Invariance principles for switching systems via hybrid systems techniques

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Abstract

Invariance principles for hybrid systems are used to derive invariance principles for nonlinear switching systems with multiple Lyapunov-like functions. Dwell-time, persistent dwell-time, and weak dwell-time solutions are considered. Asymptotic stability results are deduced under further observability assumptions or common bounds on the Lyapunov-like functions.

Key words: switching systems, hybrid systems, invariance principles, dwell-time solutions, multiple Lyapunov functions.

1. Introduction

1.1. *Background*

Switching systems are dynamical systems governed by a differential equation whose right hand side is selected from a given family of functions, based on some (time or state dependent) switching rule. These systems are a particular class of hybrid dynamical systems as they combine continuous dynamics (differential equations) with discrete dynamics (switching). Over the last fifteen years, the area of switching systems has been very active and many efforts have been made to study their stability properties. These include the early work on sufficient conditions for asymptotic stability of linear switching systems with multiple Lyapunov-like functions in [18, 17] and of nonlinear switching systems in [11, 2, 4, 14]. Asymptotic stability under particular classes of switching signals has been analyzed in [10, 14, 8, 9, 1]. For much more background, see [15, 14, 8].

In this paper, we focus on tools for convergence analysis of solutions to switching systems under certain classes of switching signals. On this topic, [8] introduced an invariance principle for switched linear systems under *persistently dwell-time* switching signals. The follow-up work, [9], extended some of the results of [8] to a family of nonlinear switching systems, while [1] presented invariance principles for nonlinear switching systems with *dwell-time* switching signals and *state-dependent* switching that, as a difference to [8], allow for locally Lipschitz Lyapunov functions. For hybrid systems, [16] extended LaSalle's principle to nonblocking, deterministic, and continuous hybrid systems, while in [3], invariance principle for left-continuous and impulsive systems without multiple jumps at an instant (and with further quasi-continuity properties including uniqueness of solutions) is presented. More recently, in [19] (with the results announced in [20]), invariance principles were shown for general hybrid systems in the framework of [7]. (That framework allows for nonuniqueness of solutions, multiple jumps at time instants, and Zeno behaviors, while only posing mild regularity conditions on the data.)

1.2. *Contribution*

The goal of this paper is to show how some of the results of [19] can be used to obtain invariance principles for switching systems, under various types of switching signals. While doing that, we recover, generalize, and/or strengthen some of the results of [8, 9, 1]. In particular:

– Corollary 5.3 strengthens [1, Theorems 1, 2] by including both forward and backward invariance conditions on the

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set to which solutions converge. Also, Corollary 4.4, while having the same invariance conditions as those in [1], also incorporates level sets of Lyapunov functions into the description of the invariant set.
- Corollary 4.7 is an invariance principle for nonlinear switching systems that generalizes [8, Theorem 8] stated for linear switching systems. Even in the linear setting, Corollary 4.7 yields smaller, in comparison to [8, Theorem 8], sets to which solutions converge. This is possible thanks to taking into account the period of persistence of solutions.
- [9, Theorem 7] is derived, in Corollary 4.13, as a consequence of the hybrid invariance principle in Theorem 4.1.

In deriving the results, we rely on invariance principles in [19, 20], but only in proving Theorems 4.1 and 5.2. (Then, several consequences of these two theorems are derived in a self-contained way.) We also use two techniques that should prove useful for purposes other than those in this paper. More specifically:

- Given a solution to a switching system, and a sequence of time intervals of length at least τ_D on which the logical mode takes on a particular value q^*, one can identify the restriction of the solution to those intervals with a function on $[0,\infty)$. The resulting object is not a solution to a switching system, as the continuous variable of the original switching system may now be only piecewise continuous. (Indeed, for the original system there is no reason for the continuous variable to have the same value at the end of an interval when the mode is q^* and at the beginning of the next interval when the mode is q^* again.) However, this resulting object is a solution to an appropriately formulated hybrid system (truly hybrid system, in which both the "continuous" variable and the logical mode may jump). That hybrid system can be given sufficient regularity properties, like those called for by [7]. Thus, invariance principles of [19, 20] can be applied to it, with implications for the original switching system. See the proofs of Corollaries 4.4, 4.7 for illustrations of this technique.
- In the case of multiple Lyapunov functions, i.e., when in logical mode q, a function V_q is decreasing at a rate W_q, it is often assumed that the value of V_{q^*} at the end of an interval with mode q^* is greater or equal than the value of V_{q^*} at the beginning of the next interval with mode q^*. It follows that the function $(x,q) \mapsto V_q(x)$ can not be used in the standard Lyapunov sense, as it is not necessarily decreasing along solutions – it can increase during switches between different logical modes. However, it can be shown that for each bounded solution (x,q) to the switching system, the function $(x,q) \mapsto W_q(x)$ is integrable. (A similar technique was used, for example, in [9, Theorem 7].) This paves way to the application of invariance principles of [19, 20] that rely on an output function that decreases sufficiently fast to 0. See the proof of Theorem 5.2 for an illustration.

In presenting the results, we clearly separate the statements only about invariance of sets to which bounded solutions of switched systems converge (Corollaries 4.4, 4.7, and 5.3) from stronger statements about asymptotic stability that rely on additional information like observability or common bounds on Lyapunov functions (Corollaries 4.10 and 4.13).

2. Preliminaries

2.1. *Switching systems*

Let $O \subset \mathbb{R}^n$ be an open set, let $Q := \{1, 2, \dots, q_{max}\}$, and for each $q \in Q$, let $f_q : O \to \mathbb{R}^n$ be a continuous function. We consider switching systems given by

$$\mathcal{SW}: \qquad \dot{x} = f_q(x). \tag{1}$$

For more background on switching systems, see [14] or [8].

A complete *solution to the switching system* $\mathcal{SW}$ consists of a locally absolutely continuous function $x : [0,\infty) \to O$ and a function $q : [0,\infty) \to Q$ that is piecewise constant and has a finite number of discontinuities in each compact time interval, and $\dot{x}(t) = f_{q(t)}(x(t))$ for almost all $t \in [0,\infty)$. We will say that a complete solution (x,q) to $\mathcal{SW}$ is precompact if x is bounded with respect to O, that is, there exists a compact set $K \subset O$ such that $x(t) \in K$ for all $t \in [0,\infty)$.

In this paper, we will consider only complete solutions to $\mathcal{SW}$ that are generated under particular classes of switching signals. Let (x,q) be a complete solution to $\mathcal{SW}$ and let $t_0 = 0$, and $t_1, t_2, \dots$ be the consecutive (positive) times at which q is discontinuous (informally, t_i is the time of the i-th switch). The solution (x,q) is a *dwell-time solution* with dwell time $\tau_D > 0$ if $t_{i+1} - t_i \geq \tau_D$ for $i = 0, 1, \dots$. (That is, jumps are separated by at least τ_D amount of time.) The solution (x,q) is a *persistent dwell-time solution* with persistent dwell time $\tau_D > 0$ and period of persistence $T > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, t_{i_2}, \dots$ of the sequence $\{t_i\}$ such that $t_{i_k+1} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \dots$ and $t_{i_{k+1}} - t_{i_k+1} \leq T$ for $k = 0, 1, \dots$. (That is, at most T amount of time passes between two consecutive intervals of length at least τ_D on which there is no jumps.) Finally, a solution (x,q) is a *weak dwell-time solution* with dwell time $\tau_D > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, t_{i_2}, \dots$ of the sequence $\{t_i\}$ such that $t_{i_k+1} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \dots$. (That is, there are infinitely many intervals of length τ_D with no switching.) These classes of solutions follow the definitions in [8], see also [10]. More precisely, in [8], dwell-time solutions to $\mathcal{SW}$ are elements of the set $\mathcal{S}_{dwell}$, persistent dwell-time solutions to $\mathcal{SW}$ are elements of the set $\mathcal{S}_{p-dwell}$, and weak dwell-time solutions to $\mathcal{SW}$ are elements of $\mathcal{S}_{weak-dwell}$.

2.2. *Hybrid systems*

We consider hybrid systems of the form

$$\mathcal{H}: \quad \begin{cases} \dot{x} \in F(x) & x \in C, \\ x^+ \in G(x) & x \in D, \end{cases} \tag{2}$$

with an associated state space $\mathcal{O} \subset \mathbb{R}^m$. Above, F (respectively, G) is the possibly set-valued map describing the flow, (respectively, the jumps) while C (respectively, D) is the set on which the flow can occur (respectively, from which the jumps can occur). For more background on hybrid systems in this framework, see [5] or [7].

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \, ... \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = \infty$. A *hybrid arc* is a function whose domain is a hybrid time domain (for a hybrid arc x, its domain will be denoted $\operatorname{dom} x$) and such that for each $j \in \mathbb{N}$, $t \to x(t, j)$ is locally absolutely continuous on $\operatorname{dom} x \cap ([0, \infty) \times \{j\})$.

A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$, $x(t, j) \in \mathcal{O}$ for all $(t, j) \in \operatorname{dom} x$, and

(S1) for all $j \in \mathbb{N}$ such that $I_j := \operatorname{dom} x \cap ([0, \infty) \times \{j\})$ is nonempty, $x(\cdot, j)$ is locally absolutely continuous in t on I_j and, for almost all $t \in I_j$,
$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j));$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,
$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j)).$$

With some abuse of terminology, we will say that x is a *dwell time solution* to $\mathcal{H}$, with dwell time $\tau_D > 0$, if $t_{j+1} - t_j \geq \tau_D$ for $j = 0, 1, \ldots$, where $\operatorname{dom} x = \bigcup ([t_j, t_{j+1}], j)$. Similarly, we will talk about *persistent dwell-time* and *weak dwell-time solutions* to $\mathcal{H}$.

Results on structural properties of solutions to $\mathcal{H}$, like (appropriately understood) sequential compactness of the space of solutions and outer/upper semicontinuous dependence of solutions on initial conditions, were obtained in [7]. These results made possible the general invariance principles of [19, 20]. The assumptions on the data $(\mathcal{O}, F, G, C, D)$ of $\mathcal{H}$ that enabled the results of [7, 19, 20] are as follows: [1]

(A0) $\mathcal{O}$ is open;

(A1) C and D are relatively closed subsets of $\mathcal{O}$;

(A2) $F : \mathcal{O} \rightrightarrows \mathbb{R}^m$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$;

(A3) $G : \mathcal{O} \rightrightarrows \mathbb{R}^m$ is outer semicontinuous, and $G(x)$ is nonempty and such that $G(x) \subset \mathcal{O}$ for all $x \in D$.

In particular, any continuous function f satisfies the assumption (A2), but so does the necessarily set-valued map F we define in (5). Similarly, given a finite set Q, the set-valued map $G(x) = Q$ for all $x \in \mathcal{O}$ satisfies (A3). Let us say that all hybrid systems we write down in this paper do satisfy the assumptions just stated.

3. Switching systems as hybrid systems

Given a switching system $\mathcal{SW}$ as presented in Section 2.1, consider the hybrid system

$$\mathcal{H}_{\mathcal{SW}}: \quad \begin{cases} \dot{x} = f_q(x) & x \in O, q \in Q \\ q^+ \in Q & x \in O, q \in Q \end{cases} \tag{3}$$

with the variable $(x, q) \in \mathbb{R}^{n+1}$. (Here and in what follows, not mentioning $\dot{q}$ in the description of flow or x^+ in the description of jumps means that q remains constant during flow while x does not change during jumps.) To view the system (3) as a special case of (2), one can take the state space to be $\mathcal{O} = O \times \mathbb{R}$, the flow set $C = \bigcup_{q \in Q} O \times \{q\}$, the flow map $F(x, q) = (f_q(x), 0)$ if $x \in O$ and $F(x, q) = \emptyset$ otherwise; the jump set $D = \bigcup_{q \in Q} O \times \{q\}$; and the (set-valued!) jump map $G(x, q) = (x, Q)$ for $(x, q) \in D$ and $G(x, q) = \emptyset$ otherwise. With such data, the conditions (A0)-(A3) are satisfied.

To every solution of the switching system $\mathcal{SW}$ there corresponds a solution to the hybrid system. Indeed, if $t_0 = 0$ and $t_1, t_2, \ldots$ are the times at which q is discontinuous, one can easily build a solution to $\mathcal{H}_{\mathcal{SW}}$ on a hybrid time domain $E = \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\})$ (with J finite or infinite) that corresponds to (x, q). Of course, there are solutions to $\mathcal{H}_{\mathcal{SW}}$ that do not correspond to any solution to $\mathcal{SW}$, for example $\mathcal{H}_{\mathcal{SW}}$ has solutions that only jump (instantaneous Zeno solutions), and other solutions with multiple jumps at an instant. While $\mathcal{H}_{\mathcal{SW}}$ satisfies conditions (A0)-(A3), using invariance principles applied to $\mathcal{H}_{\mathcal{SW}}$ to deduce convergence of, say, dwell-time solutions to it (and behavior of these reflects the behavior of dwell time solutions to $\mathcal{SW}$) may lead to invariant sets whose invariance is verified by the said Zeno solutions. This does not lead to useful conclusions for the underlying switching system. Consequently, better hybrid representations of $\mathcal{SW}$ under dwell time and other classes of switching signals are needed.

To each dwell-time solution (x, q), with dwell time $\tau_D > 0$, to $\mathcal{SW}$ there corresponds a solution (x, q, τ) to the following hybrid system:

$$\mathcal{H}_{\tau_D} \begin{cases} \dot{x} = f_q(x), \ \dot{\tau} \in \kappa_{\tau_D}(\tau) & \tau \in [0, \tau_D] \\ q^+ \in Q, \ \tau^+ = 0 & \tau \in [\tau_D, \infty) \end{cases} \tag{4}$$

Above, $\kappa_{\tau_D} : \mathbb{R} \rightrightarrows \mathbb{R}$ is the (set-valued) map given by

[1] The set-valued map $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if for every convergent sequence of x_i's with $\lim x_i \in \mathcal{O}$, and every convergent sequence of $y_i \in F(x_i)$, $\lim y_i \in F(\lim x_i)$. F is *locally bounded* if for every compact $K \subset \mathcal{O}$ there exists a compact $K' \subset \mathbb{R}^n$ such that $F(K) \subset K'$. Similarly for G.

$$\kappa_{\tau_D}(\tau) = \begin{cases} 1 & \text{if } \tau < \tau_D \\ [0,1] & \text{if } \tau = \tau_D \\ 0 & \text{if } \tau > \tau_D \end{cases}.$$

Solutions to $\dot{\tau} \in \kappa_{\tau_D}(\tau)$ increase at the rate 1 when $\tau < \tau_D$ and remain constant otherwise. The map κ_{τ_D} is introduced to keep the variable τ bounded.

In the opposite direction, some solutions to (4) may flow before the first jump for less than τ_D amount of time, but those that have $\tau(0,0) = 0$ do correspond directly to dwell time solutions, with dwell time τ_D, to $\mathcal{SW}$.

Let $F : O \rightrightarrows \mathbb{R}^n$ be the set valued map defined by

$$F(x) = \overline{\text{con}} \bigcup_{q \in Q} f_q(x) , \tag{5}$$

where $\overline{\text{con}} S$ stands for the closed convex hull of the set S. To each persistent dwell-time solution (x, q) to $\mathcal{SW}$, with dwell time $\tau_D > 0$ and period of persistence $T > 0$, there corresponds a solution (x, q, τ_1, τ_2) to the following hybrid system:

$$\mathcal{H}_{\tau_D,T} \begin{cases} \dot{x} = f_q(x), \dot{\tau}_1 \in \kappa_{\tau_D}(\tau_1) & q \in Q, \tau_1 \in [0, \tau_D] \\ \dot{x} \in F(x), \dot{\tau}_2 = 1 & q = 0, \tau_2 \in [0, T] \\ \left.\begin{array}{c} q^+ \in Q \cup \{0\}, \\ \tau_1^+ = 0, \tau_2^+ = 0 \end{array}\right\} & q \in Q, \tau_1 = \tau_D \\ \left.\begin{array}{c} q^+ \in Q, \\ \tau_1^+ = 0, \tau_2^+ = 0 \end{array}\right\} & q = 0, \tau_2 \in [0, T] \end{cases}$$

In other words, solutions x to $\dot{x} = f_q(x)$ under arbitrary switching signals q are solutions to the inclusion $\dot{x} \in F(x)$. (In fact, x is a solution to the inclusion, on some bounded time interval, if and only if it is a uniform limit of some sequence of solutions generated via switching.)

As we show in the next sections, invariance principles of [20], applied to $\mathcal{H}_{\tau_D}$, do lead to conclusions that can be translated to useful results on the behavior of x variable only, and thus, to results on the behavior of dwell-time solutions to $\mathcal{SW}$.

4. Hybrid invariance principle using a nonincreasing function, and consequences

In this section, we present invariance principles to establish convergence of dwell-time, persistent dwell-time, and weak dwell-time solutions to switching systems. The foundation to those will be an invariance principle for hybrid systems, which comes out of [20], and is based on a nonincreasing Lyapunov function.

4.1. *A hybrid invariance principle using a nonincreasing function*

Theorem 4.1 *Let $O \subset \mathbb{R}^n$ be open, $f : O \to \mathbb{R}^n$ be continuous, $K \subset O$ be nonempty and compact, $V : O \to \mathbb{R}$ be continuously differentiable, $W : O \to \mathbb{R}_{\geq 0}$ be continuous and such that*

$$\nabla V(x) \cdot f(x) \leq -W(x)$$

for all $x \in O$. Consider a hybrid system

$$\mathcal{H}_1 : \begin{cases} \dot{x} = f(x), \dot{\tau} \in \kappa_{\tau_D}(\tau) & \tau \in [0, \tau_D], \\ x^+ \in K, \tau^+ = 0 & \tau \in [\tau_D, \infty), \end{cases} \tag{6}$$

on the state space $O \times \mathbb{R}$. Let $(x, \tau) : \text{dom}(x, \tau) \to O \times \mathbb{R}_{\geq 0}$ be a complete solution to $\mathcal{H}_1$ such that $x(t, j) \in K$ for all $(t, j) \in \text{dom}\, x$ and such that

$$V(x(t, j + 1)) \leq V(x(t, j))$$

for all $(t, j) \in \text{dom}\, x$ such that $(t, j + 1) \in \text{dom}\, x$. Then, for some constant $r \in \mathbb{R}$, x approaches the largest subset M of

$$V^{-1}(r) \cap K \cap W^{-1}(0)$$

that is invariant in the following sense: for each $x_0 \in M$ there exists a solution ξ to $\dot{x} = f(x)$ on $[0, \tau_D/2]$ such that $\xi(t) \in M$ for all $t \in [0, \tau_D/2]$ and either $\xi(0) = x_0$ or $\xi(\tau_D/2) = x_0$.

Proof. The hybrid system $\mathcal{H}_1$ does satisfy assumptions (A0)-(A3). By assumptions, x is bounded (by K), while τ is bounded by the construction of $\mathcal{H}_1$. Thus, (x, τ) is precompact. [20, Corollary 4.3], specialized to dwell time solutions along the lines of [20, Corollary 4.2], (or just [19, Corollary 4.4]) implies that, for some $r \in \mathbb{R}$, (x, τ) approaches the largest subset M' of

$$\left(V^{-1}(r) \cap K \cap W^{-1}(0)\right) \times \mathbb{R}$$

that is weakly backward invariant for $\mathcal{H}_1$ in the following sense: for each $(x_0, \tau_0) \in M'$, each $R \in \mathbb{R}_{\geq 0}$, there exists a complete solution (x^b, τ^b) to $\mathcal{H}_1$ such that $(x_0, \tau_0) = (x^b(t^*, j^*), \tau^b(t^*, j^*))$ for some $(t^*, j^*) \in \text{dom}(x^b, \tau^b)$ with $t^* + j^* \geq R$ and $(x^b(t, j), \tau^b(t, j)) \in M'$ for all $(t, j) \in \text{dom}(x^b, \tau^b)$. [2]

As (x, τ) approaches M', x approaches the projection of $M' \subset O \times \mathbb{R}$ onto O. It is thus sufficient to show that this projection, let us denote it by M'', is a subset of M. Take any $x_0 \in M''$, and $\tau_0 \in \mathbb{R}$ such that $(x_0, \tau_0) \in M'$. Consider $R \geq \tau_D$ in the definition of weak backward invariance in the paragraph above, and let (x^b, τ^b) be a solution verifying that backward invariance at (x_0, τ_0). We have

[2] The conclusion that $(x^b(t, j), \tau^b(t, j)) \in M'$ for all $(t, j) \in \text{dom}(x^b, \tau^b)$ is stronger than that used in weak backward invariance definition in [20, 19], where it is only required that $(x^b(t, j), \tau^b(t, j)) \in M'$ for $(t, j) \in \text{dom}(x^b, \tau^b)$ with $t+j \leq t^*+j^*$. However, the stronger property can be immediately deduced from the proof of [19, Lemma 3.3].

$(x^b(t^*,j^*), \tau^b(t^*,j^*)) = (x_0,\tau_0)$ where $t^* \in [t_{j^*}, t_{j^*+1}]$, $[t_{j^*}, t_{j^*+1}] \times \{j^*\} \subset \mathrm{dom}(x^b,\tau^b)$, and $t_{j^*+1} - t_{j^*} \geq \tau_D$. Either $t^* - t_{j^*}$ or $t_{j^*+1} - t^*$ is greater or equal to $\tau_D/2$, and this leads to a solution (x',τ') to $\dot{x} = f(x)$, $\dot{\tau} = 1$ on $[0,\tau_D/2]$ such that $(x'(t),\tau'(t)) \in M'$ for all $t \in [0,\tau_D/2]$ and either $x(0) = x_0$ or $x(\tau_D/2) = x_0$. Then also $x'(t) \in M''$ for all $t \in [0,\tau_D/2]$, and so M'' is weakly backward invariant. So $M'' \subset M$. ◻

4.2. *Invariance principles for switching systems*

We now apply Theorem 4.1 to switching systems. The results are shown for the case of multiple Lyapunov functions, under the following assumptions.

Assumption 4.2 *$O \subset \mathbb{R}^n$ is an open set, for each $q \in Q$, $f_q : O \to \mathbb{R}^n$ is a continuous function, $V_q : O \to \mathbb{R}$ is a continuously differentiable function, $W_q : O \to \mathbb{R}_{\geq 0}$ is a continuous function, and*

$$\nabla V_q(x) \cdot f_q(x) \leq -W_q(x) \text{ for all } x \in O.$$

Assumption 4.3 *The solution (x,q) to $\mathcal{SW}$ is such that, for each $q^* \in Q$, for any two consecutive intervals (t_j,t_{j+1}), (t_k,t_{k+1}) such that $q(t) = q^*$ for all $t \in (t_j,t_{j+1})$ and all $t \in (t_k,t_{k+1})$, we have*

$$V_{q^*}(x(t_{j+1})) \geq V_{q^*}(x(t_k)).$$

In short, the value of V_{q^*} at the end of an interval on which $q = q^*$ is greater or equal to the value of V_{q^*} at the beginning of the next interval on which $q = q^*$. This assumption is usually needed when establishing convergence and stability results for switching systems, see e.g. [8],[1]. In control design, Assumption 4.3 is sometimes enforced by constructing a supervisory switching controller that selects the mode q at switches appropriately.

4.2.1. *Invariance principle for dwell-time solutions to $\mathcal{SW}$*

We begin with an application of Theorem 4.1 to dwell-time solutions of $\mathcal{SW}$.

Corollary 4.4 *Let Assumption 4.2 hold, and let (x,q) be a precompact dwell-time solution, with dwell time $\tau_D > 0$, to the switching system $\mathcal{SW}$ satisfying Assumption 4.3. Then there exist $r_1, \ldots, r_{q_{max}} \in \mathbb{R}$ such that x approaches*

$$M = \bigcup_{q \in Q} M_q(r_q,\tau_D), \tag{7}$$

where $M_q(r_q,\tau_D)$ is the largest subset of $V_q^{-1}(r_q) \cap W_q^{-1}(0)$ that is invariant in the following sense: for each $x_0 \in M_q(r_q,\tau_D)$ there exists a solution ξ to $\dot{x} = f_q(x)$ on $[0,\tau_D/2]$ such that $\xi(t) \in M_q(r_q,\tau_D)$ for all $t \in [0,\tau_D/2]$ and either $\xi(0) = x_0$ or $\xi(\tau_D/2) = x_0$.

Proof. For each $q^* \in Q$ for which there is infinitely many disjoint time intervals $(t_j, t_j + \Delta t_j)$, $j = 0,1,\ldots$, $\Delta t_j \geq \tau_D$, on which q equals q^*, consider a hybrid arc z with

$$\mathrm{dom}\, z = \bigcup_{j=0}^{\infty} \left[\sum_{i=0}^{j-1} \Delta t_j, \sum_{i=0}^{j} \Delta t_j \right] \times \{j\}$$

(with the convention that $\sum_{i=0}^{-1} \Delta t_j = 0$) defined by

$$z(t,j) = x\left(t - \sum_{i=0}^{j-1} \Delta t_j + t_j \right)$$

for $t \in \left[\sum_{i=0}^{j-1} \Delta t_j, \sum_{i=0}^{j} \Delta t_j \right]$. Such a hybrid arc is a solution to $\mathcal{H}_1$ of Theorem 4.1, and meets the assumptions of that theorem, with f, V, W replaced by f_{q^*}, V_{q^*}, W_{q^*}, and with $K \subset O$ being any compact set such that $x(t) \in K$ whenever $q(t) = q^*$. Theorem 4.1 implies the claim. ◻

If, given a continuously differentiable function $V : O \to \mathbb{R}^n$, and a continuous function $W : O \to \mathbb{R}_{\geq 0}$, we have that $V_q = V$, $W_q = W$ for all $q \in Q$, the conclusion of Corollary 4.4 is stronger than that of Theorem 1 in [1]. One of the reasons is due to [1] not taking advantage of the invariant set to which solutions converge being a subset of some level set (and not just a sublevel set) of V. (In [1], the counterpart of the set M is given by the union of the largest invariant subsets in $\bigcup_{q \in Q} W_q^{-1}(0)$.) Further strengthening of this result will be carried out in Theorem 5.2 and Corollary 5.3, where the set to which solutions converge will be shown to be both forward and backward invariant (note that the set to which solutions converge in Corollary 4.4 can be either forward or backward invariant, but not necessarily both).

Example 4.5 Consider the switching system in [1, Example 5] given by

$$f_1(x) = \begin{bmatrix} -x_1 - x_2 \\ x_1 \end{bmatrix}, \quad f_2(x) = \begin{cases} \begin{bmatrix} -x_1 - x_2 \\ x_1 \end{bmatrix} & \text{if } x_1 < 0 \\ \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & \text{if } x_1 \geq 0 \end{cases}$$

where $x = [x_1 \ x_2]^T \in \mathbb{R}^2$. Let $Q = \{1,2\}$. With the quadratic function $V(x) = x_1^2 + x_2^2$, we get $W_1(x) = -2x_1^2$ and $W_2(x) = -2x_1^2$ if $x_1 < 0$ and $W_2(x) = 0$ if $x_1 \geq 0$. Then, [1, Theorem 1] establishes that each bounded solution to the switching system starting from $x^0 \in \mathbb{R}^2$ converges to $S := \left\{ x \in \mathbb{R}^2 \mid x_1 \geq 0 \right\} \cap \left\{ x \in \mathbb{R}^2 \mid V(x) \leq V(x^0) \right\}$ since

$$W_1^{-1}(0) = \left\{ x \in \mathbb{R}^2 \mid x_1 = 0 \right\},$$
$$W_2^{-1}(0) = \left\{ x \in \mathbb{R}^2 \mid x_1 \geq 0 \right\},$$

and the largest invariant set in

$$\bigcup_{q \in Q} W_q^{-1}(0) \cap \left\{ x \in \mathbb{R}^2 \mid V(x) \leq V(x^0) \right\} = S$$

is the set S itself.

Now note that for each $q \in \{1,2\}$, the only invariant set in $V^{-1}(r) \cap W_q^{-1}(0)$ (in the sense defined in Corollary 4.4) is for $r = 0$. Hence $M_q = 0$ for $q \in \{1, 2\}$ and thus Corollary 4.4 guarantees that every precompact dwell-time solution to the switching system is such that the x component converges to the origin. △

In addition to the improvement due to using a level set of V in Corollary 4.4, the invariance properties requested in Corollary 4.4 are stronger than those in [1, Theorem 1]. The latter concludes convergence to the largest invariant subset of $\bigcup_{q\in Q} W_q^{-1}(0)$, where invariance at each point needs to be verified by a solution to $\dot{x} = f_q(x)$ for some $q \in Q$. Corollary 4.4 concludes convergence to the union, over $q \in Q$, of invariant sets in $V_q^{-1}(r_q) \cap W_q^{-1}(0)$ with respect to $\dot{x} = f_q(x)$.

Regarding the invariance principle for switching systems with multiple Lyapunov-like functions in [1, Theorem 2], the conclusion of Corollary 4.4 is also stronger since [1, Theorem 2] does not take advantage of the invariant set that the authors denote by Ω_l. (Ω_l is also not used in the more precisely stated [1, Theorem 1] that deals with a common Lyapunov function.)

4.2.2. *Invariance principle for persistent dwell-time solutions to $\mathcal{SW}$*

Given $f_1, \ldots, f_{q_{max}}$ as in Assumption 4.2, let $F : O \rightrightarrows \mathbb{R}^n$ be the set-valued map given by (5). Given sets $S_1, S_2 \subset \mathbb{R}^n$, let $\mathcal{F}_T(S_1, S_2)$ be the set of all points that can be expressed as $\xi(t)$ where $\xi : [0, T'] \to O$, with some $T' \in [0, T]$, is a solution to $\dot{\xi} \in F(\xi)$ such that $\xi(0) \in S_1$ and $\xi(T') \in S_2$. Note that considering $T' = 0$ suggests that $S \subset \mathcal{F}_T(S, S)$ for any set $S \subset \mathbb{R}^n$.

Below, $\mathrm{dist}_S(x)$ denotes the distance of the point x from the set S.

Lemma 4.6 *Let $K \subset O$ be compact. Consider two sequences of points $\xi_i, \eta_i \in K$ such that $\mathcal{F}_T(\xi_i, \eta_i) \subset K$ for $i = 1, 2, \ldots$ and such that $\mathrm{dist}_L(\xi_i) \to 0$, $\mathrm{dist}_L(\eta_i) \to 0$ as $t \to \infty$ for some closed $L \subset K$. Then, for any $\varepsilon > 0$ there exists i_ε such that for all $i \geq i_\varepsilon$,*

$$\mathcal{F}_T(\xi_i, \eta_i) \subset \mathcal{F}_T(L, L) + \varepsilon\mathbb{B}.$$

The result above is immediate from local boundedness, upper semicontinuity, and convex-valuedness of the set-valued map F.

Corollary 4.7 *Under Assumption 4.2, let (x, q) be a precompact persistent dwell-time solution to $\mathcal{SW}$, with dwell time $\tau_D > 0$ and period of persistency $T > 0$, satisfying Assumption 4.3. Then, there exist $r_1, \ldots, r_{q_{max}} \in \mathbb{R}$ such that x approaches $\mathcal{F}_T(M, M)$, where M is as in Corollary 4.4.*

Proof. Let $I \subset \mathbb{R}_{\geq 0}$ be the union of all open intervals of length at least τ_D on each of which q is constant. As in Corollary 4.4 and using Theorem 4.1, one can show that $x(t)$ approaches M when $t \in \overline{I}$ and $t \to \infty$. Let $\{(a_i, b_i)\}$ be the sequence of consecutive connected components of $\mathbb{R}_{\geq 0} \setminus \overline{I}$. Then $\mathrm{dist}_M(a_i) \to 0$, $\mathrm{dist}_M(b_i) \to 0$ as $t \to \infty$. Lemma 4.6, with K being any compact set for which $x(t) \in K$ for all $t \in \mathbb{R}_{\geq 0}$ and with $L = \overline{M} \cap K$, finishes the proof. □

Consider the case of linear vector fields $f_q(x) = A_q x$, quadratic $V_q(x) = x^T P_q x$, $W_q(x) = x^T C_q^T C_q x$. A very similar case was treated by [8, Theorem 8], with the difference that here the set Q is finite, rather than compact. [8, Theorem 8] concludes that every precompact persistent dwell-time solution (x, q) to $\mathcal{SW}$ is such that x converges to L, the smallest subspace that is A_q-invariant for each $q \in Q$ and contains the unobservable subspaces of all the pairs (A_q, C_q). Corollary 4.7 gives a more precise statement, taking into account the period of persistency T of the solution at hand. The set M of Corollary 4.4 is the union of unobservable subspaces of all the pairs (A_q, C_q). While $\mathcal{F}_T(M, M) \subset L$, the set $\mathcal{F}_T(M, M)$ is a strict subset of L (and not a subspace) for each T. In fact, $\mathcal{F}_T(M, M)$ is a subset of a neighborhood of M, the radius of which depends on T and on the matrices A_q.

Further improvement in Corollary 4.7 can be made by noting that one can replace M in that corollary by M', with M' being the union of only those sets $M_{q^*}(r_{q^*}, \tau_D)$ from Corollary 4.4 for which q^* is attained by the variable q for at least τ_D units of time, infinitely many times. This can lead to surprisingly stronger conclusions than [8, Theorem 8] as the following example illustrates.

Example 4.8 Consider the switching system with vector fields

$$f_1(x) := \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}, \quad f_2(x) := \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

where $x = [x_1 \ x_2]^T \in \mathbb{R}^2$. Let $Q = \{1, 2\}$. Consider any persistent dwell-time solution (x, q). With the quadratic function $V(x) = x_1^2 + x_2^2$, we get $W_1(x) = -2x_1^2 - 2x_2^2$ and $W_2(x) = 0$. It follows that

$$W_1^{-1}(0) = \{0\}, \quad W_2^{-1}(0) = \mathbb{R}^2.$$

The smallest subspace that is f_q-invariant for all $q \in Q$ and contains $W_1^{-1}(0) \cup W_2^{-1}(0)$ is $\mathbb{R}^2$. Then, [8, Theorem 8] does not give any useful information regarding the convergence of x.

Suppose now that the switching signal is such that $q = 2$ is attained at most finitely many times during the "dwell intervals" of no switching that are of length at least $\tau_D > 0$. (And so $q = 1$ is attained infinitely many times.) In Corollary 4.7 one can replace M by the set M', given as the largest invariance set in W_q^{-1} for $q = 1$. That is, the largest invariant set in $\left\{x \in \mathbb{R}^2 \mid x = 0\right\}$ for $\dot{x} = f_1(x)$, which turns out to be the origin itself. As $\mathcal{F}_T(0, 0) = 0$, the persistent dwell-time solutions for the switching signal above are such that x converges to the origin. △

4.2.3. *Observability and stability*

We will say that a pair of functions (f, W) is observable if, for each $a < b$, the only solution $x : [a, b] \to \mathbb{R}^n$ to $\dot{x} = f(x)$ with $W(x(t)) = 0$ for all $t \in [a, b]$ is $x(t) = 0$ for all $t \in [a, b]$.

Assumption 4.9 *For each $q \in Q$, the pair (f_q, W_q) is observable.*

This assumption implies, in particular, that the sets $M_q(r, \tau_D)$ in Corollary 4.4 all equal $\{0\}$.

Corollary 4.10 *Let Assumptions 4.2, 4.9 hold. Then, every precompact dwell-time solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3 is such that x converges to the origin. If furthermore, for each $q \in Q$, f_q is locally Lipschitz continuous and $f_q(0) = 0$, then every precompact persistent dwell-time solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3 is such that x converges to the origin.*

Proof. The first conclusion comes from Corollary 4.4, as the set M of that Corollary is just $\{0\}$. The second conclusion follows from Corollary 4.7. Indeed, the map F defined in (5) is locally Lipschitz continuous (in the sense of set-valued maps), with $F(0) = 0$, and the unique solution to it from 0 remains at 0. Thus $\mathcal{F}_T(M, M) = \mathcal{F}_T(0, 0) = \{0\}$ for any $T \geq 0$. $\square$

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$ function if $\gamma(0) = 0$ and γ is continuous, strictly increasing, and unbounded.

Assumption 4.11 *There exist class-$\mathcal{K}_\infty$ functions $\alpha, \beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for each $q \in Q$ and all $x \in O$,*

$$\alpha(|x|) \leq V_q(x) \leq \beta(|x|).$$

The following result is immediate. See, for example, [2, Theorem 2.3].

Lemma 4.12 *Under Assumptions 4.2 and 4.11 there exists a class-$\mathcal{K}_\infty$ function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for any solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3, $|x(t)| \leq \gamma(|x(0)|)$. In particular, for any $\varepsilon > 0$ there exists $\delta > 0$ such that every solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3 with $|x(0)| \leq \delta$ satisfies $|x(t)| \leq \varepsilon$ for all $t \in \mathbb{R}_{\geq 0}$.*

In particular, Assumption 4.11 implies that all solutions to $\mathcal{SW}$ are bounded. Furthermore, it guarantees stability of 0, and hence quite weak conditions are sufficient for solutions (x, q) to $\mathcal{SW}$ to be such that $x \to 0$. In particular, we have the following result, that parallels [9, Theorem 7], and also [8, Theorem 4] that was given for the case of linear systems and quadratic Lyapunov functions.

Corollary 4.13 *Let Assumptions 4.2, 4.9, and 4.11 hold. Then any weak dwell time solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3 is bounded and any such complete solution is such that x converges to the origin.*

Proof. Let (x, q) be a weak dwell-time solution to $\mathcal{SW}$. By Lemma 4.12, (x, q) is bounded. If it is complete, to show convergence of x to 0, it is enough to show that $\liminf_{t\to\infty} |x(t)| = 0$, thanks to Lemma 4.12. There exists $\tau_D > 0$, $q^* \in Q$, and infinitely many time intervals $(t_j, t_j + \tau_D)$ on which $q(t) = q^*$. As in Corollary 4.4, using these intervals one can build a solution to $\mathcal{H}_1$ of Theorem 4.1, so that assumptions of that theorem hold with f, V, W replaced by f_{q^*}, V_{q^*}, W_{q^*}, and with $K \subset O$ being any compact set such that $x(t) \in K$ whenever $q(t) = q^*$. Theorem 4.1 and the observability assumption implies that $\lim_{j\to\infty} |x(t_j)| = 0$ and so $\liminf_{t\to\infty} |x(t)| = 0$. $\square$

5. Hybrid invariance principle using a meagre function, and consequences

We now improve one of our results, Corollary 4.4, by relying on an invariance principle for hybrid systems from [19, 20] that does not involve a nondecreasing Lyapunov function, but rather, an appropriately fast vanishing output. We will rely on the following version of Assumption 4.3 which is appropriate for solutions to hybrid systems.

Assumption 5.1 *The hybrid arc (x, q), with $\mathrm{dom}(x, q) = \bigcup_{j=0}^{J} [t_j, t_{j+1}] \times \{j\}$ where $J \in \mathbb{N} \cup \{\infty\}$, is such that, for each $q^* \in Q$, for any two consecutive numbers $j_* < j^*$ such that $q(t, j_*) = q^*$ for all $t \in [t_{j_*}, t_{j_*+1}]$ and $q(t, j^*) = q^*$ for all $t \in [t_{j^*}, t_{j^*+1}]$, we have*

$$V_{q^*}(x(t_{j_*+1})) \geq V_{q^*}(x(t_{j^*})).$$

Theorem 5.2 *Let Assumption 4.2 hold. Let (x, q, τ) be a precompact solution to $\mathcal{H}_{\tau_D}$ in (4) such that (x, q) satisfies Assumption 5.1. Then x approaches the largest subset N of*

$$\bigcup_{p \in Q} W_p^{-1}(0)$$

that is invariant in the following sense: for each $x_0 \in N$ there exist $p_1, p_2 \in Q$ such that $x_0 \in W_{p_1}^{-1}(0) \cup W_{p_2}^{-1}(0)$, $t_1, t_2 > 0$ with $t_1 + t_2 \geq \tau_D$, a solution $\xi_1 : [-t_1, 0] \to W_{p_1}^{-1}(0) \cap N$ to $\dot{\xi}_1 = f_{p_1}(\xi_1)$ such that $\xi_1(0) = x_0$, and a solution $\xi_2 : [0, t_2] \to W_{p_2}^{-1}(0) \cap N$ to $\dot{\xi}_2 = f_{p_2}(\xi_2)$ such that $\xi_2(0) = x_0$.

Proof. Let $\ell(t) = W_{q(t,j)}(x(t, j))$ for each $t \in \mathbb{R}_{\geq 0}$ that is not a time of a jump, and $\ell(t) = 0$ otherwise. We will first show that ℓ is an L^1 function on $\mathbb{R}_{\geq 0}$, and thus it is weakly meagre. For a $q^* \in Q$, let $[t_{j_i}, t_{j_i+1}] \times \{j_i\}$ be the sequence of all intervals in $\mathrm{dom}(x, q, \tau)$ on which $q(t, j_i) = q^*$, with j_i increasing. (The sequence may be empty, finite, or infinite, and it is infinite for at least one $q^* \in Q$.) Let $\ell_{q^*} : \mathbb{R}_{\geq 0}$ be a function given by

$$\ell_{q^*(t)}(t) = \begin{cases} W_{q^*}(x(t, j_i)) & \text{if } t \in (t_{j_i}, t_{j_i+1}) \\ 0 & \text{otherwise} \end{cases}$$

Let $I_{q^*} = \bigcup (t_{j_i}, t_{j_i+1})$. By Assumptions 4.2 and 4.3, for all $t \in I_{q^*}$,

$$V_{q^*}(x(t, j_i)) - V_{q^*}(x(t_{j_0}, t_{j_0+1})) \leq -\int_{s \in I_{q^*}, s \leq t} W_{q^*}(x(s, j(s)))\, ds$$

where $j(s)$ is such that $(s, j(s)) \in \mathrm{dom}(x, q, \tau)$. Recall that (x, q, τ) is precompact, and thus $V_{q^*}(x(t, j(t)))$ is bounded over all $t \in I_{q^*}$. This implies that $w_{q^*} := \int_{s \in I_{q^*}} W_{q^*}(x(s, j(s)))\, ds$ exists. Now note that for each

$t \in \mathbb{R}$, $\int_0^t \ell(s)\,ds \leq \sum_{q^* \in Q} w_{q^*}$, and this is enough to conclude that l is integrable on $\mathbb{R}_{\geq 0}$.

Now, [19, Corollaries 5.4, 5.6] imply that (x, q, τ) approaches the largest subset N' of

$$\bigcup_{p \in Q} W_p^{-1}(0) \times \{p\} \times \mathbb{R}$$

that is invariant (with respect to $\mathcal{H}_{\tau_D}$) in the following sense: for each $(x^0, q^0, \tau^0) \in N'$, each $R \in \mathbb{R}$ there exists a complete solution (ξ, p, σ) of $\mathcal{H}_{\tau_D}$ such that $(\xi, p, \sigma)(t, j) \in N'$ for all $(t, j) \in \operatorname{dom}(\xi, p, \sigma)$ and $(\xi, p, \sigma)(t^*, j^*) = (x^0, q^0, \tau^0)$ for some $(t, j) \in \operatorname{dom}(\xi, p, \sigma)$ with $t + j \geq R$. Thus x approaches the projection N'' of $N' \subset \mathbb{R}^n \times Q \times \mathbb{R}$ onto $\mathbb{R}^n$. It remains to show that N'' is invariant in the sense specified in the theorem. $\square$

We note that the idea of considering functions ℓ_{q^*} is similar to what is done in the proof of [9, Theorem 7]. (However, in [9, Theorem 7], additional assumptions led to asymptotic stability of 0, not an invariance statement.)

Theorem 5.2 implies the following two invariance principles for dwell-time solutions to $\mathcal{SW}$.

Corollary 5.3 *Let Assumption 4.2 hold. Let (x, q) be a precompact dwell-time solution to $\mathcal{SW}$ that satisfies Assumption 4.3. Then the conclusions of Theorem 5.2 hold.*

When compared to [1, Theorem 2], Corollary 5.3 gives stronger invariance conditions on the set to which x must converge. In [1], it is only required that there exist either a forward or a backward solution (i.e., either ξ_1 or ξ_2) while here, Theorem 5.2 calls for the existence of both a forward and a backward solution.

Example 5.4 Consider the switching system in [1, Example 4] given by

$$f_1(x) = \begin{bmatrix} -x_1 - x_2 \\ x_1 \end{bmatrix}, \; f_2(x) = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

where $x = [x_1 \; x_2]^T \in \mathbb{R}^2$. Let $Q := \{1, 2\}$. Following [1, Example 4], with $V(x) = x_1^2 + x_2^2$ we get $W_1(x) = -2x_1^2$ and $W_2(x) = -V(x)$. Then

$$W_1^{-1}(0) = \left\{ x \in \mathbb{R}^2 \mid x_1 = 0 \right\}, \;\; W_2^{-1}(0) = \{0\},$$

and the largest invariant set in $\bigcup_{q \in Q} W_q^{-1}(0)$ is equal to $\left\{ x \in \mathbb{R}^2 \mid x_1 = 0 \right\}$. Then, via [1, Theorem 1], every solution starting from x^0 converges to $\left\{ x \in \mathbb{R}^2 \mid x_1 = 0 \right\} \cap \left\{ x \in \mathbb{R}^2 \;\middle|\; V(x) \leq V(x^0) \right\}$, which corresponds to a segment on the x_2-axis centered at the origin. Convergence to the origin can be shown using Corollary 4.4. Let us apply Corollary 5.3 instead. This corollary is more similar to [1, Theorem 1], as it does not use a level set of V_q in the characterization of the invariant set. The basic difference between [1, Theorem 1] and Corollary 5.3 is the notion of invariance.

We have $\bigcup_{p \in Q} W_p^{-1}(0) = \left\{ x \in \mathbb{R}^2 \mid x_1 = 0 \right\}$. Any point $x_0 \neq 0$ in this set is in $W_1^{-1}(0)$ but not in $W_2^{-1}(0)$. Now, the fact that no subset of $W_1^{-1}(0) = \left\{ x \in \mathbb{R}^2 \mid x_1 = 0 \right\}$ except $\{0\}$ is invariant under f_1 implies that the subset N of $\bigcup_{p \in Q} W_p^{-1}(0)$, invariant in the sense of Theorem 5.2, is exactly $\{0\}$. Hence, all solutions of the system under discussion have x converging to $\{0\}$. $\triangle$

The example above shows that one way to obtain stronger results from invariance principles is by considering invariance notions that involve both forward and backward invariance. This is, of course, the case in simpler settings.

Example 5.5 Consider the continuous-time system $\dot{x} = -\operatorname{sat}(x)$, $x \in \mathbb{R}$, where $\operatorname{sat}(x) = x$ if $|x| \leq 1$, 1 if $x > 1$, -1 if $x < -1$. Let $V(x) = 0$ if $|x| \leq 2$, $x^2 - 4$ if $|x| > 2$, so that $W^{-1}(0) = [-2, 2]$. Relying on an invariance principle with only forward invariance (a la [12, Theorem 1]) shows that solutions converge to $M = [-2, 2]$. Relying on both forward and backward invariance (a la [13, Theorem 6.4]) leads to asymptotic stability of 0. $\triangle$

To conclude, we combine Corollaries 4.4 and 5.3.

Corollary 5.6 *Let Assumption 4.2 hold. Let (x, q) be a precompact dwell-time solution to $\mathcal{SW}$ that satisfies Assumption 4.3. Then there exist $r_1, \ldots, r_{q_{max}} \in \mathbb{R}$ such that x approaches the largest subset of*

$$\bigcup_{p \in Q} W_p^{-1}(0) \cap V_p^{-1}(r_p)$$

that is invariant as stated in Theorem 5.2.

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