Model Predictive Control for Hybrid Dynamical Systems: Sufficient Conditions for Asymptotic Stability with Persistent Flows or Jumps

Berk Altın and Ricardo G. Sanfelice

***Abstract*— Recent results on asymptotically stabilizing model predictive control for hybrid dynamical systems are relaxed by exploiting basic knowledge about the structure of the set of trajectories. Specifically, it is shown that when the system to be controlled has trajectories with infinitely many discrete transitions, the cost functional of the underlying optimal control problem does not need to weight the state and the input in continuous time. An analogue of this result shows that when trajectories are defined over all ordinary time, the functional does not need to weight the state and the input during discrete transitions. Results are demonstrated with recurring examples.**

I. Introduction

Building on the foundational work detailed in [1], further detailed in [2], and following the framework therein, this paper presents model predictive control (MPC) schemes for hybrid dynamical systems. The model of a hybrid system considered here allows the state to *flow*, according to a constrained differential equation, and *jump*, according to a constrained difference equation. It is derived from the general framework of [3], where hybrid systems without inputs are given by the combination of constrained differential and difference inclusions.[1] Roughly speaking, we are interested in the MPC problem for hybrid systems where either every complete trajectory flows for an infinite duration of ordinary time, or every trajectory has infinitely many jumps. We refer to these cases as *persistent flows* and *persistent jumps*, respectively, and study asymptotic stability properties of each case under the MPC algorithm proposed in [2] (referred simply as the hybrid MPC algorithm) and recalled in Section III. For both persistent flows and jumps, we provide alternatives to the sufficient conditions in [2] guaranteeing asymptotic stability under the hybrid MPC algorithm.

To motivate the study of MPC strategies pertaining to persistent flows or jumps, we consider the hybrid model of a ball bouncing vertically on a horizontal flat surface with height x_1 and velocity x_2. When $x_1 \geq 0$, the motion of the ball can be represented by the the differential equation

$$\dot{x}_1 = x_2, \dot{x}_2 = -\gamma, \tag{1}$$

where $\gamma > 0$ is the gravitational constant. Impacts with the surface are modeled by the difference equation

$$x_1^+ = x_1, x_2^+ = -\lambda x_2 + u, \tag{2}$$

which takes effect when $x_1 = 0$ and $x_2 \leq 0$. Here, $\lambda \in [0, 1]$ is the coefficient of restitution, and $u \geq 0$ is an input that affects the velocity after impacts. For the bouncing ball, the time-to-impact from any initial state $x := (x_1, x_2)$ with $x_1 \geq 0$ is finite, which implies that trajectories of the system have persistent jumps; see [3, Example 2.12]. In the autonomous case with dissipative jumps (i.e., when $u = 0$ and $\lambda < 1$), asymptotic stability of the origin can then be certified by the total energy function $W(x) := \gamma x_1 + x_2^2/2$. Although W is constant during flows, it decreases with each jump away from the origin. Hence, it is a Lyapunov function for the system due to persistence of jumps [3, Example 3.15].

The relevance of the above observation to the hybrid MPC algorithm is multifaceted. The sufficient conditions for asymptotic stability in [2] require a terminal cost V that is a control Lyapunov function (CLF), decreasing away from the target set during both flows and jumps under an appropriate feedback law. Then, the value function of the underlying optimal control problem (OCP) inherits the descent characteristics of V, and is therefore a stability certificate for the closed loop under the hybrid MPC algorithm. From this perspective, it is of interest to relax these requirements by allowing V to be nonincreasing during jumps (respectively, flows) for persistent flows (respectively, jumps), thus simplifying the design of the terminal cost. Under the relaxed conditions on V, made precise in Section IV, descent characteristics of the value function are still inherited from V (Section V). Nevertheless, combined with persistence of jumps or flows, asymptotic stability can again be shown via the value function, as illustrated in Section VI.

A secondary interpretation comes from the functions defining the OCP. The cost functional of the OCP is defined by the terminal cost V, along with two functions L_C and L_D, weighting the state-input pair during flows and jumps, respectively. Without any information on persistence, these functions are required to be positive definite. The relaxed conditions reveal that under persistence of flows (respectively, jumps) the function L_D (respectively, L_C) can be taken as zero, leading to a simpler functional and reducing computational burden. From a physical perspective, this simplification pertains to systems where either flows or jumps can be deemed unimportant, perhaps due to being uncontrolled (e.g. an underactuated walking robot). The price to pay for these relaxations is that optimal trajectories need

*This research has been partially supported by the National Science Foundation under Grant no. ECS-1710621 and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, Grant no. FA9550-19-1-0053, and Grant no. FA9550-19-1-0169, and by CITRIS and the Banatao Institute at the University of California.

Berk Altın and Ricardo G. Sanfelice are with the Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA; berkaltin@ucsc.edu, ricardo@ucsc.edu.

[1]The rich descriptive capabilities of the hybrid inclusions framework are demonstrated in [3], and discussed in [1], [2], and [4].

to satisfy some properties that can be difficult to verify. To overcome this issue, in Section IV, we provide sufficient conditions on the system data guaranteeing that these properties hold.

It should be emphasized that to the best of our knowledge, the cost functionals appearing in the literature (see the references in [1], [2]) assume a more specific structure than the one in [2], and as such, similar relaxation results have not been reported before. The cost functional in [5] is closest in spirit to the one in [2], but relaxation results for the MPC strategy therein are not available. Due to space constraints, proofs are omitted and will be published in another venue. The main focus here is to present the essence of the relaxation results along some examples, and show how they afford flexibility in the application of CLF-based hybrid MPC. In particular, in Section VII, we revisit the numerical example from [2] from a persistent jumps perspective and show that the relaxation results lead to a more natural design.

II. Preliminaries

We use $\mathbb{R}$ to represent real numbers and $\mathbb{R}_{\geq 0}$ its nonnegative subset. The set of nonnegative integers is denoted $\mathbb{N}$. The notation $S_1 \subset S_2$ indicates S_1 is a subset of S_2, not necessarily proper. The 2-norm is denoted $|.|$. The distance of a vector $x \in \mathbb{R}^n$ to a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is denoted $|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$. We denote by $\mathcal{A} + \delta\mathbb{B}$ the set of all $x \in \mathbb{R}^n$ such that $|x - a| \leq \delta$ for some $a \in \mathcal{A}$. The interior and closure of a set $S \subset \mathbb{R}^n$ are denoted $\operatorname{int} S$ and $\operatorname{cl} S$, respectively. We denote by $\Pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the standard projection onto $\mathbb{R}^n$ such that $\Pi(x, y) = x$. A strictly increasing continuous unbounded function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if $\alpha(0) = 0$.

A. Hybrid Control Systems

This paper considers hybrid control systems $\mathcal{H}$ of the form

$$\mathcal{H} \begin{cases} \dot{x} = f(x, u) & (x, u) \in C \\ x^+ = g(x, u) & (x, u) \in D, \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The *flow map* $f : C \to \mathbb{R}^n$ describes the continuous evolution of x when (x, u) belongs to the *flow set* $C \subset \mathbb{R}^n \times \mathbb{R}^m$. The *jump map* $g : D \to \mathbb{R}^n$ describes the discrete evolution of x when (x, u) belongs to the *jump set* $D \subset \mathbb{R}^n \times \mathbb{R}^m$.

Assumption 2.1: The set C or the set $\Pi(C)$ is closed.

Example 2.2 (Bouncing Ball): The dynamics of the bouncing ball system in Section I can be represented in the form of (3) by incorporating the constraints therein to (1)-(2). The flow map is given as $f(x, u) = (x_2, -\gamma)$ on the flow set $C = \{(x, u) : x_1 \geq 0\}$. Similarly, the jump map is given as $g(x, u) = (x_1, -\lambda x_2 + u) = (0, -\lambda x_2 + u)$ on the jump set $D = \{(x, u) : x_1 = 0, x_2 \leq 0, u \geq 0\}$.

We refer to an input signal u and the corresponding state trajectory x collectively as a solution pair (x, u) of $\mathcal{H}$. A solution pair (x, u) is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes the duration of flows and j denotes the number of jumps. The domain of x, denoted $\operatorname{dom} x$, is a *hybrid time domain*: for every $(T, J) \in \operatorname{dom} x$, there exists a finite nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ so that $t_0 = 0$ and

$$\operatorname{dom} x \cap ([0, T] \times \{0, 1, \ldots, J\}) = \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}).$$

Here, for every $j \in \{1, 2, \ldots, J\}$, t_j is the ordinary time of the j-th jump of x. The domain of u is denoted $\operatorname{dom} u$ in a similar fashion, and is equal to $\operatorname{dom} x$.

Definition 2.3: Given a pair of functions $x : \operatorname{dom} x \to \mathbb{R}^n$ and $u : \operatorname{dom} u \to \mathbb{R}^m$, (x, u) is said to be a solution pair of (3) if $\operatorname{dom}(x, u) := \operatorname{dom} x = \operatorname{dom} u$ is a hybrid time domain, $(x(0, 0), u(0, 0)) \in \operatorname{cl}(C) \cup D$, and the following hold.

- For all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \operatorname{dom}(x, u)\}$ has a nonempty interior, 1) the function $t \mapsto x(t, j)$ is locally absolutely continuous, 2) $(x(t, j), u(t, j)) \in C$ for all $t \in \operatorname{int} I^j$, 3) the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded, and 4) for almost all $t \in I^j$,

$$\dot{x}(t, j) = f(x(t, j), u(t, j)). \tag{4}$$

- For all $(t, j) \in \operatorname{dom}(x, u)$ such that $(t, j + 1) \in \operatorname{dom}(x, u)$,

$$\begin{aligned} (x(t, j), u(t, j)) &\in D, \\ x(t, j + 1) &= g(x(t, j), u(t, j)). \end{aligned} \tag{5}$$

Throughout the paper, the set of solution pairs of $\mathcal{H}$ originating from a set $S \subset \mathbb{R}^n$ is denoted $\widehat{\mathcal{S}}_{\mathcal{H}}(S)$. That is, $(x, u) \in \widehat{\mathcal{S}}_{\mathcal{H}}(S)$ implies $x(0, 0) \in S$. Given a solution pair (x, u), $(T, J) \in \operatorname{dom}(x, u)$ is said to be the terminal (hybrid) time of (x, u) if $T \geq t$ and $J \geq j$ for all $(t, j) \in \operatorname{dom}(x, u)$. The pair (x, u) is said to be complete if $\operatorname{dom}(x, u)$ is unbounded. A complete (x, u) is said to have

- *persistent flows* if $\operatorname{dom}(x, u)$ is unbounded in the t-axis (i.e., there exists no $T \in \mathbb{R}_{\geq 0}$ such that $t \leq T$ for all $(t, j) \in \operatorname{dom}(x, u)$), and
- *persistent jumps* if $\operatorname{dom}(x, u)$ is unbounded in the j-axis (i.e., there exists no $J \in \mathbb{N}$ such that $j \leq J$ for all $(t, j) \in \operatorname{dom}(x, u)$).

Example 2.4 (Digital Control): Consider a continuous-time control system with state $z \in \mathbb{R}^{n_p}$ and input $\eta \in \mathbb{R}^{n_c}$, described by a function $\tilde{f} : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_p}$. When the input η of the plant is applied in a zero-order hold fashion, the resulting digital control system can be expressed in the form of (3) by treating η as a state component and introducing the clock variable $\tau_s \in \mathbb{R}$. Letting $x = (z, \eta, \tau_s)$, the data of this model is given by the flow set $C = \{(x, u) : \tau_s \in [0, T_s]\}$, jump set $D = \{(x, u) : \tau_s = T_s\}$, flow map $f(x, u) = (\tilde{f}(z, \eta), 0, 1)$, and jump map $g(x) = (z, u, 0)$. During flows, the timer τ_s counts up with a constant rate of one and η stays constant, while z evolves according to the plant dynamics. When τ_s reaches T_s, it gets reset to zero, while η is updated with the input u of the hybrid system. Complete solution pairs (x, u) of this system have persistent flows *and* jumps, as the j-th jump occurs at ordinary time $t_j = T_s j - \tau_s(0, 0)$.

We say that the hybrid system $\mathcal{H}$ has unique state trajectories $(t, j) \mapsto x(t, j)$ if two inputs that are equivalent (equal during jumps and equal almost everywhere during flows) generate the same state trajectory from the same initial condition. To ensure uniqueness for $\mathcal{H}$, we adopt the following assumption, where uniqueness is to be understood in a similar sense.

Assumption 2.5: The constrained differential equation

$$\dot{x} = f(x, u) \quad (x, u) \in C$$

has unique state trajectories $t \mapsto x(t)$.

Proposition 2.6: The hybrid control system $\mathcal{H}$ has unique state trajectories if and only if Assumption 2.5 holds.

B. Hybrid Control Systems under Static State-Feedback

For analysis purposes, we also study the closed-loop system arising from the application of given feedback controllers $\kappa_C : \mathbb{R}^n \to \mathbb{R}^m$ and $\kappa_D : \mathbb{R}^n \to \mathbb{R}^m$ to $\mathcal{H}$. This closed-loop system, denoted $\mathcal{H}_\kappa$, is represented in the form

$$\mathcal{H}_\kappa \begin{cases} \dot{x} = f_\kappa(x) := f(x, \kappa_C(x)) & x \in C_\kappa \\ x^+ = g_\kappa(x) := g(x, \kappa_D(x)) & x \in D_\kappa, \end{cases} \tag{6}$$

where

$$C_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_C(x)) \in C\},$$
$$D_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_D(x)) \in D\}.$$

We define trajectories of (6) over hybrid time domains via Definition 2.3. Namely, we say that a function x is a state trajectory of (6) if there exists a solution pair (x, u) *generated by the feedback* κ; that is, if there exists a solution pair (x, u) of $\mathcal{H}$ that satisfies (4) with $u(t, j) = \kappa_C(x(t, j))$ for all $t \in \operatorname{int} I^j$ and (5) with $u(t, j) = \kappa_D(x(t, j))$.

III. Overview of Hybrid MPC

The hybrid MPC algorithm in [2] is implemented by measuring the state of the plant $\mathcal{H}$ in (3) and solving an OCP, in a moving horizon fashion. In comparison to conventional continuous/discrete-time MPC, two key differences arise.

- Since hybrid systems can have solution pairs from nearby initial conditions with drastically different time domains, similar to free end-time optimal control, the terminal time is allowed to vary within a set.
- To account for the differences in the time domains of optimal controls, the optimization times are not assumed to be periodic. Instead, the initial optimization is performed at time $(0, 0)$, and each subsequent optimization time can be selected online, provided the re-optimization occurs before the current control expires.

Next, we detail the formulation of the underlying OCP, which comprises two explicit constraints described by

- the *prediction horizon* $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, and
- the *terminal constraint set* $X \subset \Pi(\operatorname{cl}(C) \cup D)$.

Note that while the flow set C and jump set D implicitly define mixed state-input constraints, any additional explicit state-input constraints can be embedded in the problem by modifying the system's data. For example, the bouncing ball system in Example 2.2 can be reformulated with the flow set $C = \{(x, u) : x_1 \in [0, h_{\max}], u = 0\}$ for any desired maximum height $h_{\max} \geq 0$.

A. The Cost Functional

Given a solution pair (x, u) of $\mathcal{H}$ with compact domain and terminal time (T, J), let $\{t_j\}_{j=0}^{J+1}$ be the sequence such that $\operatorname{dom}(x, u) = \cup_{j=0}^{J}([t_j, t_{j+1}] \times \{j\})$, where $t_{J+1} = T$. If $x(T, J) \in X$, then the cost of the pair (x, u) is given by

$$\mathcal{J}(x, u) := \left(\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} L_C(x(t, j), u(t, j))\, dt \right) + \left(\sum_{j=0}^{J-1} L_D(x(t_{j+1}, j), u(t_{j+1}, j)) \right) + V(x(T, J)).$$

In the definition of the cost functional $\mathcal{J}$, $L_C : C \to \mathbb{R}_{\geq 0}$ is called the *flow cost*, $L_D : D \to \mathbb{R}_{\geq 0}$ is called the *jump cost*, and $V : X \to \mathbb{R}_{\geq 0}$ is called the *terminal cost*.

B. The Prediction Horizon

To accommodate different hybrid time domains, we assume the following structure for the prediction horizon $\mathcal{T}$.

Assumption 3.1: There exists a finite nonincreasing sequence $\{t_j\}_{j=0}^{J+1}$ such that $t_{J+1} = 0$, and

$$\mathcal{T} := \bigcup_{j=0}^{J} \left([t_{j+1}, t_j] \times \{j\} \right).$$

This assumption guarantees that every solution pair that "lasts long enough" (in the sense that there exists large enough $t + j$, $(t, j) \in \operatorname{dom}(x, u)$) eventually "reaches" $\mathcal{T}$, and is used to prove recursive feasibility [2], [1].

C. The Constrained OCP

With the terminal constraint set X and prediction horizon $\mathcal{T}$ already defined, the minimization is performed over solution pairs of $\mathcal{H}$ with initial condition x_0, terminal condition belonging to X, and terminal time belonging to $\mathcal{T}$.

Problem 3.2: Given an initial condition $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} \underset{(x,u) \in \widehat{\mathcal{S}}_{\mathcal{H}}(x_0)}{\text{minimize}} \quad & \mathcal{J}(x, u) \\ \text{subject to} \quad & (T, J) \in \mathcal{T} \\ & x(T, J) \in X, \end{aligned} \tag{7}$$

where (T, J) denotes the terminal time of (x, u).

We say that a solution pair (x, u) is *feasible* if it satisfies the constraints of (7) with $x(0, 0) = x_0$. If, in addition (x, u) minimizes $\mathcal{J}$, it is said to be *optimal*. The feasible set $\mathcal{X}$ is the set of all x_0 with a feasible $(x, u) \in \widehat{\mathcal{S}}_{\mathcal{H}}(x_0)$. The *value function* $\mathcal{J}^* : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is defined as the infimum of all feasible solution pairs at a given initial condition x_0, i.e.,

$$\mathcal{J}^*(x_0) := \inf_{\substack{(x,u) \in \widehat{\mathcal{S}}_{\mathcal{H}}(x_0) \\ (T,J) \in \mathcal{T} \\ x(T,J) \in X}} \mathcal{J}(x, u) \quad \forall x_0 \in \mathcal{X}, \tag{8}$$

where (T, J) is the terminal time of (x, u).

D. Implementation

Having defined the OCP associated with the algorithm, we formalize the implementation and recall the notion of solution pairs generated by the hybrid MPC algorithm.

Definition 3.3: A solution pair (x, u) is said to be generated by the hybrid MPC algorithm if there exists a sequence $\{(T_i, J_i)\}_{i=0}^{\infty} \in \mathrm{dom}(x, u)$ with $(T_0, J_0) = (0, 0)$ such that the following hold.

- The sequence $\{T_i + J_i\}_{i=0}^{\infty}$ is strictly increasing and unbounded.
- For every $i \in \mathbb{N}$, there exists an optimal solution pair (x_i, u_i) such that for every $(t, j) \in \mathrm{dom}(x, u)$ satisfying $t + j \in [T_i + J_i, T_{i+1} + J_{i+1})$,

$$x(t, j) = x_i(t - T_i, j - J_i),$$
$$u(t, j) = u_i(t - T_i, j - J_i).$$

This definition is equivalent to [2, Definition 6.1]. Above, (T_i, J_i) is the hybrid time of the $(i + 1)$th optimization. Under this definition, an appropriate notion of asymptotic stability for the hybrid MPC algorithm is given in Section VI. In general, asymptotic stability can be certified by using the value function $\mathcal{J}^*$, without any assumptions on persistence of jumps or flows [2, Theorem 6.3], provided the cost functions defining $\mathcal{J}$ have basic positive definiteness properties. With persistence of jumps or flows, these requirements can be relaxed, as detailed in the next section.

IV. Main Assumptions

We now list the basic assumptions imposed on Problem 3.2 to ensure feasibility, and certify asymptotic stability of a given *closed set* $\mathcal{A} \subset X$ of interest. The initial assumptions are similar to their counterparts in [2], with some modifications, primarily to account for the case where $\mathcal{A}$ may be unbounded.[2] Later, we outline the additional assumptions needed for the case of persistent jumps and persistent flows, and provide sufficient conditions guaranteeing them.

Assumption 4.1: For any $x_0 \in \mathcal{X}$, an optimal solution pair $(x, u) \in \widehat{\mathcal{S}}_{\mathcal{H}}(x_0)$ exists.

A set of sufficient conditions for Assumption 4.1 is presented in [6]. The next assumption concerns the terminal cost V and terminal constraint set X.

Assumption 4.2: There exists $\varepsilon > 0$ such that the following hold.

(G1) There exist class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in X \cap (\mathcal{A} + \varepsilon\mathbb{B}).$$

(G2) The inclusion $(\mathcal{A} + \varepsilon\mathbb{B}) \cap \Pi(\mathrm{cl}\,(C) \cup D) \subset X$ holds.

The key stabilizing ingredient of the hybrid MPC algorithm is given by the following familiar CLF-like assumption. When the term $L_C(x, \kappa_C(x))$ (respectively, $L_D(x, \kappa_C(x))$) is allowed to be zero, the terminal cost V can be viewed as a Lyapunov function with the feedback κ, provided the trajectories of the closed-loop system $\mathcal{H}_\kappa$ have persistent jumps (respectively, flows). The term *forward invariant* ([7, Definition 3.1]) here means that maximal[3] closed-loop trajectories originating from X stay in X and have unbounded domains, and such trajectories exist from everywhere on X.

Assumption 4.3: There exists a feedback κ such that the terminal constraint set X is forward invariant for the hybrid system $\mathcal{H}_\kappa$ in (6). Moreover, the terminal cost V is differentiable on an open set containing $\mathrm{cl}\,(X \cap C_\kappa)$, and

$$\begin{aligned} \langle \nabla V(x), f_\kappa(x) \rangle &\leq -L_C(x, \kappa_C(x)) \quad \forall x \in X \cap C_\kappa, \\ V(g_\kappa(x)) - V(x) &\leq -L_D(x, \kappa_D(x)) \quad \forall x \in X \cap D_\kappa. \end{aligned} \tag{9}$$

A. The Case of Persistent Jumps

When the hybrid MPC algorithm generates solution pairs with persistent jumps, we do not insist on the flow cost to be positive definite with respect to the distance of the state x to the set $\mathcal{A}$. Instead, we assume a property on optimal solution pairs (x, u) originating away from $\mathcal{A}$.

Assumption 4.4: The following properties hold.

(D1) There exists a class-$\mathcal{K}_\infty$ function α_D such that for every $(x, u) \in D$, $L_D(x, u) \geq \alpha_D(|x|_{\mathcal{A}})$.

(D2) There exists a continuous positive definite function $\gamma_D : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\liminf_{r \to \infty} \gamma_D(r) > 0$, and for every optimal solution pair (x, u),

$$|x(t_{j+1}, j)|_{\mathcal{A}} \geq \gamma_D(|x(0, 0)|_{\mathcal{A}})$$

for some $j \in \{0, 1, \ldots, J\}$, where $\{t_j\}_{j=0}^{J+1}$ is the sequence satisfying $\mathrm{dom}(x, u) = \cup_{j=0}^{J}([t_j, t_{j+1}] \times \{j\})$.

Condition (D2) guarantees that x is away from $\mathcal{A}$ at a jump time or at the terminal time. Since the flow cost need not be positive definite and can be zero, (D2) is utilized to ensure that the cost of (x, u) is nonzero. This property is enforced to maintain positive definiteness of the value function $\mathcal{J}^*$ with respect to $\mathcal{A}$, as $\mathcal{J}^*$ will be used as a stability certificate for the hybrid MPC algorithm.

Verifying Condition (D2) requires knowledge of optimal solution pairs, which might not always be possible. Nevertheless, this requirement can be removed if the flows of the system satisfy a Lyapunov-like inequality that limit the rate of convergence to $\mathcal{A}$ and prohibit finite-time convergence.

Proposition 4.5: Let $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a function that is differentiable on an open set containing $\mathrm{cl}\,(\Pi(C))$. Suppose that there exist class-$\mathcal{K}_\infty$ functions $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and a constant $\lambda \in \mathbb{R}$ satisfying the following for some $\varepsilon > 0$:

$$\tilde{\alpha}_1(|x|_{\mathcal{A}}) \leq \widetilde{V}(x) \leq \tilde{\alpha}_2(|x|_{\mathcal{A}}) \quad \forall x \in \Pi(C) : |x|_{\mathcal{A}} \leq \varepsilon, \tag{10}$$

$$\langle \nabla \widetilde{V}(x), f(x, u) \rangle \geq \lambda \widetilde{V}(x) \quad \forall (x, u) \in C : |x|_{\mathcal{A}} \leq \varepsilon. \tag{11}$$

Then, (D2) holds if $\lambda \geq 0$ or there exists $T_{\max} \in \mathbb{R}_{\geq 0}$ such that $T \leq T_{\max}$ for every $(T, J) \in \mathcal{T}$.

Example 4.6 (Bouncing Ball during Flows): Consider the data of the bouncing ball system in Example 2.2, and the total energy function W defined in Section I. Let $\mathcal{A} = \{x : x_1 \geq 0, W(x) = \gamma h\}$ for some $h \geq 0$, which corresponds to the limit cycle of the autonomous bouncing

[2]The work in [1] and [2] deals with the case where $\mathcal{A}$ is compact.

[3]Closed-loop trajectories are maximal if they cannot be extended.

ball originating from $(h, 0)$ when $\lambda = 1$. Since $\mathcal{A}$ is compact and W is positive definite (on the domain $x_1 \geq 0$), the function $x \mapsto (W(x) - \gamma h)^2 =: \widetilde{V}(x)$ satisfies (10) for some $\varepsilon > 0$ and class-$\mathcal{K}_\infty$ functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, as W is continuous. In fact, these functions can be chosen independently of $\varepsilon > 0$, due to *radial unboundedness* of W. Moreover, $\langle \nabla \widetilde{V}(x), f(x, u) \rangle = 0$ for all $(x, u) \in C$, so (11) holds with arbitrarily large $\varepsilon > 0$.

B. The Case of Persistent Flows

For persistent flows, similar to Assumption 4.4, we do not impose any conditions on the jump cost, and instead assume a property on optimal solution pairs.

Assumption 4.7: The following properties hold.

- (C1) There exists a class-$\mathcal{K}_\infty$ function α_C such that for every $(x, u) \in C$, $L_C(x, u) \geq \alpha_C(|x|_{\mathcal{A}})$.
- (C2) There exist a continuous positive definite function $\gamma_C : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\liminf_{r \to \infty} \gamma_C(r) > 0$, and for every optimal solution pair (x, u) with terminal time (T, J), either of the following holds.
  - There exist $t^1, t^2 \in [0, T]$ such that $t^2 - t^1 \geq \gamma_C(|x(0,0)|_{\mathcal{A}})$, and $|x(t,j)|_{\mathcal{A}} \geq \gamma_C(|x(0,0)|_{\mathcal{A}})$ for almost every $(t, j) \in \operatorname{dom} x$ satisfying $t \in [t^1, t^2]$.
  - $|x(T, J)|_{\mathcal{A}} \geq \gamma_C(|x(0,0)|_{\mathcal{A}})$.

Again, the required knowledge of optimal solution pairs in Condition (C2) can be replaced with some assumptions on the system data. However, as opposed to the persistent jumps case, where the substitute assumptions pertained only to flows, the persistent flows case imposes requirements on both flows and jumps. Indeed, a delicate matter that needs to be taken care of now is to ensure that the optimal trajectories do not tend to $\mathcal{A}$ arbitrarily fast during flows, potentially resulting in the value function being zero outside $\mathcal{A}$ (due to the integrand in the definition of $\mathcal{J}$ becoming zero almost everywhere). While the Lyapunov-like conditions in Proposition 4.5 can be employed for such a task, a more direct way of checking this property relies on the existence of a uniform upper bound on the magnitude of the velocity $\dot{x} = f(x, u)$ away from $\mathcal{A}$.

Proposition 4.8: Suppose that there exists a continuous function $\sigma : (0, \infty) \to [0, \infty)$ and $\varepsilon > 0$ satisfying

$$|f(x, u)| \leq \sigma\left(|x|_{\mathcal{A}}\right) \quad \forall (x, u) \in C : 0 < |x|_{\mathcal{A}} \leq \varepsilon. \tag{12}$$

Moreover, suppose that there exists a continuous increasing positive definite function $\tilde{\alpha}_D : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$|g(x, u)|_{\mathcal{A}} \geq \tilde{\alpha}_D(|x|_{\mathcal{A}}) \qquad \forall (x, u) \in D : g(x, u) \in \operatorname{cl}(\Pi(C)) \cup \Pi(D).$$

Then, (C2) holds if $\tilde{\alpha}_D(r) \geq r$ for every $r \geq 0$, or there exists $J_{\max} \in \mathbb{N}$ such that $J \leq J_{\max}$ for every $(T, J) \in \mathcal{T}$.

Unlike the conditions in Proposition 4.5, (12) does allow for finite-time convergence to $\mathcal{A}$ during flows. Existence of a continuous function σ and $\varepsilon > 0$ satisfying (12) is guaranteed when $C = C' \times U$ for a closed set $C' \subset \mathbb{R}^n$ and compact set $U \subset \mathbb{R}^m$, provided $\mathcal{A}$ is compact and f is continuous.

Example 4.9 (Digital Control with Actuator Constraints): Given a compact set $U \subset \mathbb{R}^m$, we revisit the digital control system in Example 2.4 with the modified flow/jump sets

$$C = \{(x, u) : \eta \in U, \tau_s \in [0, T_s]\},$$
$$D = \{(x, u) : \eta \in U, \tau_s = T_s, u \in U\}.$$

With this, we consider the compact set $\mathcal{A} = \{0\} \times U \times [0, T_s]$. The first condition of Proposition 4.8 related to flows holds as the flow map f is affine and does not depend on u. We also note that $|g(x, u)|_{\mathcal{A}} = |z| = |x|_{\mathcal{A}}$ for every $(x, u) \in D$, so the second condition holds with $\tilde{\alpha}$ as the identity.

Remark 4.10: The condition imposed on flows in Proposition 4.8 has previously been used as part of the main stabilizing assumptions; see [2, Assumption 4.2].

V. Properties of the OCP

This section presents the properties pertinent to Problem 3.2 that are used to show asymptotic stability of $\mathcal{A}$. The initial results are similar to those in [2], stated with more generality. As opposed to [2], the results here do not require any regularity, aside from Assumption 2.1 at times.

Proposition 5.1: Under Assumption 3.1, if there exists a feedback κ such that the terminal constraint set X is forward invariant for the hybrid system $\mathcal{H}_\kappa$ in (6)*, then, $X \subset \mathcal{X}$.*

Proposition 5.2: Suppose that there exists a feedback κ such that the terminal constraint set X is forward invariant for the hybrid system $\mathcal{H}_\kappa$ in (6)*. Moreover, suppose that Assumptions 2.1 and 3.1 hold. Let (x, u) be a feasible solution pair. Then, for any $(t, j) \in \operatorname{dom}(x, u)$, $x(t, j)$ belongs to the feasible set $\mathcal{X}$; i.e., $x(t, j) \in \mathcal{X}$.*

Proposition 5.1 and 5.2 show that the terminal constraint set is contained in the feasible set, and recursive feasibility is maintained with the moving horizon implementation, respectively. The conditions in Proposition 5.1 and the inequalities in Assumption 4.3 ensure an upper bound on the value function $\mathcal{J}^*$ over the terminal constraint set X, in terms of the terminal cost V. Combining this with the class-$\mathcal{K}_\infty$ upper bound on V is the first step towards establishing $\mathcal{J}^*$ as a Lyapunov function for the hybrid MPC algorithm.

Lemma 5.3: Under Assumptions 3.1 and 4.3,

$$\mathcal{J}^*(x_0) \leq V(x_0) \quad \forall x_0 \in X \subset \mathcal{X}.$$

Next, we show that $\mathcal{J}^*$ is upper bounded by a nonincreasing function along optimal trajectories, which decreases during flows (respectively, jumps) if L_C (respectively, L_D) satisfies the lower bound in (C1) (respectively, (D1)).

Lemma 5.4: Suppose Assumptions 2.1, 3.1 and 4.3 hold. Then, for any optimal (x, u) and any $(t, j) \in \operatorname{dom}(x, u)$,

$$\mathcal{J}^*(x(t,j)) \leq \mathcal{J}^*(x(0,0)) - \sum_{i=0}^{j} \int_{s_i}^{s_{i+1}} L_C(x(s,i), u(s,i))\, ds - \sum_{i=0}^{j-1} L_D(x(s_{i+1}, i), u(s_{i+1}, i)),$$

where $\{s_i\}_{i=0}^{j+1}$ *is the sequence satisfying*

$$\mathrm{dom}(x,u)\cap([0,t]\times\{0,1,\ldots,j\}) = \cup_{i=0}^{j}\left([s_i,s_{i+1}]\times\{i\}\right).$$

The final step in establishing $\mathcal{J}^*$ as a Lyapunov function is to show that it is positive definite with respect to the target set $\mathcal{A}$. This property holds when either of Assumptions 4.4 or 4.7 are combined with Assumption 3.1.

Lemma 5.5: Under Assumption 3.1, if either of Assumptions 4.4 or 4.7 holds, there exists a continuous positive definite function $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ *with* $\liminf_{r\to\infty}\alpha(r)>0$ *such that the value function satisfies* $\mathcal{J}^*(x_0)\geq\alpha(|x_0|_{\mathcal{A}})$ *for all* $x_0\in\mathcal{X}$.

VI. Asymptotic Stability of Hybrid MPC

This section summarizes asymptotic stability of the hybrid MPC algorithm in the case of persistent flows or jumps.

Definition 6.1: The hybrid MPC algorithm is said to render the set $\mathcal{A}$ asymptotically stable (for $\mathcal{H}$) if $\mathcal{H}$ has unique state trajectories and the following hold:

- There exists $\delta>0$ such that for every $x_0\in\Pi(\mathrm{cl}\,(C)\cup D)$ satisfying $|x_0|_{\mathcal{A}}\leq\delta$, there exists a solution pair (x,u) generated by the hybrid MPC algorithm originating from x_0.
- For every $\varepsilon>0$, there exists $\delta>0$ such that given any solution pair (x,u) generated by the hybrid MPC algorithm, $|x(0,0)|_{\mathcal{A}}\leq\delta$ implies $|x(t,j)|_{\mathcal{A}}\leq\varepsilon$ for all $(t,j)\in\mathrm{dom}(x,u)$.
- Every solution pair (x,u) generated by the hybrid MPC algorithm satisfies $\lim_{t+j\to\infty}|x(t,j)|_{\mathcal{A}}=0$.

Theorem 6.2: Suppose Assumptions 2.1, 2.5, 4.1, 4.2, and 4.3 hold. Then, the hybrid MPC algorithm renders the set $\mathcal{A}$ *asymptotically stable for the hybrid system* $\mathcal{H}$ *if Assumption 3.1 holds with* $t_0>0$ *and* $J\geq 1$, *and either of the following statements are true.*

- *Every solution pair generated by the hybrid MPC algorithm has persistent jumps and Assumption 4.4 holds.*
- *Every solution pair generated by the hybrid MPC algorithm has persistent flows and Assumption 4.7 holds.*

Remark 6.3: Persistence of jumps or flows are usually inherited from the open-loop system $\mathcal{H}$. For example, for the bouncing ball in Examples 2.2 and 4.6, every complete solution pair has persistent jumps (see Section I). Consequently, every solution pair generated by the hybrid MPC algorithm has persistent jumps, regardless of the OCP formulation.

VII. Illustrative Example

In this section, we revisit the bouncing ball example in [2, Section VII] and show that exploiting persistence of jumps leads to an MPC design that is much less involved.

Recall the bouncing ball model in Example 4.6, and note that the flow set C therein is closed. Hence, Assumption 2.1 holds. Let $X=\Pi(C)$. Given the function $\widetilde{V}$ in Example 4.6, let $V=\widetilde{V}$. Then, Assumption 4.2 holds. Indeed, (G2) holds with any $\varepsilon>0$, since $X=\Pi(\mathrm{cl}\,C)$. Similarly, (G1) holds with any $\varepsilon>0$, as discussed in Example 4.6.

For the closed-loop system $\mathcal{H}_\kappa$ in (6), choose an arbitrary function κ_C, which results in the set $C_\kappa=\{x:x_1\geq 0\}$ and the mapping $f_\kappa(x)=f(x)$. For jumps, for any $x\in\mathbb{R}^2$, let $\kappa_D(x)=\max\{\lambda x_2+\sqrt{2\gamma h},0\}$, which leads to the closed-loop jump set $D_\kappa=\{x:x_1=0,x_2\leq 0\}$ and jump map $g_\kappa(x)=(0,\max\{-\lambda x_2,\sqrt{2\gamma h}\})$ for all $x\in D_\kappa$. It follows that $X=\Pi(C)=C_\kappa$ is forward invariant for $\mathcal{H}_\kappa$, since $g_\kappa(D_\kappa)\subset C_\kappa$, and flows of $\mathcal{H}_\kappa$ eventually reach D_κ; see [3, Example 2.12]. Select the flow cost L_C as the zero function. Let $L_D(x,u)=\gamma h(x_2+\sqrt{2\gamma h})^2/2$ if $x_2\geq-\sqrt{2\gamma h}/\lambda$, otherwise, let

$$L_D(x,u)=\min\Big\{\gamma h(x_2+\sqrt{2\gamma h})^2/2,\\ (x_2^2/2-\gamma h)^2-(\lambda^2x_2^2/2-\gamma h)^2\Big\}.$$

Routine algebraic manipulations show that $V(g_\kappa(x))-V(x)\leq -L_D(x,\kappa_D(x))$. Hence, Assumption 4.3 holds. Moreover, it can be seen that Assumption 4.4 holds: (D2) follows via Proposition 4.5, as shown in Example 4.6, and (D1) holds by radial unboundedness of L_D. Finally, by inspection, it can be observed that regardless of the choice of $\mathcal{T}$, optimal solution pairs are generated by the feedback κ, hence Assumption 4.1 holds, and Theorem 6.2 applies.

The cost functions L_C, L_D, and V chosen for the same problem in [2, Section VII] have much more complex expressions and depend on a parameter $\theta>0$, primarily to ensure the inequality $\langle\nabla V(x),f_\kappa(x)\rangle\leq -L_C(x,\kappa_C(x))\leq-\alpha_C(|x|_{\mathcal{A}})$. The cost functions chosen in this section are simpler and can be seen as pointwise limits of their counterparts in [2, Section VII], as θ tends to zero. In particular, the terminal cost V here has a much more natural expression and is an obvious choice to stabilize the γh-level set of W.

VIII. Conclusion

For the CLF-based stabilizing MPC algorithm in [1] and [2], we showed that the cost functions of the OCP can be taken as positive semidefinite. In particular, for systems with persistent flows (respectively, jumps), the jump (respectively, flow) cost can be taken as zero. Future work will focus on relaxing the structure of the prediction horizon under similar persistence assumptions to preserve feasibility properties.

References

[1] B. Altın, P. Ojaghi, and R. G. Sanfelice, "A model predictive control framework for hybrid dynamical system," in *6th IFAC Conference on Nonlinear Model Predictive Control NMPC 2018*, 8 2018, pp. 128–133.

[2] B. Altın and R. G. Sanfelice, "Asymptotically stabilizing model predictive control for hybrid dynamical systems," in *2019 American Control Conference (ACC)*, July 2019, pp. 3630–3635.

[3] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton, NJ: Princeton University Press, 2012.

[4] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[5] F. L. Pereira, F. A. C. C. Fontes, A. P. Aguiar, and J. B. de Sousa, *An Optimization-Based Framework for Impulsive Control Systems*. Cham: Springer International Publishing, 2015, pp. 277–300.

[6] R. Goebel, "Existence of optimal controls on hybrid time domains," *Nonlinear Analysis: Hybrid Systems*, vol. 31, pp. 153–165, 2019.

[7] J. Chai and R. G. Sanfelice, "Forward invariance of sets for hybrid dynamical systems (Part I)," *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2426–2441, June 2019.