Multiple Barrier Function Certificates for Weak Forward Invariance in Hybrid Inclusions

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***Abstract*— Weakly forward (pre-)invariant sets guarantee the existence of at least one maximal solution, when starting from any point in the set, that stays in that set. As a continuation to prior works and using multiple barrier functions, this paper studies weak forward invariance in hybrid systems modeled by constrained inclusions. We propose sufficient conditions to guarantee weak forward invariance of a closed set generated by the intersection of the zero-sublevel sets of the different components of a (vector) function called *multiple barrier function*. Our sufficient conditions are in terms of the multiple barrier function generating the set. Moreover, along the flow part of the hybrid system, our conditions are of two types. The first type of flow conditions need to hold only at the boundary of the set and weak forward invariance is shown by imposing *transversality conditions* on the intersection between the zero-sublevel sets of the components of the barrier function. The second type of conditions require both transversality and flow conditions on an external complement of the boundary of the considered set. Examples throughout the paper illustrate the results.**

I. Introduction

Weak forward invariance for dynamical systems with non-unique solutions is a useful property in many applications. For example, it enables to conclude attractivity properties using invariance principles [1]. It is also useful to assure the existence of feedback controllers that solve constrained control problems [2], [3].

Sufficient conditions for weak forward invariance for continuous-time systems are usually established using cone-based conditions [4]. This is already the case in the well-known result by Nagumo [5] for ordinary differential equations. In [2], similar conditions are proposed for differential inclusions provided that some regularity properties on the set-valued dynamics are satisfied; see [2, Proposition 3.4.2]. When the set under consideration is defined as the intersection of zero-sublevel sets of a family of scalar functions, the simplest approach to conclude weak forward invariance, while avoiding the sometimes numerically expensive computation of the contingent cones, consists of finding alternative expressions for the contingent cones using these functions. This approach is studied in [6] for the case of only one scalar function defining the set. In the general case where multiple scalar functions define the considered set, those functions form a vector function called *barrier candidate*. However, when using this approach, we immediately notice that the points on the boundary of the considerd set belonging to the zero-level set of more than one component of the multiple barrier candidate need to be treated carefully. Indeed, the contingent cone at those points depends on the geometry of the intersection between the different zero-level sets. Unfortunately, the intersection of the contingent cones with respect to the different zero-sublevel set is not necessarily the contingent cone with respect to the set defined as the intersection of those sublevel sets. To handle such a situation, one can assume a suitable intersection between the different zero-sublevel sets defining the considered set. Those additional conditions are named *transversality conditions* in [2], [7]. In [8], [9], the tightest transversality conditions are investigated in order to allow the intersection of the contingent cones with respect different sets to be the contingent cone of the intersection of those sets.

In this paper, we propose sufficient conditions for weak forward invariance of sets for general hybrid inclusions. The sufficient conditions imposed on the continuous-time dynamics (referred to as*flows*) fall into two categories. In the first category, the flow conditions concern only the boundary of the set to be rendered weakly invariant, which is denoted as K. The results, in this case, hold only when the intersections of the zero-sublevel sets satisfy the aforementioned transversality conditions. In the second category, the transversality and the flow conditions are imposed on an external complement of the boundary of K. To the best of our knowledge, this second characterization is new and offers the advantage of handling the situation where the gradient of the barrier function has zero components at the boundary of the set.

The remainder of the paper is organized as follows. Preliminaries and basic conditions are presented in Section II. Sufficient conditions for weak forward pre-invariance using barrier functions are in Section III. Due to space limitations, the proofs are omitted and will appear elsewhere.

Notation. Let $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. For x, $y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the Euclidean norm of x, $|x|_K := \inf_{y \in K} |x - y|$ defines the distance between x and the nonempty set K, and $\langle x, y \rangle = x^\top y$ the inner product between x and y. The inequalities $x \leq 0$ and $x < 0$ mean that $x_i \leq 0$ and $x_i < 0$, respectively, for all $i \in \{1, 2..., n\}$. The inequalities $x \nleq 0$ and $x \nless 0$ mean that there exists $i \in \{1, 2, \ldots, n\}$ such that $x_i > 0$ and,

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respectively, $x_i \geq 0$. For a closed set $K \subset \mathbb{R}^n$, we use int(K) to denote its interior, ∂K its boundary, cl(K) its closure, and $U(K)$ to denote an open neighborhood of K. For $O \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O. The open unit ball in $\mathbb{R}^n$ centered at the origin is denoted $\mathcal{B}$. The set of continuously differentiable functions is denoted by $\mathcal{C}^1$. For a $\mathcal{C}^1$ function $B : \mathbb{R}^n \to \mathbb{R}$, $\nabla B(x)$ denotes the gradient of the function B evaluated at x. Finally, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ denotes a set-valued map associating each element $x \in \mathbb{R}^n$ to a set $F(x) \subset \mathbb{R}^m$ and $F^{-1}(y) := \{x \in \mathbb{R}^n : F(x) = y\}$ is the reciprocal image of $y \in \mathbb{R}^m$ by the map F.

II. Preliminaries

A. Hybrid inclusions

According to [10], a general hybrid inclusions is given by

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \tag{1}$$

with state variable $x \in \mathbb{R}^n$, flow set $C \subset \mathbb{R}^n$, jump set $D \subset \mathbb{R}^n$, flow and the jump (set-valued) maps, respectively, $F : C \rightrightarrows \mathbb{R}^n$ and $G : D \rightrightarrows \mathbb{R}^n$. A solution x to $\mathcal{H}$ is defined on a hybrid time domain denoted $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$. The solution x is parametrized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a discrete jump variable $j \in \mathbb{N}$. Its domain of definition $\operatorname{dom} x$, is such that for each $(T, J) \in \operatorname{dom} x$, $\operatorname{dom} x \cap ([0, T] \times \{0, 1, \dots, J\}) = \cup_{j=0}^{J} ([t_j, t_{j+1}] \times j)$ for a sequence $\{t_j\}_{j=0}^{J+1}$, such that $t_{j+1} \geq t_j$ and $t_0 = 0$.

Definition 1: (solution to $\mathcal{H}$) A function $x : \operatorname{dom} x \to \mathbb{R}^n$ defined on a hybrid time domain $\operatorname{dom} x$ is a *solution* to $\mathcal{H}$ if

(S0) $x(0, 0) \in \text{cl}(C) \cup D$;

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t \ : \ (t, j) \in \operatorname{dom} x \ \}$ has nonempty interior, the function $t \mapsto x(t, j)$ is locally absolutely continuous, and

$$\begin{aligned} &x(t, j) \in C && \text{for all} && t \in \text{int}(I^j), \\ &\dot{x}(t, j) \in F(x(t, j)) && \text{for almost all} && t \in I^j; \end{aligned} \tag{2}$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \qquad x(t, j + 1) \in G(x(t, j)). \tag{3}$$

•

A solution x to $\mathcal{H}$ is said to be complete if $\operatorname{dom} x$ is unbounded. It is said to be trivial if $\operatorname{dom} x$ is a singleton. It is said to be maximal if there is no solution y to $\mathcal{H}$ such that $x(t, j) = y(t, j)$ for all $(t, j) \in \operatorname{dom} x$ with $\operatorname{dom} x$ a proper subset of $\operatorname{dom} y$.

B. Weak forward invariance in hybrid systems

For a set $K \subset C \cup D$, following [4], we present the weak forward invariance notions considered in this paper.

Definition 2 (Weak forward invariance): The set K is said to be weakly forward pre-invariant if for each $x_o \in K$, at least one maximal solution x starting from x_o satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. Furthermore, it is said to be weakly forward invariant if for each $x_o \in K$, at least one maximal solution x starting from x_o is complete and satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. •

Characterizing weak forward invariance is useful in many situations. We cite here two examples and we refer the reader to [2] for more examples.

- The first example concerns the analysis of stability using invariance principle for *well-posed* hybrid systems [1, Theorem 4.7]. Consider the system $\mathcal{H}$ and assume that there exists a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $u_C(x) \leq 0$ and $u_D(x) \leq 0$ for all $x \in \mathbb{R}^n$, where

$$u_C(x) := \begin{cases} \max_{s \in F(x)} V^\circ(x, s) & \text{if } x \in C \\ -\infty & \text{otherwise,} \end{cases}$$
$$u_D(x) := \begin{cases} \max_{\zeta \in G(x)} (V(\zeta) - V(x)) & \text{if } x \in D \\ -\infty & \text{otherwise,} \end{cases}$$

  and $V^\circ(x, s) := \max_{\zeta \in \partial V(x)} \langle \zeta, s \rangle$ with $\partial V(x)$ the generalized gradient of V at x in the sense of [11]. Then, any solution ϕ that is bounded and complete approaches the largest weakly (forward and backward) invariant set inside the set $V^{-1}(r) \cap \left(u_C^{-1}(0) \cup \left(u_D^{-1}(0) \cap G(u_D^{-1}(0))\right)\right)$ for some $r \geq 0$. Hence, finding such a set and showing that it is weakly forward invariant is important in order to conclude where solutions converge to.
- Another example where analyzing weak forward invariance is useful arises when controlling constrained systems of the form

$$\mathcal{H}_u : \begin{cases} x \in C, \ u_c \in \mathcal{U}_c(x) & \dot{x} = f(x, u_c) \\ x \in D, \ u_d \in \mathcal{U}_d(x) & x^+ = g(x, u_d), \end{cases} \tag{4}$$

  where $\mathcal{U}_c(x)$ and $\mathcal{U}_d(x)$ are state dependant input constraints. When designing feedback controllers $x \mapsto u_c(x)$ and $x \mapsto u_d(x)$ such that the solutions to the closed-loop system remain in a given closed set $K \subset C \cup D$, it is important to show that the set K is at least weakly forward invariant for the following hybrid inclusion:

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in f(x, \mathcal{U}_c(x)) \\ x \in D & x^+ \in g(x, \mathcal{U}_d(x)). \end{cases} \tag{5}$$

C. Background

For a set $K \subset \mathbb{R}^n$, we recall from [2] the different types of cones used in this paper.

- The *contingent* cone of K at x is given by

$$T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{|x + hv|_K}{h} = 0 \right\}. \tag{6}$$

- The *Clarke tangent* cone of K at x is given by

$$C_K(x) := \left\{ v \in \mathbb{R}^n : \limsup_{y \to x, h \to 0^+} \frac{|y + hv|_K}{h} = 0 \right\}. \tag{7}$$

- The *Dubovitskii-Milyutin* cone of K at x is given by

$$D_K(x) := \{v \in \mathbb{R}^n : \exists \epsilon > 0 : x + (0, \epsilon](v + \epsilon \mathcal{B}) \subset K\}. \quad (8)$$

We, also, recall from [2] that, for all $x \in \partial K$,

$$D_K(x) = \mathbb{R}^n \backslash T_{\mathbb{R}^n \backslash K}(x) = T_K(x) \backslash T_{\mathbb{R}^n \backslash K}(x). \quad (9)$$

The projection map with respect to the set K, $\Pi_K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, is given by

$$\Pi_K(x) := \left\{ y \in K : y = \arg\min_{z \in \partial K} |z - x| \right\}.$$

Next, the pre-image of the projection map is defined as

$$\Omega_K(p_1) := \left\{ p \in \mathbb{R}^n \backslash K : p_1 = \arg\min_{x \in \partial K} |p - x| \right\}. \quad (10)$$

The corresponding set of unitary directions is defined as

$$J_K(p_1) := \{p - p_1 : p \in \Omega_K(p_1),\ |p - p_1| = 1\}. \quad (11)$$

The following lemma shows that $J_K(x) \subset D_{\mathbb{R}^n \backslash K}(x)$ for all $x \in \partial K$ and, when the variable x sweeps the boundary ∂K, the set $x + J_K(x)$ generates an external neighborhood of ∂K; i.e $U(\partial K) \backslash K \subset \cup_{x \in \partial K}(x + J_K(x))$.

Lemma 1: For each $p_1 \in \partial K$,

1) $J_K(p_1)$ is compact,
2) $J_K(p_1) \cap T_K(p_1) = \emptyset$,
3) $\forall h > 0$, $\exists U_h(K)$ such that

$$U_h(K) \subset K \cup [\cup_{p_1 \in \partial K} (p_1 + [0, h] J_K(p_1))].$$

□

Although the set-valued map J_K enjoys the useful properties listed in Lemma 1, it might be empty at some elements of the boundary ∂K. As an example, consider the set $K := \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}$. Indeed, from the definition of J_K in (11), it is easy to see that $J_K(x_o := (0, 0)) = \emptyset$.

III. Sufficient Conditions for Weak Forward Pre-invariance

Given a hybrid system $\mathcal{H} = (C, F, D, G)$, we consider sets K, subsets of $C \cup D$, collecting points where multiple scalar functions are simultaneously nonpositive.

Definition 3: For a hybrid system $\mathcal{H}$, a (vector) function $B : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a multiple barrier candidate defining the set K if [1]

$$K = \{x \in C \cup D : B(x) \leq 0\}, \quad (12)$$

where $B(x) := [B_1(x)\ B_2(x)\ \ldots\ B_m(x)]^\top$, $B_i : \mathbb{R}^n \to \mathbb{R}$ for all $i \in \{1, 2, \ldots, m\}$. •

If B is continuous, the set K is closed relative to $C \cup D$. In addition, when $C \cup D$ is closed, K is automatically closed. Furthermore, we introduce the following sets that we use in the statements and their proofs.

[1] $B(x) \leq 0$ means that $B_i(x) \leq 0$ for all $i \in \{1, 2, ..., m\}$.

Given B as in Definition 3, $\mathcal{H}$, and the set K such that (12) holds, define

$$K_e := \{x \in \mathbb{R}^n : B(x) \leq 0\}, \quad (13)$$

and, for each $i \in \{1, 2, \ldots, m\}$,

$$K_{ei} := \{x \in \mathbb{R}^n : B_i(x) \leq 0\}, \quad (14)$$

$$M_i := \{x \in \partial K : B_i(x) = 0\}. \quad (15)$$

It is useful to notice that $K_e = \cap_{i=1}^m K_{ei}$, $K = K_e \cap (C \cup D)$, and that $\partial K = \cup_{i=1}^m M_i \cup [\partial K \cap \partial(C \cup D)]$. Note that, in general, $M_i \neq \partial K_{ei}$.

Remark 1: From (12), it is possible to construct a scalar barrier candidate defining the closed set K as the zero sublevel set of

$$\bar{B}(x) := \max_{i \in \{1, 2, \ldots, m\}} B_i(x).$$

However, by doing so, the resulting barrier candidate $\bar{B}$ is not guaranteed to be $\mathcal{C}^1$. Indeed, for $i, k \in \{1, 2, \ldots, m\}$, $i \neq k$, and x such that $B_i(x) = B_k(x)$, if the gradients $\nabla B_i(x)$ and $\nabla B_k(x)$ are not equal, then $\bar{B}(x)$ is not differentiable. •

Our results are obtained under the following standing assumptions.

Standing assumptions. The data (C, F, G, D) of $\mathcal{H}$ is such that the flow map F is outer semicontinuous and locally bounded on cl(C), $F(x)$ is nonempty and convex for all $x \in$ cl(C), and $G(x)$ is nonempty for all $x \in D$. •

A. Sufficient Conditions Under Non-vanishing Gradients

In the following result, the flow condition is a reinterpretation of the well-known Nagumo condition using multiple barrier function candidates. The proposed interpretation is valid only under a following transversality condition at the intersections between the different zero-sublevel sets defining the set K. Due to the possible intersections between the zero-level sets of the components of the barrier function and the boundary of C, the following general transversality condition is considered at the intersection points $x \in \partial K$ to be specified in the upcoming statements of our results.

Assumption 1: There exists $v \in C_C(x)$ (C_C is the Clarke cone with respect to the set C) such that if $x \in \partial K_i$ for some $i \in \{1, 2, \ldots, m\}$, then

$$\langle \nabla B_i(x), v \rangle < 0. \quad (16)$$

•

When $x \in \partial K \cap$ int(C), then $C_C(x)$ in Assumption 1 reduces to $\mathbb{R}^n$. Note that it is possible to have $x \in \partial K \cap \partial C$ and at the same time $B_i(x) < 0$ for all $i \in \{1, 2, \ldots, m\}$, in which case, Assumption 1 holds trivially.

Theorem 1: Consider a hybrid system $\mathcal{H}$ and a $\mathcal{C}^1$ barrier function candidate B defining the closed set K as in (12). The set K is weakly forward pre-invariant if the following conditions hold:

1) For any $x \in (\partial K \backslash D) \cap \text{int}(C)$, Assumption 1 holds and, for any $x_1 \in U(x) \cap \partial K \cap C$ and any $i \in \{1, 2, \ldots, m\}$ such that $x_1 \in M_i$,

$$\exists \eta \in F(x_1) \text{ such that } \langle \nabla B_i(x_1), \eta \rangle \leq 0. \tag{17}$$

2) For any $x \in K \backslash \text{cl}(C)$,

$$\exists \eta \in G(x) \cap (C \cup D) \text{ such that } B(\eta) \leq 0. \tag{18}$$

3) For any $x \in \partial K \cap \text{int}(C) \cap D$, either (18) holds, or Assumption 1 holds and, for any $x_1 \in U(x) \cap \partial K \cap C$ and $i \in \{1, 2, \ldots, m\}$ such that $x_1 \in M_i$, (17) holds.

4) For any $x \in \partial K_e \cap \partial C$,
   - If $F(x) \cap T_C(x) = \emptyset$ and $x \in D$, (18) holds.
   - If $F(x) \cap T_C(x) \neq \emptyset$, then either (18) holds, or both Assumption 1 and the following conditions hold:

$$F(x_1) \cap T_C(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial(K \cap C) \cap \partial C \tag{19}$$

   and, for any $x_1 \in U(x) \cap \partial K_e \cap \text{cl}(C)$ and for each $i \in \{1, 2, \ldots, m\}$ such that $x_1 \in M_i$,

$$\exists \eta \in F(x_1) \cap T_C(x_1) \text{ s.t. } \langle \nabla B_i(x_1), \eta \rangle \leq 0. \tag{20}$$

5) For any $x \in \text{int}(K_e) \cap \partial C \cap D$, either (18) or the following condition holds:

$$F(x_1) \cap T_C(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial C. \tag{21}$$

□

Remark 2: According to the conditions in Theorem 1, for each initial condition in $(\partial K \backslash D) \cap \text{int}(C)$, there exists a nontrivial flow that remain in K since a nontrivial flow always exists and jumps are not allowed. Such a property is guaranteed by 1). Furthermore, for each initial condition in $\partial K \cap \text{int}(C) \cap D$, $\text{int}(K_e) \cap \partial C \cap D$, or $K \backslash \text{cl}(C) \subset D$, since there exists at least one nontrivial solution that jumps, we need to guarantee that either there exists a nontrivial flow that remains in K or a jump towards the set K. Such a property is guaranteed by 2), 3), and 5). Finally, for each initial condition in $\partial K_e \cap \partial C$, if there is not a possibility of flowing but there is a possibility of a jump, then at least one of the solutions must jump towards the set K. However, if a nontrivial flow may exist, then, either there exists a nontrivial flow that remain in K or a jump towards K. Such a property is guaranteed by 4). •

Example 1: Consider the hybrid system

$$\begin{aligned} F(x) &:= \begin{bmatrix} -[0,1] x_2 x_1^2 \\ x_1^3 - 2[-1,1] x_2 \end{bmatrix} \quad \forall x \in C, \\ C &:= \left\{ x \in \mathbb{R}^2 : x_1^2 + \frac{1}{2} x_2^2 \leq 1 \right\}, \\ G(x) &:= \begin{bmatrix} x_1 \\ [0,1] \end{bmatrix} \quad \forall x \in D, \\ D &:= \left\{ x \in \mathbb{R}^2 : x_2 = x_1 \right\}. \end{aligned}$$

We will show that the set

$$K := \left\{ x \in C \cup D : \begin{bmatrix} -x_1 x_2 \leq 0 \\ (x_2 - x_1)(x_1 + x_2) \leq 0 \\ x_1^2 + x_2^2 - 1 \leq 0 \end{bmatrix} \right\} \tag{22}$$

is weakly forward pre-invariant. First, the set K admits the $\mathcal{C}^1$ barrier function candidate $B(x) := [-x_1 x_2 \quad (x_2 - x_1)(x_1 + x_2) \quad x_1^2 + x_2^2 - 1]^\top$. Note that $(\partial K \backslash D) \cap \text{int}(C) = S_1 \cup S_2$, where

$$S_1 := \left\{ x \in \mathbb{R}^2 : x_2 = 0, \quad |x_1| < 1 \right\}, \tag{23}$$

and

$$S_2 := \left\{ x \in \mathbb{R}^2 : \sqrt{2}/2 < |x_1| < 1, \quad |x| = 1 \right\}. \tag{24}$$

Assumption 1 with $C_C(x)$ therein replaced by $\mathbb{R}^n$ is satisfied for all $x_o \in \partial K \backslash \left\{ [0 \quad 0]^\top \right\}$. Indeed, for all $x_o \in \partial K \backslash \{0\}$, there exists a unique $i \in \{1, 2, 3\}$ such that $B_i(x_o) = 0$ and $\nabla B_i(x_o) \neq 0$; thus, Assumption 1 holds. Otherwise, for every $x_o \in \left\{ [\pm 1 \quad 0]^\top \right\}$, $B_1(x_o) = B_3(x_o) = 0 \neq B_2(x_o)$ and for every $x_o \in \left\{ \pm\sqrt{2} [1 \quad 1]^\top \right\}$, $B_2(x_o) = B_3(x_o) = 0 \neq B_1(x_o)$. At the latter points, $\nabla B_i(x_o) \neq 0$ for all $i \in \{1, 2, 3\}$ such that $B_i(x_o) = 0$. Consequently, for every $x_o \in \partial K \backslash \{0\}$ such that $B_i(x_o) = B_j(x_o) = 0$ and $i \neq j$, the candidate $v := -\nabla(B_i(x_o) + B_j(x_o))$ allows to verify Assumption 1.

Next, $x \in S_1$ (respectively, S_2) implies $x \in M_1$ (respectively, $x \in M_3$), so we can always find $U(x)$ small enough such that for each $y \in U(x) \cap \partial K$, $y \in M_i$ if and only if $i = 1$ (respectively, 3). Hence, (17) is satisfied for all $x \in S_1$ if, for every $y \in A_1 := \{ y \in x + \epsilon \mathcal{B} : y_2 = 0 \}$ for some $\epsilon > 0$, there exists $\eta \in F(y)$ such that $\langle \nabla B_1(y), \eta \rangle \leq 0$. Similarly, (17) is satisfied for all $x \in S_2$ if, for each $y \in A_2 := \left\{ y \in \mathbb{R}^2 : \sqrt{2}/2 \leq |y_1| \leq 1, \quad y_1^2 + y_2^2 = 1 \right\}$, there exists $\eta \in F(y)$ such that $\langle \nabla B_3(y), \eta \rangle \leq 0$.

Now, we notice that the set $K \backslash C$ is empty; hence, (18) holds trivially. Moreover, for every $x \in K \cap D$, there exists always a possibility of a jump that maintains the corresponding solution in the set K (i.e., $G(x) \cap K \neq \emptyset$). Hence, it is enough to study the case where $x \in (\partial K_e \cap \partial C) \backslash D = \left\{ [\pm 1 \quad 0]^\top \right\}$. To do so, we note that

$$T_C(x) = \left\{ v \in \mathbb{R}^2 : x_1 v_1 \leq 0 \right\} \quad \forall x \in \left\{ [\pm 1 \quad 0]^\top \right\}, \tag{25}$$

which implies that $F(x) \cap T_C(x) \neq \emptyset$ for all $x \in \left\{ [\pm 1 \quad 0]^\top \right\}$. Hence, to conclude weak forward pre-invariance of K, we need to show that Assumption 1, conditions (19), and (20) hold for all $x \in \left\{ [\pm 1 \quad 0]^\top \right\}$. Indeed, the set C is convex; thus, $C_C(x) = T_C(x)$ for all $x \in \partial C$, see [12, Proposition 2.1]. Furthermore, we already showed that the choice $v = -\nabla(B_1(x) + B_3(x)) = [-2x_1 \quad x_1]^\top$ for all $x \in \left\{ [\pm 1 \quad 0]^\top \right\}$ can be used to verify Assumption 1 when $C_C(x)$ therein is replaced by $\mathbb{R}^n$. Moreover, it is easy to check that $v \in T_C(x)$ using (25); thus, Assumption 1 holds. Next, we notice that the set $U(x) \cap \partial(K \cap C) \cap \partial C = x$ for all $x \in \left\{ [\pm 1 \quad 0]^\top \right\}$; hence, (19) follows as in the previous steps. Finally, to satisfy (20), we notice that, for all $x \in \left\{ [\pm 1 \quad 0]^\top \right\}$, $U(x) \cap \partial K_e \cap C = U(x) \cap (\{x\} \cup S_1 \cup S_2)$,

$F(y) \subset T_C(y)$ for all $y \in S_1 \cup S_2$ since $S_1 \cup S_2 \subset \text{int}(C)$, and that $F(x) \subset T_C(x)$ using (25). Moreover, when $x \in \left\{[\pm 1 \;\; 0]^\top\right\}$, for every $U(x)$ sufficiently small and for every $y \in U(x) \cap \partial K_e$, either $y \in M_1$ (hence, $y \in A_1$) or $y \in M_3$ (hence, $y \in A_2$). That is, using the previous steps, we conclude that (20) is satisfied for all $x \in \left\{[\pm 1 \;\; 0]^\top\right\}$. □

B. Sufficient Conditions Under Possibly Vanishing Gradients

Consider a $\mathcal{C}^1$ scalar function $B : \mathbb{R}^n \to \mathbb{R}$ satisfying the following assumption at a point $x \in \partial K_e$ to be specified in the statements.

Assumption 2: There exists $\epsilon_o > 0$ and $\epsilon_1 > 0$ such that, for each $x_1 \in \partial K_e \cap (x + \epsilon_o \mathcal{B})$ and for each $v \in J_{K_e}(x_1)$,

$$\langle \nabla B(x_1 + cv), v\rangle > 0 \quad \forall c \in (0, \epsilon_1], \tag{26}$$

where the set $J_{K_e}(x_1)$ is introduced in (11). •

Assumption 2 is not very restrictive. Indeed, for any $x \in \partial K_e$, the set $J_{K_e}(x)$ is a compact subset of $D_{\mathbb{R}^n \setminus K_e}(x) = \mathbb{R}^n \setminus T_{K_e}(x)$ Furthermore, by definition of B, the set $\mathbb{R}^n \setminus K_e$ corresponds to the set of points where B is strictly positive. Consequently, the function B along each element of $J_{K_e}(x)$, when starting from $x \in \partial K_e$, varies from zero to strictly positive values. Indeed, when B is polynomial in x, Assumption 2 is always satisfied.

Remark 3: The property claimed in Assumption 2 is not always satisfied; see [6, Example 3.13]. •

In the sequel, we present an original characterization of weak forward pre-invariance in terms of multiple barrier functions defining the closed set K. The proposed characterization imposes (17) to hold in a neighborhood of some special points in $\partial K \cap C$. Furthermore, we assume that, in a neighborhood of points where more than one component B_i of the barrier function candidate is zero, the set where they are equal is a union of $\mathcal{C}^1$ manifolds [2].

Moreover, in the parametric space of such a manifold, Assumption 2 needs to be satisfied. The latter is formally stated in the following assumption satisfied at a point $x \in \partial K_e$ to be specified in the result that follows.

Assumption 3: Consider the set

$$L_x := \{i \in \{1, 2, \dots, m\} : B_i(x) = 0\}.$$

- If L_x is a singleton and $\nabla B_{L_x}(x) = 0$, then Assumption 2 holds for given $x \in \partial K_e$.
- If L_x is not a singleton, then either Assumption 1 with $C_C(x)$ therein replaced by $\mathbb{R}^n$ holds, otherwise, for each set $M_x \subset L_x$ such that a solution y to

  $$B_i(y) = \epsilon \;\; \forall i \in M_x$$

  exists for all $\epsilon \in [0, \epsilon_U]$ for some $\epsilon_U > 0$, the following two properties hold:

(W1) The set of solutions to

$$B_i(y) = B_j(y) \in [0, \epsilon_U] \quad \forall i, j \in M_x^2 \tag{27}$$

is equal to $\cup_\ell \mathcal{S}_\ell$, where each set $\mathcal{S}_\ell$ is a $\mathcal{C}^1$ manifold. The diffeomorphic parametrization associated with $\mathcal{S}_\ell$ is denoted $\gamma_{\mathcal{S}_\ell} : I_\ell \to V_\ell := \mathcal{S}_\ell \cap U(x)$ for some $U(x)$, where $I_\ell \subset \mathbb{R}^k$, $k < n$, is the parametric subspace corresponding to the pre-image of V_ℓ by $\gamma_{\mathcal{S}_\ell}$, for each ℓ.

(W2) For each ℓ as in (W1), Assumption 2 holds with (K_e, B) replaced by $(I_{o\ell}, \bar{B}_\ell)$, where $I_{o\ell} := \gamma_{\mathcal{S}_\ell}^{-}(\mathcal{S}_{o\ell}) \subset I_\ell$,

$$\mathcal{S}_{o\ell} := \{y \in V_\ell : B_i(y) = 0 \quad \forall i \in M_x\},$$

and $\bar{B}_\ell : I_\ell \to \mathbb{R}$ is defined as

$$\bar{B}_\ell(p) := B_i(\gamma_{\mathcal{S}_\ell}(p)) \quad \forall i \in M_x. \tag{28}$$

•

Assumption 3 allows to handle the case where components of the gradient of the barrier functions vanish at some points in ∂K_e, while at some other points Assumption 1 may be satisfied.

Example 2: Consider the closed set $K \subset \mathbb{R}^2$ defined using the $\mathcal{C}^1$ multiple barrier candidate $B(x) := [x_1^3 \;\; x_2^5]^\top$ as in (12). We notice that the origin is the only element of ∂K_e solution of the equation $B_1(x) = B_2(x) = 0$ and that Assumption 1 is not satisfied at the origin since $\nabla B_{1,2}(0) = 0$. However, Assumption 3 holds since the solutions to the system $B_1(y) = B_2(y) \geq 0$ are the elements of the curve $x_1 = {x_2}^{5/3} \geq 0$. The latter curve defines a one dimensional $\mathcal{C}^1$ manifold and both (W1) and (W2) hold. □

We are now ready to state a new characterization of weak forward pre-invariance of a set for general hybrid systems.

Theorem 2: Consider a hybrid system $\mathcal{H}$ and a $\mathcal{C}^1$ barrier function candidate B defining the closed set $K \subset C \cup D$ as in (12). The set K is weakly forward pre-invariant if the following conditions hold:

1) For any $x \in (\partial K \setminus D) \cap \text{int}(C)$, Assumption 3 holds and, for any $x_1 \in (U(x) \setminus K) \cap C$ and $i \in \{1, 2, \dots, m\}$ such that $x_1 \in U(M_i) \setminus K_{ei}$,

$$\exists \eta \in F(x_1) \text{ such that } \langle \nabla B_i(x_1), \eta\rangle \leq 0. \tag{29}$$

2) For each $x \in K \setminus \text{cl}(C)$,

$$\exists \eta \in G(x) \cap (C \cup D) \text{ such that } B(\eta) \leq 0. \tag{30}$$

3) For any $x \in \partial K \cap \text{int}(C) \cap D$, either (30) holds or Assumption 3 holds and, for any $x_1 \in (U(x) \setminus K) \cap C$ and $i \in \{1, 2, \dots, m\}$ such that $x_1 \in U(M_i) \setminus K_{ei}$, (29) holds.
4) For any $x \in \partial K_e \cap \partial C$,
   - If $F(x) \cap T_C(x) = \emptyset$ and $x \in D$, (30) holds.

[2]The set $\mathcal{S} \subset \mathbb{R}^n$ is a $\mathcal{C}^1$ manifold if for each $x \in \mathcal{S}$, there exists a neighborhood $U(x)$ such that $V := U(x) \cap \mathcal{S}$ is diffeomorphic to an open set $I \subset \mathbb{R}^k$, $k \leq n$. A diffeomorphism $\gamma : I \to V$ is called a parametrization of the neighborhood V [13].

- If $F(x) \cap T_C(x) = \emptyset$, either (30) holds, or Assumption 3 holds and the following conditions are satisfied:

$$F(x_1) \cap T_{K \cap C}(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial(K \cap C) \cap \partial C. \tag{31}$$

For any $x_1 \in U(x) \backslash K_e \cap \text{int}(C)$ and for each $i \in \{1, 2, \ldots, m\}$ such that $x_1 \in U(M_i) \backslash K_{ei}$,

$$\exists \eta \in F(x_1) \text{ s.t. } \langle \nabla B_i(x_1), \eta \rangle \leq 0. \tag{32}$$

5) For any $x \in \text{int}(K_e) \cap \partial C \cap D$, either (30) holds or

$$F(x_1) \cap T_C(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial C. \tag{33}$$

□

Example 3: Consider the hybrid system studied in Example 1 while replacing the set D therein by $D := \{x \in \mathbb{R}^2 : x_2 = x_1,\ x_1 \neq 0\}$. We propose to establish weak forward pre-invariance for the set K introduced in (22) using Theorem 2. We start showing that Assumption 3 is satisfied for all $x_o \in \partial K$. Indeed, for each $x_o \in \partial K \backslash \{[0 \ \ 0]^\top\}$, we already showed that Assumption 1 is satisfied; thus, for any $x_o \in \partial K \backslash \{[0 \ \ 0]^\top\}$, Assumption 3 is also satisfied. Next, we notice that $x_o = [0 \ \ 0]^\top \notin D$, $B_1(0) = B_2(0) = 0 \neq B_3(0)$ and $\nabla B_1(x_o) = \nabla B_2(x_o) = 0$. The latter means that Assumption 1 is not satisfied at the origin; thus, Theorem 1 is not applicable in this case. Instead, we will show that around the origin (W1) and (W2) in Assumption 3 are both satisfied. Indeed, for $M_{x_o} = \{1, 2\}$ and for any neighborhood $U(x_o)$, the solution to the algebraic equation $B_1(x) = B_2(x) \in [0, \epsilon_U],\ \epsilon_U \geq 0$ defines the $\mathcal{C}^1$ manifold $\mathcal{S} = \left\{x \in \mathbb{R}^2 : x_2 = -\frac{\sqrt{5}+1}{2} x_1\right\}$. It is easy to see that the manifold $\mathcal{S}$ admits a diffeomorphic parametrization $\gamma_{\mathcal{S}} : I \to V$ with $I = \epsilon[-1, 1]$ for some $\epsilon > 0$, $V = \left\{x \in \mathbb{R}^2 : x_2 = -\frac{\sqrt{5}+1}{2} x_1,\ |x_1| \leq \epsilon\right\}$, and $\gamma_{\mathcal{S}}(p = x_1) = \left[p \quad -\frac{\sqrt{5}+1}{2} p\right]^\top$. Hence, (W1) is satisfied. Next, we need to show that Assumption 2 holds after replacing (K_e, B) therein by $(I_o, \bar{B})$. Indeed, $I_o = \{0\}$, $\mathcal{S}_o = \{x_o\} := \{[0 \ \ 0]^\top\}$, and $\bar{B}(p) = \frac{\sqrt{5}+1}{2} p^2$. Moreover, $\partial I_o \cap (x_o + \epsilon_o[-1, 1]) = x_o$ for all $\epsilon_o > 0$, $J_{I_o}(x_o) = \{-1, 1\}$, and $\langle \nabla \bar{B}(x_o + cv), v \rangle = (\sqrt{5} + 1) c v^2 > 0 \quad \forall c > 0$. Hence, Assumption 3 holds at $x_o = [0 \ \ 0]^\top$.

Now, we verify the conditions in Theorem 2. We start considering the set $(\partial K \cap \text{int}(C)) \cup (K \cap D)$. Indeed, according to Example 1, the set $(\partial K \cap \text{int}(C)) \backslash D$ can be decomposed as $(\partial K \cap \text{int}(C)) \backslash D = S_1 \cup S_2 \cup \{[0 \ \ 0]^\top\}$, where S_1 and S_2 are introduced in (23) and (24), respectively. Furthermore, $x \in S_1$ (respectively, S_2) implies that $x \in M_1$ (respectively, $x \in M_3$), and we can always find $U(x)$ small enough such that, for all $y \in U(x) \backslash K$, $y \in U(M_i) \backslash K_{ei}$ if and only if $i = 1$ (respectively, 3). Hence, (29) is satisfied for all $x \in S_1$ since we can show that, for any $y \in A_1 := \{y \in x + \epsilon \mathcal{B} : y_1 y_2 \leq 0\}$ for some $\epsilon > 0$, there exists $\eta \in F(y)$ such that $\langle \nabla B_1(y), \eta \rangle \leq 0$. Similarly, (29) is satisfied for every $x \in S_2$ since we can show that, for all $y \in A_2 := \{y \in \mathbb{R}^2 : \sqrt{2}/2 < |y_1| < 1 + \epsilon,\ \ y_1^2 + y_2^2 - 1 \in (0, \epsilon)\}$ for some $\epsilon > 0$, there exists $\eta \in F(y)$ such that $\langle \nabla B_3(y), \eta \rangle \leq 0$. Finally, $K \cap D = \{x \in \mathbb{R}^2 : |x_1| \leq \sqrt{2}/2,\ x_2 = x_1\}$. That is, for any $x \in K \cap D$ it is easy to verify the existence of a jump such that (30) holds. Hence, to conclude weak forward pre-invariance, it remains only to focus on the elements of the set $(\partial K_e \cap \partial C) \backslash D$. We start noticing that $(\partial K_e \cap \partial C) \backslash D = \{[\pm 1 \ \ 0]^\top\}$. Moreover, since (29) is satisfied, then (32) also holds. Furthermore, according to Example 1, $U(x) \cap \partial(K \cap C) \cap \partial C = \{x\}$ for each $x \in \{[1 \ \ 0]^\top, [-1 \ \ 0]^\top\}$, Assumption 1 holds, and (17) is satisfied. Thus, using [7, Proposition 4.3.7], $\eta = [x_2 x_1^2 \ \ x_1^3 - 2x_2]^\top \in F(x)$ belongs to $T_{K_e}(x)$. Moreover, from Example 1, we know that Assumption 1 also holds and $\eta \in T_C(x)$. Hence, using [7, Proposition 4.3.7], (31) follows.

□

Remark 4: We recall from [6, Proposition 1] that a weakly forward pre-invariant set is weakly forward invariant if the solutions cannot escape in finite time inside the set $K \cap C$ and, for any $x \in (K \cap \partial C) \backslash D$, a nontrivial flow exists. •

IV. Conclusion

This paper proposed sufficient conditions for weak forward invariance of a closed set for general hybrid inclusions. The considered sets are generated by intersecting sublevel sets of multiple scalar functions that form a (vector) barrier function candidate. The proposed conditions in terms of barrier candidates are alternatives to the existing tangent-cone conditions and those involving only scalar barrier candidates.

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