \quad x_2+1]| = 2 \right\}$. Furthermore, for any $x \in \partial K_2 \cap C = \left\{ x \in \mathbb{R}^2 : x_2 = 0, \quad |x_1| \leq \sqrt{3} \right\}$, $F(x) = [-1 \quad -2+x_1]^\top \notin T_C(x)$ since $-2+x_1 < 0$ for all $x_1 \in \partial K_2 \cap C$. Hence, (23) is satisfied. Furthermore, in order to show (24), we notice that $\partial K \cap \partial C = \left\{ [0 \quad 1]^\top \right\} \cup \partial K_2$. Moreover, when $x = [0 \quad 1]^\top$, $F(x) = [-2 \quad -2]^\top \notin T_C(x)$ and $x \in \partial K_2$, we have already shown that $F(x) \notin T_C(x)$. The two latter facts together allow to conclude that $F(x) \notin T_{\partial C \cap \partial K}(x)$ for all $x \in \partial K \cap \partial C$. Hence, the pre-contractivity of the set K follows. □

In the sequel, we propose extra conditions to conclude contractivity rather than only pre-contractivity.

Proposition 3: For a hybrid system $\mathcal{H}$, a pre-contractive set $K \subset C \cup D$ is contractive provided that the solutions do not escape in finite time along the flows inside $K \cap C$ and, starting from each element in $(\partial(K \cap C) \cap \partial C) \backslash D$, a nontrivial flow exists. □

In the following statement, as in Proposition 2, the existence of a nontrivial flow starting from $(\partial(K \cap C) \cap \partial C) \backslash D$ is expressed using a qualitative condition.

Proposition 4: For a hybrid system $\mathcal{H}$, consider a $\mathcal{C}^1$ barrier function candidate defining the closed set K as in (8). The set K is contractive if (23)-(27) hold and

$$F(x) \cap T_C(x) \neq \emptyset \quad \forall x \in U(x_o) \cap \partial(K \cap C) \cap \partial C,$$
$$\forall x_o \in (\partial(K \cap C) \cap \partial C) \backslash D. \tag{28}$$

□

Example 6: Let us reconsider the case study in Example 5. In order to conclude contractivity for the set K according to Proposition 4, we need to show that there is no possibility of a finite escape along the flows inside the set $K \cap C$, which is the case since the set K is compact. Also, (28) has to hold for all $x \in (\partial(K \cap C) \cap \partial C) \backslash D = \{(0,1)\}$. Indeed, notice that the neighborhood $U(x) \cap \partial(K \cap C) \cap \partial C$ is reduced to the element $x = \left\{ [1 \quad 0]^\top \right\}$ with $F(x) = [-2 \quad -4]^\top \in T_C(x)$, which concludes the contractivity of the set K. □

V. Conclusion

This paper proposed new characterizations of forward invariance and contractivity for closed sets for general hybrid inclusions. The considered closed sets are defined using a vector of barrier functions. The proposed conditions are alternatives to the existing cone-based conditions and those involving only a scalar barrier function.

References

[1] M. Maghenem and R. G. Sanfelice, "Barrier function certificates for forward invariance in hybrid inclusions," in *In proceedings of the 56th IEEE Conference on Decision and Control (CDC)*, pp. 759–764, 2018.

[2] A. Loría, E. Panteley, D. Popovic, and A. R. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 183–198, 2005.

[3] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[4] M. Nagumo, "Über die lage der integralkurven gewöhnlicher differentialgleichungen," *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, vol. 24, pp. 551–559, 1942.

[5] J. P. Aubin, L. Lygeros, M. Quincampoix, S. Sastry, and N. Seube, "Impulse differential inclusions: A viability approach to hybrid systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 1, pp. 2–20, 2002.

[6] J. Chai and R. G. Sanfelice, "Forward invariance of sets for hybrid dynamical systems (Part I)," *IEEE Transactions on Automatic Control*, pp. 1–1, 2019.

[7] J. P. Aubin, *Viability Theory*. Cambridge, MA, USA: Birkhauser Boston Inc., 1991.

[8] S. Prajna, A. Jadbabaie, and G. J. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, 2007.

[9] F. Blanchini, "Survey paper: Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[10] S. Prajna and A. Rantzer, "On the necessity of barrier certificates," *IFAC Proceedings Volumes*, vol. 38, no. 1, pp. 526–531, 2005.

[11] P. Wieland and F. Allgöwer, "Constructive safety using control barrier functions," *IFAC Proceedings Volumes*, vol. 40, no. 12, pp. 462–467, 2007.

[12] S. Prajna, *Optimization-based methods for nonlinear and hybrid systems verification*. PhD thesis, California Institute of Technology, 2005.

[13] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, vol. 178. Springer Science & Business Media, 2008.

[14] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

[15] J. P. Aubin and H. Frankowska, *Set-valued analysis*. Springer Science & Business Media, 2009.