# Feedback Control of Hybrid Dynamical Systems

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**Abstract**

The control of systems with hybrid dynamics requires algorithms capable of dealing with the intricate combination of continuous and discrete behavior, which typically emerges from the presence of continuous processes, switching devices, and logic for control. Several analysis and design techniques have been proposed for the control of nonlinear continuous-time plants, but little is known about controlling plants that feature truly hybrid behavior. This short article focuses on recent advances in the design of feedback control algorithms for hybrid dynamical systems. The focus is on hybrid feedback controllers that are systematically designed employing Lyapunov-based methods. The control design techniques summarized in this article include control Lyapunov function-based control, passivity-based control, trajectory tracking control, safety, and temporal logic.

**keywords** Feedback control, hybrid control, hybrid systems, asymptotic stability

## I. Definition

A *hybrid control system* is a *feedback system* whose variables may flow and, at times, jump. Such a hybrid behavior can be present in one or more of the subsystems of the feedback system: in the system to control, i.e., *the plant*; in the algorithm used for control, i.e., *the controller*; or in the subsystems needed to interconnect the plant and the controller, i.e., *the interfaces/signal conditioners*. Figure 1 depicts a feedback system in closed-loop configuration with such subsystems under the presence of environmental disturbances. Due to its hybrid dynamics, a hybrid control system is a particular type of *hybrid dynamical system*.

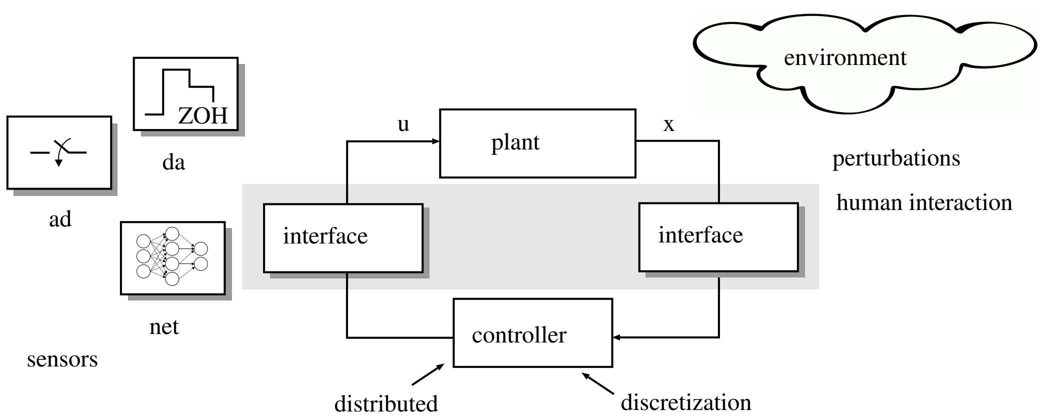


Fig. 1. **A hybrid control system:** a feedback system with a plant, controller, and interfaces/signal conditioners (along with environmental disturbances) as subsystems featuring variables that flow and, at times, jump.

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## II. Motivation

Hybrid dynamical systems are ubiquitous in science and engineering as they permit capturing the complex and intertwined continuous/discrete behavior of a myriad of systems with variables that flow and jump. The recent popularity of feedback systems combining physical and software components demands tools for stability analysis and control design that can systematically handle such a complex combination. To avoid the issues due to approximating the dynamics of a system, in numerous settings it is mandatory to keep the system dynamics as pure as possible, and to be able to design feedback controllers that can cope with flow and jump behavior in the system.

## III. Modeling Hybrid Control Systems

In this article, hybrid control systems are represented in the framework of *hybrid equations/inclusions* for the study of hybrid dynamical systems. Within this framework, the continuous dynamics of the system are modeled using a differential equation/inclusion while the discrete dynamics are captured by a difference equation/inclusion. A solution to such a system can *flow* over nontrivial intervals of time and *jump* at certain time instants. The conditions determining whether a solution to a hybrid system should flow or jump are captured by subsets of the state space and input space of the hybrid control system. In this way, a *plant* with hybrid dynamics can be modeled by the hybrid inclusion[1]

$$\mathcal{H}_P \;:\; \begin{cases} \dot z \in F_P(z,u) & (z,u) \in C_P \\ z^+ \in G_P(z,u) & (z,u) \in D_P \\ y = h(z,u) \end{cases} \tag{1}$$

where $z$ is the *state* of the plant and takes values from the Euclidean space $\mathbb{R}^{n_P}$, $u$ is the *input* and takes values from $\mathbb{R}^{m_P}$, $y$ is the *output* and takes values from the output space $\mathbb{R}^{r_P}$, and $(C_P, F_P, D_P, G_P, h)$ is the *data* of the hybrid system. The set $C_P$ is the *flow set*, the set-valued map $F_P$ is the *flow map*, the set $D_P$ is the *jump set*, the set-valued map $G_P$ is the *jump map*, and the single-valued map $h$ is the *output map*. In (1), $\dot z$ denotes time derivative and $z^+$ denotes an instantaneous change in $z$.

*Example 3.1 (controlled bouncing ball):* Consider the juggling system consisting of a ball moving vertically and bouncing on a fixed horizontal surface. The surface, located at the origin of the line of motion, is equipped with a mechanical actuator that controls the speed of the ball resulting after impacts. From a physical viewpoint, control authority may be obtained varying the viscoelastic properties of the surface and, in turn, the coefficient of restitution of the surface. The position and the velocity of the ball are denoted as $z_1$ and $z_2$, respectively. Between bounces, the free motion of the ball is given by

$$\dot z_1 = z_2, \quad \dot z_2 = -\gamma \tag{2}$$

where $\gamma > 0$ is the gravity constant. The conditions at which impacts occur are modeled as

$$z_1 = 0 \text{ and } z_2 \le 0 \tag{3}$$

while the new value of the state variables after each impact is described by the difference equations

$$z_1^+ = z_1, \qquad z_2^+ = u \tag{4}$$

where $u$, which is larger than or equal to zero, is the controlled velocity after impacts, capturing the effect of the mechanism installed on the horizontal surface. In this way, the data $(C_P, F_P, D_P, G_P, h)$ of the bouncing ball model is defined as follows:

$$F_P(z,u) := \begin{bmatrix} z_2 \\ -\gamma \end{bmatrix} \qquad \forall (z,u) \in C_P := \left\{ (z,u) \in \mathbb{R}^2 \times \mathbb{R} \;:\; z_1 \ge 0 \right\}$$

$$G_P(z,u) := \begin{bmatrix} z_1 \\ u \end{bmatrix} \qquad \forall (z,u) \in D_P := \left\{ (z,u) \in \mathbb{R}^2 \times \mathbb{R} \;:\; z_1 = 0, z_2 \le 0 \right\}$$

[1]This hybrid inclusion captures the dynamics of (constrained or unconstrained) continuous-time systems when $D_P = \emptyset$ and $G_P$ is arbitrary. Similarly, it captures the dynamics of (constrained or unconstrained) discrete-time systems when $C_P = \emptyset$ and $F_P$ is arbitrary. Note that while the output inclusion does not explicitly include a constraint on $(z,u)$, the output map is only evaluated along solutions.

and, when assuming that the state $z$ is measured, the output map is $h(z,u) = z$. $\triangle$

Other examples of hybrid plants whose dynamics can be captured by $\mathcal{H}_P$ in (1) include walking robots, network control systems, and spiking neurons. Systems with different modes of operation can also be modeled as $\mathcal{H}_P$, and, as the following example illustrates, can be captured by the constrained differential equation (or inclusion) part of $\mathcal{H}_P$. Such systems are not necessarily hybrid – at least as the term *hybrid* is used in this short article – since they can be modeled by a dynamical system with state that only evolves continuously.

*Example 3.2 (Thermostat system):* The evolution of the temperature of a room under the effect of a heater can be modeled by a differential equation with constraints on its input. The temperature of the room is denoted by $z$, and takes values from $\mathbb{R}$. The input is given by the pair $u = (u_1, u_2)$, where $u_1$ denotes whether the heater is turned on ($u_1 = 1$) or turned off ($u_1 = 0$) – these are the constraints on the inputs – while $u_2$ denotes the temperature outside the room, which can assume any value in $\mathbb{R}$. With these definitions, the evolution of the temperature $z$ is governed by

$$\dot{z} \;=\; -z + \begin{bmatrix} z_\Delta & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (z,u) \in C_P = \left\{ (z,u) \in \mathbb{R} \times \mathbb{R}^2 \;:\; u_1 \in \{0,1\} \right\} \tag{5}$$

where $z_\Delta$ is a positive constant representing the heater capacity. Note that $C_P$ captures the constraint on the input $u_1$, which restricts it to the values $0$ and $1$. $\triangle$

Given an input $u$, a *solution to a hybrid inclusion* is defined by a state trajectory $\phi$ that satisfies the inclusions. Both the input and the state trajectory are functions of $(t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} := [0,\infty) \times \{0,1,2,\ldots\}$, where $t$ keeps track of the amount of flow while $j$ counts the number of jumps of the solution. These functions are given by *hybrid arcs* and *hybrid inputs*, which are defined on *hybrid time domains*. More precisely, hybrid time domains are subsets $E$ of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T', J') \in E$,

$$E \;\cap\; ([0,T'] \times \{0,1,...J'\})$$

can be written in the form

$$\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_J$, $J \in \mathbb{N}$. A hybrid arc $\phi$ is a function on a hybrid time domain. (The set $E \;\cap\; ([0,T] \times \{0,1,\ldots,J\})$ defines a compact hybrid time domain since it is bounded and closed.) The hybrid time domain of $\phi$ is denoted by $\operatorname{dom}\phi$. A hybrid arc is such that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t,j)$ is locally absolutely continuous on intervals of flow $I^j := \{t \;:\; (t,j) \in \operatorname{dom}\phi\}$ with nonzero Lebesgue measure. A hybrid input $u$ is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $I^j$.

In this way, a solution to the plant $\mathcal{H}_P$ is given by a pair $(\phi, u)$ with $\operatorname{dom}\phi = \operatorname{dom} u$ $(= \operatorname{dom}(\phi,u))$ satisfying

*(S0)* $(\phi(0,0), u(0,0)) \in \overline{C}_P$ *or* $(\phi(0,0), u(0,0)) \in D_P$*, and* $\operatorname{dom}\phi = \operatorname{dom} u$*;*

*(S1) For each* $j \in \mathbb{N}$ *such that* $I^j$ *has nonempty interior int*$(I^j)$*, we have*

$$(\phi(t,j), u(t,j)) \in C_P \qquad \text{for all } t \in \mathit{int}(I^j)$$

*and*

$$\frac{d}{dt}\phi(t,j) \in F_P(\phi(t,j), u(t,j)) \qquad \text{for almost all } t \in I^j$$

*(S2) For each* $(t,j) \in \operatorname{dom}(\phi,u)$ *such that* $(t, j+1) \in \operatorname{dom}(\phi,u)$*, we have*

$$(\phi(t,j), u(t,j)) \in D_P$$

*and*

$$\phi(t, j+1) \in G_P(\phi(t, j), u(t, j))$$

A solution pair $(\phi, u)$ to $\mathcal{H}$ is said to be *complete* if $\mathrm{dom}(\phi, u)$ is unbounded and *maximal* if there does not exist another pair $(\phi, u)'$ such that $(\phi, u)$ is a truncation of $(\phi, u)'$ to some proper subset of $\mathrm{dom}(\phi, u)'$. A solution pair $(\phi, u)$ to $\mathcal{H}$ is said to be *Zeno* if it is complete and the projection of $\mathrm{dom}(\phi, u)$ onto $\mathbb{R}_{\geq 0}$ is bounded.

*On decomposition of inputs and outputs:* At times, it is convenient to define inputs $u_c \in \mathbb{R}^{m_{P,c}}$ and $u_d \in \mathbb{R}^{m_{P,d}}$ collecting every component of the input $u$ that affect flows and that affect jumps, respectively.[2] Similarly, one can define $y_c$ and $y_d$ as the components of $y$ that are measured during flows and jumps, respectively.

To control the hybrid plant $\mathcal{H}_P$ in (1), control algorithms that can cope with the nonlinearities introduced by the flow and jump equations/inclusions are required. In general, feedback controllers designed using classical techniques from the continuous-time and discrete-time domain fall short. Due to this limitation, hybrid feedback controllers would be more suitable for the control of plants with hybrid dynamics. Then, following the hybrid plant model above, hybrid controllers for the plant $\mathcal{H}_P$ in (1) will be given by the hybrid inclusion

$$\mathcal{H}_K \;:\; \begin{cases} \dot{\eta} & \in \; F_K(\eta, v) \qquad (\eta, v) \in C_K \\ \eta^+ & \in \; G_K(\eta, v) \qquad (\eta, v) \in D_K \\ \zeta & = \; \kappa(\eta, v) \end{cases} \tag{6}$$

where $\eta$ is the *state* of the controller and takes values from the Euclidean space $\mathbb{R}^{n_K}$, $v$ is the *input* and takes values from $\mathbb{R}^{r_P}$, $\zeta$ is the *output* and takes values from the output space $\mathbb{R}^{m_P}$, and $(C_K, F_K, D_K, G_K, \kappa)$ is the *data* of the hybrid inclusion defining the hybrid controller.

The control of $\mathcal{H}_P$ via $\mathcal{H}_K$ defines an interconnection through the input/output assignment $u = \zeta$ and $v = y$; the system in Figure 1 without interfaces represents this interconnection. The resulting closed-loop system is a hybrid dynamical system given in terms of a hybrid inclusion/equation with state $x = (z, \eta)$. We will denote such a closed-loop system by $\mathcal{H}$, with data denoted $(C, F, D, G)$, state $x \in \mathbb{R}^n$, and dynamics

$$\mathcal{H} \;:\; \begin{cases} \dot{x} & \in \; F(x) \qquad x \in C \\ x^+ & \in \; G(x) \qquad x \in D \end{cases} \tag{7}$$

Its data can be constructed from the data $(C_P, F_P, D_P, G_P, h)$ and $(C_K, F_K, D_K, G_K, \kappa)$ of each of the subsystems. Solutions to both $\mathcal{H}_K$ and $\mathcal{H}$ are understood following the notion introduced above for $\mathcal{H}_P$.

## IV. Definitions and Notions

For convenience, we use the equivalent notation $[x^\top \; y^\top]^\top$ and $(x, y)$ for vectors $x$ and $y$. Also, we denote by $\mathcal{K}_\infty$ the class of functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ that are continuous, zero at zero, strictly increasing, and unbounded.

In general, the dynamics of hybrid inclusions have right-hand sides given by set-valued maps. Unlike functions or single-valued maps, set-valued maps may return a set when evaluated at a point. For instance, at points in $C_P$, the set-valued flow map $F_P$ of the hybrid plant $\mathcal{H}_P$ might return more than one value, allowing for different values of the derivative of $z$. A particular continuity property of set-valued maps that will be needed later is lower semicontinuity. A set-valued map $S$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is lower semicontinuous if for each $x \in \mathbb{R}^n$ one has that $\liminf_{x_i \to x} S(x_i) \supset S(x)$, where $\liminf_{x_i \to x} S(x_i) = \{z \;:\; \forall x_i \to x, \exists z_i \to z \text{ s.t. } z_i \in S(x_i)\ \}$ is the so-called *inner limit* of $S$.

[2]Some of the components of $u$ can be used to define both $u_c$ and $u_d$, that is, there could be inputs that affect both flows and jumps.

A vast majority of control problems consist of designing a feedback algorithm that assures that a function of the solutions to the plant approach a desired set-point condition *(attractivity)* and, when close to it, the solutions remain nearby *(stability)*. In some scenarios, the desired set-point condition is not necessarily an isolated point, but rather a set. The problem of designing a hybrid controller $\mathcal{H}_K$ for a hybrid plant $\mathcal{H}_P$ typically pertains to the stabilization of sets, in particular, due to the hybrid state of the controller including timers that persistently evolve within a bounded time interval and logic variables that take values from discrete sets. Denoting by $\mathcal{A}$ the set of points to stabilize for the closed-loop system $\mathcal{H}$ and $|\cdot|_{\mathcal{A}}$ as the distance to such set, the following property captures the typically desired properties outlined above. A closed set $\mathcal{A}$ is said to be

(S) *Stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each maximal solution $\phi$ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $|x_\circ|_{\mathcal{A}} \leq \delta$ satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;

(pA) *Pre-attractive* if there exists $\mu > 0$ such that every maximal solution $\phi$ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $|x_\circ|_{\mathcal{A}} \leq \mu$ is bounded and if it is complete satisfies $\lim_{(t,j)\in \operatorname{dom}\phi,\ t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0$;

(FTpA) *Finite time pre-attractive* if there exists $\mu > 0$ such that every maximal solution $\phi$ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $|x_\circ|_{\mathcal{A}} \leq \mu$ is such that $|\phi(t,j)|_{\mathcal{A}} = 0$ for some $(t,j) \in \operatorname{dom}\phi$;

(pAS) *Pre-asymptotically stable* if it is stable and pre-attractive.

The basin of pre-attraction of a pre-asymptotically stable set $\mathcal{A}$ is the set of points from where the pre-attractivity property holds. The set $\mathcal{A}$ is said to be globally pre-asymptotically stable when the basin of pre-attraction is equal to the entire state space. Similarly, notions pertaining to finite time stability and basin of attraction for finite time stability can be defined. When every maximal solution is complete, then the prefix “pre” can be dropped since, in that case, the notions resemble those for continuous-time or discrete-time systems with solutions defined for all time.

At times, one is interested in asserting convergence when the state trajectory or the output remain in a set. For instance, a dynamical system (with assigned inputs) is said to be detectable when its output being held to zero implies that its state converges to the origin (or to a particular set of interest). A similar property can be defined for hybrid dynamical systems, for a general set $K$, which may not necessarily be the set of points at which the output is zero. For the closed-loop system $\mathcal{H}$, given sets $\mathcal{A}$ and $K$, the distance to $\mathcal{A}$ is said to be

(D) $0$-*input detectable relative to* $K$ if every complete solution $\phi$ to $\mathcal{H}$ is such that

$$\phi(t,j) \in K \qquad \forall (t,j) \in \operatorname{dom}\phi \qquad \Rightarrow \qquad \lim_{(t,j)\in \operatorname{dom}\phi,\ t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0$$

Note that “$\phi(t,j) \in K$” captures the “output being held to zero”-like property in the usual detectability notion.

In addition to stability, attractivity, and detectability, in this short note we are interested in hybrid controllers that guarantee that a set $K$ is forward invariant in the following sense:

(FpI) *Forward pre-invariant* if each maximal solution $\phi$ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $x_\circ \in K$, satisfies $\phi(t,j) \in K$ for all $(t,j) \in \operatorname{dom}\phi$.

(FI) *Forward invariant* if each maximal solution $\phi$ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $x_\circ \in K$, is complete and satisfies $\phi(t,j) \in K$ for all $(t,j) \in \operatorname{dom}\phi$.

## V. Feedback Control Design for Hybrid Dynamical Systems

Several methods for the design of a hybrid controller $\mathcal{H}_K$ rendering a given set $\mathcal{A}$ such that the properties defined in Section IV are given below. At the core of these methods are sufficient conditions in terms of Lyapunov-like functions guaranteeing properties such as asymptotic stability, invariance, and finite-time attractivity of a set. Some of the methods presented below exploit such sufficient conditions when applied to the closed-loop system $\mathcal{H}$, while others exploit the properties of the hybrid plant to design controllers with a particular structure.

### A. CLF-based Control Design

In simple terms, a control Lyapunov function (CLF) is a regular enough scalar function that decreases along solutions to the system for some values of the unassigned input. When such a function exists, it is very tempting to exploit its properties to construct an asymptotically stabilizing control law. Following the ideas from the literature of continuous-time and discrete-time nonlinear systems, we define control Lyapunov functions for hybrid plants $\mathcal{H}_P$ and present results on CLF-based control design. For simplicity, as mentioned in the *input and output modeling remark* in Section IV, we use inputs $u_c$ and $u_d$ instead $u$. Also, for simplicity, we restrict the discussion to sets $\mathcal{A}$ that are compact as well as hybrid plants with $F_P, G_P$ single valued and such that $h(z,u) = z$. For notational convenience, we use $\Pi$ to denote the "projection" of $C_P$ and $D_P$ onto $\mathbb{R}^{n_P}$, i.e., $\Pi(C_P) = \{z \ : \ \exists u_c \text{ s.t. } (z,u_c) \in C_P \}$ and $\Pi(D_P) = \{z \ : \ \exists u_d \text{ s.t. } (z,u_d) \in D_P \}$, and the set-valued maps $\Psi_c(z) = \{u_c \ : \ (z,u_c) \in C_P \}$ and $\Psi_d(z) = \{u_d \ : \ (z,u_d) \in D_P \}$.

Given a compact set $\mathcal{A}$, a continuously differentiable function $V : \mathbb{R}^{n_P} \to \mathbb{R}$ is a *control Lyapunov function for $\mathcal{H}_P$ with respect to $\mathcal{A}$* if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous, positive definite function $\rho$ such that

$$\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}}) \qquad \forall z \in \mathbb{R}^{n_P}$$

$$\inf_{u_c \in \Psi_c(z)} \langle \nabla V(z), F_P(z,u_c) \rangle \leq -\rho(|z|_{\mathcal{A}}) \qquad \forall z \in \Pi(C_P) \tag{8}$$

$$\inf_{u_d \in \Psi_d(z)} V(G_P(z,u_d)) - V(z) \leq -\rho(|z|_{\mathcal{A}}) \qquad \forall z \in \Pi(D_P) \tag{9}$$

With the availability of a CLF, the set $\mathcal{A}$ can be asymptotically stabilized if it is possible to synthesize a controller $\mathcal{H}_K$ from inequalities (8)-(9). Such a synthesis is feasible, in particular, for the special case of $\mathcal{H}_K$ being a static state-feedback law $z \mapsto \kappa(z)$. Sufficient conditions guaranteeing the existence of such a controller as well as a particular state-feedback law with point-wise minimum norm are given next.

Given a compact set $\mathcal{A}$ and a control Lyapunov function $V$ (with respect to $\mathcal{A}$), define, for each $r \geq 0$, the set $\mathcal{I}(r) := \{z \in \mathbb{R}^{n_P} \ : \ V(z) \geq r \}$. Moreover, for each $(z,u_c)$ and $r \geq 0$, define the function

$$\Gamma_c(z,u_c,r) := \begin{cases} \langle \nabla V(z), F_P(z,u_c) \rangle + \frac{1}{2}\rho(|z|_{\mathcal{A}}) & \text{if } (z,u_c) \in C_P \cap (\mathcal{I}(r) \times \mathbb{R}^{m_{P,c}}), \\ -\infty & \text{otherwise} \end{cases}$$

and, for each $(z,u_d)$ and $r \geq 0$, the function

$$\Gamma_d(z,u_d,r) := \begin{cases} V(G_P(z,u_d)) - V(z) + \frac{1}{2}\rho(|z|_{\mathcal{A}}) & \text{if } (z,u_d) \in D_P \cap (\mathcal{I}(r) \times \mathbb{R}^{m_{P,d}}), \\ -\infty & \text{otherwise} \end{cases}$$

It can be shown that the following conditions involving $V$ and the data $(C_P, F_P, D_P, G_P, h)$ of $\mathcal{H}_P$ guarantee that, for each $r > 0$, there exists a state-feedback law

$$z \mapsto \kappa(z) = (\kappa_c(z), \kappa_d(z))$$

with $\kappa_c$ continuous on $\Pi(C_P) \cap \mathcal{I}(r)$ and $\kappa_d$ continuous on $\Pi(D_P) \cap \mathcal{I}(r)$ rendering the compact set

$$\mathcal{A}_r := \{z \in \mathbb{R}^{n_P} \ : \ V(z) \leq r \}$$

pre-asymptotically stable for $\mathcal{H}_P$:

(CLF1) $C_P$ and $D_P$ are closed sets, and $F_P$ and $G_P$ are continuous;

(CLF2) The set-valued maps $\Psi_c(z) = \{u_c \ : \ (z,u_c) \in C_P \}$ and $\Psi_d(z) = \{u_d \ : \ (z,u_d) \in D_P \}$ are lower semicontinuous with convex values;

(CLF3) For every $r > 0$, we have that, for every $z \in \Pi(C_P) \cap \mathcal{I}(r)$, the function $u_c \mapsto \Gamma_c(z,u_c,r)$ is convex on $\Psi_c(z)$ and that, for every $z \in \Pi(D_P) \cap \mathcal{I}(r)$, the function $u_d \mapsto \Gamma_c(z,u_d,r)$ is convex on $\Psi_d(z)$;

In addition to guaranteeing the existence of a (continuous) state-feedback law practically pre-asymptotically stabilizing the set $\mathcal{A}$, these conditions also lead to the following natural definition of the feedback: for every $r > 0$, the state-feedback law pair

$$\kappa_c : \Pi(C_P) \to \mathbb{R}^{m_{P,c}}, \qquad \kappa_d : \Pi(D_P) \to \mathbb{R}^{m_{P,d}}$$

can be defined on $\Pi(C_P)$ and $\Pi(D_P)$ as

$$\begin{aligned} \kappa_c(z) &:= \arg\min \{ |u_c| \ : \ u_c \in \mathcal{T}_c(z) \ \} \qquad \forall z \in \Pi(C_P) \cap \mathcal{I}(r) \\ \kappa_d(z) &:= \arg\min \{ |u_d| \ : \ u_d \in \mathcal{T}_d(z) \ \} \qquad \forall z \in \Pi(D_P) \cap \mathcal{I}(r) \end{aligned}$$

where $\mathcal{T}_c(z) = \Psi_c(z) \cap \{u_c \ : \ \Gamma_c(z, u_c, V(z)) \le 0\}$ and $\mathcal{T}_d(z) = \Psi_d(z) \cap \{u_d \ : \ \Gamma_d(z, u_d, V(z)) \le 0\}$.

The stability property guaranteed by this feedback is also practical. Under further properties, similar results hold when the input $u$ is not partitioned into $u_c$ and $u_d$. To achieve asymptotic stability (or stabilizability) of $\mathcal{A}$ with a continuous state-feedback law, extra conditions are required to hold nearby the compact set, which for the case of stabilization of continuous-time systems are the so-called *small control properties*. Furthermore, the continuity of the feedback law assures that the closed-loop system has closed flow and jump sets as well as continuous flow and jump maps, which, in turn, due to the compactness of $\mathcal{A}$, implies that the asymptotic stability property is robust. Robustness follows from results for hybrid systems without inputs.

*Example 5.1 (controlled bouncing ball revisited):* For the juggling system in Example 3.1, the feedback law $u = \kappa_d \equiv 0$ asymptotically stabilizes its origin – note that for this system, $u_d = u$. In fact, with this feedback, after the first impact (or jump) every solution remains at the origin.

### *B. Passivity-based Control Design*

Dissipativity and its special case, passivity, provide a useful physical interpretation of a feedback control system as they characterize the exchange of energy between the plant and its controller. For an open system, passivity (in its very pure form) is the property that the energy stored in the system is no larger than the energy it has absorbed over a period of time. The energy stored in a system is given by the difference between the initial and final energy over a period of time, where the energy function is typically called the *storage function*. Hence, conveniently, passivity can be expressed in terms of the derivative of a storage function (i.e., the rate of change of the internal energy) and the product between inputs and outputs (i.e., the power flow of the system). Under further observability conditions, this power inequality can be employed as a design tool by selecting a control law that makes the rate of change of the internal energy negative. This method is called *passivity-based control design*.

The passivity-based control design method can be employed in the design of a controller for a "passive" hybrid plant $\mathcal{H}_P$, in which energy might be dissipated during flows, jumps, or both. Passivity notions and a passivity-based control design method for hybrid plants are given next. Since the form of the output of the plant plays a key role in asserting a passivity property, and this property may not necessarily hold both during flows and jumps, as mentioned in the *input and output modeling remark* in Section IV, we define outputs $y_c$ and $y_d$, which, for simplicity, are assumed to be single-valued: $y_c = h_c(x)$ and $y_d = h_d(x)$. Moreover, we consider the case when the dimension of the space of the inputs $u_c$ and $u_d$ coincide with that of the outputs $y_c$ and $y_d$, respectively, i.e., a "duality" of the output and input space.

Given a compact set $\mathcal{A}$ and functions $h_c$, $h_d$ such that $h_c(\mathcal{A}) = h_d(\mathcal{A}) = 0$, a hybrid plant $\mathcal{H}_P$ for which there exists a continuously differentiable function $V : \mathbb{R}^{n_P} \to \mathbb{R}_{\ge 0}$ satisfying for some functions $\omega_c : \mathbb{R}^{m_{P,c}} \times \mathbb{R}^{n_P} \to \mathbb{R}$ and $\omega_d : \mathbb{R}^{m_{P,c}} \times \mathbb{R}^{n_P} \to \mathbb{R}$

$$\langle \nabla V(z), F_P(z, u_c) \rangle \le \omega_c(u_c, z) \qquad \forall (z, u_c) \in C \tag{10}$$

$$V(G_P(z, u_d)) - V(z) \le \omega_d(u_d, z) \qquad \forall (z, u_d) \in D \tag{11}$$

is said to be *passive with respect to a compact set* $\mathcal{A}$ if

$$(u_c, z) \mapsto \omega_c(u_c, z) = u_c^\top y_c \quad (12)$$

$$(u_d, z) \mapsto \omega_d(u_d, z) = u_d^\top y_d \quad (13)$$

The function $V$ is the so-called *storage function*. If (10) holds with $\omega_c$ as in (12), and (11) holds with $\omega_d \equiv 0$, then the system is called *flow-passive*, i.e., the power inequality holds only during flows. If (10) holds with $\omega_c \equiv 0$, and (11) holds with $\omega_d$ as in (13), then the system is called *jump-passive*, i.e., the energy of the system decreases only during jumps.

Under additional detectability properties, these passivity notions can be used to design static output feedback controllers. In fact, given a hybrid plant $\mathcal{H}_P = (C_P, F_P, D_P, G_P, h)$ satisfying

(PBC1) $C_P$ and $D_P$ are closed sets; $F_P$ and $G_P$ are continuous; and $h_c$ and $h_d$ are continuous;

and a compact set $\mathcal{A}$, it can be shown that if $\mathcal{H}_P$ is flow-passive with respect to $\mathcal{A}$ with a storage function $V$ that is positive definite with respect to $\mathcal{A}$ and has compact sublevel sets, and if there exists a continuous function $\kappa_c : \mathbb{R}^{m_{P,c}} \to \mathbb{R}^{m_{P,c}}$, $y_c^\top \kappa_c(y_c) > 0$ for all $y_c \neq 0$, such that the resulting closed-loop system with $u_c = -\kappa_c(y_c)$ and $u_d \equiv 0$ has the following properties:

(PBC2) The distance to $\mathcal{A}$ is detectable relative to

$$\left\{ z \in \Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P) \,:\, h_c(z)^\top \kappa_c(h_c(z)) = 0,\, (z, -\kappa_c(h_c(z))) \in C_P \right\};$$

(PBC3) Every complete solution $\phi$ is such that, for some $\delta > 0$ and some $J \in \mathbb{N}$, we have $t_{j+1} - t_j \geq \delta$ for all $j \geq J$;

then the output-feedback law

$$u_c = -\kappa_c(y_c), \qquad u_d \equiv 0$$

renders $\mathcal{A}$ globally pre-asymptotically stable.

In a similar manner, an output-feedback law can be designed when, instead of being flow-passive, $\mathcal{H}_P$ is jump-passive with respect to $\mathcal{A}$. In this case, if the storage function $V$ is positive definite with respect to $\mathcal{A}$ and has compact sublevel sets, and if there exists a continuous function $\kappa_d : \mathbb{R}^{m_{P,d}} \to \mathbb{R}^{m_{P,d}}$, $y_d^\top \kappa_d(y_d) > 0$ for all $y_d \neq 0$, such that the resulting closed-loop system with $u_c \equiv 0$ and $u_d = -\kappa_d(y_d)$ has the following properties:

(PBC4) The distance to $\mathcal{A}$ is detectable relative to

$$\left\{ z \in \Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P) \,:\, h_d(z)^\top \kappa_d(h_d(z)) = 0,\, (z, -\kappa_d(h_d(z))) \in D_P \right\};$$

(PBC5) Every complete solution $\phi$ is Zeno;

then the the output-feedback law

$$u_d = -\kappa_d(y_d), \qquad u_c \equiv 0$$

renders $\mathcal{A}$ globally pre-asymptotically stable. Such a feedback design can be employed to globally asymptotically stabilize the controlled bouncing ball in Example 3.1.

Strict passivity notions can also be formulated for hybrid plants, including the special cases where the power inequalities hold only during flows or jumps. In particular, strict passivity and output strict passivity can be employed to assert asymptotic stability with zero inputs.

## *C. Tracking Control Design*

While numerous control problems pertain to the stabilization of a set-point condition, at times, it is desired to stabilize the solutions to the plant to a time-varying trajectory. In this section, we consider the problem of designing a hybrid controller $\mathcal{H}_K$ for a hybrid plant $\mathcal{H}_P$ to *track* a given reference trajectory $r$ (a hybrid arc). The notion of tracking is introduced below. We propose sufficient conditions that general hybrid plants and controllers should satisfy to solve such a problem. For simplicity, we consider tracking of state trajectories and that the hybrid controller can measure both the state of the plant $z$ and the reference trajectory $r$; hence $v = (z, r)$.

The particular approach used here consists of recasting the tracking control problem as a set stabilization problem for the closed-loop system $\mathcal{H}$. To do this, we embed the reference trajectory $r$ into an augmented hybrid model for which it is possible to define a set capturing the condition that the plant tracks the given reference trajectory. This set is referred to as *the tracking set*. More precisely, given a reference $r : \operatorname{dom} r \to \mathbb{R}^{n_p}$, we define the set $\mathcal{T}_r$ collecting all of the points $(t, j)$ in the domain of $r$ at which $r$ jumps, that is, every point $(t^r_j, j) \in \operatorname{dom} r$ such that $(t^r_j, j+1) \in \operatorname{dom} r$. Then, the state of the closed loop $\mathcal{H}$ is augmented by the addition of states $\tau \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$. The dynamics of the states $\tau$ and $k$ are such that $\tau$ counts elapsed flow time while $k$ counts the number of jumps of $\mathcal{H}$; hence, during flows $\dot{\tau} = 1$ and $\dot{k} = 0$, while at jumps $\tau^+ = \tau$ and $k^+ = k + 1$. These new states are used to parameterize the given reference trajectory $r$, which is employed in the definition of the tracking set

$$\mathcal{A} = \{(z, \eta, \tau, k) \in \mathbb{R}^{n_P} \times \mathbb{R}^{n_K} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \ : \ z = r(\tau, k), \ \eta \in \Phi_K \ \} \tag{14}$$

This set is the target set to be stabilized for $\mathcal{H}$. The set $\Phi_K \subset \mathbb{R}^{n_K}$ in the definition of $\mathcal{A}$ is some closed set capturing the set of points asymptotically approached by the state of the controller $\eta$.

Using results to certify pre-asymptotic stability of closed sets for hybrid systems, sufficient conditions guaranteeing that a hybrid controller $\mathcal{H}_K$ stabilizes the tracking set $\mathcal{A}$ in (14) for the hybrid closed-loop system $\mathcal{H}$ can be formulated. In particular, with $\mathcal{H}$ having state $x = (z, \eta, \tau, k)$ and data

$$\begin{aligned}
C &= \big\{x \ : \ (z, \kappa_c(\eta, z, r(\tau, k))) \in C_P, \tau \in [t^r_k, t^r_{k+1}], (\eta, z, r(\tau, k)) \in C_K \ \big\} \\
F(z, \eta, \tau, k) &= (F_P(z, \kappa_c(\eta, z, r(\tau, k))), F_K(\eta, z, r(\tau, k)), 1, 0) \\
D &= \{x \ : \ (z, \kappa_c(\eta, z, r(\tau, k))) \in D_P, (\tau, k) \in \mathcal{T}_r \ \} \cup \big\{x \ : \ \tau \in [t^r_k, t^r_{k+1}), (\eta, z, r(\tau, k)) \in D_K \ \big\} \\
G_1(z, \eta, \tau, k) &= (G_P(z, \kappa_c(\eta, z, r(\tau, k))), \eta, \tau, k+1), \quad G_2(z, \eta, \tau, k) \ = \ (z, G_K(\eta, z, r(\tau, k)), \tau, k)
\end{aligned}$$

given a complete reference trajectory $r : \operatorname{dom} r \to \mathbb{R}^{n_P}$ and associated tracking set $\mathcal{A}$, a hybrid controller $\mathcal{H}_K$ with data $(C_K, F_K, D_K, G_K, \kappa)$ guaranteeing that

(T1) The jumps of $r$ and $\mathcal{H}_P$ occur simultaneously;

(T2) For some continuously differentiable function $V : \mathbb{R}^{n_P} \times \mathbb{R}^{n_K} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$; and continuous, positive definite functions $\rho_1, \rho_2, \rho_3$, the following hold:

(T2a) For all $(z, \eta, \tau, k) \in C \cup D \cup G_1(D) \cup G_2(D)$

$$\alpha_1(|(z, \eta, \tau, k)|_{\mathcal{A}}) \leq V(z, \eta, \tau, k) \leq \alpha_2(|(z, \eta, \tau, k)|_{\mathcal{A}})$$

(T2b) For all $(z, \eta, \tau, k) \in C$ and all $\zeta \in F(z, \eta, \tau, k)$,

$$\langle \nabla V(z, \eta, \tau, k), \zeta \rangle \leq -\rho_1 \left(|(z, \eta, \tau, k)|_{\mathcal{A}}\right)$$

(T2c) For all $(z, \eta, \tau, k) \in D_1$ and all $\zeta \in G_1(z, \eta, \tau, k)$

$$V(\zeta) - V(z, \eta, \tau, k) \leq -\rho_2 \left(|(z, \eta, \tau, k)|_{\mathcal{A}}\right)$$

(T2d) For all $(z, \eta, \tau, k) \in D_2$ and all $\zeta \in G_2(z, \eta, \tau, k)$

$$V(\zeta) - V(z, \eta, \tau, k) \leq -\rho_3 \left(|(z, \eta, \tau, k)|_{\mathcal{A}}\right)$$

renders $\mathcal{A}$ is globally pre-asymptotically stable for $\mathcal{H}$.

Note that condition (T1) imposes that the jumps of the plant and of the reference trajectory occur simultaneously. Though restrictive, at times, this property can be enforced by proper design of the controller.

### D. Forward Invariance-based Control Design

As defined in Section IV, a set $K$ is forward invariant if every solution to the system from $K$ stays in $K$. Also known as *flow-invariance*, *positively invariance*, *viability*, or just *invariance*, this property is very important in feedback control design. In fact, asymptotically stabilizing feedback laws induce forward invariance of the set $\mathcal{A}$ that is asymptotically stabilized. Forward invariance is also key to guarantee safety properties, since safety can be typically recast as forward invariance of the set that excludes every point (and, for robustness, a neighborhood of it) for which the system is considered to be unsafe. In this section, forward invariance (or, equivalently, safety) is guaranteed by infinitesimal conditions that involve functions of the state known as barrier functions.

Given a hybrid closed-loop system $\mathcal{H} = (C, F, D, G)$, a function $B : \mathbb{R}^n \to \mathbb{R}$ is a *barrier function candidate* defining a set $K \subset C \cup D$ if

$$K = \{x \in C \cup D \ : \ B(x) \leq 0 \ \} \tag{15}$$

In some settings, the set $K$ might be given a priori and then one would seek for a barrier function $B$ such that (15) holds; namely, find a function $B$ that is nonpositive at points in $K$ only. In some other settings, one may generate the set $K$ from the given sets $C, D$ and the given function $B$.

A function candidate $B$ is a *barrier function* if, in addition, its change along every solution $\phi$ to $\mathcal{H}$ that starts from $K$ is such that

$$(t, j) \mapsto B(x(t, j))$$

is nonpositive. One way to guarantee such property is as follows. Given a hybrid system $\mathcal{H} = (C, F, D, G)$, suppose the barrier function candidate $B$ defines a closed $K$ as in (15). Furthermore, suppose $B$ is continuously differentiable. Then, $B$ is said to be a barrier function if, for some $\rho > 0$,

$$\langle \nabla B(x), \xi \rangle \ \leq \ 0 \qquad \forall x \in ((K + \rho\mathbb{B})\backslash K) \cap C, \ \forall \xi \in F(x) \cap T_C(x) \tag{16}$$

$$B(\xi) \ \leq \ 0 \qquad \forall \xi \in G(D \cap K) \tag{17}$$

$$G(D \cap K) \ \subset \ C \cup D \tag{18}$$

The barrier function notion introduced above for $\mathcal{H}$ can be formulated for a hybrid plant $\mathcal{H}_P$ given as in (1). In such a setting, since the input to $\mathcal{H}_P$ is not yet assigned, the conditions in (16)-(18) would depend on the input – similar to the conditions that control Lyapunov functions in Section V-A have to satisfy. Such an extension is illustrated in the next example for the bouncing ball system, which, since the input only affects the jumps, conditions (17)-(18) become

$$\forall z \in \Pi(D_P) \ \ \exists u_d \ \text{ such that } \ (z, u_d) \in D_P \text{ and } \left\{ \begin{array}{ll} B(\xi) \leq 0 & \forall \xi \in G_P(z, u_d) \\ G_P(z, u_d) \subset \Pi(C_P) \cup \Pi(D_P) & \end{array} \right. \tag{19}$$

*Example 5.2 (controlled bouncing ball revisited):* Consider the problem of keeping the total energy of the juggling system in Example 3.1 less than or equal to a constant $V^* \geq 0$. The total energy of the hybrid plant therein is given by

$$V(z) = \gamma z_1 + \frac{1}{2} z_2^2$$

Then, the desired set $K$ to render invariant is defined by the continuously differentiable barrier candidate

$$B(z) := V(z) - V^* \qquad \forall z \in \mathbb{R}^2$$

In fact, for the case of a hybrid plant, the set $K$ in (15) collects all points in $z \in \Pi(C_P) \cup \Pi(D_P)$ such that $V(z) \leq V^*$. It follows that

$$\langle \nabla B(z), F_P(z) \rangle = 0 \qquad \forall z \in \Pi(C_P)$$

since, during flows, the total energy remains constant. At jumps, the following hold – recall that $u_d = u$: for each $z \in \Pi(D_P) = \{z \in \mathbb{R}^2 \ : \ z_1 = 0, z_2 \leq 0 \ \}$,

$$G_P(z,u) = \begin{bmatrix} 0 \\ u \end{bmatrix}$$

and

$$B(G_P(z,u)) = V(G_P(z,u)) - V^* = \gamma z_1 + \frac{1}{2}u^2 - V^* = \frac{1}{2}u^2 - V^*$$

Hence, for $B$ to be a barrier function, for each $z \in \Pi(D_P)$ we pick $u$ to satisfy

$$|u| \leq \sqrt{2V^*}$$

In this way, any state feedback $z \mapsto \kappa_d(z)$ such that $|\kappa(z)| \leq \sqrt{2V^*}$ for each $z \in \Pi(D_P)$ leads to a hybrid closed-loop system with $K$ forward invariant[3]. Note that assigning $u$ to a feedback that is positive (when possible) leads to solutions that, after a jump, flow for some time.

### *E. Temporal Logic*

Design specifications for control design typically include requirements that go beyond asymptotic stability properties, such as finite time properties and safety constraints, some of which need to be satisfied at specific times rather than in the limit. A framework suitable for handling such specifications is Linear Temporal Logic (LTL). LTL permits the formulation of desired properties such as *safety*, or equivalently, "something bad never happens," and *liveness*, namely "something good eventually happens" in finite time.

In LTL, a formula (or sentence) is given in terms of atomic propositions that are combined using boolean and temporal operators. An atomic proposition is a function of the state that, for each possible value of the state, is either true or false. More precisely, for $\mathcal{H}$ in (7), a proposition $\mathfrak{a}$ is such that $\mathfrak{a}(x)$ is either `True` (1 or $\top$) or `False` (0 or $\perp$). Boolean operators include the following: $\neg$ is the *negation* operator; $\vee$ is the *disjuction* operator; $\wedge$ is the *conjunction* operator; $\Rightarrow$ is the *implication* operator; and $\Leftrightarrow$ is the *equivalence* operator. A way to reason about a solution defined over hybrid time is needed to introduce temporal operators. This is defined by the semantics of LTL, as follows.

Given a solution $\phi$ to $\mathcal{H}$, a proposition $\mathfrak{a}$ being `True` at $(t,j) \in \operatorname{dom}\phi$ is denoted by

$$\phi(t,j) \Vdash \mathfrak{a}$$

If $\mathfrak{a}$ is `False` at $(t,j) \in \operatorname{dom}\phi$, then we write

$$\phi(t,j) \nVdash \mathfrak{a}$$

Similarly, given an LTL formula $\mathfrak{f}$, we say that it is satisfied by $\phi$ at $(t,j)$ if

$$(\phi,(t,j)) \vDash \mathfrak{f}$$

while $\mathfrak{f}$ not being satisfied at $(t,j)$ is denoted by

$$(\phi,(t,j)) \nvDash \mathfrak{f}$$

The temporal operators are defined as follows: with $\mathfrak{a}$ and $\mathfrak{b}$ being two atomic propositions

- $\bigcirc$ is the *next* operator: $(\phi,(t,j)) \vDash \bigcirc\mathfrak{a}$ if and only if

$$(t,j+1) \in \operatorname{dom}\phi \text{ and } (\phi,(t,j+1)) \vDash \mathfrak{a}$$

- $\Diamond$ is the *eventually* operator: there exists $(t',j') \in \operatorname{dom}\phi$, $t'+j' \geq t+j$ such that $(\phi,(t',j')) \vDash \mathfrak{a}$

[3]It is easy to show that every maximal solution to such a closed loop is complete.

- $\square$ is the *always* operator: $(\phi,(t,j)) \vDash \square\mathfrak{a}$ if and only if, for each $t'+j' \geq t+j$, $(t',j') \in \operatorname{dom}\phi$,
$$(\phi,(t',j')) \vDash \mathfrak{a}$$
- $\mathcal{U}_s$ is the strong *until* operator: $(\phi,(t,j)) \vDash \mathfrak{a}\mathcal{U}_s\mathfrak{b}$ if and only if there exists $(t',j') \in \operatorname{dom}\phi$, $t'+j' \geq t+j$, such that
$$(\phi,(t',j')) \vDash \mathfrak{b}$$
and for all $(t'',j'') \in \operatorname{dom}\phi$ such that $t+j \leq t''+j'' < t'+j'$,
$$(\phi,(t'',j'')) \vDash \mathfrak{a}$$
- $\mathcal{U}_w$ is the weak *until* operator: $(\phi,(t,j)) \vDash \mathfrak{a}\mathcal{U}_w\mathfrak{b}$ if and only if either
$$(\phi,(t',j')) \vDash \mathfrak{a}$$
for all $(t',j') \in \operatorname{dom}\phi$ such that $t'+j' \geq t+j$, or
$$(\phi,(t,j)) \vDash \mathfrak{a}\mathcal{U}_s\mathfrak{b}$$

Similar semantics apply to a formula $\mathfrak{f}$.

*Example 5.3 (Thermostat system revisited):* Consider the thermostat system in Example 3.2. Suppose that the goal is to keep the temperature within the range $[z_{\min}, z_{\max}]$ when the temperature starts in that region, and when it does not start from that range, steer it to that range in finite time and, after that, remain in that range for all time. For simplicity, suppose that the second input is constant and given by $u_2 \equiv z_{\text{out}}$, with $z_{\text{out}} \in (-\infty, z_{\max}]$ and $z_{\text{out}} + z_{\Delta} \in [z_{\min}, \infty)$. It can be shown that the following hybrid controller $\mathcal{H}_K$ accomplishes the state goal: with $\eta \in \{0,1\}$, and with dynamics

$$\mathcal{H}_K \ : \ \begin{cases} \dot{\eta} & \in & F_K(\eta,v) := 0 & (\eta,v) \in C_K := (\{0\}\times C_{K,0}) \cup (\{1\}\times C_{K,1}) \\ \eta^+ & \in & G_K(\eta,v) := 1-\eta & (\eta,v) \in D_K := (\{0\}\times D_{K,0}) \cup (\{1\}\times D_{K,1}) \\ \zeta & = & \kappa(\eta,v) := \eta & \end{cases} \tag{20}$$

where

$$\begin{array}{lllllll} C_{K,0} & := & \{v \ : \ v \geq z_{\min}\}, & \quad & C_{K,1} & := & \{v \ : \ v \leq z_{\max}\} \\ D_{K,0} & := & \{v \ : \ v \leq z_{\min}\}, & \quad & D_{K,1}) & := & \{v \ : \ v \geq z_{\max}\} \end{array}$$

The input of the controller is assigned via $v = z$ and its output assigned $u$ via $u = \zeta = \eta$. Furthermore, it can be shown that the hybrid closed-loop system satisfies the following LTL formulae: with

$$\mathfrak{a}(z,\eta) = 1 \qquad \text{if } z \in [z_{\min}, z_{\max}], \qquad \mathfrak{a}(z,\eta) = 0 \qquad \text{if } z \notin [z_{\min}, z_{\max}],$$

the following hold:

- $\square\mathfrak{a}$ for every solution with initial temperature in $[z_{\min}, z_{\max}]$, regardless of the initial value of $\eta$.
- $\Diamond\mathfrak{a}$, $\Diamond\square\mathfrak{a}$, and $\square\Diamond\mathfrak{a}$ for every solution.

Sufficient conditions involving Lyapunov functions for finite-time attractivity and barrier functions can be employed to guarantee that certain formulas are satisfied. The following table provides pointers to such results.

## VI. Summary and Future Directions

Advances over the last decade on modeling and robust stability of hybrid dynamical systems (without control inputs) have paved the road for the development of systematic methods for the design of control algorithms for hybrid plants. The results selected for this short expository article, along with recent efforts on multi-mode/logic-based control, event-based control, and backstepping, which were not covered here, are scheduled to appear. Future research directions include the development of more powerful tracking control design methods, state observers, and optimal controllers for hybrid plants.

| $\mathfrak{f}$ | Sufficient Conditions in the Literature |
|---|---|
| $\bigcirc\mathfrak{a}$ | Properties of the data of $\mathcal{H}$ – [18, Sections 4.3 and 5.3] |
| $\square\mathfrak{a}$ | Forward invariance – [15], [16], and [18, Section 5.1] |
| $\diamond\mathfrak{a}$ | Finite-time attractivity – [17] and [18, Section 5.2] |
| $\mathfrak{a}\mathcal{U}_s\mathfrak{b}$ | Forward invariance and finite-time attractivity – [18, Section 5.4] |
| $\mathfrak{a}\mathcal{U}_w\mathfrak{b}$ | Forward invariance or finite-time attractivity – [18, Section 5.4] |

TABLE I

SUFFICIENT CONDITIONS FOR LTL FORMULAE INVOLVING TEMPORAL OPERATORS $\bigcirc$, $\square$, $\diamond$, $\mathcal{U}_s$, AND $\mathcal{U}_w$.

## VII. CROSS REFERENCES

Modeling Hybrid Systems
Stability Theory for Hybrid Dynamical Systems
Simulation of Hybrid Systems
Hybrid Observers
Hybrid Model Predictive Control

## VIII. FURTHER READING


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