Regularity Properties of Reachability Maps for Hybrid Dynamical Systems with Applications to Safety

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***Abstract*— In this paper, motivated by the safety problem in hybrid systems, two set-valued reachability maps are introduced. The outer semicontinuity, the continuity, and the local boundedness of the proposed reachability maps with respect to their arguments are analyzed under mild regularity conditions. This study is then used to revisit and improve the existing converse safety statements in terms of barrier functions. In particular, for safe hybrid systems satisfying the aforementioned regularity conditions, we construct time-varying barrier functions that depend on the proposed reachability maps. Consequently, we show that the constructed barrier functions inherit the continuity properties established for the proposed reachability maps.**

I. INTRODUCTION

For continuous-time systems, the reachable set (or equivalently the attainable set) over a finite window of time $[0, T]$ can be seen as a set-valued map that maps an initial condition x_o to the collection of points reached by the solutions starting from x_o during the window $[0, T]$ [1], [2]. Reachable sets are very useful in predictive and finite-horizon optimal control problems, as such problems can be expressed as standard optimization problems over reachable sets [3]. For hybrid systems modeled according to the framework in [4], depending on the context of study, different reachability maps have been used in the literature, see, e.g., [4, Section 6.3.2] and [5].

One of the fundamental questions related to reachability maps concern their continuity with respect to their arguments. Such results are already well-established in the particular case of continuous-time systems, see e.g., [6], [7]. The continuity of the reachability maps is key, for example, when analyzing discretization of continuous-time systems [8], [9]. Furthermore, in this paper, it is shown that the regularity properties of reachability maps play a key role in the converse safety problem, particularly for hybrid systems. The converse safety problem consists of showing the existence of a barrier function that satisfies sufficient conditions for safety provided that the system is safe; namely, the solutions starting from a given initial set never reach a given unsafe set [10]. One of the challenges in this context is to show the existence of a barrier function with the best possible degree of smoothness. Indeed, the smoothness of the barrier function allows the characterization of safety using infinitesimal conditions without any knowledge about the solutions [11].

In this paper, motivated by the safety problem, we introduce two reachability maps for hybrid systems modeled as hybrid inclusions. A hybrid inclusion is defined as a differential inclusion with a constraint, which models the *flow* or continuous evolution of the system, and a difference inclusion with a constraint, modeling the *jumps* or discrete events. Both reachability maps admit as arguments a given number of jumps J, a given amount of flow T, and an initial condition x_o. The first map, denoted by R, is defined as the set reached by the solutions starting from x_o during the interval of flow $[0, T]$ with at most J many jumps. In the particular case of continuous-time systems, this map reduces to the one used in [1]. It also reduces to the union, up to T, of the one studied in [6], [12], [8]. The second map, denoted by $\widehat{R}$, is a prolongation of R using the solutions to the system. The map $\widehat{R}$ includes not only the points reached during the interval of flow $[0, T]$ with at most J jumps, but also those reached after time T until the maximal possible jump, without exceeding J, is achieved.

In the first part of this paper, we analyze the continuity properties for the proposed reachability maps under mild regularity conditions. In particular, we show that the map R is outer semicontinuous and locally bounded with respect to its arguments. Furthermore, we show that the map $\widehat{R}$ shares the same properties only when an extra condition is satisfied. Finally, under the same conditions, we show that $\widehat{R}$ enjoys a stronger continuity property as a function of the ordinary (flow) time. In the second part of this paper, the properties established for R and $\widehat{R}$ are used to revisit the converse safety results in [11]. We re-introduce the barrier functions constructed in [11] as functions of the map R. Consequently, we prove that these barrier functions inherit the regularity properties established for the reachability map R. Finally, we propose a new barrier function using the map $\widehat{R}$ and show that it inherits the relatively stronger regularity properties of the map $\widehat{R}$.

The rest of the paper is organized as follows. Notions related to set-valued maps are in Section II. Preliminaries are in Section III. The proposed reachability maps are in Section IV. The main results are in Section V. Finally, the application of the results in Section V to the converse safety problem is in Section VI. Due to space constraints, the proofs are omitted and will be published elsewhere.

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Notation. Let $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots, \infty\}$. For $x, y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the Euclidean norm of x, $|x|_K := \inf_{y \in K} |x - y|$ defines the distance between x and the nonempty set K, and $\langle x, y \rangle = x^\top y$ denotes the scalar product of x and y. For a set $K \subset \mathbb{R}^n$, we use $\mathrm{int}(K)$ to denote its interior, ∂K to denote its boundary, $\mathrm{cl}(K)$ to denote its closure, and $U(K)$ to denote any open neighborhood of K. For a set $O \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O. By $\mathbb{B}$, we denote the closed unit ball in $\mathbb{R}^n$ centered at the origin. Finally, $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ denotes a set-valued map associating each element $x \in \mathbb{R}^m$ to a set $F(x) \subset \mathbb{R}^n$.

II. Background

We start this section by recalling the following continuity notions for set-valued and single-valued maps.

Definition 1 (Semicontinuous set-valued maps): Consider a set-valued map $F : K \rightrightarrows \mathbb{R}^n$, where $K \subset \mathbb{R}^m$.

- The map F is said to be *outer semicontinuous* at $x \in K$ if, for every sequence $\{x_i\}_{i=0}^{\infty} \subset K$ and for every sequence $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ with $\lim_{i\to\infty} x_i = x$, $\lim_{i\to\infty} y_i = y \in \mathbb{R}^n$, and $y_i \in F(x_i)$ for all $i \in \mathbb{N}$, we have $y \in F(x)$; see [4, Definition 5.9].
- The map F is said to be *lower semicontinuous* (or, equivalently, *inner semicontinuous*) at $x \in K$ if for each $\epsilon > 0$ and $y_x \in F(x)$, there exists $U(x)$ satisfying the following property: for each $z \in U(x) \cap K$, there exists $y_z \in F(z)$ such that $|y_z - y_x| \leq \epsilon$; see [13, Proposition 2.1].
- The map F is said to be *upper semicontinuous* at $x \in K$ if, for each $\epsilon > 0$, there exists $U(x)$ such that for each $y \in U(x) \cap K$, $F(y) \subset F(x) + \epsilon\mathbb{B}$; see [14, Definition 1.4.1].
- The map F is said to be *continuous* at $x \in K$ if it is both upper and lower semicontinuous at x.

Furthermore, the map F is said to be upper, lower, outer semicontinuous, or continuous if it is upper, lower, outer semicontinuous, or continuous for all $x \in K$, respectively. •

Definition 2 (Semicontinuous single-valued maps): Consider a scalar function $B : K \to \mathbb{R}$, where $K \subset \mathbb{R}^m$.

- The scalar function B is said to be *lower semicontinuous* at $x \in K$ if, for every sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that $\lim_{i\to\infty} x_i = x$, we have $\liminf_{i\to\infty} B(x_i) \geq B(x)$.
- The scalar function B is said to be *upper semicontinuous* at $x \in K$ if, for every sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that $\lim_{i\to\infty} x_i = x$, we have $\limsup_{i\to\infty} B(x_i) \leq B(x)$.
- The scalar function B is said to be *continuous* at $x \in K$ if it is both upper and lower semicontinuous at x.

Furthermore, B is said to be upper, lower semicontinuous, or continuous if it is upper, lower semicontinuous, or continuous for all $x \in K$, respectively. •

Definition 3 (Locally bounded set-valued maps): A set-valued map $F : K(\subset \mathbb{R}^m) \rightrightarrows \mathbb{R}^n$ is said to be *locally bounded* if for any $x \in K$ there exist $U(x)$ and $\beta > 0$ such that

$$|\zeta| \leq \beta \qquad \forall \zeta \in F(y) \quad \forall y \in U(x) \cap K.$$

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III. Hybrid Inclusions and Basic Conditions

Following [4], a hybrid dynamical system is modeled by a hybrid inclusion $\mathcal{H} = (C, F, D, G)$ given by

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \tag{1}$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$, the jump set $D \subset \mathbb{R}^n$, the flow and jump maps $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, respectively.

A hybrid arc ϕ is defined on a hybrid time domain denoted $\mathrm{dom}\,\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$. The hybrid arc ϕ is parametrized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a discrete jump variable $j \in \mathbb{N}$. Its domain of definition $\mathrm{dom}\,\phi$ is such that for each $(T, J) \in \mathrm{dom}\,\phi$, $\mathrm{dom}\,\phi \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\})$ for a sequence $\{t_j\}_{j=0}^{J+1}$, such that $t_{j+1} \geq t_j$, $t_0 = 0$, and $t_{j+1} = T$.

Definition 4 (Concept of solutions to $\mathcal{H}$): A hybrid arc $\phi : \mathrm{dom}\,\phi \to \mathbb{R}^n$ is a *solution* to $\mathcal{H}$ if

(S0) $\phi(0, 0) \in \mathrm{cl}(C) \cup D$;

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \mathrm{dom}\,\phi\}$ has nonempty interior, $t \mapsto \phi(t, j)$ is absolutely continuous and

$$\begin{aligned} \phi(t, j) &\in C && \text{for all} && t \in \mathrm{int}(I^j), \\ \dot{\phi}(t, j) &\in F(\phi(t, j)) && \text{for almost all} && t \in I^j; \end{aligned} \tag{2}$$

(S2) for all $(t, j) \in \mathrm{dom}\,\phi$ such that $(t, j + 1) \in \mathrm{dom}\,\phi$,

$$\phi(t, j) \in D, \qquad \phi(t, j + 1) \in G(x(t, j)). \tag{3}$$

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A solution to $\mathcal{H}$ is said to be maximal if there is no solution ψ to $\mathcal{H}$ such that $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \mathrm{dom}\,\phi$ and $\mathrm{dom}\,\phi$ is a proper subset of $\mathrm{dom}\,\psi$. It is said to be trivial if $\mathrm{dom}\,\phi$ contains only one element. It is said to be continuous if it never jumps. Finally, it is said to be non-Zeno if it has a finite number of jumps on each finite interval of (ordinary) time. The system $\mathcal{H}$ is said to be forward complete if the domain of each maximal solution is unbounded. It is said to be pre-forward complete if the domain of each maximal solution is closed. Finally, we use $\hat{\mathcal{S}}_{\mathcal{H}}(x_o)$ to denote the set of solutions to $\mathcal{H}$ starting from x_o.

Well-posed [4, Definition 6.2] hybrid systems refer to a class of hybrid inclusions with very useful structural properties [4, Chapter 6]. A hybrid inclusion $\mathcal{H} = (C, F, D, G)$ is well-posed if the following conditions known as the hybrid basic conditions are satisfied [4, Assumption 6.5], [4, Theorem. 6.8].

(A1) Both C and D are closed.

(A2) The flow map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C, and $F(x)$ is nonempty and convex for all $x \in C$.

(A3) The jump map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D and $G(x)$ is nonempty for all $x \in D$.

IV. Reachability Maps in Hybrid Systems

In this section, we introduce the reachability maps considered for hybrid dynamical systems modeled as hybrid inclusions. Given $x_o \in \mathrm{cl}(C) \cup D$, $T \in \mathbb{R}_{\geq 0}$, and $J \in \mathbb{N}$, we define the reachable set from the initial condition x_o along the hybrid interval $\mathcal{T}(T,J) := [0,T] \times \{0,1,\ldots,J\}$ as the set-valued map $R : \mathbb{R}_{\geq 0} \times \mathbb{N} \times (\mathrm{cl}(C) \cup D) \rightrightarrows \mathrm{cl}(C) \cup D$ given by

$$R(T,J,x_o) := \Big\{\phi(t,j) : \phi \in \hat{\mathcal{S}}_{\mathcal{H}}(x_o), (t,j) \in \mathrm{dom}\,\phi \cap \mathcal{T}(T,J)\Big\}. \quad (4)$$

In [4] and [5], the slightly different reachability map $R_{\mathcal{H}} : \mathbb{R}_{\geq 0} \times (\mathrm{cl}(C) \cup D) \rightrightarrows \mathrm{cl}(C) \cup D$ is proposed. It is given by

$$R_{\mathcal{H}}(\tau,x_o) := \Big\{\phi(t,j) : \phi \in \hat{\mathcal{S}}_{\mathcal{H}}(x_o),\ (t,j) \in \mathrm{dom}\,\phi,\ t+j \leq \tau\Big\}. \quad (5)$$

Note that for all $(\tau,x_o) \in \mathbb{R}_{\geq 0} \times (\mathrm{cl}(C) \cup D)$,

$$R_{\mathcal{H}}(\tau,x_o) = \bigcup_{j=0}^{[\tau]} R(\tau - j, j, x_o), \quad (6)$$

where $[\tau]$ is the integer part of τ. The relationship between R and $R_{\mathcal{H}}$ in (6) is very useful as it allows us to conclude continuity properties of the map $R_{\mathcal{H}}$ as a straightforward consequence of the continuity properties of the map R.

Next, we introduce a new reachability map, which is denoted by $\widehat{R}$ and includes more points than those included in R (and $R_{\mathcal{H}}$). Given $x_o \in \mathrm{cl}(C) \cup D$, $T \in \mathbb{R}_{\geq 0}$, and $J \in \mathbb{N}$, the set $\widehat{R}(T,J,x_o)$ is an extension of the set $R(T,J,x_o)$ that is given by

$$\widehat{R}(T,J,x_o) := \Big\{\phi(t,j) : \phi \in \hat{\mathcal{S}}_{\mathcal{H}}(x_o),\ (t,j) \in \mathrm{dom}\,\phi \cap \mathcal{T}_\phi(T,J)\Big\}, \quad (7)$$

where $\mathcal{T}_\phi$ is an extension of $\mathcal{T}$ that depends on ϕ and is given by

$$\mathcal{T}_\phi(T,J) := [0, T + \delta_\phi(T,J)] \times \{0,1,\ldots,J\}, \quad (8)$$

$$\delta_\phi(T,J) := \begin{cases} \min\{\delta \geq 0 : (T+\delta, \mathbb{J}_\phi(J)) \in \mathrm{dom}\,\phi\} & \text{if } I^{\mathbb{J}_\phi(J)} \cap [0,T] = \emptyset \\ 0 & \text{otherwise,} \end{cases} \quad (9)$$

$$\mathbb{J}_\phi(J) := \max\{j \leq J : \exists t \geq 0 : (t,j) \in \mathrm{dom}\,\phi\}, \quad (10)$$

and

$$I^{\mathbb{J}_\phi(J)} := \{t \geq 0 : (t, \mathbb{J}_\phi(J)) \in \mathrm{dom}\,\phi\}. \quad (11)$$

The reachability map $\widehat{R}(T,J,x_o)$ includes not only the elements reached by the solutions starting from x_o over the hybrid interval $\mathcal{T}(T,J)$, but also the elements reached by each solution ϕ starting from x_o after ordinary time T up to the time the $\mathbb{J}_\phi(J)$-th jump happens, if that jump happens after T. The $\mathbb{J}_\phi(J)$-th is the last jump with $\mathbb{J}_\phi(J)$ less than or equal to J that the solution ϕ achieves.

Remark 1: Note that $\widehat{R} \equiv R$ when $\mathcal{H} = (C,F,\emptyset,\star)$. Furthermore, when there is a unique maximal solution ϕ starting from $x \in \mathrm{cl}(C) \cup D$, it follows that $\widehat{R}(T,J,x) = R(T + \delta_\phi(T,J), J, x)$ for all $(T,J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$. •

Example 1 (Bouncing ball): Consider the hybrid system $\mathcal{H} = (C,F,D,G)$ modeling the so-called bouncing ball, where $F(x) := (x_2, -\gamma)$ for each $x \in C := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$ and $G(x) := (0, -\lambda x_2)$ for each $x \in D := \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$. The constants $\gamma > 0$ and $\lambda \in [0,1]$ are the gravitational acceleration and the restitution coefficient, respectively. Let $x_o := (x_{o1}, x_{o2}) \in \mathrm{int}(C) \backslash D$ (i.e., $x_{o1} > 0$) and let ϕ be the maximal solution starting from x_o. Furthermore, let $T_o \geq 0$ be the time at which the solution ϕ achieves the first jump, which is given as $T_o = \big(x_{o2} + \sqrt{x_{o2}^2 + 2\gamma x_{o1}}\big)/\gamma$. Now, for each $T \in (0, T_o)$, according to (4), we conclude that

$$\begin{aligned} &R(T,1,x_o) = R(T,0,x_o) = \phi([0,T],0) \\ &= \bigcup_{s=0}^{T} \left\{ \left[-\frac{1}{2}\gamma s^2 + x_{o2}s + x_{o1} \quad -\gamma s + x_{o2} \right]^\top \right\}. \end{aligned} \quad (12)$$

On the other hand, to compute $\widehat{R}(T,1,x_o)$, we use (9) to conclude that $\delta_\phi(T,1) = T_o - T$; hence, using (7), we obtain

$$\begin{aligned} \widehat{R}(T,1,x_o) &= R(T_o,1,x_o) = \phi([0,T_o],0) \bigcup \{\phi(T_o,1)\} \\ &= \phi([0,T_o],0) \cup \{G(\phi(T_o,0))\} \\ &= \bigcup_{s=0}^{T_o} \left\{ \left[-\gamma s^2/2 + x_{o2}s + x_{o1} \quad -\gamma s + x_{o2} \right]^\top \right\} \\ &\quad \cup \left\{ \left[0 \quad \lambda \left(x_{o2}^2 + 2\gamma x_{o1}\right)^{\frac{1}{2}} \right]^\top \right\}. \end{aligned} \quad (13)$$

We also notice that $\widehat{R}(T,1,x_o) = R(T,1,x_o)\ \forall T \geq T_o$. $\square$

V. Semicontinuity, Boundedness, and Continuity of the Reachable Sets

In this section, key continuity properties of the maps R and $\widehat{R}$ are analyzed for well-posed hybrid systems satisfying (A1)-(A3). Under appropriate extra conditions, we show that the map $\widehat{R}$ has better continuity properties with respect to ordinary (flow) time than the map R. The usefulness of the continuity properties of $\widehat{R}$ is shown in Section VI as we revisit the converse safety problem using barrier functions.

A. Outer Semicontinuity and Local Boundedness of R

In the following result, we provide mild conditions under which the reachability map R in (4) is locally bounded and outer semicontinuous.

Proposition 1: Consider a pre-forward complete hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying (A1)-(A3). Then, R is outer semicontinuous and locally bounded. □

Outer semicontinuity and local boundedness of $R_{\mathcal{H}}$ in (5) follows as a consequence of Proposition 1.

Corollary 1: Consider a pre-forward complete hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying (A1)-(A3). Then, the set-valued map $R_{\mathcal{H}}$ is outer semicontinuous and locally bounded. □

B. Continuity of the Map $T \mapsto R(T, J_o, x_o)$

In this section, we show that, when J_o and x_o are fixed, the set-valued map $T \mapsto R(T, J_o, x_o)$ is actually continuous at $T_o \geq 0$ provided that (T_o, J_o, x_o) satisfies the following condition:

($\star$) If $J_o > 0$ and $T_o > 0$, then, for each $j \in \{1, 2, \ldots, J_o\}$, no solution to $\mathcal{H}$ starting from x_o achieves the j-th jump at ordinary time $t = T_o$.

Proposition 2: Consider a pre-forward complete hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying (A1)-(A3). Then, the set-valued map $T \mapsto R(T, J_o, x_o)$ is continuous at $T_o \geq 0$ provided that (T_o, J_o, x_o) satisfies ($\star$). □

In the following result, we establish continuity of the map $\tau \mapsto R_{\mathcal{H}}(\tau, x_o)$ as a consequence of Proposition 2.

Corollary 2: Consider a pre-forward complete hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying (A1)-(A3). Then, the set-valued map $\tau \mapsto R_{\mathcal{H}}(\tau, x_o)$ is continuous at $\tau_o \in \mathbb{R}_{\geq 0} \backslash \{1, 2, ...\}$ if, for each $J \in \{0, 1, \ldots, [\tau_o]\}$, $(\tau_o - J, J, x_o)$ satisfies ($\star$). □

Assumption ($\star$) is enforced to avoid the discontinuities in the map $T \mapsto R(T, J_o, x_o)$ at $T_o > 0$, which are caused by jumps occurring at T_o. Continuity of the map $T \mapsto R(T, J_o, x_o)$ at $T_o = 0$ is a consequence of Proposition 1 and the nondecrease of $T \mapsto R(T, J_o, x_o)$. When ($\star$) does not hold, it is easy to find examples of hybrid systems for which the map $T \mapsto R(T, J_o, x_o)$ is discontinuous at $T_o > 0$, as the following example shows.

Example 2 (Bouncing ball): Consider the bouncing-ball system in Example 1. It is easy to see that the hybrid basic conditions (A1)-(A3) are satisfied and that the system has unique maximal solutions. However, $(T_o, 1, x_o)$ does not satisfy ($\star$), and the map $T \mapsto R(T, 1, x_o)$ is discontinuous at T_o, with $T_o \geq 0$ be the time at which the solution starting from $x_o := (x_{o1}, x_{o2})$, with $x_{o1} > 0$, achieves the first jump. From Example 1, we conclude that this time is given by $T_o = \left(x_{o2} + \sqrt{x_{o2}^2 + 2\gamma x_{o1}}\right)/\gamma$, $R(T_o, 1, x_o)$ satisfies (13), and, for every $T \in [T_o/2, T_o)$, $R(T, 1, x_o)$ satisfies (12). Thus, $\lim_{T \to T_o^-} R(T, 1, x_o) = \bigcup_{s=0}^{T_o} \left\{ \left[-\gamma s^2/2 + x_{o2} s + x_{o1} \quad -\gamma s + x_{o2}\right]^\top \right\}$ is not equal to $R(T_o, 1, x_o)$. □

C. Outer Semicontinuity and Local Boundedness of $\widehat{R}$

In this section, we show that the map $\widehat{R}$ is locally bounded and outer semicontinuous with respect to its arguments provided the following extra condition holds:

(A4) If, from x_o, there exists a continuous or trivial maximal solution, then there exists a neighborhood $U(x_o)$ such that every maximal solution starting from $U(x_o) \cap C$ is continuous or trivial.

Proposition 3: Consider a forward pre-complete system $\mathcal{H} = (C, F, D, G)$ satisfying (A1)-(A4). Then, for each $J_o \in \mathbb{N}$, the set-valued map $(T, x) \mapsto \widehat{R}(T, J_o, x)$ is outer semicontinuous and locally bounded. □

When (A4) is not satisfied, $\widehat{R}$ may fail to be outer semicontinuous. The following example illustrates such a situation.

Example 3: Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ with

$$F(x) := -\left(\frac{\rho(x)x_1}{2(\rho(x)+1)}, \frac{\rho(x)(x_2+1)}{2(\rho(x)+1)}\right)$$

for each $x \in C := \left\{x \in \mathbb{R}^2 : x_2 \geq 0\right\}$, $\rho(x) := x_1^2 + (x_2+1)^2 - 1$, $\psi(x) := \arctan\left((x_2+1)/x_1\right)$, and $G(x) := (-x_1, x_2)$ for each $x \in D := \left\{x \in \mathbb{R}^2 : x_2 = 0\right\}$. In polar coordinates (ρ, ψ), the flow dynamics are given by $(\dot{\rho}, \dot{\psi}) = (-\rho, 0)$. Hence, the solutions starting from $\{x \in \mathbb{R}^2 : x_2 > 0\}$ tend to converge radially and asymptotically to the circle μ until they hit the jump set D, where $\mu := \left\{x \in \mathbb{R}^2 : x_1^2 + (x_2+1)^2 - 1 = 0\right\}$. Furthermore, since the convergence to μ is radial (i.e. $\dot{\psi} = 0$) and asymptotic, it follows that, for all $\alpha > 0$, the solution ϕ starting from $x_o := (0, \alpha)$ never reaches the set D; hence, the solution ϕ is continuous with $\operatorname{dom}\phi$ unbounded. However, from all the remaining initial conditions within the set $C \backslash D$, the corresponding maximal solution reaches the jump set D. As a consequence, (A4) is not satisfied.

Next, we show that the map $x \mapsto \widehat{R}(1, 1, x)$ is not outer semicontinuous at $x_o = (0, 1)$. Indeed, consider the sequence $\{x_{oi}\}_{i=1}^{\infty}$ that converges to x_o with $x_{oi} := (1/i, \sqrt{4 - 1/i^2} - 1)$. Let ϕ_i be the nontrivial and continuous solution starting from x_{oi}. Since the solution ϕ is continuous, we conclude that

$$\widehat{R}(1, 1, x_o) = R(1, 1, x_o) = R(1, 0, x_o) =$$
$$\phi([0, 1], 0) = \left\{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \in [\sqrt{3e^{-1} + 1} - 1, 1]\right\}.$$

Furthermore, for all $i \in \{1, 2, ...\}$,

$$\widehat{R}(1, 1, x_{oi}) = \phi_i([0, 1 + \delta_{\phi_i}(1, 1)], 0) \cup \phi_i(1 + \delta_{\phi_i}(1, 1), 1)$$
$$= \left\{x \in \mathbb{R}^2 : x_1 = \frac{1}{\sqrt{4i^2 - 1}}(x_2 + 1),\right.$$
$$\left. x_2 \in \left[0, \sqrt{4i^2 - 1}/i - 1\right]\right\} \cup \left\{[-1/\sqrt{4i^2 - 1} \quad 0]^\top\right\}.$$

Next, we consider the sequence $\{y_i\}_{i=1}^{\infty}$ with $y_i = (-1/\sqrt{4i^2-1}, 0) \in \widehat{R}(1,1,x_i)$. We notice that $\lim_{i\to\infty} y_i = (0,0) =: y \notin \widehat{R}(1,1,x_o)$, which shows that the map $x \mapsto \widehat{R}(1,1,x)$ is not outer semicontinuous. □

D. Continuity of the Map $T \mapsto \widehat{R}(T, J_o, x_o)$

In the next result, we show that when x_o and J_o are fixed, the set-valued map $T \mapsto \widehat{R}(T, J_o, x_o)$ is continuous, instead of being only outer semicontinuous, provided that (A4) holds.

Proposition 4: Consider a pre-forward complete system $\mathcal{H} = (C, F, D, G)$ satisfying (A1)-(A4). Then, for each $(J_o, x_o) \in \mathbb{N} \times (C \cup D)$, the set-valued map $T \mapsto \widehat{R}(T, J_o, x_o)$ is continuous. □

VI. Application to Converse Safety Problem

In this section, using the continuity properties of the maps R and $\widehat{R}$ established in Propositions 1, 3, and 4, we revisit and improve the converse safety theorems in [11] by adopting a new point of view.

A. Safety Analysis Using Barrier Functions

Following [11], given a hybrid system $\mathcal{H} = (C, F, D, G)$ and two sets $X_o \subset \mathrm{cl}(C) \cup D$ and $X_u \subset \mathbb{R}^n$ with $X_o \cap X_u = \emptyset$, the hybrid system $\mathcal{H}$ is said to be safe with respect to (X_o, X_u) if, for every $x_o \in X_o$, every solution ϕ starting from X_o satisfies $\phi(t,j) \in \mathbb{R}^n \backslash X_u$ for all $(t,j) \in \mathrm{dom}\,\phi$. Furthermore, a barrier function candidate for safety with respect to the sets (X_o, X_u) is defined as a scalar function $B : \mathrm{cl}(C) \cup D \to \mathbb{R}$ satisfying

$$\begin{aligned} &B(x) > 0 && \forall x \in X_u \cap (\mathrm{cl}(C) \cup D) \\ &B(x) \le 0 && \forall x \in X_o. \end{aligned} \tag{14}$$

Such a barrier candidate certifies safety if it allows us to conclude that the set $K := \{x \in \mathrm{cl}(C) \cup D : B(x) \le 0\}$ is forward pre-invariant for $\mathcal{H}$; namely, each maximal solution to $\mathcal{H}$ starting from K[1] stays in K. In particular, when B is lower semicontinuous, the set K is forward pre-invariant if the following conditions are satisfied [11]:

M1) For all $x \in K \cap D$,

$$G(x) \subset \mathrm{cl}(C) \cup D \text{ and } B(\eta) \le 0 \ \forall \eta \in G(x).$$

M2) The function B is monotonically nonincreasing along flows; namely, there exists $U(K)$ such that for every solution ϕ to $\mathcal{H}$ satisfying $\phi(t,0) \in (U(K)\backslash \mathrm{int}(K)) \cap \mathrm{cl}(C)$ for all $(t,0) \in \mathrm{dom}\,\phi$, the map $t \mapsto B(\phi(t,0))$ is nonincreasing.

The converse safety problem consists of showing the existence of a barrier function B satisfying (14) and M1)-M2) provided that $\mathcal{H}$ is safe with respect to (X_o, X_u). One of the challenges, in this case, is to show the existence of a barrier function B with the best possible degree of smoothness. The smoothness of B allows us to replace the solution dependent condition M2) with infinitesimal conditions involving only the flow set C and the flow map F.

It is shown in [15] that in some cases of smooth and safe differential equations, it is not possible to find a continuous and autonomous barrier function satisfying (14) and M1)-M2). To address this issue, in [15] and [11], time-varying barrier-like functions are introduced for safe differential equations and hybrid systems, respectively.

Before recalling the barrier-like functions used in [11], we define the notion of *backward solutions* to $\mathcal{H}$. We say that ϕ is a backward solution to $\mathcal{H}$ if ϕ is a trivial hybrid arc with $\phi(0,0) \in D$, or there exists a solution ψ to the hybrid system $\mathcal{H}^- = (C, -F, G(D), G_D^{-1})$ such that $\mathrm{dom}\,\phi = -\mathrm{dom}\,\psi$ and $\psi(t,j) = \phi(-t,-j)$ for all $(t,j) \in \mathrm{dom}\,\psi$, where $G_D^{-1} : G(D) \rightrightarrows \mathbb{R}^n$ is the reciprocal map of the jump map G restricted to the set D; namely, $G_D^{-1}(y) := \{x \in D : y \in G(x)\}$. Furthermore, for each $x \in \mathrm{cl}(C) \cup D$, the set of backward solutions to $\mathcal{H}$ is denoted by $\hat{\mathcal{S}}_{\mathcal{H}}^-(x_o)$.

The non-autonomous barrier-like function used in [11] is given by:

$$\begin{aligned} B(T,J,x) := \inf\{&|\phi(t,j)|_{X_o} : \phi \in \hat{\mathcal{S}}_{\mathcal{H}}^-(x), \\ &(t,j) \in \mathcal{T}(-T,-J),\ (t,j) \in \mathrm{dom}\,\phi\}, \end{aligned} \tag{15}$$

where $\mathcal{T}(-T,-J) := [-T,0] \times \{-J, ..., -1, 0\}$. Moreover, when the backward solutions to $\mathcal{H}$ are non-Zeno, the following time-varying barrier function is proposed in [11].

$$\begin{aligned} B(T,\, x) := \inf\{&|\phi(t,j)|_{X_o} : \phi \in \hat{\mathcal{S}}_{\mathcal{H}}^-(x), \\ &(t,j) \in \mathcal{T}(-T, -\kappa(-T,\phi)),\ (t,j) \in \mathrm{dom}\,\phi\}, \end{aligned} \tag{16}$$

where $\kappa : \mathbb{R}_{\le 0} \times \hat{\mathcal{S}}_{\mathcal{H}}^-(x) \to \mathbb{N}$ determines the number of jumps in the backward solution ϕ over the ordinary-time interval $[-T, 0]$. The function κ is well-defined when the backward solutions to $\mathcal{H}$ are non-Zeno. In particular, when $\mathcal{H}$ is a continuous-time system (i.e. $\mathcal{H} = (C, F, \emptyset, \star)$), $\kappa \equiv 0$.

The very important and yet challenging question concerning the regularity of the barrier functions in (15) and (16) arises. Indeed, it is shown in [11] that the latter functions are lower semicontinuous provided that $\mathcal{H}^-$ is well posed; namely, in addition to (A1)-(A2), the data of $\mathcal{H}$ satisfies:

(A5) The reciprocal jump map $G_D^{-1} : G(D) \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded.

Next, we re-express the barrier functions in (15) and (16) as functions of the reachability map R. Then, we prove that these barrier functions inherit the regularity properties established for R in Proposition 1. Finally, we propose a new construction of a barrier function that uses $\widehat{R}$ instead of R. We show that this new construction inherits the regularity properties established for $\widehat{R}$ in Propositions 3 and 4.

B. Using the Reachability Map R

First, we note that, for each $(T, J, x) \in \mathbb{R}_{\ge 0} \times \mathbb{N} \times (\mathrm{cl}(C) \cup D)$, the barrier function in (15) satisfies

$$B(T,J,x) = \inf\{|y|_{X_o} : y \in R(-T,-J,x)\}, \tag{17}$$

[1] By convention, points not in $\mathrm{cl}(C) \cup D$ are considered unsafe, which implies that $\mathbb{R}^n \backslash (\mathrm{cl}(C) \cup D) \subset X_u$.

where the *backward reachable set* $R(-T,-J,x)$ is defined as in (4) while replacing $\hat{\mathcal{S}}$ therein by $\hat{\mathcal{S}}^-$ and for $\mathcal{T}(-T,-J) := [-T,0] \times \{-J,\ldots,-1,0\}$. Furthermore, for each $(T,x) \in \mathbb{R}_{\geq 0} \times (\mathrm{cl}(C) \cup D)$, the barrier candidate in (16) satisfies

$$B(T,x) = \inf\{|y|_{X_o} : y \in R_z(-T,x)\}, \tag{18}$$

$$R_z(-T,x) := \Big\{\phi(t,j) : \phi \in \hat{\mathcal{S}}^-_{\mathcal{H}}(x), (t,j) \in \operatorname{dom}\phi \cap [-T,0] \times \{-\kappa(-T,\phi),\ldots,-1,0\}\Big\}. \tag{19}$$

Note that, when $\mathcal{H}^-$ has unique solutions,

$$R_z(-T,x) = R(-T,-\kappa(-T,\phi),x),$$

where $\phi \in \hat{\mathcal{S}}^-_{\mathcal{H}}(x)$, if nontrivial, is the maximal solution to $\mathcal{H}^-$ starting from x. Proposition 1 leads to the following result:

Proposition 5: Consider a system $\mathcal{H} = (C,F,D,G)$ such that (A1)-(A2) and (A5) hold, and suppose that $\mathcal{H}^-$ is pre-forward complete. Then,

- The barrier function in (15) is lower semicontinuous.
- If, additionally, every solution to $\mathcal{H}^-$ is non-Zeno, then the barrier function in (16) is lower semicontinuous.

□

C. Using the Reachability Map $\widehat{R}$

For each $(T,J,x) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times (\mathrm{cl}(C) \cup D)$, we propose the new barrier function given by

$$B(T,J,x) = \inf\{|y|_{X_o} : y \in \widehat{R}(-T,-J,x)\}, \tag{20}$$

where $\widehat{R}(-T,-J,x)$ is as in (7) while replacing $\hat{\mathcal{S}}_{\mathcal{H}}$ therein by $\hat{\mathcal{S}}^-_{\mathcal{H}}$ and with $\mathcal{T}_\phi(-T,-J) = (0,0)$ if ϕ is trivial, otherwise, $\mathcal{T}_\phi(-T,-J) = -\mathcal{T}_\psi(T,J)$, where ψ is the solution to $\mathcal{H}^-$ such that $\operatorname{dom}\phi = -\operatorname{dom}\psi$ and $\psi(t,j) = \phi(-t,-j)$ for all $(t,j) \in \operatorname{dom}\psi$. We are now ready to improve the converse safety theorem in [11, Theorem 3.5].

Theorem 1: Let $\mathcal{H} = (C,F,D,G)$ be a system such that (A1)-(A2) and (A5) hold, $\mathcal{H}^-$ is pre-forward complete, and the solutions to $\mathcal{H}^-$ satisfy (A4). Consider a set $X_o \subset C \cup D$ and a set $X_u \subset \mathbb{R}^n$ with X_o closed, $X_o \cap X_u = \emptyset$, and $\mathbb{R}^n \setminus (C \cup D) \subset X_u$. Then, the system $\mathcal{H}$ is safe with respect to (X_o, X_u) if and only if there exists a lower semicontinuous barrier function candidate $B : \mathbb{R}_{\geq 0} \times \mathbb{N} \times (C \cup D) \to \mathbb{R}$ such that $T \mapsto B(T,J,x)$ is continuous for each $(J,x) \in \mathbb{N} \times (C \cup D)$, B is nonincreasing along the flows, and the following hold:

$$B(T,J,x) \leq 0 \quad \forall (T,J,x) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times X_o, \tag{21}$$

$$B(T,J,x) > 0 \quad \forall (T,J,x) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times (X_u \cap (C \cup D)), \tag{22}$$

$$B(T,J+1,\eta) \leq 0 \quad \forall \eta \in G(x) \quad \text{and} \quad \forall (T,J,x) \in K \cap (\mathbb{R}_{\geq 0} \times \mathbb{N} \times D), \tag{23}$$

$$G(x) \subset C \cup D \quad \forall x : (T,J,x) \in (\mathbb{R}_{\geq 0} \times \mathbb{N} \times D) \cap K, \tag{24}$$

where

$$K := \{(T,J,x) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times (C \cup D) : B(T,J,x) \leq 0\}.$$

□

The sufficiency part of the proof follows using [11, Theorem 3.5]. To prove the necessity part, it can be shown that the barrier candidate in (20) satisfies the conditions in Theorem 1 when $\mathcal{H}$ is safe with respect to (X_o, X_u).

In [16], under further restrictions on $\mathcal{H}$, it is shown that the barrier candidate in (20) is locally Lipschitz.

VII. Conclusion

In this paper, we introduced finite-horizon reachable sets for hybrid dynamical systems. Those reachable sets are viewed as set-valued maps for which we established useful continuity properties. The continuity properties are analyzed in the context of well-posed hybrid systems. The usefulness of this study is illustrated when revisiting and improving the existing converse safety theorems in terms of barrier functions.

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