Lipschitzness of Minimal-Time Functions in Constrained Continuous-Time Systems with Applications to Reachability Analysis

Mohamed Maghenem and Ricardo G. Sanfelice

***Abstract*— The minimal-time function with respect to a closed set for a constrained continuous-time system provides the first time that a solution starting from a given initial condition reaches that set. In this paper, we propose infinitesimal necessary and sufficient conditions for the minimal-time function to be locally Lipschitz. As an application of our results, we show that, in constrained continuous-time systems, the Lipschitz continuity of the minimal-time function with respect to the boundary of the set where the solutions are defined plays a crucial role on the Lipschitz continuity of the reachable set.**

I. Introduction

In constrained continuous-time systems of the form $\dot{x} \in F(x)$ $x \in C$, the minimal-time function with respect to a closed subset $K \subset C$ provides the first time that a solution reaches the set K. The minimal-time function, denoted t_K^{min}, has a close relationship to the solution of the well-studied Hamilton-Jacobi equation [1], [2]. Moreover, it is very useful in minimal-time control problems, where the objective is to steer the solutions towards a given target in minimal time [3]. Furthermore, when the set K coresponds to the jump set of a hybrid system, t_K^{min} is shown in [4] and [5] to be key when characterizing orbital stability. The name *time-to-impact* function is used in the latter two references.

One of the most interesting questions related to t_K^{min} concerns the analysis of its continuity properties. Such a problem has been widely studied in the literature, see, e.g., [1], [6], [7], where different continuity properties have been established under particular assumptions on the data defining the system. In particular, the case when the vector field is constant is treated in [8]. The case when the vector field is a general set-valued map satisfying mild continuity conditions is treated in [6]. In addition, the case when the vector field is set-valued and time dependent is treated in [7]. On the other hand, to the best of our knowledge, the case when the system is subject to constraints has not been considered in the literature.

For constrained continuous-time systems, the Lipschitz continuity of $t_{\partial C}^{min}$, where ∂C is the boundary of the constraint set C, plays a key role when analyzing the Lipschitz continuity of the reachable sets of such systems. Reachable sets are often viewed as set-valued maps [9]. One of the most standard reachable sets studied in the literature, denoted R, maps an initial condition x_o and a time t_o to all the points reached by the solutions starting from x_o before time exceeds t_o. Another widely used reachable set, denoted R^b, includes only the final values reached by each maximal solution starting from x_o before time exceeds t_o. The Lipschitz continuity of R and R^b allows quantification of the separation between the solutions starting from different initial conditions, see [10], [11]. This property is useful in many contexts including discretization of continuous-time systems [12], controllability analysis [13], and finite-time optimization problems [1]. In unconstrained systems, Lipschitz continuity of R and R^b can be established using the well-known Filippov Lemma [10]. However, in the constrained case, where the solutions are defined only within the set C, even if the solutions are unique, the maps R and R^b can fail to be locally Lipschitz. One of the scenarios preventing such a property is when the function $t_{\partial C}^{min}$ is not locally Lipschitz, see the forthcoming Example 1.

In the first part of this paper, we analyze the Lipschitz continuity of $t_{\partial C}^{min}$ for constrained continuous-time systems. More precisely, we propose necessary and sufficient infinitesimal conditions to guarantee that $t_{\partial C}^{min}$ is locally Lipschitz. We assume that the set C is closed and that F is single valued and locally Lipschitz. Our approach is mainly inspired by the results proposed in [1] and [6] for unconstrained differential inclusions $\dot{x} \in F(x)$ $x \in \mathbb{R}^n$. As we shall see, even when the solutions are unique, the results in [1] and [6] are not directly applicable to the constrained setting. Moreover, their extension offers some technical challenges. In particular, since the solutions are defined only on the set C, the proposed infinitesimal conditions must be satisfied on C. However, by doing so, we will show that the arguments used in the aforementioned references are not enough to prove Lipschitz continuity of t_K^{min}. To handle this situation, extra assumptions are proposed in this paper, see (C1)-(C3). Relaxing (or showing the necessity) of the proposed assumptions as well as considering the general case of nonunique solutions are interesting open questions, which will be the subject of our future work.

In the second part of this paper, as an application of our results, we use the Lipschitz continuity of $t_{\partial C}^{min}$ to formulate sufficient conditions to conclude the Lipschitz continuity of the maps R and R^b in the constrained setting. To the best of our knowledge, this is an original contribution that constitutes a key step to analyze the continuity of the reachable sets for general hybrid systems [14].

M. Maghenem and R. G. Sanfelice are with Department of Electrical and Computer Engineering, University of California, Santa Cruz. Email: mmaghene,ricardo@ucsc.edu. This research has been partially supported by the National Science Foundation under Grant no. ECS-1710621 and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, Grant no. FA9550-19-1-0053, and Grant no. FA9550-19-1-0169, and by CITRIS and the Banatao Institute at the University of California.

The rest of this paper is organized as follows. Preliminaries on constrained continuous-time systems, reachable sets, and minimal-time functions are in Section II. A general motivation to our work is in Section III. Necessary and sufficient conditions for Lipschitzness of t_K^{min}, in the constrained setting, are in Section IV. The Lipschitz continuity of R and R^b, in the constrained setting, is analyzed in Section V. Examples are introduced to illustrate our results.

Due to space constraints, the proofs are omitted and will be published elsewhere.

Notation. For x, $y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the norm of x, $|x|_K := \inf_{y\in K} |x-y|$ defines the distance between x and the nonempty set K, and $\langle x, y\rangle = x^\top y$ denotes the inner product between x and y. For a set $K \subset \mathbb{R}^n$, we use $\mathrm{int}(K)$ to denote its interior, ∂K to denote its boundary, $\mathrm{cl}(K)$ to denote its closure, and $U(K)$ to denote an open neighborhood of K. For a set $O \subset \mathbb{R}^n$, $K\backslash O$ denotes the subset of elements of K that are not in O. For the sets $(O, K, I) \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $O + K := \{x_1 + x_2 : (x_1, x_2) \in O \times K\}$ and $IK := \{x_1x_2 : (x_1, x_2) \in I \times K\}$. By $\mathcal{B}$, we denote the open unit ball in $\mathbb{R}^n$ centered at the origin. Finally, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes a set-valued map associating each element $x \in \mathbb{R}^n$ to a subset $F(x) \subset \mathbb{R}^n$.

II. Preliminaries: Constrained Systems, Reachability Maps, and Minimal-Time Functions

A constrained differential inclusion $\mathcal{H}_f := (C, F)$ is defined as the continuous-time system

$$\mathcal{H}_f : \qquad \dot{x} \in F(x) \quad x \in C \subset \mathbb{R}^n, \tag{1}$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$ and the dynamics $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. The set C in (1) is not necessarily open and does not neccessarily correspond to $\mathbb{R}^n$, as opposed to the existing literature dealing with unconstrained differential inclusions where $C \equiv \mathbb{R}^n$ [15], [16].

Next, we introduce the concept of solutions to $\mathcal{H}_f$.

Definition 1: (Solution to $\mathcal{H}_f$) A function $x : \mathrm{dom}\, x \to \mathbb{R}^n$ with $\mathrm{dom}\, x \subset \mathbb{R}_{\geq 0}$ and $t \mapsto x(t)$ locally absolutely continuous is a *solution* to $\mathcal{H}_f$ if

(S1) $x(0) \in \mathrm{cl}(C)$,

(S2) $x(t) \in C$ for all $t \in \mathrm{int}(\mathrm{dom}\, x)$,

(S3) $\dot{x}(t) \in F(x(t))$ for almost all $t \in \mathrm{dom}\, x$.

•

A solution to $\mathcal{H}_f$ is said to be maximal if there is no solution z to $\mathcal{H}_f$ such that $x(t) = z(t)$ for all $t \in \mathrm{dom}\, x$ with $\mathrm{dom}\, x$ a proper subset of $\mathrm{dom}\, z$. It is said to be trivial if the set $\mathrm{dom}\, x$ contains only one element. The system $\mathcal{H}_f$ is said to be forward complete if every maximal solution to $\mathcal{H}_f$ is defined on an unbounded hybrid time domain. Finally, the system $\mathcal{H}_f$ is said to be pre-forward complete, if every maximal solution to $\mathcal{H}_f$ is either forward complete or bounded.

Remark 1: Constrained differential inclusions $\mathcal{H}_f = (C, F)$ constitute a key component in the modeling of hybrid systems. Indeed, according to [17], a general hybrid system modeled as a hybrid inclusion is given by

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \tag{2}$$

where, in addition to the continuous dynamics $\mathcal{H}_f = (C, F)$, the *discrete dynamics* is defined by the jump set $D \subset \mathbb{R}^n$ and the jump map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. The solutions to $\mathcal{H}_f = (C, F)$ according to Definition 1, correspond to the solutions to $\mathcal{H}$, according to [17, Definition 2.6], that never jump. •

For a constrained system $\mathcal{H}_f = (C, F)$, the reachability maps R and R^b from x_o over the interval $[0, t_o]$ are given by

$$R(t_o, x_o) := \{\phi(t) : \phi \in \mathcal{S}(x_o),\ t \in \mathrm{dom}\, \phi \cap [0, t_o]\}\,, \tag{3}$$

$$R^b(t_o, x_o) := \{\phi(t) : \phi \in \mathcal{S}(x_o),\ t \in \mathrm{dom}\, \phi \cap [0, t_o], \nexists t' \in [0, t_o] \cap \mathrm{dom}\, \phi \text{ s.t. } t' > t\}\,, \tag{4}$$

where $\mathcal{S}(x_o)$ is the set of maximal solutions to $\mathcal{H}_f$ starting from $x_o \in \mathrm{cl}(C)$. Finally, we use $\mathrm{reach}(x_o)$ to denote the set generated by the maximal solutions starting from $x_o \in \mathrm{cl}(C)$; namely,

$$\mathrm{reach}(x_o) := \{x(t) : t \in \mathrm{dom}\, x,\ x \in \mathcal{S}(x_o)\}\,. \tag{5}$$

Finally, we define the minimal-time function for constrained differential inclusions.

Definition 2 (Minimal-time function): The minimal-time function $t_K^{min} : \mathrm{cl}(C) \to \mathbb{R}^n$ with respect to a closed set $K \subset C$ and for a system $\mathcal{H}_f = (C, F)$ is given by

$$t_K^{min}(x_o) := \begin{cases} +\infty & \text{if } \mathrm{reach}(x_o) \cap K = \emptyset \\ \min\limits_{\substack{x(t) \in K \\ t \in \mathrm{dom}\, x \\ x \in \mathcal{S}(x_o)}} t & \text{otherwise.} \end{cases} \tag{6}$$

•

The minimal-time function t_K^{min} in Definition 2 provides the first time that a maximal solution starting from x_o reaches the set K. If all the solutions starting from x_o never reach the set K, the minimal-time function is set to infinity.

III. Motivation

In unconstrained systems $\mathcal{H}_f = (\mathbb{R}^n, F)$, one can use the well-known Filippov Theorem [10, Theorem 5.3.1] to conclude that, when F is a Lipschitz set-valued map with closed values, the maps R^b and R are locally Lipschitz[1].

Proposition 1: Suppose the differential inclusion $\mathcal{H}_f = (\mathbb{R}^n, F)$ is pre-forward complete and such that F is locally

[1] A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *locally Lipschitz* if for each compact $K \subset \mathbb{R}^n$ there exists $k > 0$ such that, for all $x \in K$ and $y \in K$,

$$F(y) \subset F(x) + k|x - y|\mathcal{B}. \tag{7}$$

Lipschitz. Then, the set-valued maps R^b and R are locally Lipschitz. □

However, in the general case of constrained systems $\mathcal{H}_f = (C, F)$, the maps R^b and R may fail to be locally Lipschitz even if $\mathcal{H}_f$ is pre-forward complete, the solutions are unique, and F is locally Lipschitz. As we show in this paper, the Lipschitz continuity of the minimal-time function $t^{min}_{\partial C}$ plays a key role in this case. Indeed, one of the scenarios preventing Lipschitz regularity of R and R^b is described below.

($\star$) The function $t_{\partial C} : \mathrm{cl}(C) \to \mathbb{R}_{\geq 0}$ is not locally Lipschitz on the set $S_{\partial C}$ defined as

$$S_{\partial C} := \{x \in \mathrm{cl}(C) : t_{\partial C}(x) < \infty\}, \tag{8}$$

where, given a closed set $K \subset \mathrm{cl}(C)$, the function t_K is given by

$$t_K(x_o) := \begin{cases} t^{min}_K(x_o) & \text{if } x_o \notin K \text{ or } \mathcal{S}(x_o) \text{ is trivial}, \\ \inf_{x \in \mathcal{S}(x_o)} \liminf_{t \to 0^+} t^{min}_K(x(t)) & \text{otherwise}. \end{cases} \tag{9}$$

Remark 2: The only difference between t_K and t^{min}_K is that, when $x_o \in K$ and the maximal solutions starting from x_o immediately leave the set K, $t_K(x_o)$ provides the next time, after the initial time, at which a maximal solution reaches the set K. The latter is captured by the "otherwise" in (9). •

The scenario ($\star$) is illustrated in the following simple example.

Example 1: Consider the constrained system $\mathcal{H}_f = (C, F)$ with

$$F(x) := [1 \quad 0]^\top \ \forall x \in C := \left\{x \in \mathbb{R}^2 : x_1 \leq \sqrt{|x_2|} + 2\right\}.$$

Note that F is single-valued and locally Lipschitz and that $\mathcal{H}_f$ is pre-complete. Hence, the conditions in Proposition 1 hold. Moreover, $t_{\partial C}(x) = t^{min}_{\partial C}(x)$ for all $x \in C$. Furthermore, let $(x_o, y_o) \in C \times C$ with $x_o := [1 \quad 0]^\top$ and $y_o := [1 \quad \beta]^\top$, for some $\beta \in [0, 1]$. After some computations, we obtain that $t_{\partial C}(y_o) = t^{min}_{\partial C}(y_o) = 1 + \sqrt{\beta}$, $t_{\partial C}(x_o) = t^{min}_{\partial C}(x_o) = 1$, $y(t_{\partial C}(y_o)) = [2 + \sqrt{\beta} \quad \beta]^\top$, and $x(t_{\partial C}(x_o)) = [2 \quad 0]^\top$. First, it is easy to see that the function $t_{\partial C}$ is not locally Lipschitz since $|t_{\partial C}(y_o) - t_{\partial C}(x_o)| = \sqrt{\beta}$ and $|y_o - x_o| = \beta$. Furthermore, for $t^* = 2$, we obtain

$$R^b(t^*, y_o) = y(t_{\partial C}(y_o)) \text{ and } R^b(t^*, x_o) = x(t_{\partial C}(x_o)).$$

The latter implies that $x \mapsto R^b(t^*, x)$ is not locally Lipschitz on C since $|y_o - x_o| = \beta$ and

$$\begin{aligned} |R^b(t^*, y_o) - R^b(t^*, x_o)| &= |y(t_{\partial C}(y_o)) - x(t_{\partial C}(x_o))| \\ &= \sqrt{\beta + \beta^2}. \end{aligned}$$

□

Analyzing the Lipschitz continuity of t_K reduces to analyzing the same property for the minimal-time function $t^{min}_{K_a}$, where $K_a \subset K$ is a closed set such that

$$t_K(x) = t^{min}_{K_a}(x) \quad \forall x \in \mathrm{cl}(C). \tag{10}$$

For example, in the following lemma, we show that (10) holds when the following assumption is satisfied:

(M) The set K_a is closed and such that
1) $t_K(x) = 0$ for all $x \in K_a$,
2) $t_{\mathrm{cl}(K \setminus K_a)}(x) = \infty$ for all $x \in \mathrm{cl}(C) \setminus K_a$.

Lemma 1: Consider a constrained differential inclusion $\mathcal{H}_f = (C, F)$, a closed set $K \subset \mathrm{cl}(C)$, and a subset $K_a \subset K$ such that (M) holds. Then, (10) holds. □

The latter facts are illustrated in the following example.

Example 2: [Bouncing ball] The continuous dynamics of the bouncing ball hybrid model is given by $\mathcal{H}_f := (C, F)$, where $F(x) := [x_2 \ -\gamma]^\top$, $C := \left\{x \in \mathbb{R}^2 : x_1 \geq 0\right\}$, and the constant $\gamma > 0$ is the gravitational acceleration. Furthermore, we consider the following useful set:

$$D := \left\{x \in \mathbb{R}^2 : x_1 = 0, \ x_2 \leq 0\right\}. \tag{11}$$

Next, we will show that (M) is satisfied with $(K, K_a) = (\partial C, D)$. Indeed, the set D is closed and is a subset of ∂C. Furthermore, it is easy to see that each solution starting from D is trivial; hence, $t_{\partial C}(x) = 0$ for all $x \in D$. Furthermore, $t_{\mathrm{cl}(\partial C \setminus D)}(x) = \infty$ for all $x \in \partial C \setminus D$ since the solutions starting from $\partial C \setminus D$ leave $\partial C \setminus D$ immediately and never reach it again. □

IV. Lipschitzness of Minimal-Time Functions in Constrained Systems

In this section we investigate necessary and sufficient conditions such that the minimal-time function t^{min}_K in Definition 2 is locally Lipschitz on the set

$$S^{min}_K := \left\{x \in C : t^{min}_K(x) < \infty\right\}. \tag{12}$$

The proposed conditions are infinitesimal, i.e., they involve only the sets K and C, and the map F.

A. Assumptions

Throughout this paper, we are concerned with the particular class of constrained nonlinear systems $\mathcal{H}_f = (C, F)$ satisfying the following condition:

(SA) The set C is closed and F is single valued and locally Lipschitz on C.

Furthermore, given a closed set $K \subset C$, we impose the following assumptions in some of our results[2]:

(C1) There exists $U(K)$ such that, for each $(x_o, t_1, t_2) \in S^{min}_K \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with $0 \leq t_1 < t_2 < t^{min}_K(x_o)$, the solution x to $\mathcal{H}_f$ from x_o does not satisfy

$$x(t) \in (\partial C \cap U(K)) \setminus K \quad \forall t \in [t_1, t_2].$$

[2]So far, we are not assuming any of these assumptions to hold.

(C2) For a given $(x_o, \eta_o) \in K \times \mathbb{R}_{>0}$, $(x_o + \eta_o \mathcal{B}) \cap C \subset S_K^{min}$.

(C3) For a given $(x_o, \eta_o) \in K \times \mathbb{R}_{>0}$, for all $x \in (x_o + \eta_o \mathcal{B}) \cap K$ and $y \in C$ such that $x = \text{proj}_K(y)$, $y - x \in M_C(x)$ with

$$M_C(x) := \{v \in \mathbb{R}^n : \exists \epsilon > 0 : x + [0, \epsilon]v \subset C\}. \quad (13)$$

Remark 3: Property (C1) prevents the solutions that reach the set K from sliding in $(\partial C \cap U(K)) \backslash K$. Lemma 2 in the Appendix proposes sufficient infinitesimal conditions such that (C1) holds. •

Remark 4: Condition (C2) is trivially satisfied if, for example, the set K is locally attractive. Moreover, condition (C3) is satisfied if the set C is locally convex around x_o; namely, there exists $\epsilon > 0$ such that $C \cap (x_o + \epsilon \mathcal{B})$ is convex. •

Example 3: [Bouncing ball] Consider the constrained system $\mathcal{H}_f = (C, F)$ studied in Example 2 and assume that $K = D$, where D is introduced in (11). To verify (C1), notice that the nontrivial solutions flowing from ∂C are only those starting from $\partial C \backslash D = \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 > 0\}$, and are given by $x(t) = [-\frac{1}{2}\gamma t^2 + x_{o2}t \quad -\gamma t + x_{o2}]^\top$ for all $t \geq 0$ and for all $x_o := [x_{o1} \quad x_{o2}]^\top \in \partial C \backslash D$. Note that $x_1(t) > 0$ for all $t \in (0, 2x_{o2}/\gamma)$; hence, $x(t) \in \text{int}(C)$ for all $t \in (0, 2x_{o2}/\gamma)$. Hence, (C1) is verified with $S_D^{min} = C$. Another way to conclude (C1) consists in using Lemma 2 and the fact $F(x_o) \in D_C(x_o)$ for all $x_o \in \partial C \backslash D$, where D_C is introduced in (20). To show that (C2) and (C3) hold, notice that $t_D^{min}(x) < \infty$ for all $x \in C$; hence, (C2) holds. Moreover, the set C is convex; hence, (C3) is also satisfied. □

B. Main Result

We are now ready to establish the main result of this section.

Theorem 1: Suppose that the constrained system $\mathcal{H}_f = (C, F)$ satisfies (SA).

1) If (C1) holds and, for all $x_o \in K$, there exist $\beta > 0$ and $\eta_1 > 0$ such that

$$\langle F(x), \zeta_x \rangle \leq -\beta |\zeta_x| \quad \forall x \in ((x_o + \eta_1 \mathcal{B}) \backslash K) \cap C, \\ \forall \zeta_x \in \{x - y : y \in \text{proj}_K(x)\}, \quad (14)$$

then, the minimal-time function t_K^{min} is locally Lipschitz on S_K^{min}.

2) If (C1) holds and there exist $x_o \in K$, $\beta > 0$, and $\eta_1 > 0$ such that (14) holds, then there exists $\eta_2 > 0$ such that t_K^{min} is locally Lipschitz on $S_K^{min} \cap (x_o + \eta_2 \mathcal{B})$.

3) Conversely, if t_K^{min} is locally Lipschitz on $S_K^{min} \cap (x_o + \eta_o \mathcal{B})$ for some $\eta_o > 0$, $x_o \in K$, and (C2)-(C3) hold with (x_o, η_o), then there exists $\eta_1 > 0$ and $\beta > 0$ such that (14) holds.

□

Example 4: [Bouncing ball] Consider the continuous dynamics of the bouncing-ball hybrid model introduced in (3) and let the set D be as introduced in (11). In this example, we propose first to show analytically that the minimal-time function t_D^{min} is locally Lipschitz around any element $z_o \in C \backslash \{0\}$. However, it is not locally Lipschitz around the origin. As a next step, we propose to validate such a claim using Theorem 1. Note that $\partial C = \{x \in \mathbb{R}^2 : x_1 = 0\}$. Furthermore, according to Definition 2, we conclude that $S_D^{min} = C$. Indeed, from any element $x_o \in C$, either there exists a nontrivial solution to $\mathcal{H}_f$ that reaches ∂C and D at the same time, otherwise, the solution starting from $\partial C \cap D$ and is trivial. Also, according to Definition 2 and after some easy computations, we conclude that $t_D^{min}(x_o) = \left(x_{o2} + \sqrt{x_{o2}^2 + 2\gamma x_{o1}}\right)/\gamma$ for all $x \in C$. Hence, t_D^{min} are $\mathcal{C}^1$, thus locally Lipschitz, except at the origin. Now, we propose to confirm the latter result using Theorem 1; that is, without computing the system's solutions. Let $z_o \in C \backslash \{0\}$, let z be the solution flowing from z_o, and $K = D$. Furthermore, let $\eta_o > 0$ such that $(z_o + \eta_o \mathcal{B}) \cap \{0\} = \emptyset$. We start by noticing that, for all $y_o \in (z_o + \eta_o \mathcal{B}) \cap C$, $t_K^{min}(y_o) < \infty$; thus, $y_o \in S_K^{min}$. Next, since the system's nontrivial flows never reach the origin, we conclude the existence of $\eta_2 > 0$ such that $(z(t_D^{min}(z_o)) + \eta_2 \mathcal{B}) \cap \{0\} = \emptyset$ and, according to Proposition 1, each solution y starting from $y_o \in (z_o + \eta_o \mathcal{B}) \cap C$ reaches $(z(t_D^{min}(z_o)) + \eta_2 \mathcal{B}) \cap C$ before reaching the set D, for $\eta_o > 0$ sufficiently small. We also notice that $\eta_2 > 0$ can be made sufficiently small by taking $\eta_o > 0$ sufficiently small. That is, using the second statement in Theorem 1, we conclude that the minimal-time function t_D^{min} is locally Lipschitz around z_o since we can always show the existence of $\eta_1 > 0$ and $\beta > 0$ such that (14) holds for all $x_o := z(t_D^{min}(z_o)) \in D \backslash \{0\}$. Indeed, it suffices to take $\eta_1 > 0$ sufficiently small such that, for all $x := [x_1 \quad x_2]^\top \in ((x_o + \eta_1 \mathcal{B}) \backslash D) \cap C$, $x_2 \leq 0$; hence, $\zeta_x = x - \text{proj}_D(x) = [x_1 \quad 0]^\top$ with $x_1 > 0$; thus, $\langle \zeta_x, F(x) \rangle = -\lambda x_1 = -\lambda |\zeta_x|$.

Next, we propose to use the third statement in Theorem 1 in order to conclude that t_D^{min} is not locally Lipschitz around the origin. Indeed, we already established in Example 3 that (C2)-(C3) are satisfied. Now, in order to confirm the claim, we show that, for all $\eta_1 > 0$ and for all $\beta > 0$, there exists $x \in (\eta_1 \mathcal{B} \backslash D) \cap C$ such that $\langle F(x), \zeta_x \rangle > -\beta |\zeta_x|$. That is, let $x = [\eta_1/2 \quad 0]^\top$, it is easy to see that $x \in (\eta_1 \mathcal{B} \backslash D) \cap C$, $\zeta_x = x$, and $F(x) = [0 \quad -\gamma]^\top$; hence, $\langle F(x), \zeta_x \rangle = 0 > -\beta \eta_1 = -\beta |\zeta_x|$, which confirms that t_D^{min} is not locally Lipschitz around the origin. □

C. Discussion

- The first and second items in Theorem 1 are inspired by [6, Theorem 3.2]. Assumption (C1) plays a role when adapting the arguments used in the aforementioned reference. Indeed, since the proof is based on [6, Theorem 2.1], condition (C1) is assumed to guarantee that the z in [6, Theorem 2.1] belongs to the set C. The latter does not hold for free in the constrained case. Showing the necessity of (C1), or relaxing it, is an interesting open question.

- If we consider the general case where F is set valued with compact and convex images, a key step to extend Theorem 1 consists in extending the condition (14). Indeed, it has been shown in [6, Theorem 3.1] that, when $C = \mathbb{R}^n$, (14) can be replaced by

$$\langle \eta_x, \zeta_x \rangle \le -\beta |\zeta_x| \quad \text{for some } \eta_x \in F(x) \text{ and } \forall x \in (x_o + \eta_1 \mathcal{B}) \setminus K, \ \forall \zeta_x \in \{x - y : y \in \text{proj}_K(x)\}. \quad (15)$$

Under (15) and starting from each $x_o \in \mathbb{R}^n$ close enough to K, we can prove the existence of a nontrivial solution $x \in \mathcal{S}(x_o)$ that reaches K while satisfying

$$|x(t)|_K - |x_o|_K \le -t/c_o \qquad \forall t \in [0, t_K^{min}(x_o)]. \quad (16)$$

However, in the constrained case, the solution x satisfying (16) can be cut by the set C before reaching K. At the same time, it is possible to have a different solution starting from the same x_o that reaches K while not satisfying (16).
- The proof of the third statement in Theorem 1 is inspired by the proof in [6, Theorem 6.2] and the arguments used therein do not apply to the constrained case when we remove (C2)-(C3).

V. Lipschitzness of Reachability Maps in Constrained Systems

In this section, we analyze the Lipschitz continuity of the reachability maps R^b and R in constrained systems. In addition to ($\star$), the following scenarios prevent such a regularity of the reachability maps R^b and R.

A. When the Solutions are Nontrivial After Reaching ∂C

Assume the existence of a solution x starting from $x_o \in \text{int}(C)$ such that $0 < t_{\partial C}(x_o) < \infty$ and $\text{dom}\, x := [0, t^*]$ with $t^* > t_{\partial C}$, see Example 5. In this case, it is possible to find an example where there exists a sequence of initial conditions $\{x_{oi}\}_{i=0}^{\infty} \subset \text{int}(C)$ with $\lim_{i \to \infty} x_{oi} = x_o$ such that each maximal solution x_i staring from x_{oi} satisfies $\text{dom}\, x_i := [0, t_{\partial C}(x_{oi})]$ with $t_{\partial C}(x_{oi}) \le t_{\partial C}(x_o) < t^*$. Hence, in such a scenario, the map $x \mapsto R^b(t^*, x)$ fails to be locally Lipschitz since

$$|R^b(t^*, x_o) - R^b(t^*, x_{oi})| = |x(t^*) - x_i(t_{\partial C}(x_{oi}))|,$$

and the time mismatch in the right-hand side of the previous equality will not allow the map R^b to be locally Lipschitz.

Example 5: Consider the constrained system $\mathcal{H}_f$ with $F(x) := [-1 \quad 0]^\top$, $C := \mathbb{R}^2 \setminus \{x \in \mathbb{R}_{<0} \times \mathbb{R} : |x_2| < 1\}$. It is easy to see that (SA) is satisfied. Furthermore, let x be the solution starting from $x_o := [1 \quad 1]^\top$, and let x_i be the solution starting from $x_{oi} := [1 \quad 1 - (1/i)]^\top$. It is easy to see that $t_{\partial C}(x_{oi}) = t_{\partial C}(x_o) = 1$, ($t_{\partial C}$ is locally Lipschitz) and $\text{dom}\, x_i = [0, t_{\partial C}(x_{oi})] = [0, 1]$ for all $i \in \mathbb{N}$. However, $\text{dom}\, x = [0, +\infty]$; hence, when $t^* = 2$ and for any $i \in \mathbb{N}$,

$$\begin{aligned} |R^b(2, x_o) - R^b(2, x_{oi})| &= |x(2) - x_i(1)| \\ &= |[-1 \quad 1]^\top - [0 \quad 1 - (1/i)]^\top| > 1, \end{aligned}$$

which shows that the map $x \mapsto R^b(2, x)$ is not locally Lipschitz on K. □

B. When the Solutions Start From ∂C

We assume, in this case, the existence of $x_o \in \partial C$ such that a nontrivial solution x starting from x_o exists; namely, $\text{dom}\, x = [0, t^*]$, for some $t^* > 0$. In this case, the following two situations prevent the maps R and R^b from being locally Lipschitz. The first situation is when there exists a sequence of initial conditions $\{x_{oi}\}_{i=0}^{\infty} \subset \partial C$ with $\lim_{i \to \infty} x_{oi} = x_o$ such that each maximal solution x_i starting from x_{oi} is trivial, i.e., $\text{dom}\, x_i = \{0\}$. For example, consider the constrained system $\mathcal{H}_f$ in Example 5, and let $x_o := [0 \quad 1]^\top$ and $x_{oi} := [0 \quad 1 - 1/(i+1)]^\top$ for all $i \in \mathbb{N}$. In such a scenario, the map $x \mapsto R^b(t^*, x)$ fails to be locally Lipschitz since $|R^b(t^*, x_o) - R^b(t^*, x_{oi})| = |x(t^*) - x_i(0)|$, and the time mismatch in the right-hand side of the previous equality will not allow the map to be locally Lipschitz.

The second situation is when the solution x starting from $x_o \in \partial C$ remains in ∂C and its domain is unbounded. However, there exists a sequence of initial conditions $\{x_{oi}\}_{i=0}^{\infty} \subset \partial C$ with $\lim_{i \to \infty} x_{oi} = x_o$ such that each maximal solution x_i starting from x_{oi} is nontrivial but its domain is bounded, i.e., $\text{dom}\, x_i = [0, t_{\partial C}(x_{oi})]$ and $\sup_{i \in \mathbb{N}} \{t_{\partial C}(x_{oi})\} < \infty$. For example, consider the constrained system $\mathcal{H}_f = (C, F)$ with $F(x) := [1 \quad 0]^\top$, $C := \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 : x_1 \in (1,2), \ |x_2| > 0\}$, and let $x_o := [0 \quad 0]^\top$ and $x_{oi} := [0 \quad 1/(i+1)]^\top$ for all $i \in \mathbb{N}$. In such a scenario, the map $x \mapsto R^b(t^*, x)$ fails to be locally Lipschitz for sufficiently large $t^* > 0$ since

$$|R^b(t^*, x_o) - R^b(t^*, x_{oi})| = |x(t^*) - x_i(t_{\partial C}(x_{oi}))|.$$

C. Assumptions and Main Result

To avoid the scenarios in ($\star$) and Sections V-A-V-B, we assume the following to hold on a given set $X \subset C$.

- (M1) The set $X \subset C$ is forward pre-invariant (i.e., $x(\text{dom}\, x) \subset X$ for all $x \in \mathcal{S}(X)$) and, for all $x \in \mathcal{S}(X)$, $\text{dom}\, x$ is closed.
- (M2) There exists $K_a \subset \partial C$ such that (M) holds with K therein replaced by ∂C and $t_{K_a}^{min}$ is locally Lipschitz on $S_{\partial C} \cap X$.
- (M3) $\mathcal{S}(x_o)$ is trivial for all $x_o \in X \cap \partial C$ reachable by a solution staring from $y_o (\neq x_o) \in X$.
- (M4) For any $x_o \in \partial C \cap X$ from which solutions are nontrivial, there exists $U(x_o) \subset \mathbb{R}^n$ such that

$$\forall y \in \mathcal{S}(U(x_o) \cap X \cap C), \ \exists t_y > 0 : y((0, t_y]) \subset \text{int}(C). \quad (17)$$

Remark 5: Infinitesimal conditions to check (M3) and (M4) are provided in Lemmas 3 and 4, respectively. •

Theorem 2: Suppose that the constrained system $\mathcal{H}_f = (C, F)$ satisfies (SA). Let $X \subset C$ be such that (M1)-(M4) hold. Then, the maps R^b and R are locally Lipschitz on $\mathbb{R}_{\ge 0} \times X$. □

Example 6: Consider the constrained system $\mathcal{H}_f = (C, F)$ studied in Example 2. Consider the set $X := C \setminus \{0\}$.

Note that the set X is forward pre-invariant since the solutions starting from X never reach the origin. Note, also, that $\mathcal{H}_f$ is pre-complete as the solutions cannot escape in finite time under the linearity of F; hence, (M1) holds. Furthermore, we already established in Examples 2 and 4 that $t_{\partial C} \equiv t_D^{min}$ is locally Lipschitz on $C \backslash \{0\} = X$, which means that (M2) holds. Next, to verify (M3), we notice that the set C is convex; thus, regular, see Definition 3. Furthermore, (21) holds for all $x_o \in \partial C \backslash D$ and $\partial C \backslash D$ is the only set from which nontrivial solutions exist. Hence, (M3) follows using the second statement in Lemma 3. Finally, in order to verify (M4), we use Lemma 4 since the set C is convex and we already showed in Example 3 that $F(x_o) \subset D_C(x_o)$ for all $x_o \in \partial C$ such that $\mathcal{S}(x_o)$ is nontrivial. □

VI. CONCLUSION

Necessary and sufficient (infinitesimal) conditions for the minimal-time function to be locally Lipschitz for constrained continuous-time systems modeled as constrained differential equations were formulated. It was shown that Lipschitz continuity of the reachable sets plays a key role in establishing such a property. Establishing necessity of conditions (C1)-(C3), and relaxing them if needed, is part of future work. The more general case of constrained differential inclusions is also part of future research.

APPENDIX

Here, we recall some useful tangent cones and the notion of regular closed sets [11], [18], [16].

- The *contingent* cone of K at x is given by

$$T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{|x + hv|_K}{h} = 0 \right\}. \quad (18)$$

- The *Clarke tangent* cone of K at x is given by

$$C_K(x) := \left\{ v \in \mathbb{R}^n : \limsup_{y \to x, h \to 0^+} \frac{|y + hv|_K}{h} = 0 \right\}. \quad (19)$$

- The *Dubovtsky-Miliutin* cone of K at x is given by

$$D_K(x) := \{ v \in \mathbb{R}^n : \exists \epsilon > 0 : x + (0, \epsilon](v + \epsilon \mathcal{B}) \subset K \}. \quad (20)$$

Definition 3: A set $K \subset \mathbb{R}^n$ is said to be regular if $T_K(x) = C_K(x)$ for all $x \in K$. •

Lemma 2: Consider a constrained system $\mathcal{H}_f = (C, F)$ such that (SA) holds. Property (C1) is satisfied if, for all $x_o \in (\partial C \cap U(K)) \backslash K$ such that $\mathcal{S}(x_o)$ is nontrivial, either $F(x_o) = 0$ or

$$F(x_o) \notin T_{\partial C}(x_o). \quad (21)$$

□

Lemma 3: Consider a constrained system $\mathcal{H}_f = (C, F)$ such that (SA) holds. Property (M3) is satisfied if, for any initial condition $x_o \in \partial C \cap X$ such that $\mathcal{S}(x_o)$ is nontrivial, either $F(x_o) = 0$ or

$$-F(x_o) \notin T_C(x_o). \quad (22)$$

Moreover, when C is regular, (22) can be relaxed to (21). □

Lemma 4: Consider a constrained system $\mathcal{H}_f = (C, F)$ such that (SA) holds. The condition (M4) is satisfied if, for each $x_o \in \partial C \cap X$ such that $\mathcal{S}(x_o)$ is nontrivial,

$$F(y_o) \in D_C(y_o) \quad \forall y_o \in U(x_o) \cap \partial C. \quad (23)$$

Moreover, if the set C is *regular*, condition (23) can be relaxed to

$$F(x_o) \in D_C(x_o). \quad (24)$$

□

REFERENCES

[1] P. R. Wolenski and Y. Zhuang. Proximal analysis and the minimal time function. *SIAM Journal on Control and Otimization*, 36(3):1048–1072, 1998.

[2] H. Frankowska. Lower semicontinuous solutions of Hamilton–Jacobi–Bellman equations. *SIAM Journal on Control and Optimization*, 31(1):257–272, 1993.

[3] O. Carja, F. Mignanego, and G. Pieri. Lower semicontinuous solutions of the bellman equation for the minimum time problem. *Journal of optimization theory and applications*, 85(3):563–574, 1995.

[4] J. W. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Transactions on Automatic Control*, 46(1):51–64, 2001.

[5] X. Lou and R. G. Sanfelice. Asymptotic stability of limit cycles in hybrid systems with explicit logic states. In *Proceedings of the IEEE 2019 American Control Conference*, 2019.

[6] F. H. Clarke and P. R. Wolenski. Control of systems to sets and their interiors. *Journal of Optimization Theory and Applications*, 88(1):3–23, 1996.

[7] VM Veliov. Lipschitz continuity of the value function in optimal control. *Journal of Optimization Theory and Applications*, 94(2):335–363, 1997.

[8] B. Mordukhovich and N. M. Nam. Subgradients of minimal time functions under minimal requirements. *arXiv preprint arXiv:1009.1585*, 2010.

[9] P. R. Wolenski. A uniqueness theorem for differential inclusions. *Journal of differential equations*, 84(1):165–182, 1990.

[10] J. P. Aubin. *Viability Theory*. Birkhauser Boston Inc., Cambridge, MA, USA, 1991.

[11] J.P. Aubin. Differential calculus of set-valued maps. an update. 1987.

[12] R. P. Wolenski. The exponential formula for the reachable set of a lipschitz differential inclusion. *SIAM Journal on Control and Optimization*, 28(5):1148–1161, 1990.

[13] H. Frankowska. Local controllability of control systems with feedback. *Journal of Optimization Theory and Applications*, 60(2):277–296, 1989.

[14] M. Maghenem and R. Sanfelice. Local lipschitzness of reachability maps for hybrid systems with applications to safety. *To be preseted at the international conference on Hybrid Systems: Computation and Control (HSCC)*, 2020. Sydnay, Australia.

[15] J. P. Aubin and A. Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*, volume 264. Springer Science & Business Media, 2012.

[16] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*, volume 178. Springer Science & Business Media, 2008.

[17] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: :Modeling, stability, and robustness*. Princeton University Press, 2012.

[18] M. Ciligot-Travain. An intersection formula for the normal cone associated with the hypertangent cone. *Journal of Applied Analysis*, 5(2):239–247, 1999.