Hybrid Model Predictive Control

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Abstract

Model predictive control is a powerful technique to solve problems with constraints and optimality requirements. Hybrid model predictive control further expands its capabilities by allowing for the combination of continuous-valued with discrete-valued states, and continuous dynamics with discrete dynamics or switching. This short article summarizes techniques available in the literature addressing different aspects of hybrid MPC, including those for discrete-time piecewise affine systems, discrete-time mixed logical dynamical systems, linear systems with periodic impulses, and hybrid dynamical systems.

keywords model predictive control, receding horizon control, hybrid systems, hybrid control, optimization

I. Definition

Model predictive control (MPC) ubiquitously incorporates mathematical models, constraints, and optimization schemes to solve control problems. Through the use of a mathematical model describing the evolution of the system to control and the environment, MPC strategies determine the value of the control inputs to apply by solving an optimization problem over a finite time horizon. The system to control, which is known as the *plant*, is typically given by a state-space model with state, input, and outputs. Most MPC strategies **measure** the output of the plant and, using a mathematical model of the plant, **predict** the possible values of the state and the output in terms of free input variables to be determined. The prediction is performed over a *prediction horizon* which is defined as a finite time interval in the future. With such parameterized values of the state and input over the prediction horizon, MPC **solves** an optimization problem. To formulate the optimization control problem (OCP), a cost functional is to be defined. The OCP can also incorporate constraints, such as those on the state, the input, and the outputs of the plant. Then, MPC **selects** an input signal that solves the OCP and **applies** it to the plant for a finite amount of time, which is defined by the *control horizon*. After the input is applied over the control horizon, the process is repeated.

II. Motivation

Over the past two decades, MPC has emerged as one of the preferred planning and control engines in robotics and control applications. Such grown interest in MPC has been propelled by recent advances in computing and optimization, which have led to significantly smaller computation times in the solution to certain optimization problems. However, the complexity of the dynamics, constraints, and control objectives in the MPC has increased so as to accommodate the specifics of the applications of interest. In particular, the dynamics of the system to control might substantially change over the region of operation, the constraints or the control objectives may abruptly change upon certain self-triggered or externally-triggered events, and the optimization problems may involve continuous-valued and discrete-valued variables. MPC problems with such features have been studied in the literature under the name *hybrid model predictive control* [17]. Within a common framework and notation, this entry summarizes several hybrid MPC approaches available in the literature.

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III. NOTATION AND PRELIMINARIES

A. Notation

The set of nonnegative integers is denoted as $\mathbb{N} := \{0, 1, 2 \ldots\}$ and the set of positive integers as $\mathbb{N}_{>0} := \{1, 2, \ldots\}$. Given $N \in \mathbb{N}_{>0}$, we define $\mathbb{N}_{<N} := \{0, 1, 2, \ldots, N-1\}$ and $\mathbb{N}_{\leq N} := \{0, 1, 2, \ldots, N\}$. The set of real numbers is denoted $\mathbb{R}$ and, given $n \in \mathbb{N}_{>0}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space. The set of nonnegative real numbers is $\mathbb{R}_{\geq 0} := [0, \infty)$ and the set of positive real numbers is denoted as $\mathbb{R}_{>0} := (0, \infty)$. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm and given $p \in [1, \infty]$, $|x|_p$ denotes its p-norm.

B. Discrete-time, Continuous-time, and Hybrid Systems

A general nonlinear discrete-time system takes the form

$$x^+ = f(x, u) \tag{1}$$
$$y = h(x, u) \tag{2}$$

where x is the state, u the input, and y the output. The symbol $^+$ on x is to indicate, when a solution to the system is computed, the new value of the state after each discrete-time step. The variable $k \in \mathbb{N}$ denotes discrete time. Given an input signal $\mathbb{N} \ni k \mapsto u(k)$, the solution to (1) is given by the function $\mathbb{N} \ni k \mapsto x(k)$ that satisfies

$$x(k+1) = f(x(k), u(k)) \qquad \forall k \in \mathbb{N}$$

Similarly, the output associated to this solution is given by

$$y(k) = h(x(k), u(k)) \qquad \forall k \in \mathbb{N}$$

A general nonlinear continuous-time system takes the form

$$\dot{x} = f(x, u) \tag{3}$$
$$y = h(x, u) \tag{4}$$

where x is the state, u the input, and y the output. The dot on x is to indicate the change of x with respect to ordinary time. The discrete variable t denotes discrete time. Given an input signal $\mathbb{R}_{\geq 0} \ni t \mapsto u(t)$, a solution to (3) is given by a function $\mathbb{R}_{\geq 0} \ni t \mapsto x(t)$ that is regular enough to satisfy

$$\frac{d}{dt} x(t) = f(x(t), u(t))$$

at least for almost all $t \in \mathbb{R}_{\geq 0}$. The output associated to this solution is given by

$$y(t) = h(x(t), u(t)) \qquad \forall t \in \mathbb{R}_{\geq 0}$$

A hybrid dynamical system takes the form

$$\mathcal{H} \ : \ \begin{cases} \dot{x} = F(x, u) & (x, u) \in C \\ x^+ = G(x, u) & (x, u) \in D \end{cases} \tag{5}$$

where x is the state and u the input. The set C defines the set of points in the state and input space from which flows are possible according to the differential equation $\dot{x} = F(x, u)$. The function F is called the flow map. The set D defines the set of points in the state and input space from where jumps are possible according to the difference equation $x^+ = G(x, u)$. The function G is called the jump map G. Similar to the model for continuous-time systems in (3)-(4), the dot on x is to indicate the change of x with respect to ordinary time. Ordinary time is denoted by the parameter t, which takes values from $\mathbb{R}_{\geq 0}$. As a difference to the model for discrete-time systems in (1)-(2), the symbol $^+$ on x denotes the new value of the state after a jump is triggered. Every time that a jump occurs, the discrete counter j is incremented by one. A solution to the hybrid system can flow or jump, according to the values of the state and of the input relative to C and D, respectively. Solutions to $\mathcal{H}$ are parameterized by t and j, and are defined on hybrid time domains. The hybrid time domain of a solution x is denoted $\operatorname{dom} x$, which is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ and has the following structure: there exist a real nondecreasing sequence $\{t_j\}_{j=0}^{J}$ with $t_0 = 0$, and when J is finite, $t_{J+1} \in [t_J, \infty]$ such that $\operatorname{dom} x = \cup_{j=0}^{J} I_j \times \{j\}$. Here, $I_j = [t_j, t_{j+1}]$ for all $j < J$. If J is finite, I_{J+1} can take the form $[t_J, t_{J+1}]$ (when $t_{J+1} < \infty$), or possibly $[t_J, t_{J+1})$. It is convenient to consider the solution and the input as a pair (x, u) defined on the same hybrid time domain, namely, $\operatorname{dom}(x, u) = \operatorname{dom} x = \operatorname{dom} u$. Given an input signal $(t, j) \mapsto u(t, j)$, a function $(t, j) \mapsto x(t, j)$ is a solution to the hybrid system if, over intervals of flow, it is locally absolutely continuous and satisfies

$$\frac{d}{dt}x(t,j) \in F(x(t,j), u(t,j))$$

when

$$(x(t,j), u(t,j)) \in C$$

and, at jump times, it satisfies

$$x(t,j+1) \in G(x(t,j), u(t,j))$$

when

$$(x(t,j), u(t,j)) \in D$$

IV. Hybrid Model Predictive Control Strategies

Most MPC strategies in the literature are for plants given in terms of discrete-time models of the form $x^+ = f(x, u)$, where x is the state and u is the input. In particular, the case of f being linear has been widely studied and documented Cite Encyclopedia entry about pure MPC. The cases in which the function f is discontinuous, or when either x or u have continuous-valued and discrete-value components require special treatment. In the literature, such cases are referred to as *hybrid*: either the system or the MPC strategy are said to be hybrid. Another use of the term hybrid is when the control algorithm obtained from MPC is nonsmooth; e.g., when the MPC strategy selects a control law from a family or when the MPC strategy is implemented using sample and hold. Finally, the term hybrid has been also employed in the literature to indicate that the plant exhibits state jumps at certain events. In such settings, the plant is modeled by the combination of continuous dynamics and discrete dynamics. In this section, after introducing basic definitions and notions, we present hybrid MPC strategies in the literature, starting from those that are for purely discrete-time systems (albeit with nonsmooth right-hand side) and ending at those for systems that are hybrid due to the combination of continuous and discrete dynamics.

A. *MPC for discrete-time piecewise affine systems*

Piecewise affine (PWA) systems in discrete time take the form

$$x^+ = A_i x + B_i u + f_i \tag{6}$$

$$y = C_i x + D_i u \tag{7}$$

$$\text{subject to } x \in \Omega_i, u \in \mathcal{U}_i(x), i \in S \tag{8}$$

where x is the state, u the input, and y the output. The origin of (6)-(8) is typically assumed to be an equilibrium state for zero input u. The set $S := \{1, 2, \ldots, \overline{s}\}$ with $\overline{s} \in \mathbb{N}_{>0}$ is a finite set to index the different values of the matrices and constraints. For each $i \in S$, the constant matrices $(A_i, B_i, f_i, C_i, D_i)$ define the dynamics of x and the output y. The elements f_i of the collection is such that $f_i = 0$ for all $i \in S$ such that $0 \in \Omega_i$. For each $i \in S$, the state constraints are determined by Ω_i and the state-dependent input constraints by $\mathcal{U}_i(x)$. The collection $\{\Omega_i\}_{i=1}^{\overline{s}}$ is a collection of polyhedra such that

$$\bigcup_{i \in S} \Omega_i = \mathcal{X}$$

where $\mathcal{X} \subset \mathbb{R}^n$ is the region of operation of interest. Moreover, the polyhedra are typically such that their interiors do not intersect, namely,

$$\text{int}(\Omega_i) \cap \text{int}(\Omega_j) = \emptyset \qquad \forall i, j \in S \ : \ i \neq j$$

For each $x \in \Omega_i$, the set $\mathcal{U}_i(x)$ defines the allowed values for the input u.

For this class of systems, an MPC strategy consists of

1) At the current state of the plant, solve the OCP over a discrete prediction horizon;
2) Apply the optimal control input over a discrete control horizon;
3) Repeat.

The OCP associated to this strategy is as follows: given

- the current state x_0 of (6)-(8),
- a prediction horizon $N \in \mathbb{N}_{>0}$,
- a terminal constraint set $\mathcal{X}_f$,
- a stage cost $\mathcal{L}$, and
- a terminal cost $\mathcal{F}$

the MPC problem consists of minimizing the cost functional

$$\mathcal{J}(x, i, u) := \sum_{k=0}^{N-1} \mathcal{L}(x(k), i(k), u(k)) + \mathcal{F}(x(N))$$

over solutions $k \mapsto x(k)$, indices sequence $k \mapsto i(k)$, and inputs $k \mapsto u(k)$ subject to

$$x(0) = x_0 \tag{9}$$

$$x(N) \in \mathcal{X}_f \tag{10}$$

$$\forall k \in \mathbb{N}_{<N} : \qquad x(k+1) = A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \tag{11}$$

$$\forall k \in \mathbb{N}_{\leq N} : \qquad y(k) = C_{i(k)}x(k) + D_{i(k)}u(k) \tag{12}$$

$$x(k) \in \Omega_{i(k)} \tag{13}$$

$$u(k) \in \mathcal{U}_{i(k)}(x(k)) \tag{14}$$

$$i(k) \in S \tag{15}$$

A few observations about the OCP above are in order. The solution $k \mapsto x(k)$ is uniquely defined by x_0 and $k \mapsto (i(k), u(k))$. The condition in (9) imposes the initial condition for the solution $k \mapsto x(k)$, while (10) restricts the value of x at the end of the prediction horizon. The condition in (11) is to enforce that $k \mapsto x(k)$ is a solution for the input $k \mapsto u(k)$. Similarly, (12) imposes that an output is generated according to the dynamics of the PWA system in (6)-(8). The conditions in (13) and (14) enforce the state and (state-dependent) input constraints associated with the PWA system. Finally, (15) forces the indexing signal $k \mapsto i(k)$ to belong to the finite set of possible indices S. Typical choices of the functions $\mathcal{L}$ and $\mathcal{F}$ in the cost functional $\mathcal{J}$ are

$$\mathcal{L}(x, i, u) := |Q_i x|_p + |R_i u|_p, \qquad \mathcal{F}(x) := |Px|_p$$

for some $p \in [1, \infty]$, where, for each $i \in S$, Q_i and R_i, and P are matrices of appropriate dimensions.

Further Reading: For key properties of the MPC problem for PWA systems in this section the reader is referred to [11]. In particular, sufficient conditions for recursive feasibility and asymptotic stability of the origin of the PWA system appeared in [11, Theorem III.2]. In that reference, the reader can also find insight on how to select the terminal cost and the terminal constraint set; see [11, Section IV and Section V]. The same reference also provides insight on how to solve the OCP using off-the-shelf tools. Very importantly, when $p = 1$ or $p = \infty$, the OCP can be rewritten as a mixed integer linear program (MILP). Furthermore, when the stage and terminal costs are quadratic, the OCP can be rewritten as a mixed integer quadratic program (MIQP).

B. MPC for discrete-time systems with continuous and discrete-valued states

Mixed Logical Dynamical (MLD) systems are discrete-time systems involving states, inputs, and outputs that have continuous-valued and discrete-valued components. MLD systems are given by

$$x^+ = Ax + B_1u + B_2\delta + B_3z + B_4 \tag{16}$$
$$y = Cx + D_1u + D_2\delta + D_3z + D_4 \tag{17}$$
$$\text{subject to } E_2\delta + E_3z \leq E_1u + E_4x + E_5 \tag{18}$$

where x is the state, u the input, y the output, z continuous-valued auxiliary variables, and δ discrete-valued auxiliary variables. The state, the input, and the output have continuous-valued and discrete-valued components; for example, certain components take values from Euclidean spaces while others from discrete sets, like $\{0, 1\}$. The partition of the state x is typically given by $x = (x_c, x_\ell)$, with x_c being the continuous-valued components and x_ℓ the discrete-valued components of x; similarly for the input $u = (u_c, u_\ell)$ and for the output $y = (y_c, y_\ell)$. The matrices A, $\{B_i\}_{i=1}^3$, B_4, C, $\{D_i\}_{i=1}^3$, and D_4 define the dynamics of x and the output y. The matrices $\{E_i\}_{i=1}^5$ are used in (18) to define constraints coupling the continuous-valued and discrete-valued states. The latter matrices can be properly defined to recast constraints into properties of discrete-valued states. For instance, when $x \in [-x_{\max}, x_{\max}]$ with $x_{\max} \geq 0$, the constraint $x \geq 0$ is equivalent to $\delta = 1$ (and, in turn, $x < 0$ is equivalent to $\delta = 0$) when, for some $\varepsilon > 0$, the following inequalities hold:

$$x_{\max}\delta \leq x + x_{\max}, \qquad -(x_{\max} + \varepsilon)\delta \leq -x - \varepsilon \tag{19}$$

This constraint can be written as (18) with $E_1 = 0$, $E_2 = \begin{bmatrix} x_{\max} & -(x_{\max} + \varepsilon) \end{bmatrix}^\top$, $E_3 = 0$, $E_4 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$, and $E_5 = \begin{bmatrix} x_{\max} & \varepsilon \end{bmatrix}^\top$.

For this class of systems, the MPC strategy in Section IV-A is typically employed, and the OCP associated to it is as follows: given

- the current state x_0 of (16)-(18),
- a prediction horizon $N \in \mathbb{N}_{>0}$,
- a terminal constraint set $\mathcal{X}_f$,
- a stage cost $\mathcal{L}$, and
- a terminal cost $\mathcal{F}$

the MPC problem consists of minimizing the cost functional

$$\mathcal{J}(x, z, \delta, u)$$

over solutions $k \mapsto x(k)$, auxiliary variables $k \mapsto z(k)$ and $k \mapsto \delta(k)$, and inputs $k \mapsto u(k)$ subject to

$$x(0) = x_0 \tag{20}$$
$$x(N) \in \mathcal{X}_f \tag{21}$$
$$\forall k \in \mathbb{N}_{<N}: \quad x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_4 \tag{22}$$
$$\forall k \in \mathbb{N}_{\leq N}: \quad y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_4 \tag{23}$$
$$E_2\delta(k) + E_3z(k) \leq E_1u(k) + E_4x(k) + E_5 \tag{24}$$

A particular choice of the cost functional $\mathcal{J}$ used in the MLD literature is

$$\mathcal{J}(x, z, \delta, u) = \sum_{k=0}^{N-1} \mathcal{L}(x(k), z(k), \delta(k), u(k)) + \mathcal{F}(x(N))$$

where the stage and terminal costs are given by

$$\mathcal{L}(x, z, \delta, u) := |Qx|_p + |Q_z z|_p + |Q_\delta \delta|_p + |Ru|_p, \qquad \mathcal{F}(x) := |Px|_p$$

for some $p \in [1, \infty]$ and constant matrices Q, R, Q_δ, and Q_z of appropriate dimension.

Further Reading: Chapter 18 of [6] provides a detailed presentation of MPC for MLD systems as in (16)-(18). Following the ideas in recasting state constraints using discrete-valued auxiliary states as in (19), the MPC problem for MLD systems is formulated therein as mixed integer (linear and quadratic) problems. Previous articles about MLD systems include [4], [3], [5], [11], [6]. Mixed-integer optimization methods were also used in [10] to solve feasibility and optimization problems for MLD systems with temporal logic specifications.

C. Periodic MPC for continuous-time systems

MPC can be directly applied to plants given by continuous-time nonlinear systems modeled as in (3)-(4), without discretization of the dynamics. The price to pay is that the optimal control input has to be determined over a finite-time window of continuous time. An MPC strategy for such systems consists of

1) At the current state of the plant, solve the OCP over a continuous-time prediction horizon;
2) Apply the optimal control input over a continuous-time control horizon;
3) Repeat.

The OCP to solve is as follows: given

- the current state x_0 of (3),
- a prediction horizon $T \in \mathbb{R}_{>0}$,
- a terminal constraint set $\mathcal{X}_f$,
- a stage cost $\mathcal{L}$, and
- a terminal cost $\mathcal{F}$

minimize the cost functional

$$\mathcal{J}(x,u) := \int_0^T \mathcal{L}(x(t),u(t))dt + \mathcal{F}(x(T))$$

over solutions $t \mapsto x(t)$ and inputs $t \mapsto u(t)$ subject to

$$x(0) = x_0 \tag{25}$$

$$x(T) \in \mathcal{X}_f \tag{26}$$

$$\forall t \in (0,T): \quad \frac{d}{dt}x(t) = f(x(t),u(t)) \tag{27}$$

$$\forall t \in [0,T]: \quad y(t) = h(x(t),u(t)) \tag{28}$$

$$u(t) \in \mathcal{U} \tag{29}$$

Typically, the right-hand side f is assumed to be twice continuously differentiable to assure uniqueness of solutions. Furthermore, the zero state is typically assumed to be an equilibrium point for (3) with zero input. The input constraint set, which is denoted by $\mathcal{U}$, is commonly assumed to be compact, convex, and to have the property that the origin belongs to its interior.

Further Reading: One of the first articles on MPC for continuous-time systems is [13]. A reference that is more closely related to the MPC algorithm presented in this section is [8]. The approach in this article is to select the terminal constraint set $\mathcal{X}_f$ as a forward invariant neighborhood of the origin. The invariance-inducing feedback law is taken to be linear. In [8], a method to design the feedback, the terminal constraint set, and the terminal cost is provided. Due to their design approach leading to a cost functional that upper bounds the cost of the associated infinite horizon control problem, the MPC strategy in [8] is called quasi-infinite horizon nonlinear MPC. While the work in [8] allows for general piecewise continuous inputs as candidates for the optimal control, the particular case studied in [12] considers only piecewise-constant functions that change values periodically. The algorithm in the latter reference results in a type of sample-and-hold MPC strategy. Similar ideas were employed in [14] for the purposes of finding a sampled version of a continuous-time controller via MPC.

D. *MPC for linear systems with periodic impulses*

Linear systems with periodic impulses on the state are given by

$$\dot{x}(t) = Ax(t) \qquad \forall t \in (k\delta, (k+1)\delta] \tag{30}$$

$$x(t^+) = x(t) + Bu_k \qquad \forall t = k\delta \tag{31}$$

for each $k \in \mathbb{N}$, where $t \mapsto x(t)$ is a left-continuous function and, for each impulse time, namely, each $t \in \{0, \delta, 2\delta, \ldots\}$, $x(t^+)$ is the right limit of $x(t)$. Similarly to the MPC strategies presented in the earlier sections, an output can be attached to the model in (30)-(31). The constant $\delta > 0$ is the sampling period and u_k is the constant input applied at the impulse time $t = k\delta$. As shown in the literature, the sequence of constant inputs $\{u_k\}_{k\in\mathbb{N}}$ can be determined using the following MPC strategy: for each $k \in \mathbb{N}$,

1) At the current state of the plant at $t = k\delta$, solve the OCP over a prediction horizon including $N-1$ future impulse times;
2) Apply the first entry of the optimal input to the plant until the next impulse time.

The OCP to solve is as follows: given

- the current state x_0 of (30)-(31),
- a prediction horizon $N \in \mathbb{N}_{>0}$,
- a terminal constraint set $\mathcal{X}_f$, and
- a stage cost $\mathcal{L}$

minimize the cost functional

$$\mathcal{J}(x, u) = \sum_{k=0}^{N-1} \mathcal{L}(x(\tau_k), u(\tau_k))$$

over the value of the solutions and inputs at the impulse times, which are denoted as $x(k)$ and $u(k)$, respectively, subject to

$$x(0) = x_0 \tag{32}$$

$$x(N\delta) \in \mathcal{X}_f \tag{33}$$

$$\forall k \in \mathbb{N}_{<N}: \quad \dot{x}(t) = Ax(t) \qquad \forall t \in (k\delta, (k+1)\delta] \tag{34}$$

$$x(t^+) = x(t) + Bu(t) \qquad \forall t = k\delta \tag{35}$$

$$x(t) \in \mathcal{X} \qquad \forall t \in [k\delta, (k+1)\delta] \tag{36}$$

$$u(\tau_k) \in \mathcal{U} \tag{37}$$

The cost functional given above is a particular choice used in the literature, which does not include a terminal cost. The set $\mathcal{X}$ denotes the state constraints and the set $\mathcal{U}$ the input constraints.

Further Reading: The formulation above is based on [18]. The developments therein further exploit the periodicity of the setting, and employ the fact that the solutions to the system in (30)-(31) can be directly obtained at the impulse times from the computation of the solutions to the discrete-time system $x^+ = \exp(A\delta)(x + Bu)$. In addition, [18] employs over approximation techniques with the goal of reducing the number of constraints associated with the OCP above. For this purpose, polytopic over-approximations of the continuous dynamics of the impulsive system are proposed therein so as to arrive at a convex quadratic program. It should be pointed out that the stability concept used in [18] is weak in the sense that closeness and convergence of the values of the solution are only required to occur at the impulse times. Due to the combination of features of impulsive systems and of sample-data systems, the MPC strategy in [18] is one of the MPC approaches found in the literature that is closest to hybrid dynamical systems, as introduced in the next section.

E. MPC for hybrid dynamical systems

Hybrid dynamical systems are systems with state variables that are allowed to evolve continuously and, at times, jump. A mathematical model of a hybrid system, denoted $\mathcal{H}$, was given in (5). This model is general enough to capture the main features of other hybrid system modeling formalisms. For the purposes of MPC, it should be noted that the flow set C and the jump set D in (5) can already accommodate state and input constraints. Furthermore, compared to the MPC strategies for discrete-time systems and for continuous-time systems introduced earlier, the fact that hybrid systems have solution pairs (x, u) defined on a hybrid time domain $\mathrm{dom}(x, u)$, which is parameterized by ordinary time t and jump time j, requires a prediction horizon that includes both prediction of ordinary time and of jump times. One such prediction horizon is given by the set

$$\mathcal{T} := \{(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \ : \ \max\{t/\delta, j\} = \tau \ \} \tag{38}$$

for a positive integer τ and some positive constant δ. These parameters determine the "length" of $\mathcal{T}$ and the step size during flow for prediction. With this construction, the OCP to solve in an MPC strategy for hybrid systems will require that the terminal time, which now will be given by a pair (T, J), satisfies $\max\{T/\delta, J\} = \tau$. A hybrid control horizon, parameterized by a positive integer $\tau_c < \tau$ and positive constant $\delta_c < \delta$, with the same structure as the prediction horizon is defined.

With such a horizon structure, an MPC strategy for a plant modeled as a hybrid dynamical system $\mathcal{H}$ is as follows:

1) At the current state of the plant, solve the OCP over the hybrid prediction horizon;
2) Apply the optimal hybrid input to the plant over the hybrid control horizon;
3) Repeat.

The OCP to solve as part of this MPC strategy is as follows: given

- the current state x_0 of (5),
- prediction horizon parameters $\tau \in \mathbb{N}_{>0}$ and $\delta > 0$,
- control horizon parameters $\tau_c \in \mathbb{N}_{<\tau} \setminus \{0\}$ and $\delta_c \in (0, \delta)$,
- a terminal constraint set $\mathcal{X}_f$,
- stage costs $\mathcal{L}_C$ and $\mathcal{L}_D$, and
- a terminal cost $\mathcal{F}$

the MPC problem consists of minimizing the cost functional

$$\mathcal{J}(x, u) \ := \ \left(\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} \mathcal{L}_C(x(t, j), u(t, j))\, dt \right) + \left(\sum_{j=0}^{J-1} \mathcal{L}_D(x(t_{j+1}, j), u(t_{j+1}, j)) \right) + \mathcal{F}(x(T, J))$$

over solution pairs $(t, j) \mapsto (x(t, j), u(t, j))$ with hybrid time domain that is a compact subset of $[0, T] \times \mathbb{N}_{\leq J}$, subject to

$$x(0, 0) = x_0 \tag{39}$$

$$(T, J) \in \mathcal{T} \tag{40}$$

$$x(T, J) \in \mathcal{X}_f \tag{41}$$

$$\forall j \in \mathbb{N} \text{ with } \mathrm{int}(I_j) \neq \emptyset : \quad \frac{d}{dt} x(t, j) = F(x(t, j), u(t, j)) \quad (x(t, j), u(t, j)) \in C \text{ for almost all } t \in I_j, \tag{42}$$

$$\forall (t, j) \in \mathrm{dom}\, x \text{ s.t. } (t, j+1) \in \mathrm{dom}\, x : \quad x(t, j+1) = G(x(t, j), u(t, j)) \quad (x(t, j), u(t, j)) \in D \tag{43}$$

where (T, J) denotes the terminal time of the solution pair (x, u) and $\{t_j\}_{j=0}^{J+1}$ is the sequence defining $\mathrm{dom}(x, u)$, with $t_{J+1} = T$. The function $\mathcal{L}_C$ defines the cost of flowing on the flow set C and the function $\mathcal{L}_D$ defined the cost of jumping from the jump set.

Further Reading: For more details about the MPC strategy for hybrid dynamical systems presented in this section the reader is referred to [1], [2]. The case when the hybrid dynamics are discretized is treated in [15]. It should be noted that when the flow and jump sets overlap, the OCP may have multiple optimal inputs. Nonuniqueness of optimal inputs already appears in nonlinear MPC (see, e.g., in [9], [16], [6], [13], [7], [12]) but the source of nonuniqueness of solutions for the case of hybrid dynamical systems is conceptually different. For these systems, conditions guaranteeing that the value function, which at every point x_0 is given by $\mathcal{J}^\star(x_0) := \mathcal{J}(x_\star, u_\star)$ with $(x_\star, u_\star)$ being minimizers of $\mathcal{J}$ from x_0, implies an asymptotic stability property for the plant are given in [2]. The same reference also highlights key properties of the feasibility set, recursive feasibility, monotonicity properties of the cost functional, and regularity properties of the value function.

V. Cross References

Model Predictive Control
Model Predictive Control in Practice
Robust Model Predictive Control
Modeling Hybrid Systems
Stability Theory for Hybrid Dynamical Systems
Simulation of Hybrid Systems
Hybrid Dynamical Systems, Feedback Control of

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