Hybrid systems: limit sets and zero dynamics with a view toward output regulation

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Summary. We present results on omega-limit sets and minimum phase zero dynamics for hybrid dynamical systems. Moreover, we give pointers to how these results may be useful in the future for solving the output regulation problem for hybrid systems. We highlight the main attributes of omega-limit sets and we show, under mild conditions, that they are asymptotically stable. We define a minimum phase notion in terms of omega-limit sets and establish an equivalent Lyapunov characterization. Then we study the feedback stabilization problem for a class of minimum phase, relative degree one hybrid systems. Finally, we discuss output regulation for this class of hybrid systems. We illustrate the concepts with examples throughout the paper.

1 Introduction

This paper is written as a tribute to Professor Alberto Isidori for all of the important concepts and results he has introduced in the nonlinear control systems area over his illustrious career. Following the adage that imitation is the highest form of flattery, we have chosen for this tribute to emulate some of Professor Isidori's recent results on limit sets, zero dynamics, and output regulation [4, 7, 5, 3, 6]. The novelty of our results comes from the setting that we consider: hybrid dynamical systems. These systems contain state variables that are capable of evolving continuously (flowing) and/or evolving discontinuously (jumping). In particular, systems with logical modes that interact with continuous states can be modeled in this framework. Hybrid systems have been studied in the literature for multiple decades (early notable references include [31, 28]), with the majority of progress having occurred since the early 1990s as codified, for example, in the books [30, 23, 19]. Recently, we have established mild sufficient conditions for robustness in hybrid dynamical systems [13, 14]. Along the way, these conditions have led to a generalization of results on ω-limit sets of trajectories and of LaSalle's invariance principle [26], and to general results on the existence of smooth Lyapunov functions (converse theorems) for hybrid systems [10, 9]. These results come together in the present paper, where we take inspiration from Isidori and Byrnes to establish results on Ω-limit sets (limit sets of sets of initial conditions) for hybrid systems, to show under mild conditions that these sets are asymptotically stable, to show how this notion can lead to a non-equilibrium characterization of asymptotically stable zero dynamics for hybrid systems, including converse Lyapunov theorems for a "minimum phase" property, and to give a stabilization result, related to nonlinear output regulation, for a class of minimum phase, relative degree one hybrid systems. Perhaps eventually, following the trail blazed by Professor Isidori, these results will be used for a more general theory of output regulation for hybrid systems and/or output regulation using hybrid controllers. We conclude this short introduction by noting that hybrid controllers have already appeared in the context of output regulation; see, for example, [27].

2 Hybrid dynamical systems

For the purposes of this paper, a hybrid system $\mathcal{H}$ is specified by the data (F, G, C, D) and a state space $O \subset \mathbb{R}^n$ where F is a set-valued mapping from O to $\mathbb{R}^n$ called the "flow map", G is a set-valued mapping from O to $\mathbb{R}^n$ called the "jump map", $C \subset O$ is called the "flow set" and indicates where in the state space flows may occur, $D \subset O$ is called the "jump set" and indicates from where in the state space jumps may occur.

We denote by x the state of the hybrid system $\mathcal{H}$ which can include both the so-called "continuous variables" and the so-called "discrete variables", or modes. A hybrid system $\mathcal{H}$ can be expressed as

$$\mathcal{H} \begin{cases} \dot{x} \;\; \in F(x) & x \in C \\ x^+ \in G(x) & x \in D, \end{cases}$$

which is suggestive of the meaning of solution to $\mathcal{H}$. Following [13, 14] and also [11] (cf. [1], and [21]), a solution to a hybrid system is a function defined on a hybrid time domain satisfying certain conditions. Let $\mathbb{R}_{\geq 0} := [0, +\infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \; ... \leq t_J$. The set S is a *hybrid time domain* if for all $(T, J) \in S$,

$$S \; \cap \; ([0, T] \times \{0, 1, ...J\})$$

is a compact hybrid domain. By a *hybrid arc* we understand a pair consisting of a hybrid time domain dom ϕ and a function ϕ : dom $\phi \to \mathbb{R}^n$ such that

$t \mapsto \phi(t,j)$ is locally absolutely continuous for fixed j and $(t,j) \in \text{dom}\,\phi$. We will not mention dom ϕ explicitly, but always assume that given a hybrid arc ϕ, the set dom ϕ is exactly the set on which ϕ is defined.

A hybrid arc $\phi : \text{dom}\,\phi \to O$ is a *solution to* $\mathcal{H}$ if $\phi(0,0) \in C \cup D$ and:

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t,j) \in \text{dom}\,\phi$,

$$\phi(t,j) \in C, \quad \dot{\phi}(t,j) \in F(\phi(t,j));$$

(S2) for all $(t,j) \in \text{dom}\,\phi$ such that $(t,j+1) \in \text{dom}\,\phi$,

$$\phi(t,j) \in D, \quad \phi(t,j+1) \in G(\phi(t,j)).$$

A solution is called *nontrivial* if dom ϕ contains at least one point different from $(0,0)$, *complete* if dom ϕ is unbounded, *Zeno* if it is complete but the projection of dom ϕ onto $\mathbb{R}_{\geq 0}$ is bounded, and *maximal* if it is not a truncation of another solution ϕ' to some proper subset of dom ϕ'. The notation $\mathcal{S}_{\mathcal{H}}(\mathcal{X})$ indicates the set of maximal solutions to $\mathcal{H}$ from the set of initial conditions $\mathcal{X}$. Note that when $x^0 \notin C \cup D$, $\mathcal{S}_{\mathcal{H}}(x^0) = \emptyset$.

Standing Assumption 1 (Hybrid Basic Conditions) *State space $O \subset \mathbb{R}^n$ is open; sets C and D are closed relative*[3] *to O; mappings F and G are outer semicontinuous and locally bounded*[4] *on O; $F(x)$ is nonempty and convex for all $x \in C$; $G(x)$ is nonempty and contained in O for all $x \in D$.*

These (mild) assumptions on the data of $\mathcal{H}$ are needed to guarantee that, among other properties, the sets of solutions to $\mathcal{H}$ have good sequential compactness properties.

Theorem 1. *(sequential compactness, [14, Theorem 4.4]) Let $\phi_i : \text{dom}\,\phi_i \to \mathbb{R}^n$, $i = 1, 2, ...$, be a locally eventually bounded with respect to O sequence of solutions*[5] *to $\mathcal{H}$. Then there exists a subsequence of ϕ_i's graphically converging to a solution of $\mathcal{H}$. Such a limiting solution is complete if each ϕ_i is complete, or more generally, if no subsequence of ϕ_i's has uniformly bounded domains (i.e. for any $m > 0$, there exists $i_m \in \mathbb{N}$ such that for all $i > i_m$, there exists $(t,j) \in \text{dom}\,\phi_i$ with $t + j > m$).*

We refer the reader to [14] (see also [13]) for more details on and consequences of Standing Assumption 1.

[3] A set C is closed relative to O if $C = O \cap \overline{C}$.

[4] A set-valued mapping G defined on an open set O is *outer semicontinuous* if for each sequence $x_i \in O$ converging to a point $x \in O$ and each sequence $y_i \in G(x_i)$ converging to a point y, it holds that $y \in G(x)$. It is *locally bounded* if, for each compact set $K \subset O$ there exists $\mu > 0$ such that $G(K) := \cup_{x \in K} G(x) \subset \mu\mathbb{B}$, where $\mathbb{B}$ is the open unit ball in $\mathbb{R}^n$. For more details, see [25, Chapter 5].

[5] A sequence $\{\phi_i\}_{i=1}^{\infty}$ of hybrid trajectories is *locally eventually bounded* with respect to an open set O if for any $m > 0$, there exists $i_0 > 0$ and a compact set $K \subset O$ such that for all $i > i_0$, all $(t,j) \in \text{dom}\,\phi_i$ with $t + j < m$, $\phi_i(t,j) \in K$.

A more general approach to the study of hybrid systems is to consider abstract hybrid systems given by a collection of hybrid arcs satisfying certain properties but not associated to any particular data. These abstract hybrid systems have been introduced in [26] and are called *sets of hybrid trajectories*. Sets of hybrid trajectories parallel the concept of generalized semiflows, but with elements, given by hybrid arcs (or equivalently, following [26], given by *hybrid trajectories*), that can flow and/or jump. When they satisfy the sequential compactness property stated in Theorem 1, convergence results for sets of hybrid trajectories have been presented in [26]. For the sake of simplicity, in this paper we present results for hybrid systems $\mathcal{H}$ with data (F, G, C, D) and state space O, but extensions to sets of hybrid trajectories are possible.

Regarding existence of solutions to $\mathcal{H}$, conditions were given in [14] (see also [1]) for the existence of nontrivial solutions from $C \cup D$ that are either complete or "blow up". In words, these conditions are that at every point in $C \setminus D$ flowing should be possible and at every point in D, the map G maps to $C \cup D$. These conditions are automatically satisfied when $C \cup D = O$.

In what follows, we do not necessarily assume that solutions are either complete or blow up. Moreover, given a hybrid system $\mathcal{H}$ and a set $\mathcal{Y} \subset O$ that is closed relative to O, we denote the restriction of $\mathcal{H}$ to $\mathcal{Y}$ by the hybrid system $\mathcal{H}_{|_{\mathcal{Y}}}$ which has data $(F, G, C \cap \mathcal{Y}, D \cap \mathcal{Y})$ and state space O. Note that $\mathcal{H}_{|_{\mathcal{Y}}}$ still satisfies the hybrid basic conditions.

3 Ω-limit sets

The results in this section pertain to Ω-limit sets of sets of initial conditions for hybrid dynamical systems satisfying Standing Assumption 1. They extend to these systems some of the results in [16] as specialized to finite-dimensional systems. Since the solutions to hybrid systems are often not unique, the results here resemble those for generalized semiflows in [2] and [22], where nonuniqueness of solutions to continuous-time systems is permitted.

Consider a hybrid system $\mathcal{H}$ with state space O and data (F, G, C, D) satisfying Standing Assumption 1. For a given set $\mathcal{X} \subset O$, we define the Ω-limit set of $\mathcal{X}$ for $\mathcal{H}$ as:

$$\Omega_{\mathcal{H}}(\mathcal{X}) := \Big\{ y \in \mathbb{R}^n : y = \lim_{i \to \infty} \phi_i(t_i, j_i),\ \phi_i \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}),\ (t_i, j_i) \in \operatorname{dom} \phi_i \ ,\ t_i + j_i \to \infty \Big\}.$$

Clearly, there are connections between Ω-limit sets of sets of initial conditions for hybrid systems and ω-limit sets of solutions to hybrid systems, as pursued together with various hybrid invariance principles in [26]. We do not pursue such connections here other than to observe that, letting $\omega(\phi)$ denote the ω-limit set of the solution ϕ to the hybrid system $\mathcal{H}$, we have

$$\bigcup_{x \in \mathcal{X}, \phi \in \mathcal{S}_{\mathcal{H}}(x)} \omega(\phi) \subset \Omega_{\mathcal{H}}(\mathcal{X})$$

but that the opposite set containment does not necessarily hold.

We also define, for each $i \in \mathbb{N}$,

$$\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) := \{y \in O : y = \phi(t,j),\ \phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}),\ (t,j) \in \mathrm{dom}\ \ \phi\ ,\ t+j \geq i\}\ .$$

We note that if $i' > i$ then $\mathcal{R}^{i'}_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$. Because of this, we say that the sequence of sets $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is *nested*. Below, $\mathbb{B}$ denotes the open unit ball in $\mathbb{R}^n$.

Lemma 1. *Let $\mathcal{X} \subset O$. Then*[6]

$$\Omega_{\mathcal{H}}(\mathcal{X}) = \lim_{i\to\infty} \mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) = \bigcap_i \overline{\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})}\ . \tag{1}$$

Equivalently, for each $\varepsilon > 0$ and $\rho > 0$ there exists i^* such that for all $i \geq i^*$

1. $\Omega_{\mathcal{H}}(\mathcal{X}) \cap \rho\overline{\mathbb{B}} \subset \mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) + \varepsilon\overline{\mathbb{B}}$
2. $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) \cap \rho\overline{\mathbb{B}} \subset \Omega_{\mathcal{H}}(\mathcal{X}) + \varepsilon\overline{\mathbb{B}}$.

Proof. From the very definition of the outer limit of a sequence of sets, $\Omega_{\mathcal{H}}(\mathcal{X}) = \limsup_{i\to\infty} \mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$. As the sequence $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is nested, [25, Exercise 4.3] implies that the limit $\lim_{i\to\infty} \mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ exists, and by its definition, it equals $\limsup_{i\to\infty} \mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) = \Omega_{\mathcal{H}}(\mathcal{X})$. By Exercise 4.3b in [25], it follows that $\lim_{i\to\infty} \mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is equal to $\bigcap_i \overline{\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})}$. Theorem 4.10 in [25] implies the equivalent characterization of convergence of $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ to $\Omega_{\mathcal{H}}(\mathcal{X})$. □

In what follows, we aim to clarify various attributes of the set $\Omega_{\mathcal{H}}(\mathcal{X})$. All of the subsequent attributes will be established under the assumption that the sets $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ are uniformly bounded with respect to O for large i:

Assumption 1 *The set $\mathcal{X} \subset O$ is such that the hybrid system $\mathcal{H}$ is* eventually uniformly bounded *from $\mathcal{X}$, i.e., there exist a compact set $K \subset O$ and a nonnegative integer i^* such that $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) \subset K$ for all $i \geq i^*$.*

Remark 1. The notion of eventual uniform boundedness agrees with the property defined in [16, p. 8] of a compact set (contained in O) attracting $\mathcal{X}$ under the solutions of the system $\mathcal{H}$. The papers [4, 5, 3] use the term "uniformly attracts".

Remark 2. Assumption 1 does not necessarily imply that $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is nonempty for all i. Under Assumption 1, $\Omega_{\mathcal{H}}(\mathcal{X})$ is nonempty if and only if $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is nonempty for all i.

[6] A sequence of sets $S_i \subset \mathbb{R}^n$ converges to $S \subset \mathbb{R}^n$ (i.e. $\lim_{i\to\infty} S_i = S$) if for all $x \in S$ there exists a convergent sequence of $x_i \in S_i$ such that $\lim_{i\to\infty} x_i = x$ and, for any sequence of $x_i \in S_i$ and any convergent subsequence x_{i_k}, we have $\lim_{k\to\infty} x_{i_k} \in S$. For more details, see [25, Chapter 4].

Since the sequence of sets $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is nested, it is enough to verify that $\mathcal{R}^{i^*}_{\mathcal{H}}(\mathcal{X}) \subset K$ for some nonnegative integer i^* in order to establish Assumption 1. In particular, if $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$, i.e., the reachable set from $\mathcal{X}$, is contained in a compact subset of O then $\mathcal{H}$ is eventually uniformly bounded from $\mathcal{X}$. The following examples show that it is possible for Assumption 1 to hold without $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ being bounded.

Example 1. Consider the (hybrid) system with data $F(x) = -x^3$ and $\mathcal{X} = C := \mathbb{R}$ (and $D := \emptyset$). The solutions from $\mathcal{X}$ are unique, with $|x(t)| = \frac{|x(0)|}{\sqrt{1+2x(0)^2 t}}$. Thus, $\mathcal{R}^1_{\mathcal{H}}(\mathcal{X}) \subset \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. It follows that $\mathcal{H}$ is eventually uniformly bounded from $\mathcal{X}$. △

Example 2. In the preceding example, the set $\mathcal{X}$ was not bounded whereas in this example it is. In [29] an example was given of a scalar, time-varying, locally Lipschitz, differential equation $\dot{x} = f(t,x)$, with $|f(t,x)| \leq c|x|^3$ for some real number $c > 0$, where the origin is uniformly globally attractive (meaning that for each $R > 0$ and $\varepsilon > 0$ there exists $T > 0$ such that $|x(t_\circ)| \leq R$ and $t \geq t_\circ + T$ implies $|x(t)| \leq \varepsilon$) but not uniformly globally stable. In particular, the overshoots from the set $|x(t_\circ)| = 1$ grow to infinity with $t_\circ$. Define the set-valued map $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$F(\xi) := \begin{bmatrix} -\xi_1^2 \\ f(\xi_1^{-1}, \xi_2) \end{bmatrix} \quad \forall \xi_1 \neq 0 \ , \qquad F(0,\xi_2) := \begin{bmatrix} 0 \\ [-c,c]\,\xi_2^3 \end{bmatrix}$$

and set $C = \mathbb{R}^2$, $D = \emptyset$, so that the hybrid basic conditions are satisfied. Let $\mathcal{X} = (0,1] \times [-1,1]$. Note that $\frac{d}{dt}\left(\xi_1^{-1}(t)\right) = 1$ and thus the behavior of ξ_2 matches that of the system $\dot{x} = f(t,x)$ with $t_\circ = \xi_1(0)^{-1}$. Due to the results in [29], $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is not contained in a compact subset of $\mathbb{R}^2$. Now, since $\xi_1(0) \in (0,1]$ and $\dot{\xi}_1(t) = -\xi_1^2$, we have $\xi_1(t) \in (0,1]$ for all $t \geq 0$. Using uniform global attractivity, there exists an integer i^* such that $|\xi_2(t)| \leq 1$ for all $t \geq i^*$ and all $\xi(0) \in \mathcal{X}$. Thus, $\mathcal{R}^{i^*}_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{X}$, the latter being contained in the compact set $[0,1] \times [-1,1]$. It follows that $\mathcal{H}$ is eventually uniformly bounded from $\mathcal{X}$. △

Example 3. In the preceding example, the set $\mathcal{X}$ was not compact and the system did not exhibit finite escape times from $\mathcal{X}$. Consider the hybrid system $F(x) = x^3$, $C = \{0\} \cup [1/2, \infty)$, $G(1/8) = \{0,1\}$, $G(x) = 0$ otherwise, $D = (-\infty, 1/2]$ and take $\mathcal{X} = [-1/4, 1/4]$. The solutions first jump either to zero or, if initialized at 1/8, possibly to one. From $x = 0$, the solution remains at zero for all hybrid time, either flowing or jumping. From $x = 1$, the solutions escape to infinity in one unit of time. It follows that $\mathcal{R}^2_{\mathcal{H}}(\mathcal{X}) = \{0\}$ and thus $\mathcal{H}$ is eventually uniformly bounded from $\mathcal{X}$. △

The following proposition gives a realistic scenario in which Assumption 1 is equivalent to the assumption that $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is contained in a compact subset of O.

Proposition 1. *Under Assumption 1, if $\overline{\mathcal{X}}$ is a compact subset of O and every maximal solution starting in $\overline{\mathcal{X}}$ either has an unbounded hybrid time domain or is bounded with respect to O then $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is contained in a compact subset of O.*

Proof. By Assumption 1, there exist a compact set $K \subset O$ and a nonnegative integer i^* such that $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) \subset K$ for all $i \geq i^*$. Then $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \subset R_{\leq i^*}(\overline{\mathcal{X}}) \cup K$ where

$$R_{\leq i^*}(\overline{\mathcal{X}}) := \left\{ \phi(t,j) : \phi(t,j) \in \mathcal{S}_{\mathcal{H}}(\overline{\mathcal{X}}),\ (t,j) \in \text{dom } \ \phi\ ,\ t+j \leq i^* \right\} \ .$$

To finish the proof, it suffices to show that $R_{\leq i^*}(\overline{\mathcal{X}})$ is contained in a compact subset of O. Suppose that this fails, i.e., that there exists a sequence of solutions ϕ_i with dom ϕ_i such that $(t,j) \in$ dom ϕ_i implies $t+j \leq i^*$ and such that ϕ_i are not uniformly bounded (with respect to O). Now, the arguments from the proof of Theorem 4.6 in [14] can be repeated essentially with no change, to build a solution that is not complete yet is unbounded (with respect to O). This is a contradiction. □

We will now focus on invariance properties for $\Omega_{\mathcal{H}}(\mathcal{X})$. We say that a set $O_1 \subset O$ is *weakly backward invariant* if for each $q \in O_1$, $N > 0$, there exist $x^0 \in O_1$ and at least one $\phi \in \mathcal{S}_{\mathcal{H}}(x^0)$ such that for some $(t^*, j^*) \in$ dom ϕ, $t^* + j^* \geq N$, we have $\phi(t^*, j^*) = q$ and $\phi(t,j) \in O_1$ for all $(t,j) \preceq (t^*, j^*)$, $(t,j) \in$ dom ϕ. This definition was used to characterize invariance properties for the ω-limit set of a hybrid trajectory in [26]. A similar property, but for continuous-time systems, is called "negative semi-invariance" in [22, Definition 5].

We say that a set $O_1 \subset O$ is *strongly pre-forward invariant* if, for each $x^0 \in O_1$ and each $\phi \in \mathcal{S}_{\mathcal{H}}(x^0)$, we have $\phi(t,j) \in O_1$ for all $(t,j) \in$ dom ϕ. The prefix "pre" is used here since we do not assume that maximal solutions starting in O_1 have an unbounded hybrid time domain.

The next theorem asserts that, under Assumption 1, the properties of weak backward invariance and uniform attractivity from $\mathcal{X}$ are generic for $\Omega_{\mathcal{H}}(\mathcal{X})$. These results parallel some of the results in [22, Theorem 1] for continuous-time, generalized semiflows. The result below also gives a condition for strong pre-forward invariance, which parallels a part of [2, Lemma 3.4] for continuous-time, generalized semiflows.

Theorem 2. *Under Assumption 1, the set $\Omega_{\mathcal{H}}(\mathcal{X})$ is contained in O, compact, weakly backward invariant, and for each $\varepsilon > 0$ there exists i^* such that, for all $i \geq i^*$, $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) + \varepsilon\overline{\mathbb{B}}$ and $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) \subset \Omega_{\mathcal{H}}(\mathcal{X}) + \varepsilon\overline{\mathbb{B}}$. If, in addition, $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \cup \mathcal{X}$ then $\Omega_{\mathcal{H}}(\mathcal{X})$ is strongly pre-forward invariant.*

Remark 3. Note that if $\mathcal{X} \subset C \cup D$ then $\mathcal{X} \subset \mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$. Otherwise, neither the containment $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{X}$ nor the containment $\mathcal{X} \subset \mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ necessarily holds.

Proof. (Theorem 2) Since the sequence of sets $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is nested, its limit exists and is given by (1). Then, $\Omega_{\mathcal{H}}(\mathcal{X})$ is closed. By Assumption 1, $\Omega_{\mathcal{H}}(\mathcal{X})$ is bounded with respect to O. Then, it follows that $\Omega_{\mathcal{H}}(\mathcal{X})$ is compact and a subset of O. By Assumption 1, using Theorem 4.10 in [25], for each $\varepsilon > 0$ there exists i^* such that, for all $i \geq i^*$, $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) + \varepsilon\overline{\mathbb{B}}$ and $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) \subset \Omega_{\mathcal{H}}(\mathcal{X}) + \varepsilon\overline{\mathbb{B}}$.

To show that $\Omega_{\mathcal{H}}(\mathcal{X})$ is weakly backward invariant, let $x^* \in \Omega_{\mathcal{H}}(\mathcal{X})$ be arbitrary (note that when $\Omega_{\mathcal{H}}(\mathcal{X}) = \emptyset$ there is nothing to check). By Assumption 1, there exists a compact set $K \in O$ and a nonnegative index i^* such that $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) \subset K$ for all $i \geq i^*$. We will not relabel this sequence and assume that $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) \subset K$ for all $i > 0$. Then, with the definition of $\Omega_{\mathcal{H}}(\mathcal{X})$ in (1), there exists a sequence $x_i \in \mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ with $x_i \to x^*$ as $i \to \infty$. Let $N > 0$ be arbitrary. Then, for each $l = i - N$, $i > N$, there exists a sequence of solutions $\phi_l \in \mathcal{S}_{\mathcal{H}}(\mathcal{X})$ such that $\phi_l(t_i, j_i) = x_i$ with $t_i + j_i \geq i$. From ϕ_l, generate another sequence of solutions, which we will not relabel, by truncating the hybrid time domain of each solution so that $\phi_l(t, j) \in \mathcal{R}^l_{\mathcal{H}}(\mathcal{X})$ for all $(t, j) \in \text{dom}\ \phi_l$. Note that ϕ_l is nontrivial for every l. Note also that by the construction above, the sequence $\{\phi_l\}_{l=1}^{\infty}$ is an uniformly bounded sequence of solutions; in particular, it is locally eventually bounded. By Theorem 1 , there exists a graphically convergent subsequence, that we will not relabel, converging to a solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{X})$. By construction, ϕ has the property that for some $(t^*, j^*) \in \text{dom}\ \phi$ such that $t^* + j^* \geq N$, $\phi(t^*, j^*) = x^*$ and $\phi(t, j) \in \Omega_{\mathcal{H}}(\mathcal{X})$ for all $(t, j) \in \text{dom}\ \phi$ satisfying $(0, 0) \preceq (t, j) \preceq (t^*, j^*)$. Weak backward invariance of $\Omega_{\mathcal{H}}(\mathcal{X})$ is shown since this holds for every point in $\Omega_{\mathcal{H}}(\mathcal{X})$ and every $N > 0$.

We now show that under the assumption that $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \cup \mathcal{X}$, $\Omega_{\mathcal{H}}(\mathcal{X})$ is strongly pre-forward invariant. By contradiction, suppose that there exist $q \in \Omega_{\mathcal{H}}(\mathcal{X})$ and $\bar{\phi} \in \mathcal{S}_{\mathcal{H}}(q)$ so that for some $(\bar{t}, \bar{j}) \in \text{dom}\ \bar{\phi}$, $\bar{z} = \bar{\phi}(\bar{t}, \bar{j}) \notin \Omega_{\mathcal{H}}(\mathcal{X})$. By weak backward invariance of $\Omega_{\mathcal{H}}(\mathcal{X})$, for each $N > 0$ there exists $\phi'(0, 0) \in \Omega_{\mathcal{H}}(\mathcal{X})$ and $\phi' \in \mathcal{S}_{\mathcal{H}}(\phi'(0, 0))$ such that for some $(t', j') \in \text{dom}\ \phi'$, $t' + j' \geq N$, we have $\phi'(t', j') = q$ and $\phi'(t, j) \in \Omega_{\mathcal{H}}(\mathcal{X})$ for all $(t, j) \preceq (t', j')$, $(t, j) \in \text{dom}\ \phi'$. Define $\phi(t, j) := \phi'(t, j)$ for each $(t, j) \in \text{dom}\ \phi'$, $(t, j) \prec (t', j')$, and $\phi(t, j) := \bar{\phi}(t, j)$ for each (t, j) such that $(t - t', j - j') \in \text{dom}\ \bar{\phi}$, $(t - t', j - j') \preceq (\bar{t}, \bar{j})$. Let $t^* = t' + \bar{t}$, $j^* = j' + \bar{j}$ and note that $\phi(t^*, j^*) = \bar{z}$. By construction, ϕ is bounded. Construct in this way a sequence of bounded solutions ϕ_i with $\phi_i(0, 0) \in \Omega_{\mathcal{H}}(\mathcal{X})$ and $\phi_i(t^*_i, j^*_i) = \bar{z}$ where $t^*_i + j^*_i \geq i$ for each i. By construction, $\lim_{i\to\infty} \phi_i(t^*_i, j^*_i) = \bar{z}$. Let $x^0 = \lim_{i\to\infty} \phi_i(0, 0)$. By compactness of $\Omega_{\mathcal{H}}(\mathcal{X})$, $x^0 \in \Omega_{\mathcal{H}}(\mathcal{X})$. Then, by assumption, $x^0 \in \mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \cup \mathcal{X}$. Suppose that $x^0 \in \mathcal{X}$. By definition of $\Omega_{\mathcal{H}}(\mathcal{X})$, $\bar{z} \in \Omega_{\mathcal{H}}(\mathcal{X})$ which is a contradiction. Suppose instead that $x^0 \in \mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$. By definition of $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ there exists $\tilde{x}^0 \in \mathcal{X}$ and $\tilde{\phi} \in \mathcal{S}_{\mathcal{H}}(\tilde{x}^0)$ such that $\tilde{\phi}(\tilde{t}, \tilde{j}) = x^0$ for some $(\tilde{t}, \tilde{j}) \in \text{dom}\ \tilde{\phi}$. Define $\phi''_i(t, j) := \tilde{\phi}(t, j)$ for each $(t, j) \in \text{dom}\ \tilde{\phi}$, $(t, j) \prec (\tilde{t}, \tilde{j})$, and $\phi''_i(t, j) := \phi_i(t - \tilde{t}, j - \tilde{j})$ for each $(t - \tilde{t}, j - \tilde{j}) \in \text{dom}\ \phi_i$, $(t, j) \succeq (\tilde{t}, \tilde{j})$, $(t - \tilde{t}, j - \tilde{j}) \in \text{dom}\ \phi_i$. By construction, $\tilde{x}^0 \in \mathcal{X}$ and $\lim_{i\to\infty} \phi''(t^*_i + \tilde{t}, j^*_i + \tilde{j}) = \bar{z}$. Then $\bar{z} \in \Omega_{\mathcal{H}}(\mathcal{X})$ which is also a contradiction. □

The next examples show that if the extra condition for strong pre-forward invariance, $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \cup \mathcal{X}$, is removed, then strong pre-forward invariance may fail.

Example 4. Consider the hybrid system with data $F(x) = -x$, $C = \mathbb{R}$, $G(x) = 1$, $D = \{0\}$ and take $\mathcal{X} = \{-1\}$. It is not difficult to verify that $\Omega_{\mathcal{H}}(\mathcal{X}) = \{0\}$. However, there is a solution starting at the origin that jumps to the value one, thus leaving $\Omega_{\mathcal{H}}(\mathcal{X})$, before flowing back toward the origin. Thus, $\Omega_{\mathcal{H}}(\mathcal{X})$ is not strongly (pre-)forward invariant. △

Example 5. This example is a purely continuous-time system. Consider the (hybrid) system with data $F(x) = -x$ for $x < 0$ and $F(x) = x^{1/3}$ for $x \geq 0$, $C = \mathbb{R}$ and $D = \emptyset$. With $\mathcal{X} = \{-1\}$, we again have $\Omega_{\mathcal{H}}(\mathcal{X}) = \{0\}$. However, there is a solution starting at the origin satisfying $\phi(t, 0) = (2t/3)^{3/2}$ for all $t \geq 0$. Thus, $\Omega_{\mathcal{H}}(\mathcal{X})$ is not strongly (pre-)forward invariant. △

The preceding examples motivate considering a weaker notion of forward invariance, as an alternative to strong (pre-)forward invariance. It would make sense to define weak *pre*-forward invariance, i.e., to require the existence of a solution remaining in the set that is nontrivial but not necessarily complete, at least at points where nontrivial solutions exist. However, the condition on completeness of solutions we give below actually implies more than what a weak *pre*-forward invariance notion would; it actually guarantees the existence of a complete solution remaining in $\Omega_{\mathcal{H}}(\mathcal{X})$ (this follows by weak backward invariance of $\Omega_{\mathcal{H}}(\mathcal{X})$). Therefore, our definition will actually insist on the existence of one *complete* solution remaining in the set. In this way, following [26], we say that a set $O_1 \subset O$ is *weakly forward invariant* if for each $x \in O_1$, there exists at least one complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ with $\phi(t, j) \in O_1$ for all $(t, j) \in \operatorname{dom} \phi$. We note that, in the context of continuous-time, generalized semiflows, the reference [2] combines weak forward invariance and weak backward invariance into a single property called quasi-invariance.

The next examples show that weak forward invariance can fail without extra assumptions, beyond Assumption 1.

Example 6. This example shows that there may not be any nontrivial solutions from points in $\Omega_{\mathcal{H}}(\mathcal{X})$. Consider the system with data $F(x) = x-1$, $C = [1, 2]$, $D = \emptyset$ and take $\mathcal{X} = C$. Then it is not difficult to verify that $\Omega_{\mathcal{H}}(\mathcal{X}) = \mathcal{X}$ but that from the point $x = 2$, which belongs to $\Omega_{\mathcal{H}}(\mathcal{X})$, there are no nontrivial solutions. △

Example 7. This example shows that it is possible to have the existence of nontrivial solutions but not one that remains in $\Omega_{\mathcal{H}}(\mathcal{X})$. Consider the hybrid system with data $F(x) = -x$, $C = [-1, 1]$, $G(x) = 10 + x$, $D = [-1, 1] \cup \{10\}$ and $\mathcal{X} = \{1\}$. It is not difficult to verify that $\Omega_{\mathcal{H}}(\mathcal{X}) = \{0, 10\}$ but from the point $x = 10$ there is only one solution and it jumps to the value 20, i.e., leaves $\Omega_{\mathcal{H}}(\mathcal{X})$, which doesn't belong to $C \cup D$. △

Example 8. This example shows that weak forward invariance can fail even when the system is a purely continuous-time system with constraints. Consider the system with data $F(x) = \begin{bmatrix} 0 & x_2 - 1 \end{bmatrix}^T$, $C = \{x \in \mathbb{R}^2 : x_1 \geq 0 \text{ or } x_2 \geq 0\}$, $D = \emptyset$ and take $\mathcal{X} = \{x \in C : x_1 < 0\}$. It is not difficult to verify that $\Omega_{\mathcal{H}}(\mathcal{X}) = \{x \in C : x_1 \leq 0\ , x_2 \geq 0\}$. Thus, the origin belongs to $\Omega_{\mathcal{H}}(\mathcal{X})$. There is only one solution starting at the origin and it immediately leaves $\Omega_{\mathcal{H}}(\mathcal{X})$ by virtue of the x_2 component of the solution becoming negative. △

In order to guarantee weak forward invariance of $\Omega_{\mathcal{H}}(\mathcal{X})$, we will assume that the hybrid system $\mathcal{H}$ is *eventually complete from* $\mathcal{X}$, i.e., there exists a nonnegative integer i^* such that, for all $i \geq i^*$, every maximal solution starting in $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ has an unbounded hybrid time domain. (Note: this still doesn't guarantee that $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is nonempty for all i and thus still doesn't guarantee that $\Omega_{\mathcal{H}}(\mathcal{X})$ is nonempty.) Since the sequence of sets $\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})$ is nested, it is enough to verify this property for solutions starting in $\mathcal{R}^{i^*}_{\mathcal{H}}(\mathcal{X})$ for some nonnegative integer i^*. Example 3 has already shown that it is possible for $\mathcal{H}$ to be eventually complete from $\mathcal{X}$ without being complete from $\mathcal{X}$.

We will see that eventual completeness combined with the previous condition for strong pre-forward invariance will guarantee strong pre-forward invariance with complete solutions. We say that a set $O_1 \subset O$ is *strongly forward invariant* if it is strongly pre-forward invariant and each maximal solution starting in O_1 is complete, i.e., has an unbounded hybrid time domain.

The next theorem establishes weak forward invariance under Assumption 1 and the assumption that the system $\mathcal{H}$ is eventually complete from $\mathcal{X}$. This parallels a part of [22, Lemma 3.4] on continuous-time, generalized semiflows.

Theorem 3. *Under Assumption 1, if the hybrid system $\mathcal{H}$ is eventually complete from $\mathcal{X}$ then $\Omega_{\mathcal{H}}(\mathcal{X})$ is weakly forward invariant. If, in addition, $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \cup \mathcal{X}$ then $\Omega_{\mathcal{H}}(\mathcal{X})$ is strongly forward invariant.*

Proof. Let $x^* \in \Omega_{\mathcal{H}}(\mathcal{X})$ be arbitrary. By definition of the limit and the assumptions, there exists a sequence $x_{i_k} \in \mathcal{R}^{i_k}_{\mathcal{H}}(\mathcal{X})$, $k = 1, 2, \ldots$, with $x_{i_k} \to x^*$ as $k \to \infty$, complete solutions $\phi_{i_k} \in \mathcal{S}_{\mathcal{H}}(x_{i_k})$ satisfying $\operatorname{rge} \phi_{i_k} \subset \mathcal{R}^{i_k}_{\mathcal{H}}(\mathcal{X})$ for each k, and a compact set $K \subset O$ such that $\mathcal{R}^{i_k}_{\mathcal{H}}(\mathcal{X}) \subset K$ for each k. Then, the sequence of solutions $\{\phi_{i_k}\}_{k=1}^{\infty}$ is an uniformly bounded sequence of solutions; in particular, it is locally eventually bounded. By Theorem 1, there exists a graphically convergent subsequence, that we will not relabel, converging to a complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(x^*)$. Let $(\tilde{t}, \tilde{j}) \in \operatorname{dom} \phi$ be arbitrary. By the graphical convergence of ϕ_{i_k} to ϕ, there exists a sequence $(\tilde{t}_{i_k}, \tilde{j}_{i_k}) \in \operatorname{dom} \phi_{i_k}, (\tilde{t}_{i_k}, \tilde{j}_{i_k}) \to (\tilde{t}, \tilde{j})$ such that $\phi_{i_k}(\tilde{t}_{i_k}, \tilde{j}_{i_k}) \to \phi(\tilde{t}, \tilde{j})$ as $k \to \infty$. By construction, $\lim_{k \to \infty} \phi_{i_k}(\tilde{t}_{i_k}, \tilde{j}_{i_k}) \in \Omega_{\mathcal{H}}(\mathcal{X})$. Therefore, for every $(\tilde{t}, \tilde{j}) \in \operatorname{dom} \phi$, $\phi(\tilde{t}, \tilde{j})$ is in $\Omega_{\mathcal{H}}(\mathcal{X})$. Thus $\Omega_{\mathcal{H}}(\mathcal{X})$ is weakly forward invariant.

Strong forward invariance of $\Omega_{\mathcal{H}}(\mathcal{X})$ with the additional assumption $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) \cup \mathcal{X}$ follows from Theorem 2 and the eventually completeness assumption. □

Below we state results on the Ω-limit sets of the restriction of $\mathcal{H}$ to some subset in the state space O.

Proposition 2. *If the set $\mathcal{Y} \subset O$ is closed relative to O and strongly pre-forward invariant for $\mathcal{H}$, then $\Omega_{\mathcal{H}|_{\mathcal{Y}}}(\mathcal{X}) = \Omega_{\mathcal{H}}(\mathcal{X} \cap \mathcal{Y})$.*

Proof. Clearly, $\Omega_{\mathcal{H}|_{\mathcal{Y}}}(\mathcal{X} \cap \mathcal{Y}) = \Omega_{\mathcal{H}|_{\mathcal{Y}}}(\mathcal{X})$. Since the set $\mathcal{Y} \subset O$ is closed relative to O and strongly pre-forward invariant for $\mathcal{H}$, we have $\Omega_{\mathcal{H}}(\mathcal{X} \cap \mathcal{Y}) = \Omega_{\mathcal{H}|_{\mathcal{Y}}}(\mathcal{X} \cap \mathcal{Y})$.

Theorem 4. *Suppose Assumption 1 holds. Define $M := \Omega_{\mathcal{H}}(\mathcal{X})$. Then $M \subset \Omega_{\mathcal{H}|_M}(M)$.*

Remark 4. The opposite containment, $\Omega_{\mathcal{H}|_M}(M) \subset M$, does not necessarily hold as demonstrated by Example 4 or Example 7. Clearly, the only way this containment can fail is if M is not forward invariant for the system $\mathcal{H}|_M$. This requires jumps from M that leave M, as in the referenced examples.

Proof. (Theorem 4) Let $x \in \Omega_{\mathcal{H}}(\mathcal{X})$. Since $\Omega_{\mathcal{H}}(\mathcal{X})$ is weakly backward invariant by Theorem 2, for each $i > 0$ there exists a solution ϕ_i of $\mathcal{H}$ and a hybrid time $(t_i, j_i) \in \operatorname{dom} \phi_i$ with $t_i + j_i \geq i$ such that $\phi_i(t_i, j_i) = x$ and $\phi_i(\tau, k) \in \Omega_{\mathcal{H}}(\mathcal{X})$ for all $(\tau, k) \in \operatorname{dom} \phi_i$ with $\tau + k \leq t_i + j_i$. By definition, we verify that $x \in \Omega_{\mathcal{H}|_M}(M)$. □

The following corollaries of Theorem 4 are related to [4, Lemma 4.1] and the reduction principle for Ω-limit sets given in [3, Lemma 5.2].

Corollary 1. *Suppose Assumption 1 holds, $\mathcal{Z} \subset O$, and that $\mathcal{Y} \subset O$ is closed relative to O. If $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y} \cap \mathcal{Z}$ then $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \Omega_{\mathcal{H}|_{\mathcal{Y}}}(\mathcal{Z})$.*

Proof. Let $M = \Omega_{\mathcal{H}}(\mathcal{X})$. Theorem 4 gives $M \subset \Omega_{\mathcal{H}|_M}(M)$. By the assumption $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y} \cap \mathcal{Z}$, we have $\Omega_{\mathcal{H}|_M}(M) \subset \Omega_{\mathcal{H}|_{\mathcal{Y}}}(\mathcal{Z})$. These relationships establish the results. □

Corollary 2. *Suppose Assumption 1 holds, $\mathcal{Z} \subset O$ is compact, and that $\mathcal{Y} \subset O$ is closed relative to O. Suppose, for the system $\mathcal{H}$ that, for each $\varepsilon > 0$ there exists $T > 0$ such that for each $x \in \mathcal{X}$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$, and each $(t, j) \in$ dom ϕ with $t + j \geq T$, we have $|\phi(t, j)|_{\mathcal{Y} \cap \mathcal{Z}} \leq \varepsilon$ [7]. Then $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y} \cap \mathcal{Z}$. In particular, we have $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \Omega_{\mathcal{H}|_{\mathcal{Y}}}(\mathcal{Z})$.*

Remark 5. Corollary 2 is a useful tool for stability analysis in cascade-connected systems (for example, recovering Corollaries 10.3.2 and 10.3.3 in [18]).

[7] In what follows, given $x \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$, $|x|_S := \inf\{|x - s| : s \in S\}$.

Proof. By assumption, $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \overline{\mathcal{Z}}$. Since $\mathcal{Z} \subset O$ is compact, we have $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Z}$. Next we argue $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y}$ by contradiction. Suppose there exists $z \in \Omega_{\mathcal{H}}(\mathcal{X})$ such that $z \notin \mathcal{Y}$. Then we have two cases to consider. If $z \in \overline{\mathcal{Y}} \setminus \mathcal{Y}$, then $z \in \partial O$ since $\mathcal{Y}$ is closed relative to O, and then we have $z \in \Omega_{\mathcal{H}}(\mathcal{X}) \cap \partial O$, which contradicts Assumption 1. If $z \notin \overline{\mathcal{Y}}$, then there exists $\varepsilon > 0$ such that $(z+2\varepsilon\overline{B}) \cap \overline{\mathcal{Y}} = \emptyset$; by assumption, there exists $T > 0$ such that for each $x \in \mathcal{X}$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$, and each $(t, j) \in \text{dom } \phi$ with $t + j \geq T$, we have $|\phi(t, j)|_{\overline{\mathcal{Y}}} \leq \varepsilon$, which contradicts to the combination of $z \in \Omega_{\mathcal{H}}(\mathcal{X})$ and $(z + 2\varepsilon\overline{B}) \cap \overline{\mathcal{Y}} = \emptyset$. Therefore, we conclude that $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y}$. By Corollary 1 we establish the results. □

The properties established in Theorems 2, 3 and 4 above parallel analogous results for continuous-time dynamical systems, as summarized in [16, Chapter 2], that have been fundamental to the work in [4, 7, 5, 3]. In subsequent sections, we will use these properties in a manner that parallels how results for continuous-time systems were used in these latter references. In particular, we will show how these results impact robust stability and control results for hybrid systems. The next section addresses the notion of asymptotic stability that we use.

4 Pre-asymptotically stable compact sets

Pre-asymptotic stability (pre-AS) is a generalization of standard asymptotic stability to the setting where completeness or even existence of solutions is not required. Pre-AS was introduced in [9] as an equivalent characterization of the existence of a smooth Lyapunov function for a hybrid system. It is a natural stability notion for hybrid systems, since often the set $C \cup D$ does not cover the state space O and because local existence of solutions is sometimes not guaranteed. As we will see subsequently, not insisting on local existence of solutions can make it easier to characterize certain dynamic properties, such as the minimum phase property, and to give stronger converse Lyapunov theorems for such properties.

Consider the hybrid system $\mathcal{H}$. Let $\mathcal{A} \subset O$ be compact. We say that

- $\mathcal{A}$ is *pre-stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \text{dom } \phi$;
- $\mathcal{A}$ is *pre-attractive* for $\mathcal{H}$ if there exists $\delta > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is bounded with respect to O and if it is complete then $\phi(t,j) \to \mathcal{A}$ as $t + j \to \infty$;
- $\mathcal{A}$ is *uniformly pre-attractive* if there exists $\delta > 0$ and for each $\varepsilon > 0$ there exists $T > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is bounded with respect to O and $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \text{dom } \phi$ satisfying $t + j \geq T$;
- $\mathcal{A}$ is *pre-asymptotically stable* if it is both pre-stable and pre-attractive;

- ($\mathcal{A}$ is *asymptotically stable* if it is pre-asymptotically stable and there exists $\delta > 0$ such that any maximal solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is complete.)

The set of all $x \in C \cup D$ from which all solutions are bounded with respect to O and the complete ones converge to $\mathcal{A}$ is called the *pre-basin of attraction* of $\mathcal{A}$.

Clearly, these stability definitions cover classical stability notions. They also cover some unexpected situations, such as in the following example.

Example 9. Consider the (hybrid) system $\dot{x} = Ax, x \in C$ where $A \in \mathbb{R}^{2\times 2}$ has complex eigenvalues with positive real part and $C := \left\{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\right\}$ (and $D := \emptyset$). Because of the structure of the matrix A, there is a number $T > 0$ such that solutions to $\dot{x} = Ax$ starting on the unit circle in the set C can flow for no more than T units of time before leaving C. It follows from homogeneity that no solutions are complete and thus the origin is pre-attractive, in fact, uniformly pre-attractive. Moreover, defining $c := \exp(AT)$, we have $|\phi(t,0)| \leq c|\phi(0,0)|$. Thus the origin is pre-stable. In summary, the origin is pre-asymptotically stable with pre-basin of attraction given as C. △

The following results come from [9] and are used to establish many of the subsequent statements in this paper.

Lemma 2. *For system $\mathcal{H}$, if the compact set $\mathcal{A} \subset O$ is strongly pre-forward invariant and uniformly pre-attractive, then $\mathcal{A}$ is pre-asymptotically stable.*

Lemma 3. *Let the set $O_1 \subset O$ be open, and let the set $\mathcal{A} \subset O_1$ be nonempty and compact. For system $\mathcal{H}$, the following statements are equivalent:*

- *The set $\mathcal{A}$ is pre-asymptotically stable with pre-basin of attraction containing $O_1 \cap (C \cup D)$, and O_1 is strongly pre-forward invariant;*
- *For each function $\omega : O_1 \to \mathbb{R}_{\geq 0}$ that is a proper indicator*[8] *for $\mathcal{A}$ on O_1, there exists a smooth Lyapunov function $V : O_1 \to \mathbb{R}_{\geq 0}$ for $(O_1, F, G, C, D, \omega)$ on O_1, that is, there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that*

$$\begin{aligned} \alpha_1(\omega(x)) \leq V(x) &\leq \alpha_2(\omega(x)) && \forall x \in O_1 \ , \\ \max_{f \in F(x)} \langle \nabla V(x), f \rangle &\leq -V(x) && \forall x \in O_1 \cap C \ , \\ \max_{g \in G(x)} V(g) &\leq e^{-1} V(x) && \forall x \in O_1 \cap D \ . \end{aligned}$$

Lemma 4. *For system $\mathcal{H}$, if the compact set $\mathcal{A} \subset O$ is pre-asymptotically stable, then its pre-basin of attraction is open relatively to $C \cup D$, and there exists an open set $O_1 \subset O$ that is strongly pre-forward invariant and equals to $O_1 \cap (C \cup D)$.*

[8] Given an open set O_1 containing a compact set $\mathcal{A}$, a continuous function $\omega : O_1 \to \mathbb{R}_{\geq 0}$ is *proper* on O_1 if $\omega(x_i) \to \infty$ when x_i converge to the boundary of O_1 or $|x_i| \to \infty$, and is a *proper indicator for $\mathcal{A}$ on O_1* if it is proper on O_1 and satisfies $\{x \in O_1 : \omega(x) = 0\} = \mathcal{A}$.

The combination of Lemmas 3 and 4 not only provides Lyapunov characterizations of pre-asymptotic stability (and even a strong result on converse Lyapunov theorems for pre-asymptotic stability), but also allows us to establish an equivalent Lyapunov characterization of hybrid systems with pre-asymptotically stable zero-dynamics in the next section.

In the following result we give sufficient conditions for $\Omega_{\mathcal{H}}(\mathcal{X})$ to be pre-asymptotically stable for hybrid systems.

Theorem 5. *Suppose Assumption 1 holds. If the set $\mathcal{X} \subset O$ is such that each solution starting in $\mathcal{X}$ is bounded with respect to O and $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathrm{int}(\mathcal{X})$ then $\Omega_{\mathcal{H}}(\mathcal{X})$ is a compact pre-asymptotically stable set with pre-basin of attraction containing $\mathcal{X} \cap (C \cup D)$.*

Corollary 3. *Suppose for the system $\mathcal{H}$ that there exist $T > 0$ and compact sets $\mathcal{X} \subset O$ and $\mathcal{X}_o \subset O$ such that $\mathcal{X}_o \subset \mathrm{int}(\mathcal{X})$ and, each solution ϕ starting in $\mathcal{X}$ is bounded with respect to O and $\phi(t,j) \in \mathcal{X}_o$ for all $(t,j) \in \mathrm{dom}\ \phi$ with $t+j \geq T$. Then Assumption 1 holds and $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{X}_o \subset \mathrm{int}(\mathcal{X})$. In particular, $\Omega_{\mathcal{H}}(\mathcal{X})$ is a compact pre-asymptotically stable set with pre-basin of attraction containing $\mathcal{X} \cap (C \cup D)$.*

Proof. (Theorem 5) Since $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathrm{int}(\mathcal{X})$, Theorem 2 says that $\Omega_{\mathcal{H}}(\mathcal{X})$ is compact and strongly pre-forward invariant, and that for each $\varepsilon > 0$ there exists i^* such that, for all $i \geq i^*$, we have $\mathcal{R}^{i^*}_{\mathcal{H}}(\mathcal{X}) \subset \Omega_{\mathcal{H}}(\mathcal{X}) + \varepsilon\overline{\mathbb{B}}$. Since each solution starting in $\mathcal{X}$ is bounded with respect to O, we verify that $\Omega_{\mathcal{H}}(\mathcal{X})$ is uniformly pre-attractive with pre-basin of attraction containing $\mathcal{X} \cap (C \cup D)$. Lemma 2 implies the result. □

Theorem 5 parallels the stability result for omega limit sets of sets in [16, Lemma 2.0.1] for continuous-time nonlinear systems (see also [4, Lemma 2.1]).

It is obvious that the weaker condition $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{X}$ does not imply pre-asymptotic stability for $\Omega_{\mathcal{H}}(\mathcal{X})$. For example, consider any Lipschitz differential equation where the origin is an unstable equilibrium point and take $\mathcal{X} = \{0\}$. It is even possible to have $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{X}$ and have $\Omega_{\mathcal{H}}(\mathcal{X})$ globally attractive without having $\Omega_{\mathcal{H}}(\mathcal{X})$ pre-asymptotically stable. For example, consider the system in [15, pp. 191-194] where the origin is globally attractive but not stable.

The following result gives sufficient conditions for pre-asymptotic stability of $\Omega_{\mathcal{H}_{|_\mathcal{Y}}}(\mathcal{X})$ (recall that $\mathcal{H}_{|_\mathcal{Y}} = (F, G, C \cap \mathcal{Y}, D \cap \mathcal{Y})$).

Proposition 3. *Suppose Assumption 1 holds and let $\mathcal{Y}$ be closed relative to O. Suppose, for the hybrid system $\mathcal{H}$, each solution starting in $\mathcal{X} \cap \mathcal{Y}$ is bounded with respect to O, and $\Omega_{\mathcal{H}}(\mathcal{X} \cap \mathcal{Y}) \subset \mathrm{int}(\mathcal{X})$. Then the hybrid system $\mathcal{H}_{|_\mathcal{Y}}$ is eventually uniformly bounded from $\mathcal{X}$, each solution of $\mathcal{H}_{|_\mathcal{Y}}$ starting in $\mathcal{X}$ is bounded with respect to O and $\Omega_{\mathcal{H}_{|_\mathcal{Y}}}(\mathcal{X}) \subset \mathrm{int}(\mathcal{X})$. In particular, for the* *system* $\mathcal{H}_{|_{\mathcal{Y}}}$*,* $\Omega_{\mathcal{H}_{|_{\mathcal{Y}}}}(\mathcal{X})$ *is a compact pre-asymptotically stable set with pre-basin of attraction containing* $\mathcal{X} \cap \mathcal{Y} \cap (C \cup D)$ [9]*.*

Proof. Since the solution set of $\mathcal{H}_{|_{\mathcal{Y}}}$ is contained in that of $\mathcal{H}$, Assumption 1 immediately gives that $\mathcal{H}_{|_{\mathcal{Y}}}$ is eventually uniformly bounded from $\mathcal{X}$, and the assumption that each solution starting in $\mathcal{X} \cap \mathcal{Y}$ is bounded with respect to O implies that each solution of $\mathcal{H}_{|_{\mathcal{Y}}}$ starting in $\mathcal{X}$ is bounded with respect to O. Moreover, we observe that $\Omega_{\mathcal{H}_{|_{\mathcal{Y}}}}(\mathcal{X}) = \Omega_{\mathcal{H}_{|_{\mathcal{Y}}}}(\mathcal{X} \cap \mathcal{Y}) \subset \Omega_{\mathcal{H}}(\mathcal{X} \cap \mathcal{Y})$, which, as well as the assumption $\Omega_{\mathcal{H}}(\mathcal{X} \cap \mathcal{Y}) \subset \mathrm{int}(\mathcal{X})$, gives $\Omega_{\mathcal{H}_{|_{\mathcal{Y}}}}(\mathcal{X}) \subset \mathrm{int}(\mathcal{X})$. By Theorem 5 we establish the results. □

We emphasize that none of the assumptions in the results above have guaranteed that $\Omega_{\mathcal{H}}(\mathcal{X})$ is nonempty. One may ask, in the case when $\Omega_{\mathcal{H}}(\mathcal{X})$ is empty, when one can still guarantee the existence of a compact, pre-asymptotically stable set contained in the interior of $\mathcal{X}$ with pre-basin of attraction containing $\mathcal{X}$. Such a characterization is given next.

Proposition 4. *Let Assumption 1 hold. Suppose the set* $\mathcal{X} \subset O$ *is such that each solution starting in* $\mathcal{X}$ *is bounded with respect to* O *and* $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathrm{int}(\mathcal{X})$*. There exists a nonempty, compact, pre-asymptotically stable set* $\mathcal{A} \subset \mathrm{int}(\mathcal{X})$ *with pre-basin of attraction containing* $\mathcal{X} \cap (C \cup D)$ *if and only if there exists a point* $x \in \mathrm{int}(\mathcal{X})$ *such that either* $x \notin C \cup D$ *or* $\mathcal{R}^0_{\mathcal{H}}(x) \subset \mathrm{int}(\mathcal{X})$*.*

Proof. We observe that if $\Omega_{\mathcal{H}}(\mathcal{X})$ is empty, then there are no complete solutions from $\mathcal{X}$ and yet they all remain bounded with respect to O.

First we show necessity. If $\Omega_{\mathcal{H}}(\mathcal{X})$ is nonempty then the conclusion follows from Theorem 2. Suppose $\Omega_{\mathcal{H}}(\mathcal{X})$ is empty. Let $\mathcal{A} \subset \mathrm{int}\mathcal{X}$ come from the assumptions. Then we have two cases to consider. If $\mathcal{A} \cap (C \cup D) = \emptyset$, then there exists $x \in \mathrm{int}(\mathcal{X})$ such that $x \notin C \cup D$. If there exists $x \in \mathcal{A} \cap (C \cup D)$, then $\mathcal{R}^0_{\mathcal{H}}(x) \subset \mathcal{A} \subset \mathrm{int}(\mathcal{X})$.

Next we establish sufficiency. If $\Omega_{\mathcal{H}}(\mathcal{X})$ is nonempty then by Theorem 2 the conclusion holds automatically with the compact pre-asymptotically stable set $\mathcal{A}$ given by $\Omega_{\mathcal{H}}(\mathcal{X})$. Suppose $\Omega_{\mathcal{H}}(\mathcal{X})$ is empty. By assumption, let $x \in \mathrm{int}(\mathcal{X})$. Then we have two cases to consider. If $x \notin C \cup D$, we take $\mathcal{A} = \{x\}$ and this set is pre-stable and pre-attractive from $\mathcal{X} \cap (C \cup D)$. Otherwise, we take $\mathcal{A} := \mathcal{R}^0_{\mathcal{H}}(x)$ which is forward invariant, stable and pre-attractive from $\mathcal{X} \cap (C \cup D)$. □

[9] The assumptions used in Proposition 3 are related to [4, Assumption 1]. In particular, the set $\mathcal{Y}$ in the proposition below should be associated with the set $\{(z, w) : w \in W\}$ where W is characterized in [4, Assumption 0] and $\mathcal{X}$ should be associated with the set $\{(z, w) : z \in Z\}$ where Z is a set of initial conditions given in [4]. Furthermore, if $\mathcal{Y}$ is strongly (pre-)forward invariant (cf. [4, Assumption 0]), then Proposition 2 says that $\Omega_{\mathcal{H}}(\mathcal{X} \cap \mathcal{Y}) = \Omega_{\mathcal{H}_{|_{\mathcal{Y}}}}(\mathcal{X})$.

5 Minimum phase zero dynamics

In this section, we address the concept of zero dynamics and the minimum phase property for hybrid systems with inputs. For an introduction to these concepts for non-hybrid nonlinear control systems, see [17, Chapter 6].

Consider the control-hybrid system

$$\mathcal{H}^u \begin{cases} \dot{x} = f(x,u) & (x,u) \in C \\ x^+ = g(x,u) & (x,u) \in D \end{cases} \tag{2}$$

with state space $O \subset \mathbb{R}^n$, where $f : C \to \mathbb{R}^n$ and $g : D \to O$ are continuous, and $C, D \subset O \times \mathbb{R}^m$ are closed relative to $O \times \mathbb{R}^m$. Solutions of $\mathcal{H}^u$ are defined in a manner that is analogous to the definition of solutions for $\mathcal{H}$ in Section 2. The signal u is a hybrid control signal, i.e., like a hybrid arc but instead of being locally absolutely continuous in t, it only needs to be locally bounded and measurable. A solution is a pair (x, u) consisting of a hybrid arc and a hybrid control signal that share the same hybrid time domain. In particular, it is not possible to pick the domain of the hybrid control signal independently from the domain of the state trajectory.

Associate to (2) an additional constraint $(x, u) \in \mathcal{Y}$, i.e., consider the control-hybrid system $\mathcal{H}^u_{|\mathcal{Y}}$. The "zero dynamics" (of $\mathcal{H}^u$ relative to $\mathcal{Y}$) is given by the hybrid system $\mathcal{H}^u_{|\mathcal{Y}}$. Let $\mathcal{N}$ denote the class of functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ that are continuous and nondecreasing. Given $\gamma \in \mathcal{N}$, we use $\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}$ to denote the hybrid system with the data

$$\begin{aligned}
F_{\gamma,\mathcal{Y}}(x) &:= \overline{\text{co}}\,\{z \in \mathbb{R}^n : z = f(x,u),\ (x,u) \in C \cap \mathcal{Y},\ |u| \leq \gamma(|x|)\}\ , \\
G_{\gamma,\mathcal{Y}}(x) &:= \{z \in \mathbb{R}^n : z = g(x,u),\ (x,u) \in D \cap \mathcal{Y},\ |u| \leq \gamma(|x|)\}\ , \\
C_{\gamma,\mathcal{Y}} &:= \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ such that } (x,u) \in C \cap \mathcal{Y},\ |u| \leq \gamma(|x|)\}\ , \\
D_{\gamma,\mathcal{Y}} &:= \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ such that } (x,u) \in D \cap \mathcal{Y},\ |u| \leq \gamma(|x|)\}\ .
\end{aligned} \tag{3}$$

Let $\mathcal{Y}_* := \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ such that } (x,u) \in \mathcal{Y},\ |u| \leq \gamma(|x|)\}$. The zero dynamics of $\mathcal{H}^u$ relative to $\mathcal{Y}$ is said to be *(robustly) pre-asymptotically stable from the compact set* $\mathcal{X} \subset O$ if, for each $\gamma \in \mathcal{N}$, the system $\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}$ is such that each solution starting in $\mathcal{X}$ is bounded with respect to O and $\Omega_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(\mathcal{X}) \subset \text{int}(\mathcal{X}) \cap \mathcal{Y}_*$; moreover, if $\Omega_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(\mathcal{X})$ is empty then there exists $x \in \text{int}(\mathcal{X}) \cap \mathcal{Y}_*$ such that either $x \notin C_{\gamma,\mathcal{Y}} \cup D_{\gamma,\mathcal{Y}}$ or $\mathcal{R}^0_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(x) \subset \text{int}(\mathcal{X}) \cap \mathcal{Y}_*$. When the zero dynamics of $\mathcal{H}^u$ relative to $\mathcal{Y}$ is (robustly) pre-asymptotically stable from the compact set $\mathcal{X} \subset O$, we will say that $\mathcal{H}^u_{|\mathcal{Y}}$ is *strongly minimum phase* relative to $\mathcal{X}$.

Example 10. Consider the nonlinear control system defined on $\mathbb{R}^3 \times \mathbb{R}$

$$\begin{aligned}\dot{x}_1 &= x_1^3 + x_1 u + x_2^2 u^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= q(x_1, x_2, x_3) + u \\ y &= x_2\end{aligned} \tag{4}$$

where $q : \mathbb{R}^3 \to \mathbb{R}$ is continuous such that

$$|x_1| > 1 \qquad \Longrightarrow \qquad x_1\left(x_1^3 - x_1 q(x_1, 0, 0)\right) < 0 \ . \tag{5}$$

To check the minimum phase property, we must consider, for each function $\gamma \in \mathcal{N}$, the behavior of the (hybrid) system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} \in \left\{ \begin{bmatrix} x_1^3 + x_1 u + x_2^2 u^2 \\ x_3 \\ q(x_1, x_2, x_3) + u \end{bmatrix} \ , \ |u| \le \gamma(|x|) \right\} \quad (x, u) \in \mathcal{Y} \tag{6}$$

where $\mathcal{Y} := \left\{(x, u) \in \mathbb{R}^3 \times \mathbb{R} : x_2 = 0\right\}$. We check the minimum phase property relative to a compact set $\mathcal{X}$ containing the set $[-1, 1] \times \{0\} \times \{0\}$ in its interior. We note that, regardless of the function γ, in order to flow in the set $\mathcal{Y}$ we must have $x_2(t) = x_3(t) = 0$ for all t in the maximal interval of definition and $\dot{x}_3(t) = 0$ for almost all such t. From this it follows that flowing is only possible from points $(x_1, 0, 0)$ such that $|q(x_1, 0, 0)| \le \gamma(|x_1|)$. Whenever flowing is possible, it must be the case that, for almost all t,

$$\dot{x}_1(t) = x_1(t)^3 - x_1(t) q(x_1(t), 0, 0) \ .$$

Using (5), the set $[-1, 1] \times \{0\} \times \{0\}$ is strongly forward invariant for (6). It also follows from (5) that, when the Ω-limit set of (6) is nonempty, it is contained in the set $[-1, 1] \times \{0\} \times \{0\}$ which, by assumption, is contained in $\mathrm{int}(\mathcal{X}) \cap \mathcal{Y}_*$. Then, the system (4) is minimum phase relative to $\mathcal{X}$. When the Ω-limit set is empty, which is the case for some functions $\gamma \in \mathcal{N}$ and q satisfying (5) (for example, consider $q(x_1, 0, 0) = 1 + 2x_1^2$ and $\gamma \equiv 0$), there are no complete solutions to (6). In turns out that, due to (5), we can take any point x the set $[-1, 1] \times \{0\} \times \{0\}$ and get that $\mathcal{R}^0_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(x) \in \mathrm{int}(\mathcal{X}) \cap \mathcal{Y}_*$. This shows that the system (4) is minimum phase relative to $\mathcal{X}$. △

Following the ideas in the example above, one can compare the zero dynamics notion given above to the description used in [4]. In the latter case, one identifies a subset of $\mathcal{Y}$, called the zero dynamics kernel, that is viable at every point and a (unique) feedback control selection that makes the zero dynamics kernel viable. In contrast, we work with the dynamics on all of $\mathcal{Y}$ and do not insist on viability. A possible advantage of the latter approach is that it leads to an equivalent Lyapunov characterization of the minimum phase property where the Lyapunov function is shown to be decreasing on all of $\mathcal{Y}$, not just on the zero dynamics kernel and not just for certain control values. This result is provided next and is a consequence of Lyapunov characterizations of pre-asymptotic stability in the last section.

Theorem 6. *Let $\mathcal{X} \subset O$ be compact. For system (2), the following statements are equivalent.*

1. *$\mathcal{H}^u_{|\mathcal{Y}}$ is strongly minimum phase relative to $\mathcal{X}$;*
2. *For each $\gamma \in \mathcal{N}$ there exists a nonempty open set $O_1 \subset O$ containing $\mathcal{X}$, and a nonempty compact set $\mathcal{A} \subset \mathrm{int}(\mathcal{X})$ such that for each proper indicator $\omega : O_1 \to \mathbb{R}_{\geq 0}$ for $\mathcal{A}$ on O_1 there exists a smooth function $V : O_1 \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that*

$$
\begin{aligned}
\alpha_1(\omega(x)) \leq V(x) &\leq \alpha_2(\omega(x)) && \forall x \in O_1 \ , \\
\langle \nabla V(x), f(x,u) \rangle &\leq -V(x) && \forall (x,u) \in C \cap \mathcal{Y},\ |u| \leq \gamma(|x|) \ , \\
V(g(x,u)) &\leq e^{-1} V(x) && \forall (x,u) \in D \cap \mathcal{Y},\ |u| \leq \gamma(|x|) \ .
\end{aligned}
$$

Proof. First we show the implication $2 \Rightarrow 1$. Let $\gamma \in \mathcal{N}$ be given. Define

$$F^o_{\gamma,\mathcal{Y}}(x) := \{ z : z = f(x,u),\ (x,u) \in C \cap \mathcal{Y},\ |u| \leq \gamma(|x|) \} \, .$$

By assumption, we know there exists a smooth Lyapunov function $V : O_1 \to \mathbb{R}_{\geq 0}$ for $(O_1, F^o_{\gamma,\mathcal{Y}}, G_{\gamma,\mathcal{Y}}, C_{\gamma,\mathcal{Y}}, D_{\gamma,\mathcal{Y}}, \omega)$. Using Carathéodory Theorem and the fact $\overline{\mathrm{co}} F^o_{\gamma,\mathcal{Y}}(x) = F_{\gamma,\mathcal{Y}}(x)$, we verify that V is also a smooth Lyapunov function for $(O_1, F^o_{\gamma,\mathcal{Y}}, G_{\gamma,\mathcal{Y}}, C_{\gamma,\mathcal{Y}}, D_{\gamma,\mathcal{Y}}, \omega)$. By the implication $2 \Rightarrow 1$ in Lemma 3, we use V to show that $\mathcal{A}$ is pre-asymptotically stable for system $\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}$, its pre-basin of attraction contains $O_1 \cap (C_{\gamma,\mathcal{Y}} \cup D_{\gamma,\mathcal{Y}})$, and O_1 is strongly pre-forward invariant. Finally, combining the fact $\Omega_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(\mathcal{X}) \subset \mathcal{A}$ and Proposition 4, we establish the statement 1.

Next we show the implication $1 \Rightarrow 2$. Let $\gamma \in \mathcal{N}$ be given. By the assumption on the strongly minimum phase property, combining Theorem 5 and Proposition 4 we infer for the hybrid system $\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}$ that there exists a nonempty compact set $\mathcal{A} \subset \mathrm{int}\mathcal{X}$ that is pre-asymptotically stable with pre-basin of attraction containing $\mathcal{X} \cap (C_{\gamma,\mathcal{Y}} \cup D_{\gamma,\mathcal{Y}})$. Combining the implication $1 \Rightarrow 2$ in Lemma 3 and Lemma 4 implies that there exists an open set $O_1 \subset O$ containing $\mathcal{X}$, and for each proper indicator ω for $\mathcal{A}$ on O_1 there exists a smooth Lyapunov function $V : O_1 \to \mathbb{R}_{\geq 0}$ for $(O_1, F_{\gamma,\mathcal{Y}}, G_{\gamma,\mathcal{Y}}, C_{\gamma,\mathcal{Y}}, D_{\gamma,\mathcal{Y}}, \omega)$. By definition, we use this V to verify the conclusions. □

There have been several alternative characterizations of minimum phase zero dynamics that have appeared in the literature. In [24], the authors provide a notion of minimum phase (relative to an equilibrium point) that again asks for viability of a zero dynamics kernel and the existence of a stabilizing control selection, but allows for other control selections that are destabilizing. The system in [24, Example 1] is minimum phase in the sense of [24] but it is not strongly minimum phase in the sense of the current paper. Compared to what we have proposed, one could call the notion in [24] a *weak* minimum phase property (like the distinction between weak and strong invariance.) It is easy to define a weak minimum phase property in the context of Ω-limit sets for hybrid systems by replacing u by a stabilizing, locally bounded feedback, forming the convex hull and considering Ω-limit sets. Lyapunov characterizations of this property would also be straightforward, with the Lyapunov function decreasing everywhere that the output is zero but only for control values close to those of the stabilizing feedback.

The minimum phase property is also addressed in [20, Definition 3] where more general notions, output-input stability [20, Definition 1] and weak uniform 0-detectability, are introduced. In output-input stability, the state and the input should be bounded by the output and its derivatives plus a function of the norm of the state that decays with time. When evolving in the set where the output is zero, so that the derivatives are also zero, this asks for convergence of the input and state to zero, which is a property that is similar to our strong minimum phase property in the case where the Ω-limit set is the origin and the functions γ are required to be zero at zero. Many interesting phenomena appear by considering dynamics outside of the output zeroing set, and this is in large part the focus of the paper [20]. Included in this work is a Lyapunov characterization of weak uniform 0-detectability, which is like output-input stability but without imposing a bound on the input. The authors of [20] also provide an example that partially motivates bounding the inputs by some function of the state in the Lyapunov characterization of the minimum phase property.

The work [12] also considers a strong minimum phase property, much like the one we have presented but for equilibria, and discusses its Lyapunov characterization. In [12], the decrease condition for the Lyapunov function is in the set where the output *and* all of its derivatives are zero, a set related to the zero dynamics kernel mentioned above. This is in contrast to our result when the Lyapunov function decreases everywhere in the set where the output is zero. Like the example mentioned in [20], [12, Example 2] again motivates restricting the size of the input as a function of the size of the state in order to get a converse Lyapunov theorem.

6 Feedback stabilization for a class of strongly minimum phase, relative degree one hybrid systems

Consider the control-hybrid system

$$\mathcal{H}^u \left\{ \begin{array}{ll} \left.\begin{array}{l} \dot{z} = \hat{f}(z,\zeta) \\ \dot{\zeta} = q(z,\zeta) + u \end{array}\right\} & (z,\zeta) \in \hat{C} \\ \\ \left.\begin{array}{l} z^+ = \hat{g}(z,\zeta) \\ \zeta^+ = r(z,\zeta) \end{array}\right\} & (z,\zeta) \in \hat{D} \end{array} \right.$$

where $z \in \mathbb{R}^{n_1}$; $\zeta, u \in \mathbb{R}^{n_2}$; $O = \mathbb{R}^{n_1+n_2}$ is the state space; $\hat{f}, \hat{g} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $q, r : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ are continuous functions and the sets $\hat{C}$ and $\hat{D}$ are closed relatively to O (per the Standing Assumption 1). We investigate the effect of the feedback control algorithm $u = -k\zeta$ with $k > 0$ to be specified. We make suitable assumptions, made explicit below, that guarantee this feedback steers ζ to zero while keeping the entire state bounded. The zero dynamics corresponding to the output $y = \zeta$ play a crucial role.

To match the notation of the previous section, we define $f = [\hat{f}\ q]^T$, $g = [\hat{g}\ r]^T$, $C := \hat{C} \times \mathbb{R}^m$ and $D := \hat{D} \times \mathbb{R}^m$. We also define $\hat{\mathcal{Y}} := \{(z,\zeta) : \zeta = 0\}$ and $\mathcal{Y} := \hat{\mathcal{Y}} \times \mathbb{R}^m$. We note that $\mathcal{Y}_\star = \hat{\mathcal{Y}}$.

Assumption 2 *Let $\mathcal{X} \subset O$ be compact. The system $\mathcal{H}^u_{|\mathcal{Y}}$ is strongly minimum phase relative to $\mathcal{X}$.*

Regardless of $\gamma \in \mathcal{N}$, as long as the solutions of $\mathcal{H}^{u\to\gamma}_{|\mathcal{Y}}$ exist, they are solutions of the hybrid system

$$\mathcal{H}_\circ \begin{cases} \left.\begin{array}{l} \dot{z} = \hat{f}(z,0) \\ \dot{\zeta} = 0 \end{array}\right\} & (z,\zeta) \in \hat{C} \cap \hat{\mathcal{Y}} \\ \\ \left.\begin{array}{l} z^+ = \hat{g}(z,0) \\ \zeta^+ = r(z,0) \end{array}\right\} & (z,\zeta) \in \hat{D} \cap \hat{\mathcal{Y}}\ . \end{cases} \tag{7}$$

Thus, to check the conditions for the strong minimum phase property, it is enough to check them for the system (7). It is worth noting that, depending on γ, the system (7) may have more solutions than the zero dynamics. This is because, for a given $\gamma \in \mathcal{N}$ and a certain z, the zero value may not belong to the set $\{q(z,0)\} + \gamma(|z|)\overline{\mathbb{B}}$. However, when $\gamma \in \mathcal{N}$ is such that $|q(z,0)| \le \gamma(|z|)$ for all z then the solutions of the zero dynamics agree with the solutions of (7).

Define $\mathcal{A}_\circ := \Omega_{\mathcal{H}_\circ}(\mathcal{X})$, where $\mathcal{X}$ is given in Assumption 2, for the case where $\Omega_{\mathcal{H}_\circ}(\mathcal{X})$ is nonempty. Otherwise, one can take $\mathcal{A}_\circ$ to be either the point in $\mathrm{int}(\mathcal{X}) \cap \hat{\mathcal{Y}}$ that is not in $\hat{C} \cup \hat{D}$ or else the reachable set for $\mathcal{H}_\circ$ from the point in $\mathrm{int}(\mathcal{X}) \cap \hat{\mathcal{Y}}$ having the property that this reachable set is contained in $\mathrm{int}(\mathcal{X}) \cap \hat{\mathcal{Y}}$. Necessarily the set $\mathcal{A}_\circ$ is pre-asymptotically stable for $\mathcal{H}_\circ$. Let O_1 be the largest open set such that, for the system $\mathcal{H}_\circ$, the pre-basin of attraction for $\mathcal{A}_\circ$ is $O_1 \cap (\hat{C} \cup \hat{D}) \cap \hat{\mathcal{Y}}$. We note that $(z,\zeta) \in O_1$ does not put a restriction on ζ.

In addition to the strong minimum phase assumption, we make some simplifying assumptions on the functions q, r, $\hat{f}$ and $\hat{g}$ in order to give the flavor for the kinds of results that are possible. With a good knowledge of the nonlinear control literature, the reader may be able to see the directions in which these assumptions can be relaxed, especially in light of the converse Lyapunov theorem for the strong minimum phase property, as given in Theorem 6. (Also see the discussion in Section 7.)

We let $K \subset O_1$ denote a compact set over which we expect the closed-loop system to operate. It can, and should, be chosen to contain a neighborhood of $\mathcal{A}_\circ$. In order to state the assumptions succinctly, we make the definitions

$$F(z,\zeta,u) := \begin{bmatrix} \hat{f}(z,\zeta) \\ q(z,\zeta)+u \end{bmatrix} , \qquad G(z,\zeta) := \begin{bmatrix} \hat{g}(z,\zeta) \\ r(z,\zeta) \end{bmatrix} .$$

Assumption 3 *There exist $c > 0$ and $\delta > 0$ such that*

1. *a)* $(z,\zeta) \in K \cap \hat{D} \quad \Longrightarrow \quad |r(z,\zeta)| \le c|\zeta|$;
   b) $(z,\zeta) \in \left(K \cap \hat{C}\right) + \delta\mathbb{B} \quad \Longrightarrow \quad |q(z,\zeta)| \le c|\zeta|$;
2. *There exists a closed set $\hat{D}_e \subset O$ such that $\hat{D} \subset \hat{D}_e$ and $G(\hat{D}) \cap \hat{D}_e = \emptyset$;*
3. *for almost all $(z,\zeta) \in \left(K \cap \hat{C}\right) + \delta\mathbb{B}$ and all u such that $\langle \zeta, u\rangle \le 0$,*

$$-\langle \nabla |(z,\zeta)|_{\hat{D}_e}, F(z,\zeta,u)\rangle \le c . \tag{8}$$

Remark 6. The condition (8) is certainly satisfied when $|(z,\zeta)|_{\hat{D}_e}$ is independent of ζ, i.e., when the jump condition depends only on z. The condition in the third item guarantees that the flow for ζ, which can be controlled, is given enough time to dominate the jump behavior of ζ. In particular, it rules out Zeno solutions for the closed-loop control system.

Theorem 7. *Under Assumptions 2-3, there exists $k^* \ge 0$ such that for each $k \ge k^*$, using the feedback control law $u = -k\zeta$ in the control system $\mathcal{H}^u$ results in the following property: The set $\mathcal{A}_\circ$ is pre-asymptotically stable with pre-basin of attraction containing the set of all initial conditions having the property that the ensuing solutions remain in the set K.*

Proof. Using the continuity of G and the closedness of D, we note that $G(D)$ is a closed set. It then follows from Assumption 3 that there exists $\varepsilon > 0$ such that

$$|G(z,\zeta)|_{D_e} \ge \varepsilon \qquad (z,\zeta) \in K \cap D . \tag{9}$$

Let $\rho : \mathbb{R}_{\ge 0} \to [1/(2\exp(1)c^2), 1]$ (we assume, without loss of generality, that $1/(2\exp(1)c^2) \le 1$) be globally Lipschitz and non-increasing, with $\rho(0) = 1$ and

$$\rho(\varepsilon) \le \exp(-1)/c^2 . \tag{10}$$

It follows that there exists $m > 0$ such that, for almost all $s \in \mathbb{R}_{\ge 0}$,

$$-m\rho(s) \le \frac{d}{ds}\rho(s) \le 0 . \tag{11}$$

Now consider the locally Lipschitz, partial control-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ defined as

$$V(z,\zeta) := \rho\left(|(z,\zeta)|_{D_e}\right)|\zeta|^2 . \tag{12}$$

Using (10), (9), and the facts that $\rho(0) = 1$ and ρ is non-increasing, we get, for all $(z,\zeta) \in D \cap K$,

$$V(G(z,\zeta)) = \rho\left(|G(z,\zeta)|_{D_e}\right)|r(z,\zeta)|^2 \le \rho(\varepsilon)c^2|\zeta|^2 \le \exp(-1)|\zeta|^2 \\ \le \exp(-1)\rho\left(|(z,\zeta)|_{D_e}\right)|\zeta|^2 = \exp(-1)V(z,\zeta)\ . \tag{13}$$

Also, using (11) and the second item of Assumption 3 and the control $u = -k\zeta$, we get that, for almost all $(z,\zeta) \in (K \cap C) + \delta\mathbb{B}$,

$$\langle \nabla V(z,\zeta), F(z,\zeta,u)\rangle \le cm\rho\left(|(z,\zeta)|_{D_e}\right)|\zeta|^2 - 2\rho\left(|(z,\zeta)|_{D_e}\right)[k-c]|\zeta|^2 \\ = -\left[2k - c(m+2)\right]V(z,\zeta)\ . \tag{14}$$

The conditions (13)-(14) together with the definition (12) and the range of the function ρ give that, for any

$$k \ge \frac{1+c(m+2)}{2} =: k^*\ ,$$

for all solutions and all time for which solutions are defined and remain in K, we have

$$\begin{aligned}|\zeta(t,j)|^2 &\le 2\exp(1)c^2 V(z(t,j),\zeta(t,j)) \\ &\le 2\exp(1)c^2\exp(-t-j)V(z(0,0),\zeta(0,0)) \\ &\le 2\exp(1)c^2\exp(-t-j)|\zeta(0,0)|^2\ .\end{aligned}$$

By construction of K, there exists $\varepsilon' > 0$ such that $\mathcal{A}_\circ + \varepsilon'\mathbb{B} \subset K$. Note that Assumption 2 and Theorem 5 imply that the set $\mathcal{A}_\circ$ is pre-asymptotically stable for $\mathcal{H}^{u\to\gamma}_{|\mathcal{Y}}$ with open pre-basin of attraction $O_1 \cap (C \cup D) \cap \mathcal{Y}$ (Assumption 1 holds for $\mathcal{H}^{u\to\gamma}_{|\mathcal{Y}}$ by Assumption 2). Then, there exists $\delta' > 0$ such that all solutions to $\mathcal{H}_\circ$ starting from $O_1 \cap (C \cup D) \cap \mathcal{Y}$ stay ε'-close to $\mathcal{A}_\circ$. Now, consider a perturbation to $\mathcal{H}_\circ$ as given in (7) on the ζ component of the state with perturbation magnitude equal to $\rho > 0$. The perturbed hybrid system, denoted by $\mathcal{H}^\rho_\circ$, satisfies the (CP) conditions in [14], and by Theorem 6.6 there in , there exists $\delta'' > 0$ such that all solutions to $\mathcal{H}^\rho_\circ$ from $K \cap (C \cup D)$ are ε'-close to $\mathcal{A}_\circ$. Then, the set $\mathcal{A}_\circ$ is pre-stable for $\mathcal{H}^u$ by picking $\tilde{\delta} < \min(\delta', \delta'', \sqrt{\varepsilon'/(2\exp(1)c^2)})$. Finally, pre-attractivity from the specified set of initial conditions follows from Corollary 2. □

In the proof of this result, we use the positive semidefinite Lyapunov function $V(z,\zeta) := \rho(|(z,\zeta)|_{\hat{D}_e})|\zeta|^2$ to establish, under the stated assumptions, that there exist $k^* \ge 0$ and an uniform bound on the ζ component of all solutions to $\mathcal{H}^u$ remaining in K when using the control law $u = -k\zeta$, $k \ge k^*$ and that trajectories remaining in K converge uniformly to $\hat{\mathcal{Y}}$. Then Corollary 2 and Theorem 5 are used to draw the stated conclusion.

Example 11. Consider the hybrid system given by

$$\mathcal{H}^u \left\{ \begin{array}{ll} \left.\begin{array}{l} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -g \end{bmatrix} \\ \begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} u \\ u \end{bmatrix} \end{array} \right\} & (z,\zeta) \in \hat{C} := \{(z,\zeta) : z_1 \geq 0, \zeta_1 = \zeta_2\} \\ \\ \left.\begin{array}{l} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^+ = \begin{bmatrix} a \\ 0 \end{bmatrix} \\ \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^+ = \begin{bmatrix} \zeta_1 + \eta(\zeta_1) \\ \zeta_2 + \eta(\zeta_2) \end{bmatrix} \end{array} \right\} & (z,\zeta) \in \hat{D} := \{(z,\zeta) : z_1 = 0, z_2 \leq 0\} \ , \end{array} \right.$$

where $z, \zeta \in \mathbb{R}^2$, $u \in \mathbb{R}$, $\eta : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying $\eta(0) = 0$, $a > 0, g > 0$, and the state space is given by $O := \mathbb{R}^4$. Let the control law be given by $u = -k\zeta$. The hybrid system $\mathcal{H}_u$ can be interpreted as a simplified model of an actuated particle, with horizontal position given by ζ_1, moving on a concave-shaped surface and experiencing impacts with a free-falling particle, with height z_1, vertical velocity z_2, and horizontal position ζ_2. In this setting, the goal is to stabilize the horizontal position of the actuated particle to $\zeta_1 = 0$ under the effect of the impacts with the free-falling particle, which occur when $z_1 = 0$ and $z_2 \leq 0$, and affect the position of the actuated particle by $\eta(\zeta_1)$. The horizontal position of the free-falling particle (ζ_2) tracks the position of the actuated particle (ζ_1) to guarantee the collision. At impacts, the free-falling particle is repositioned to the height given by a with zero vertical velocity. In the simplified model given above, the actuated particle moves only horizontally but the effect of the free-falling particle impacting with the actuated particle on a concave-shaped surface are captured in the function η. In this particular physical situation, the function η will be such that it has the same sign as its argument. Note that we do not need to assume this as our result hold for more general functions η.

The solutions to the zero dynamics of $\mathcal{H}^u$, denoted by $\mathcal{H}^u_{|\mathcal{Y}}$ with $\mathcal{Y} = \hat{\mathcal{Y}} \times \mathbb{R}$, $\hat{\mathcal{Y}} = \{(z,\zeta) : \zeta = 0\}$, are such that $\zeta = 0$ and the z-component of the solutions is reset to $[a\ 0]^T$, then flows until the jump set is reached, and then it is reset to $[a\ 0]^T$ from where this evolution is repeated. (For the illustration given by the physical system above, the solutions to $\mathcal{H}^u_{|\mathcal{Y}}$ are such that the actuated particle stays at $\zeta_1 = 0$ and the free-falling particle, with $\zeta_2 = 0$, falls from $z_1 = a$ with zero velocity, then impacts with the actuated particle, and then is reset to $z_1 = a$, $z_2 = 0$ again for another free fall.) We now check that Assumption 2 holds. For any compact set $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \subset O$, $\mathcal{X}_1, \mathcal{X}_2 \subset \mathbb{R}^2$, such that $\mathcal{X}_1$ contains a neighborhood of $[0, a] \times [-\sqrt{2ag}, 0]$ and $\mathcal{X}_2$ contains a neighborhood of $\{0\} \subset \mathbb{R}^2$, the zero dynamics of the system $\mathcal{H}^u$ relative to $\mathcal{Y}$ is (robustly) pre-asymptotically stable (in fact, the omega limit set $\Omega_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(\mathcal{X})$ can be explicitly computed to check that it is nonempty and satisfies $\Omega_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(\mathcal{X}) \subset \text{int}(\mathcal{X}) \cap \hat{\mathcal{Y}}$). Let $K \subset O$ be compact. We now check Assumption 3. Items 1.a and 1.b hold by inspection. Item 2 holds with $\hat{D}_e = \hat{D}$ since after every jump we have $z_1 = a > 0$. Item 3 automatically holds since $|(z,\zeta)|_{\hat{D}_e}$ is independently of ζ. Note that Assumption 3 holds for every set $K \subset O$. It follows by Theorem 7 that there exists $k^* > 0$ such that the set $\mathcal{A}_\circ = \Omega_{\mathcal{H}^{u \to \gamma}_{|\mathcal{Y}}}(\mathcal{X})$ is pre-asymptotically stable for the hybrid system $\mathcal{H}^u$ with $u = -k\zeta$, $k \geq k^*$.

7 Comments on output regulation and conclusions

We conclude this paper by comparing the assumptions of the previous section to the assumptions that are in place in the (non-hybrid) output regulation [4] problem after a preliminary compensator is introduced to cancel the term $q(z,0)$, found in the $\dot{\zeta}$ equation, as the term evolves along solutions of the zero dynamics. (See the immersion assumptions in [4, 5] and the relaxation in [6]; we acknowledge that we have not given any thought to accomplishing this preliminary step in the context of hybrid systems).

First we note that, even in the presence of a Poisson stable exosystem, our strong minimum phase assumption holds under [4, Assumption 1]. This can be achieved by restricting the flow (and jump) sets to the forward invariant set W of [4, Assumption 0] and recognizing that our strong minimum phase property is expressed in terms of *pre*-asymptotic stability, so that there is nothing to check for solutions that start outside of W. Moreover, with the forward invariance assumption on W and the other assumptions in [4] the Ω-limit set for $\mathcal{H}_\circ$ (see (7)) is non-empty the dynamics restricted to the zero dynamics kernel is complete and bounded.

Thus, the main extra condition we are assuming is that

$$|q(z,\zeta)| = 0 \qquad \forall (z,\zeta) \in C \cap \mathcal{Y}$$

whereas the preliminary steps in output regulation only provide that

$$|q(z,\zeta)| = 0 \qquad \forall (z,\zeta) \in C \cap \mathcal{Y} \cap \mathcal{A}_\circ \ .$$

(See, for example, the assumptions in [7, Proposition 4.1].) With such a relaxed assumption, and using the control $u = -k\zeta$, the interconnection of z and ζ will not behave like a cascade of systems, like it did in the previous section. Like in the continuous-time case, the analysis for the full interconnection would then require either a small gain argument or a full-state Lyapunov argument. We do not pursue such an approach here for hybrid systems, but we do mention that such arguments for general hybrid systems are in preparation. We also add here that, unlike in the non-hybrid case (see [7]), exponential stability for hybrid systems with Lipschitz data does not necessarily imply local input-to-state stability with finite gain (see the counterexample in [8, Example 1]). Thus, exponential stability for the zero dynamics will not guarantee that one can achieve asymptotic stability for $\mathcal{A}_\circ$ using a feedback of the form $u = -k\zeta$. Nevertheless, even without exponential stability for the zero dynamics, a nonlinear feedback of ζ should be able to achieve the goals of output regulation: driving ζ to zero while keeping the full state bounded.

We conjecture that the preliminary steps of output regulation can be solved for a class of minimum phase, relative degree one hybrid systems, like those considered in the previous section, and that emerging tools for the analysis of interconnected hybrid systems will permit concluding output regulation results that parallel what is known in the continuous-time case.

The present paper should at least put into place the pieces related to the characterization of Ω-limit sets that are required to start tackling output regulation for hybrid systems.

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