Monotonicity Along Solutions to Constrained Differential Inclusions

Mohamed Maghenem, Alessandro Melis, and Ricardo G. Sanfelice

Abstract— **In this paper, a general framework is proposed to determine when a scalar function is nonincreasing along solutions to differential inclusions defined on constrained sets. To the best of our knowledge, this problem has not been yet treated in the literature, and is important, for example, for the analysis of hybrid systems modeled by hybrid inclusions. The proposed characterizations are infinitesimal and do not require any knowledge about the system's solutions. Furthermore, the problem is addressed under different regularity properties of the considered scalar function, including the case of lower semi-continuous functions, the case of locally Lipschitz and regular functions, and finally the case of continuously differentiable functions.**

I. Introduction

The study of monotonic behavior of scalar functions along solutions to continuous-time dynamical systems consists in establishing necessary and sufficient conditions such that the considered scalar function is nonincreasing along solutions. The proposed characterization must be *infinitesimal*; namely, involving only the scalar function and the system's dynamics; hence, no explicit knowledge about the solutions is required.

The latter problem is one of the fundamental problems in calculus [1], and has attracted mathematicians' attention since the works of *Pierre de Fermat* in the 17^{th} century on local extrema for differentiable functions [2]. The difficulty when addressing such a problem depends on both the system's dynamics and the regularity of the considered scalar function. The first attempts to address this problem concerned the particular case where both the system's dynamics and the considered scalar function are sufficiently smooth. It is well known in this case, that the problem is solved if and only if the scalar product between the gradient of the function and the system's dynamics is nonpositive when evaluated at points of the set where the dynamics is defined. When the considered scalar function is not continuously differentiable, the problem becomes more difficult since the gradient in this case may not exist. The first successful characterization for nonsmooth functions dates back to what is considered as one of the first results in nonsmooth analysis. This result uses certain constructs alternative to the gradient, called the *directional subderivatives*, proposed by *Ulisse Dini* in 1878. Since then, many extensions were presented in the literature, see [3], [4], [5], [6], in order to cover the general case where the system's dynamics is a differential inclusion and the scalar function is only locally Lipschitz or lower semicontinuous.

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To the best of our knowledge, in all the existing literature, the system's dynamics is defined in $\mathbb{R}^n$ or in an open set such that the solutions cannot start from its boundary. The latter requirement is customarily used when studying continuous-time dynamical systems. However, such is not necessarily the case when studying hybrid systems modeled according to the framework proposed in [7]. Indeed, for general hybrid systems, the continuous-time dynamics is usually defined in a closed set, or at least in a closed set relative to a given open set [8], and the solutions are allowed to start from its boundary. The set where the continuous-time dynamics is defined is called the *flow set*. The latter requirement is important in order to guarantee some important structural properties for the set of solutions [7]. In such a scenario of constrained differential inclusions, the existing characterizations of monotonicity are not applicable and extra complexity is added to the problem. Indeed, assume that the system's solutions are defined in a closed set. In this case, it is possible to find elements of the dynamics (vector fields) not generating solutions, for example, elements pointing towards the complement of the flow set. Those elements cannot be considered in the characterization, otherwise, the characterization will not be necessary, see Example 1. At the same time, those vector fields, although not generating solutions, may affect the global behavior of the solutions; hence, they should be considered in the characterization, otherwise, it will not be sufficient, see Example 2. In order to handle such a compromise, extra assumptions relating the system's dynamics to the boundary of the flow set must be considered.

The monotonicity problem along solutions to continuous-time systems defined in closed sets finds a natural motivation when studying stability [9] and safety [10] in hybrid systems. Indeed, many theoretical solution-based constructions of Lyapunov and barrier functions are proposed in the literature. Some of those constructions, although shown to be continuous and locally Lipschitz when the system's solutions are unique and the flow set is the whole space, are only lower semicontinuous in the general case where the solutions are nonunique or the flow set is restricted.

In this paper, we investigate elements needed to address the monotonicity problem for constrained differential inclusions. This problem is studied under different regularity properties of the considered scalar function including:

- The case of lower semicontinuous functions, where we transform the problem into characterizing *forward pre-invariance* of a closed set. After that, infinitesimal conditions involving the dynamics, normal, and tangent cones with respect to the considered closed set are proposed.
- The case of locally Lipschitz and regular functions, where the Clarke generalized gradient is used.
- Finally, the case of continuously differentiable functions is invoked as a particular case of locally Lipschitz and regular functions where the generalized gradient reduces to the classical gradient.

To the best of our knowledge, our results are original and provide useful tools to study stability and safety for hybrid systems or for general constrained continuous dynamics. Due to space limitations, proofs and intermediate results are omitted and will appear elsewhere.

Notation. For x, $y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the norm of x, $|x|_K := \inf_{y \in K} |x-y|$ defines the distance between x and the nonempty set $K \subset \mathbb{R}^n$, $\langle x, y \rangle := x^\top y$ the inner product between x and y, and $\text{co}\,\{x, y\}$ the set of all convex combinations between x and y. We use $\text{int}(K)$ to denote the interior of K, ∂K its boundary, $\text{cl}(K)$ its closure, and $U(K)$ to denote a sufficiently small open neighborhood around K. For $O \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O. By $\mathbb{B}$ we denote the closed unit ball in $\mathbb{R}^n$ centered at the origin. For a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$, $\nabla B(x)$ denotes the gradient of the function B evaluated at x. By $\mathcal{C}^p$, we denote the set of p-times differentiable functions with continuous p-th derivative.

II. Preliminaries and Problem Statement

A. Constrained differential inclusions

A constrained differential inclusion $\mathcal{H}_f := (C, F)$ is defined as the continuous-time system

$$\mathcal{H}_f : \qquad \dot{x} \in F(x) \quad x \in C \subset \mathbb{R}^n, \tag{1}$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$ and the map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. The set C in (1) is not necessarily open and does not neccessarily correspond to $\mathbb{R}^n$, as opposed to the existing literature dealing with unconstrained differential inclusions where $C \equiv \mathbb{R}^n$ [6], [5].

Next, we introduce the concept of solutions to $\mathcal{H}_f$.

Definition 1: (Solution to $\mathcal{H}_f$) A function $x : \text{dom}\, x \to \mathbb{R}^n$ with $\text{dom}\, x \subset \mathbb{R}_{\geq 0}$ and $t \mapsto x(t)$ locally absolutely continuous is a *solution* to $\mathcal{H}_f$ if

(S1) $x(0) \in \text{cl}(C)$,

(S2) $x(t) \in C \quad$ for all $t \in \text{int}(\text{dom}\, x)$,

(S3) $\frac{dx(t)}{dt} \in F(x(t)) \quad$ for almost all $\ t \in \text{dom}\, x$.

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Remark 1: Condition (S1) allows solutions starting from $\partial C \backslash C$ to flow into C such that (S2) is satisfied. Furthermore, (S2) allows solutions starting from C to reach $\partial C \backslash C$. Hence, the symmetry between the forward and the backward solutions is preserved. •

Remark 2: Constrained differential inclusions $\mathcal{H}_f = (C, F)$ constitute a key component in the modeling of hybrid systems. Indeed, according to [7], a general hybrid system modeled as a hybrid inclusion is given by

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \tag{2}$$

where, in addition to the continuous dynamics $\mathcal{H}_f = (C, F)$, the *discrete dynamics* are defined by the jump set $D \subset \mathbb{R}^n$ and the jump map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Furthermore, the solutions to $\mathcal{H}_f = (C, F)$, according to Definition 1, correspond to the solutions to $\mathcal{H}$, according to [7, Definition 2.6], that never jump. •

A solution x to $\mathcal{H}_f$ is said to be maximal if there is no solution z to $\mathcal{H}_f$ such that $x(t) = z(t)$ for all $t \in \text{dom}\, x$ with $\text{dom}\, x$ a proper subset of $\text{dom}\, z$. Finally, it is said to be trivial if the set $\text{dom}\, x$ contains only one element.

Throughout this paper, we assume that the set-valued map F satisfies the following conditions:

(i) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally Lipschitz,

(ii) $F(x)$ is convex and closed for all $x \in C$.

Definition 2: A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *locally Lipschitz* if for each compact set $K \subset \mathbb{R}^n$ there exists $k > 0$ such that, for all $x \in K$ and $y \in K$,

$$F(y) \subset F(x) + k|x - y|\mathbb{B}. \tag{3}$$

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B. The monotonicity problem along solutions

Problem 1: Given a constrained differential inclusion $\mathcal{H}_f = (C, F)$, provide necessary and sufficient conditions such that the following property holds:

($\star$) A scalar function $B : \mathbb{R}^n \to \mathbb{R}$ is nonincreasing along the solutions to $\mathcal{H}_f$; namely, for every solution $t \mapsto x(t)$, the map $t \mapsto B(x(t))$ is nonincreasing [1].

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It is to be noted that the required conditions to solve Problem 1 need to be infinitesimal; namely, involving only the scalar function B, the map F, and the flow set C. Indeed, Problem 1 is a fundamental problem in calculus [4]. The first attempts to address it were generally concerned with the case where: $n = 1$, the function $B \in \mathcal{C}^1$, $F(x) = 1$ for all $x \in C$, and $C \equiv \mathbb{R}$ [1]. It is well known, in this case, that B is nonincreasing along the solutions to $\mathcal{H}_f$ if and only if

$$\nabla B(x) \leq 0 \qquad \forall x \in C \equiv \mathbb{R}^n.$$

A similar statement can be derived when $n > 1$ and F a general n-dimensional map. Indeed, the monotonicity

[1] Or, equivalently, $B(x(t_1)) \leq B(x(t_2))$ for all $(t_1, t_2) \in \text{dom}\, x \times \text{dom}\, x$ with $t_1 \geq t_2$.

problem is solved in this case, if and only if

$$\langle \nabla B(x), v \rangle \leq 0 \quad \forall v \in F(x), \quad \forall x \in C \equiv \mathbb{R}^n. \tag{4}$$

When the function B is not continuously differentiable, the problem becomes more difficult. Indeed, the gradient ∇B cannot be used in this case. Useful tools are proposed in [5] to cover the general case where the function B is only lower semicontinuous, F satisfies (i)-(ii), and the set C is open.

Definition 3 (Lower semicontinuous scalar function): A function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be lower semicontinuous at $x \in \mathbb{R}^n$ if, for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$ with $\lim_{n\to\infty} x_n = x \in \mathbb{R}^n$, we have $\lim_{n\to\infty} B(x_n) \geq B(x)$. The function B is said to be lower semicontinuous if it is lower semicontinuous at each $x \in \mathbb{R}^n$. •

When the set C is not $\mathbb{R}^n$ and nontrivial solutions to $\mathcal{H}_f$ start from ∂C, the existing characterizations of monotonicity addressing Problem 1 are not applicable. Indeed, let us assume that C is closed, then an extra complexity is added to the problem since, when $x \in \partial C \cap C$, only vectors in $F(x)$ that generate nontrivial solutions should be considered in the characterization, otherwise, the characterization will not be necessary (this is clarified later in Example 1). However, the vectors in $F(x)$ not generating solutions may affect the global behavior of the solutions in a way that renders the map $t \mapsto B(x(t))$ fails to be nonincreasing. The latter is more likely to happen when B is discontinuous. Consequently, the elements of $F(x)$ not generating solutions should be included in the characterization, otherwise, it may not be sufficient (see forthcoming Example 2). To manage such a compromise, in the general case where B is lower semicontinuous and (C, F) satisfy (i)-(ii), extra assumptions need to be assumed for $F(x)$ for x's nearby the set ∂C.

C. Motivation

Problem 1 finds a natural motivation when studying stability and safety for hybrid systems and constrained differential inclusions; see, e.g., [10] and [9]. To illustrate this point, for a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold, and C is closed, consider the safety problem defined by the initial and unsafe sets $\chi_o \subset C$ and $\chi_u \subset C$, respectively, with $\chi_o \cap \chi_u = \emptyset$. The system $\mathcal{H}_f$ is safe with respect to (χ_o, χ_u) provided that each solution starting from χ_o never reaches χ_u. It has been shown recently in [10] that safety of $\mathcal{H}_f$ with respect to (χ_o, χ_u) is satisfied if and only if there exists a lower semicontinuous (barrier) function $B : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ that is nonincreasing along solutions to $\mathcal{H}_f$ and that satisfies

$$\begin{cases} B(t,x) > 0 & \forall x \in \chi_u \quad \forall t \geq 0, \\ B(t,x) \leq 0 & \forall x \in \chi_o \quad \forall t \geq 0. \end{cases} \tag{5}$$

That is, solving Problem 1 for a lower semicontinuous function B along the solutions to the constrained differential inclusion $\tilde{\mathcal{H}}_f = (\tilde{C}, \tilde{F})$, with $\tilde{x} := [x \quad t]^\top$, $\tilde{C} := \mathbb{R}_{\geq 0} \times C$ and $\tilde{F}(\tilde{x}) := [1 \quad F(x)]^\top$, allows infinitesimal characterization of safety in terms of barrier functions.

III. Background

In this section, we recall useful notions from the context of set-valued and non-smooth analysis [2], [5].

Definition 4 (Lower semicontinuous set-valued map): A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *lower semicontinuous* or *inner semicontinuous* at $x \in \mathbb{R}^n$ if, for any $\epsilon > 0$ and any $y_x \in F(x)$, there exists $U(x)$ such that, for any $z \in U(x)$, there exists $y_z \in F(z)$ such that $|y_z - y_x| \leq \epsilon$. Furthermore, it is said to be lower semicontinuous or inner semicontinuous if it is so for all $x \in \mathbb{R}^n$. •

When a scalar function $B : \mathbb{R}^n \to \mathbb{R}$ is at least lower semicontinuous, its epigraph, given by

$$\text{epi}B := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geq B(x)\}, \tag{6}$$

can be used to address Problem 1. Indeed, the epigraph of a lower semicontinuous function is always closed and the following property holds [5].

Lemma 1: A lower semicontinuous function $B : \mathbb{R}^n \to \mathbb{R}$ is nonincreasing along the solutions to $\mathcal{H}_f$ if and only if the closed set $\text{epi}B \cap (\text{cl}(C) \times \mathbb{R})$ is *forward pre-invariant* for the differential inclusion

$$\begin{bmatrix} \dot{x} \\ \dot{r} \end{bmatrix} \in \begin{bmatrix} F(x) \\ 0 \end{bmatrix} \qquad (x, r) \in C \times \mathbb{R}; \tag{7}$$

namely, the maximal solutions to $\mathcal{H}_f$ starting from the set $\text{epi}B \cap (\text{cl}(C) \times \mathbb{R})$ remain in it. □

Lemma 1 transforms Problem 1 into characterizing forward pre-invariance of a closed set for the augmented dynamics (7). Forward pre-invariance has been extensively studied in the literature, see, e.g., [11], [5]. Infinitesimal conditions for forward pre-invariance involving F and tangent cones with respect to the considered closed set are shown to be necessary and sufficient when $C \equiv \mathbb{R}^n$.

Definition 5: The *contingent* cone of K at x is given by

$$T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h\to 0^+} \frac{|x + hv|_K}{h} = 0 \right\}. \tag{8}$$

•

Definition 6: The *Clarke tangent* cone of K at x is given by

$$C_K(x) := \left\{ v \in \mathbb{R}^n : \limsup_{y\to x, h\to 0^+} \frac{|y + hv|_K}{h} = 0 \right\}. \tag{9}$$

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Definition 7: A set $K \subset \mathbb{R}^n$ is said to be regular if $T_K(x) = C_K(x)$ for all $x \in K$. •

Definition 8: The *proximal normal cone* N_S^P associated with the set $S \subset \mathbb{R}^n$ evaluated at $x \in \text{cl}(S)$ is given by

$$N_S^P(x) := \{\zeta \in \mathbb{R}^n : \exists t > 0 \text{ so that } |x + t\zeta|_S = t|\zeta|\}. \tag{10}$$

•

The map N_P^S is a cone; hence, closed and $\{0\} \in N_S^P(x)$ for all $x \in \text{cl}(S)$.

Definition 9: The *proximal subdifferential* of a lower semicontinuous function $B : \mathbb{R}^n \to \mathbb{R}$ is the set-valued map $\partial_P B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that, for all $x \in \mathbb{R}^n$,

$$\partial_P B(x) := \left\{ \zeta \in \mathbb{R}^n : [\zeta^\top \;\; -1]^\top \in N^P_{\text{epi}B}(x, B(x)) \right\}. \tag{11}$$

Moreover, each vector $\zeta \in \partial_P B(x)$ is said to be a *proximal subgradient* of B at x. •

When the function B is locally Lipschitz, its *generalized gradient*, denoted by ∂B, constitutes a useful tool to address Problem 1. The following definition is meaningful due to the equivalence in [5, Theorem 8.1].

Definition 10: Let $B : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Let Ω be any subset of zero measure in $\mathbb{R}^n$, and let Ω_B be the set of points in $\mathbb{R}^n$ at which B fails to be differentiable. Then

$$\partial B(x) := \text{co}\left\{ \lim_{i\to\infty} \nabla B(x_i) : x_i \to x,\; x_i \notin \Omega_B,\; x_i \notin \Omega \right\}. \tag{12}$$

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Next, we introduce the notion of regular functions. The definition we provide here is meaningful due to [5, Proposition 7.3].

Definition 11: A locally Lipschitz function $B : \mathbb{R}^n \to \mathbb{R}$ is regular if epi$B$ is regular according to Definition 7. •

IV. Main Results

In this section, inspired by [5], we formulate necessary and sufficient infinitesimal conditions solving Problem 1 when the set C in $\mathcal{H}_f$ is not necessarily $\mathbb{R}^n$ and nontrivial solutions start from ∂C. Before going further we introduce the following useful set $\tilde{C}$:

$$\tilde{C} := \{x_o \in \text{cl}(C) : \exists x \in \mathcal{S}(x_o),\;\; \text{dom}\, x \neq \{0\}\}, \tag{13}$$

where $\mathcal{S}(x_o)$ is the set of solutions starting from x_o.

A. The case of lower semicontinuous functions

Our approach in this case is based on characterizing forward pre-invariance of the set epi$B \cap (\text{cl}(C) \times \mathbb{R})$ according to Lemma 1 using infinitesimal conditions. We consider the following assumptions on the data (C, F) of $\mathcal{H}_f$.

(M1) For any $x_o \in \partial C \cap \tilde{C}$, if $F(x_o) \cap T_C(x_o) \neq \emptyset$, then, for any $v_o \in F(x_o) \cap T_C(x_o)$, there exists a continuous selection $v : \partial C \cap U(x_o) \to \mathbb{R}^n$ such that $v(x) \in F(x) \cap T_C(x)$ for all $x \in \partial C \cap U(x_o)$ and $v(x_o) = v_o$.

(M2) For any $x_o \in \partial C \cap \tilde{C}$, $F(x) \subset T_C(x)$ for all $x \in U(x_o) \cap \partial C$.

The relevance of (M1) and (M2) are discussed in the sequel. Furthermore, we consider the following condition.

$$\begin{aligned} \langle \zeta, v \rangle \leq 0 \quad & \forall [\zeta^\top \;\; \alpha]^\top \in N^P_{\text{epi}B \cap (C \times \mathbb{R})}(x, B(x)), \\ & \forall v \in F(x) \cap T_C(x),\; \forall x \in \tilde{C}. \end{aligned} \tag{14}$$

The following result solves Problem 1 and its proof is inspired from [11, Theorem 5.3.4] and [5, Theorem 3.8].

Theorem 1: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold and let $B : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. Then,

- (14) + (M2) $\Rightarrow$ ($\star$).
- Conversely, ($\star$) + (M1) $\Rightarrow$ (14).

Consequently, when (M1)-(M2) hold, ($\star$) $\Leftrightarrow$ (14). □

Remark 3: (M1) ensures the existence of a nontrivial solution (i.e., solution whose domain is not a singleton) along each direction in the intersection between the images of F and the contingent cone T_C. The latter requirement is necessary in order to prove the necessary part of the statement in the general case where C is not $\mathbb{R}^n$ and nontrivial solutions start from ∂C. •

To illustrate the observation about (M1) in Remark 3, in the following example, we consider the particular situation where (i)-(ii) hold, the function B is nonincreasing along the solutions but (M1) is not satisfied and, for some $x \in \tilde{C}$, there exist $v \in F(x) \cap T_C(x)$ such that (14) is not satisfied.

Example 1: Consider the system $\mathcal{H}_f = (C, F)$ with $x \in \mathbb{R}^2$,

$$F(x) := \text{co}\left\{ [1 \;\; 0]^\top,\; [-\cos(x_1^2) \;\;\; \sin(x_1^2)]^\top \right\} \quad \forall x \in C,$$

and $C := \left\{ x \in \mathbb{R}^2 : x_2 = 0 \right\}$. Furthermore, consider the function $B(x) := -x_1$. It is easy to see that F satisfies both (i) and (ii). Furthermore, starting from any $x_o := [x_{o1} \;\; x_{o2}]^\top \in C$, the only nontrivial solution is given by $x(t) = [x_{o1} + t \;\; 0]^\top$ for all $t \geq 0$; hence, $\tilde{C} = C$ and B is nonincreasing along each nontrivial solution. However, when $x_o = 0$, we show that, for $v_o := [-1 \;\; 0]^\top \in F(0) \cap T_C(0)$, condition (14) is not satisfied. Indeed, it is easy to see that

$$N^P_{\text{epi}B \cap (C \times \mathbb{R})}(x, B(x)) = \left\{ -[1 \;\alpha\; 1]^\top : \alpha \in \mathbb{R} \right\} \quad \forall x \in C.$$

Hence,

$$\langle \zeta, v_o \rangle = 1 > 0 \;\; \forall [\zeta \;\; -1]^\top \in N^P_{\text{epi}B \cap (C \times \mathbb{R})}(0, 0).$$

□

The following result shows that in some situations (M1) is not needed. However, such situations require that the set $\tilde{C}$ and the function B satisfy the following extra properties:

(p1) $x \mapsto$ blckdiag$\{I_n, 0\}\, N^P_{\text{epi}B \cap (C \times \mathbb{R})}(x, B(x))$ is lower semicontinuous on $\tilde{C}$,

(p2) $U(x) \cap \text{int}(C) \neq \emptyset$ for all $x \in \partial C \cap C$ and for all $U(x)$.

Theorem 2: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold and let $B : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. Then,

- ($\star$) + (p1) + (p2) $\Rightarrow$ (14).

Consequently, when (M2) and (p1)-(p2) hold, ($\star$) $\Leftrightarrow$ (14). □

In the following example, we present a constrained system $\mathcal{H}_f = (C, F)$ where (M2) is not satisfied, the condition in Theorem 1 is satisfied, and there exists a lower semicontinuous function B that fails to be nonincreasing along solutions.

Example 2: Consider the system $\mathcal{H}_f = (C, F)$ with $x \in \mathbb{R}^2$,

$$F(x) := \begin{Bmatrix} [1 \quad [-1,1]x_1]^\top & \text{if } x_1 \geq 0 \\ [1 \quad 0]^\top & \text{if } x_1 < 0 \end{Bmatrix} \quad \forall x \in C,$$

$$C := \left\{x \in \mathbb{R}^2 : |x_2| \geq x_1^2\right\} \cup \left\{x \in \mathbb{R}^2 : x_1 \leq 0\right\} \\ \cup \left\{x \in \mathbb{R}^2 : x_2 = 0\right\}.$$

Furthermore, consider the lower semicontinuous function

$$B(x) := \begin{cases} 0 & \text{if } x_2 \leq 0 \\ 1 & \text{if } x_2 > 0. \end{cases}$$

We will show that in this case all of the conditions in Theorem 1 are satisfied except (M2) and that the set $\text{epi}B \cap (\text{cl}(C) \times \mathbb{R})$ is not forward pre-invariant; thus, B is not nonincreasing along the solutions to $\mathcal{H}_f$. Indeed, we start noting that

$$\begin{aligned} \text{epi}B = &\left\{(x,r) \in \mathbb{R}^3 : x_2 \leq 0,\ r \geq 0\right\} \cup \\ &\left\{(x,r) \in \mathbb{R}^3 : x_2 > 0,\ r \geq 1\right\}, \\ \text{epi}B \cap (C \times \mathbb{R}) = &\\ &\left\{(x,r) \in \mathbb{R}^3 : x_2 \leq 0,\ r \geq 0,\ x_1 \leq \sqrt{-x_2}\right\} \cup \\ &\left\{(x,r) \in \mathbb{R}^3 : x_2 > 0,\ r \geq 1,\ x_1 \leq \sqrt{x_2}\right\} \cup \\ &\left\{(x,r) \in \mathbb{R}^3 : x_1 \geq 0,\ r \geq 0,\ x_2 = 0\right\}. \end{aligned}$$

Furthermore, it is easy to see that F is locally Lipschitz and $F(x)$ is convex and compact for all $x \in \mathbb{R}^2$, hence, (i) and (ii) are both satisfied. Now, in order to show that (M1) is satisfied, we pick $x_o \in \partial C$, which implies that

$$x_o := [x_{o1} \quad x_{o2}]^\top \in \left\{x \in \mathbb{R}^2 : x_1 = \sqrt{|x_2|}\right\} \cup \\ \left\{x \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 = 0\right\},$$

and

$$F(x_o) \cap T_C(x_o) = \begin{cases} [1 \quad x_{o1}]^\top & \text{if } x_{o2} > 0 \\ [1 \quad 0]^\top & \text{if } x_{o2} = 0 \\ [1 \quad -x_{o1}]^\top & \text{if } x_{o2} < 0. \end{cases}$$

Hence, when $x_{o2} > 0$, $v_o = [1 \quad x_{o1}]^\top$ and the continuous selection is given by $v(x) = [1 \quad x_1]^\top \in F(x) \cap T_C(x)$ for all $x \in \partial C \cap U(x_o)$ and for $U(x_o)$ sufficiently small. Next, when $x_{o2} < 0$, $v_o = [1 \quad -x_{o1}]^\top$ and the continuous selection is given by $v(x) = [1 \quad -x_1]^\top \in F(x) \cap T_C(x)$ for all $x \in \partial C \cap U(x_o)$ and for $U(x_o)$ sufficiently small. Finally, when $x_{o2} = 0$, $v_o = [1 \quad 0]^\top$ and the continuous selection is given by

$$v(x) = \begin{Bmatrix} [1 \quad x_1]^\top & \text{if } x_2 > 0 \\ [1 \quad 0]^\top & \text{if } x_2 = 0 \\ [1 \quad -x_1]^\top & \text{if } x_2 < 0 \end{Bmatrix} \in F(x) \cap T_C(x)$$

for all $x \in \partial C \cap U(x_o)$ and for $U(x_o)$ sufficiently small. Thus, (M1) is satisfied. Finally, to show that (14) is satisfied, we start noticing that

$$\begin{aligned} \partial(\text{epi}B) \cap (C \times \mathbb{R}) = &\\ &\left\{(x,r) \in \mathbb{R}^3 : x_2 < 0,\ r = 0,\ x_1 \leq \sqrt{-x_2}\right\} \cup \\ &\left\{(x,r) \in \mathbb{R}^3 : x_2 > 0,\ r = 1,\ x_1 \leq \sqrt{x_2}\right\} \cup \\ &\left\{(x,r) \in \mathbb{R}^3 : x_2 = 0,\ 0 \leq r \leq 1\right\}. \end{aligned}$$

That is, it is easy to see that for any $x \in C$, which implies that $(x, B(x)) \in \partial(\text{epi}B) \cap (C \times \mathbb{R})$, we have $[F(x) \cap T_C(x) \quad 0]^\top \subset T_{\partial(\text{epi}B) \cap (C \times \mathbb{R})}(x, B(x))$; hence, (14) follows from [11, Proposition 3.2.3]. Finally, in order to show that $\text{epi}B \cap (C \times \mathbb{R})$ is not forward pre-invariant for (7), we consider the function $(x(t), B(x_o)) := [t \quad t^2 \quad 0]^\top \in (C \times \mathbb{R})$ for all $t \geq 0$, which is absolutely continuous and solution to the differential equation $(\dot{x}, \dot{r}) = ([1 \quad x_1]^\top, 0) \in (F(x), 0)$. Hence, using Lemma 1, we conclude that B is not nonincreasing along the solutions to $\mathcal{H}_f$. □

In the following, we show how Theorem 1 applies to address Problem 1 on a concrete example.

Example 3: The continuous dynamics of the bouncing-ball hybrid model is given by $\mathcal{H}_f := (C, F)$ with $F(x) := [x_2 \quad -\gamma]^\top$ and $C := \left\{x \in \mathbb{R}^2 : x_1 \geq 0\right\}$. The constant $\gamma > 0$ is the gravitational acceleration. First, F is single valued and continuously differentiable; hence, (i) and (ii) hold. Second, note that $\widetilde{C} := C \backslash \left\{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\right\}$. Hence, starting from $x_o \in \widetilde{C} \cap \partial C = \left\{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 > 0\right\}$, $F(x_o) = [x_{o2} \quad -\gamma]^\top \in T_C(x_o)$; thus, (M2) follows. Moreover, (M1) is also satisfied since $\partial C \cap \widetilde{C}$ is open and, for each $x_o \in \partial C \cap \widetilde{C}$, $F(x_o) \in T_C(x_o)$. Finally, using Theorem 1, we conclude that a lower semicontinuous function $B : \mathbb{R}^2 \to \mathbb{R}$ satisfies ($\star$) if and only if (14) holds. In particular, the energy function of the bouncing ball satisfies (14) since, by definition, it cannot increase along the solutions. □

In the sequel, we will show that the inequality in (14) does not need to be checked for all $[\zeta^\top \quad \alpha]^\top \in N^P_{\text{epi}B \cap (C \times \mathbb{R})}(x, B(x))$ when $x \in \text{int}(C)$. That is, when $x \in \text{int}(C)$, we will show that it is enough to verify the inequality in (14) only for the vectors $[\zeta^\top \quad \alpha]^\top \in N^P_{\text{epi}B \cap (\text{cl}(C) \times \mathbb{R})}(x, B(x))$ with $\alpha = -1$ to conclude that it holds for all $[\zeta^\top \quad \alpha]^\top \in N^P_{\text{epi}B \cap (C \times \mathbb{R})}(x, B(x))$.

Proposition 1: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold, and let $B : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. The inequality in (14) is satisfied at $x \in \text{int}(C)$ if

$$\langle \zeta, \eta \rangle \leq 0 \qquad \forall \zeta \in \partial_P B(x),\ \forall \eta \in F(x). \tag{15}$$

□

When the set $C \equiv \mathbb{R}^n$ or when $\widetilde{C}$ is open, in [5, Theorem 6.3], a necessary and sufficient infinitesimal condition involving only $\partial_P B$ and the map F is provided such that ($\star$) holds. Indeed, consider the condition

$$\langle \zeta, \eta \rangle \leq 0 \quad \forall \zeta \in \partial_P B(x),\ \forall \eta \in F(x),\ \forall x \in \text{int}(C). \tag{16}$$

Also, we consider the following assumptions on the solutions to $\mathcal{H}_f$ and the considered scalar function B.

(a1) For every nontrivial solution x starting from $x_o \in \partial C$, there exists $\epsilon > 0$ such that $x((0, \epsilon], x_o) \subset \text{int}(C)$.
(a2) B is continuous on $\partial C \cap \widetilde{C}$.

In the following statement, we recover [5, Theorem 6.3] as a direct consequence of Theorem 1 and Proposition 1.

Corollary 1: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold and let $B : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. Then,

- $(\star) \Rightarrow (16)$.
- When $\widetilde{C}$ is open or when (a1)-(a2) hold, $(\star) \Leftrightarrow (16)$.

□

B. The case of Lipschitz and regular functions

When the function B is locally Lipschitz, we show that (M2) is not required. Indeed, such a relaxation is possible since the generalized gradient ∂B introduced in Definition 10 will be used instead of the proximal subdifferential. Consider the following condition:

$$\langle \zeta, \eta \rangle \leq 0 \quad \forall \zeta \in \partial B(x), \ \forall \eta \in F(x) \cap T_C(x), \ \forall x \in \widetilde{C}. \tag{17}$$

Theorem 3: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) and let $B : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then,

- $(17) \Rightarrow (\star)$.
- When (M1) holds and B is regular, $(\star) \Leftrightarrow (17)$.

□

As in Corollary 1, when the solutions to $\mathcal{H}_f$ do not flow in ∂C (i.e., (a1) holds), the following condition is used.

$$\langle \zeta, \eta \rangle \leq 0 \quad \forall \zeta \in \partial B(x), \ \forall \eta \in F(x), \ \forall x \in \text{int}(C). \tag{18}$$

Corollary 2: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold and let a locally Lipschitz function $B : \mathbb{R}^n \to \mathbb{R}$. Then,

- When B is regular, $(\star) \Rightarrow (18)$.
- Conversely, when (a1) holds, $(18) \Rightarrow (\star)$.

Consequently, when B is regular and (a1) holds, $(\star) \Leftrightarrow (18)$. □

C. The case of $\mathcal{C}^1$ functions

When a function $B : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1$, $\partial B \equiv \nabla B$; hence, (17) becomes

$$\langle \nabla B(x), \eta \rangle \leq 0 \quad \forall \eta \in F(x) \cap T_C(x), \ \forall x \in \widetilde{C}. \tag{19}$$

Similarly, (18) becomes

$$\langle \nabla B(x), \eta \rangle \leq 0 \quad \forall \eta \in F(x) \cap T_C(x), \ \forall x \in \text{int}(C). \tag{20}$$

The following corollaries are in order.

Corollary 3: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) and let $B : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Then,

- $(19) \Rightarrow (\star)$.
- When (M1) holds, $(\star) \Leftrightarrow (19)$.

□

Next, using the continuity argument in Theorem 2 under (p2), we will show that (M1) is also not required.

Corollary 4: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold and let $B : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Assume further that (p2) holds. Then,

- $(\star) \Leftrightarrow (19)$.

□

Example 4: Consider the constrained system $\mathcal{H}_f = (C, F)$ introduced in Example 2. We already showed that (i) and (ii) hold. Moreover, we will show that (p2) is also satisfied. Indeed, for any $x_o \in \partial C \cap C$, i.e. $x_o = [x_{o1} \ \ 0]^\top$ for some $x_{o1} \in \mathbb{R}$, there exists $\epsilon > 0$ such that $x_\epsilon = [x_{o1} \ \ \epsilon]^\top \in \text{int}(C)$ can be made arbitrary close to x_o; thus, (p2) follows. Hence, using Corollary 4, we conclude that a $\mathcal{C}^1$ function $B : \mathbb{R}^2 \to \mathbb{R}$ satisfies $(\star)$ if and only if (19) is satisfied. □

Finally, Corollary 2 reduces to the following statement.

Corollary 5: Consider a system $\mathcal{H}_f = (C, F)$ such that (i)-(ii) hold and let a $\mathcal{C}^1$ function $B : \mathbb{R}^n \to \mathbb{R}$. Then,

- $(\star) \Rightarrow (20)$.
- When (a1) holds, $(20) \Leftrightarrow (\star)$.

□

V. Conclusion

This paper proposed characterizations of nonincreasing behavior of scalar functions along solutions to differential inclusions defined in a non-necessarily open flow set. Such a problem arises naturally from the context of study of hybrid systems. Therefore, different classes of scalar functions are considered, including lower semicontinuous, locally Lipschitz and regular, and continuously differentiable functions.

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