On the optimality of Dubins paths across heterogeneous terrain

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Abstract. We derive optimality conditions for the paths of a Dubins vehicle when the state space is partitioned into two patches with different vehicle's forward velocity. We recast this problem as a hybrid optimal control problem and solve it using optimality principles for hybrid systems. Among the optimality conditions, we derive a "refraction" law at the boundary of the patches which generalizes the so-called Snell's law of refraction in optics to the case of paths with bounded maximum curvature.

1 Introduction

Control algorithms that are capable of steering autonomous vehicles to satisfy a given set of specifications, like initial and final constraints, and at the same time, guarantee certain optimality conditions are very appealing to applications in robotics and aerospace. This has led researchers to strive for control design tools that adequately incorporate both trajectory constraints and measures of optimality. As a consequence, many results from the theory of optimal control, in particular, those that guarantee time optimality, have found wide applicability in autonomous vehicle control problems.

Perhaps, the earliest result on time-optimal control laws for autonomous vehicles modeled as a particle moving with constant, positive forward velocity and with constrained minimum turning radius is the work by Dubins [1]. While Dubins used only geometric arguments to establish his results, a few years later, the appearance of Pontryagin's Maximum Principle in [2] enabled the authors in [3] to systematically recover Dubins results. Moreover, building from the work of Reeds and Shepp [4], the application of Pontryagin's optimality principle permitted the authors in [5, 3] to derive similar results for a vehicle model without forward velocity constraints.

In this paper, we consider autonomous vehicles with dynamics governed by

$$|u| \leq 1 \ , \qquad \begin{cases} \dot{x} &= v \sin\theta \\ \dot{y} &= v \cos\theta \\ \dot{\theta} &= u \end{cases} \ , \tag{1}$$

where (x, y) is the vehicle's position, θ is the angle between the vehicle and the vertical axis determining the vehicle's orientation, u is the angular acceleration input for the vehicle, and v is the vehicle's forward velocity. This vehicle model is usually referred to as Dubins vehicle. We consider the case of heterogeneous velocity along the terrain where the vehicle is deployed. Two different velocities, v_1 and v_2, define the constant, forward velocity of Dubins vehicle on two patches of the plane, patch $\mathcal{P}_1$ and patch $\mathcal{P}_2$, depicted in Figure 1. We are interested in the following problem:

> *Find the minimum-time path for Dubins vehicle from an initial point and angle in patch $\mathcal{P}_1$ to a final point and angle in patch $\mathcal{P}_2$.*

Figure 1 shows possible initial and final vehicle configurations, which are denoted by (x^0, y^0, θ^0) and (x^1, y^1, θ^1), respectively, for which a minimum-time path is to be found. To the best of our knowledge, the problem described above has not been addressed in the past, perhaps due to the fact that the classical Pontryagin's Maximum Principle is not applicable because of the discontinuous behavior at the common boundary between the patches.

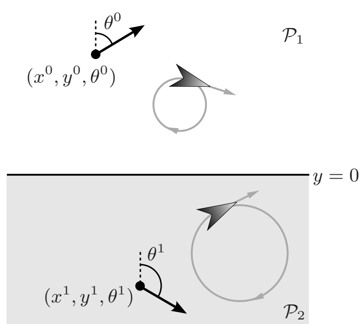


Fig. 1. Dubins vehicle on an heterogeneous terrain. The initial configuration is given by (x^0, y^0, θ^0) and the final configuration by (x^1, y^1, θ^1). The forward velocity in patch $\mathcal{P}_1$ is smaller than the forward velocity in patch $\mathcal{P}_2$.

By recasting this problem into an optimal hybrid control problem and applying principles of optimality for hybrid systems, we establish the following conditions that illuminate important characteristics of optimal paths:

- *The portions of the paths that remain in either patch are Dubins optimal.*
- *Optimal paths are such that, at the boundary between the patches, their type does not change; that is, the type of path right before and after crossing the boundary are the same.*
- *Optimal paths that cross the boundary describing a straight line are orthogonal to the boundary.*

– *The angles of the path pieces before and after crossing the boundary satisfy a "refraction" law, which consists of a generalization of Snell's law of refraction in optics.*

Applications of these results include optimal motion planning of autonomous vehicles in environments with obstacles, different terrains properties, and other topological constraints. Strategies that steer autonomous vehicles across heterogeneous terrain using Snell's law of refraction have already been recognized in the literature and applied to point-mass vehicles; see, e.g., [6, 7]. Our results extend those to the case of autonomous vehicles with Dubins dynamics.

The remainder of the paper is organized as follows. Section 2 discusses related background to the optimal control problem outlined above and introduces general notation. In Section 3, we present a hybrid model which, as shown in that same section, enable us to formulate the problem of study in an optimal hybrid control framework. In Section 4, we establish necessary conditions for optimality of paths including a refraction law at the boundary of the patches. Due to space constraints, the technical proofs are omitted and will be published elsewhere.

2 Background

Pontryagin's Maximum Principle [2] is a very powerful tool to derive necessary conditions for optimality of solutions to a dynamical system. In words, this principle establishes the existence of an adjoint function with the property that, along optimal system solutions, the Hamiltonian obtained by combining the system dynamics and the cost function associated to the optimal control problem is minimized. In its original form, this principle is applicable to optimal control problems with dynamics governed by differential equations with continuously differentiable right-hand sides.

The shortest path problem between two points with specific tangent direction and bounded maximum curvature has received wide attention in the literature. In his pioneer work in [1], by means of geometric arguments, Dubins showed that optimal paths to this problem consist of a smooth concatenation of no more than three pieces, each of them describing either a straight line, denoted by $\mathcal{L}$, or a circle, denoted by $\mathcal{C}$ (when the circle is traveled clockwise, we write $\mathcal{C}^+$, while when the circle is traveled counter-clockwise, we write $\mathcal{C}^-$), and are either of type $\mathcal{CCC}$ or $\mathcal{CLC}$, that is, they are among the following six types of paths

$$\mathcal{C}^-\mathcal{C}^+\mathcal{C}^-,\ \mathcal{C}^+\mathcal{C}^-\mathcal{C}^+,\ \mathcal{C}^-\mathcal{L}\mathcal{C}^-,\ \mathcal{C}^+\mathcal{L}\mathcal{C}^+,\ \mathcal{C}^+\mathcal{L}\mathcal{C}^-,\ \mathcal{C}^-\mathcal{L}\mathcal{C}^+, \tag{2}$$

in addition to any of the subpaths obtained when some of the pieces (but not all) have zero length. More recently, the authors in [3] recovered Dubins' result by using Pontryagin's Maximum Principle; see also [5]. Further investigations of the properties of optimal paths to this problem and other related applications of Pontryagin's Maximum Principle include [8–10], to just list a few.

Optimal control problems exhibiting discontinuous/impulsive behavior, like the heterogeneous version of Dubins' problem outlined in Section 1, cannot be solved using the classical Pontryagin's Maximum Principle. Extensions of this principle to systems with discontinuous right-hand side appeared in [11] while extensions to hybrid systems include [12], [13], and [14]. These principles establish the existence of an adjoint function which, in addition to conditions that parallel the necessary optimality conditions in the principle by Pontryagin, satisfies certain conditions at times of discontinuous/jumping behavior. The applicability of these principles to relevant problems have been highlighted in [12, 15, 16]. These will be the key tool in deriving the results in this paper.

2.1 Notation

We use the following notation throughout the paper. $\mathbb{R}^n$ denotes n-dimensional Euclidean space. $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. Given $k \in \mathbb{N}$, $\mathbb{N}_{\leq k}$ denotes $\{0, 1, \ldots, k\}$. Given a set S, $\overline{S}$ denotes its closure and S° denotes its interior. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given $U := [-1, 1]$, $\mathcal{U}$ denotes the set of all piecewise-continuous functions u from subsets of $\mathbb{R}_{\geq 0}$ to U.

3 Problem Statement

In this section, we formulate the problem of steering Dubins vehicle across heterogeneous terrain as a hybrid optimal control problem. We present a hybrid model and introduce the optimal control problem. An alternative approach is to treat this problem as a differential equation with discontinuous right-hand side and use the results in [11]. However, a hybrid control systems approach is not only more convenient from a modeling point of view as it enables the use of a sound concept of solution but also facilitates the application of more explicit optimality principles for hybrid systems, like the ones in [12].

3.1 Hybrid model

We denote by $\mathcal{H}_v$ the hybrid system that captures the dynamics of Dubins vehicle along the patches. Let $v_1, v_2 \in \mathbb{R}_{>0}$, $v_1 \neq v_2$, be the forward velocity of the vehicle on patch $\mathcal{P}_1$ and patch $\mathcal{P}_2$, respectively, where

$$\mathcal{P}_1 := \left\{ [x\ y\ \theta]^\top \in \mathbb{R}^3 \ \middle|\ y \geq 0 \right\} , \qquad \mathcal{P}_2 := \left\{ [x\ y\ \theta]^\top \in \mathbb{R}^3 \ \middle|\ y \leq 0 \right\} ,$$

which share a common boundary $\mathcal{P}_1 \cap \mathcal{P}_2 = \left\{ [x\ y\ \theta]^\top \in \mathbb{R}^3 \ \middle|\ y = 0 \right\}$; see Figure 1. Let q be a discrete state taking value in $Q := \{1, 2\}$ that indicates the current patch to which the vehicle belongs to. Following the vehicle's dynamics in (1),

$$\begin{bmatrix} \dot{\xi} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} f_q(\xi, u) \\ 0 \end{bmatrix} \qquad \xi \in \mathcal{P}_q \tag{3}$$

with

$$\xi := \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \in \mathbb{R}^3 \qquad \text{and} \qquad f_q(\xi, u) := \begin{bmatrix} v_q \sin\theta \\ v_q \cos\theta \\ u \end{bmatrix}$$

define the continuous dynamics (or *flows*) of $\mathcal{H}_v$, where ξ is the continuous state and $u \in \mathcal{U}$ is the control input. Then, during flows, ξ captures the vehicle dynamics on the q-th patch while q remains constant. We model the change of patch so that it occurs when y is zero and the vehicle is moving away from the current patch. Then, defining a function $s : Q \to \{-1, 1\}$ where $s(1) = -1$ and $s(2) = 1$, the discrete dynamics (or *jumps*) of $\mathcal{H}_v$ are given by

$$\begin{bmatrix} \xi^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} \xi \\ 3 - q \end{bmatrix} \qquad \xi \in \mathcal{P}_1 \cap \mathcal{P}_2 \;\text{ and }\; s(q) v_q \cos\theta > 0\ , \tag{4}$$

which implies that at jumps ξ does not change while q is toggled between 1 and 2. Finally, we denote by $\zeta := [\xi^\top\ q]^\top$ the full state of $\mathcal{H}_v$.

Following the hybrid systems framework outlined in [17] and further established in [18, 19], we can rewrite $\mathcal{H}_v$ as

$$\mathcal{H}_v : \begin{cases} \dot{\zeta} = f(\zeta, u) & \zeta \in C \\ \zeta^+ = g(\zeta) & \zeta \in D \end{cases}$$

by defining

$$f(\zeta, u) := \begin{bmatrix} f_q(\xi, u) \\ 0 \end{bmatrix}, C := \bigcup_{q \in Q} (C_q \times \{q\}), g(\zeta) := \begin{bmatrix} \xi \\ 3 - q \end{bmatrix}, D := \bigcup_{q \in Q} (D_q \times \{q\}),$$

where $C_q := \mathcal{P}_q$ and $D_q := \{\xi \in \mathbb{R}^3 \mid y = 0, s(q) v_q \cos\theta > 0\}$ for each $q \in Q$. Then, $\mathcal{H}_v$ is determined by the data (f, C, g, D), where f is the *flow map*, C is the *flow set*, g is the *jump map*, and D is the *jump set*. As in [17], solutions to $\mathcal{H}_v$ are given by *hybrid arcs* on *hybrid time domains*. Hybrid time domains use a variable t to indicate flow time and an index j to keep track of the number of jumps, and hence, parametrize solutions by (t, j). A subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_0 \leq t_1 \leq t_2 \leq \ldots$, or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$. (Note that the t component of elements $(t, j) \in E$ does not uniquely define the index j since, in this framework, multiple jumps at the same t are possible.) Then, given a control input $u \in \mathcal{U}$, solutions to $\mathcal{H}_v$ are given by functions, called *hybrid arcs*, $\zeta : \operatorname{dom}\zeta \to \mathbb{R}^4$, where $\operatorname{dom}\zeta$ is a hybrid time domain, $t \mapsto \xi(t, j)$ is a locally absolutely continuous function for each fixed j, $t \mapsto q(t, j)$ is a piecewise constant function for each fixed j, and ζ satisfies the flow and jump conditions mentioned above. More precisely, given an input $u \in \mathcal{U}$, a hybrid arc ζ is a *solution to the hybrid system* $\mathcal{H}_v$ if $\zeta(0, 0) \in C \cup D$, $\operatorname{dom}\zeta = \operatorname{dom} u$, and:

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} \zeta$ [1],

$$\zeta(t, j) \in C, \quad \dot{\zeta}(t, j) = f(\zeta(t, j), u(t, j)) \ .$$

(S2) For all $(t, j) \in \operatorname{dom} \zeta$ such that $(t, j+1) \in \operatorname{dom} \zeta$,

$$\zeta(t, j) \in D, \quad \zeta(t, j+1) = g(\zeta(t, j)) \ .$$

Inputs u given as signals $t \mapsto u(t)$ for each $t \in \mathbb{R}_{\geq 0}$ can be rewritten on a hybrid time domain E by defining, with some abuse of notation, $u(t, j) := u(t)$ for each $(t, j) \in E$. Note that solutions to $\mathcal{H}_v$ exist from every point in $C \cup D = \mathbb{R}^3 \times Q$. In particular, solutions are allowed to flow in the boundary $\mathcal{P}_1 \cap \mathcal{P}_2$ with either $q = 1$ or $q = 2$; such a feature cannot be captured with a differential equation with discontinuous right-hand side or with a (regular) differential inclusion without adding extra solutions. Also note that since the sets D_q are not closed subsets of $\mathbb{R}^3$, the regularity property for D required in [18, 19] does not hold (the flow map, jump map, and jump set of $\mathcal{H}_v$ satisfy the properties therein). While such a regularity is not required for the results in this paper to be true, it turns out that, as shown in [19], it highlights the presence of undesirable solutions if the sets D_q were to be closed or small noise entered through the state.

3.2 Hybrid optimal control problem

We consider the following hybrid optimal control problem. Given $(x^0, y^0, \theta^0) \in C_1^\circ$ and $(x^1, y^1, \theta^1) \in C_2^\circ$:

($\star$) Minimize the transfer time $T \in \mathbb{R}_{\geq 0}$ subject to:

- (C1) Dynamical constraint: dynamics of $\mathcal{H}_v$ given in (3)-(4).
- (C2) Input constraint: $u \in \mathcal{U}$.
- (C3) Initial and terminal constraints: every optimal solution (ξ, q) to $\mathcal{H}_v$ satisfies the initial constraint $(x(0,0), y(0,0), \theta(0,0)) = (x^0, y^0, \theta^0)$ and the terminal constraint $(x(T, J), y(T, J), \theta(T, J)) = (x^1, y^1, \theta^1)$ for some $(T, J) \in \operatorname{dom}(\xi, q)$.

The number of jumps required to solve ($\star$) is finite, given by $J - 1$, and no smaller than one; hence, optimal solutions to ($\star$) are not Zeno. The initial and final constraints are such that solutions can flow from some time before their first jump and after their final jump (that is, the first jump is at some $(t_1, 0)$ with $t_1 > 0$ and the last jump is at some $(t_J, J-1)$ with $t_J < T$). This is a technical requirement for the application of the hybrid maximum principle in [12] in the next section.

[1] $\dot{\zeta}(t, j)$ denotes the derivative of $t \mapsto \zeta(t, j)$ with respect to t for a fixed j, which exists for almost every t such that $(t, j) \in \operatorname{dom} \zeta \cap ([t_j, t_{j+1}] \times \{j\})$.

4 Necessary conditions for optimality

Necessary optimality conditions for solutions to $\mathcal{H}_v$ solving ($\star$) can be obtained using the principle of optimality for hybrid systems in [12] (see also [20] and [15]). Under further technical assumptions, Theorem 1 in [12] establishes that there exists an *adjoint pair* $(\lambda, \lambda_\circ)$, where λ is a function and $\lambda_\circ$ is a constant, which, along optimal solutions to ($\star$), satisfies certain *Hamiltonian maximization*, *nontriviality*, *transversality*, and *Hamiltonian value conditions*. In particular, [12, Theorem 1] can be applied to the optimal control problem ($\star$) to deduce the following optimality conditions for the paths.

Proposition 1 (properties of ($\star$)). *For each optimal solution (ξ, q) to ($\star$) with optimal control u, minimum transfer time T, and $J-1$ number of jumps, there exists a function $\lambda : \operatorname{dom}\lambda \to \mathbb{R}^3$, $\lambda := [\alpha\ \beta\ \gamma]^\top$, $\operatorname{dom}\lambda = \operatorname{dom}(\xi, q)$, where $t \mapsto \lambda(t, j)$ is absolutely continuous for each j, $(t, j) \in \operatorname{dom}\lambda$, and a constant $\lambda_\circ \in \mathbb{R}$ defining the adjoint pair $(\lambda, \lambda_\circ)$ satisfying:*

a) $\lambda_\circ \geq 0$ *and* $\dot{\lambda}(t, j) = -\dfrac{\partial H_{q(t,j)}}{\partial \xi}(\xi(t, j), \lambda(t, j), \lambda_\circ, u(t, j))$ *for almost every* $t \in [t_j, t_{j+1}]$, $(t, j) \in \operatorname{dom}\lambda$, *where, for each* $q \in \{1, 2\}$, $H_q : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times$ $\times U \to \mathbb{R}$ *is the Hamiltonian associated with the continuous dynamics of* $\mathcal{H}_v$, *which is given by*

$$H_q(\xi, \lambda, \lambda_\circ, u) = \alpha v_q \sin\theta + \beta v_q \cos\theta + \gamma u - \lambda_\circ$$

for each $q \in Q$.

b) There exist $\overline{\alpha}, \overline{\beta} \in \mathbb{R}$ *and, for each* $j \in \mathbb{N}_{\leq J}$, *there exists* $p_j \in \mathbb{R}$ *such that* $\alpha(t, j) := \overline{\alpha}$ *for all* $(t, j) \in \operatorname{dom}(\xi, q)$, $\beta(t, j) := \overline{\beta} + p_j$ *for almost all* $t \in [0, T]$, $(t, j) \in \operatorname{dom}(\xi, q)$, *and* $\gamma(t, j) = \gamma(t, j+1)$ *for each* (t, j) *such that* $(t, j), (t, j+1) \in \operatorname{dom}\lambda$.

c) For every $(t, j) \in \operatorname{dom}(\xi, q)$ *such that* $\gamma(t, j) \neq 0$, $u(t, j) =$ *sgn*$(\gamma(t, j))$*; and for every* $(t, j) \in \operatorname{dom}(\xi, q)$ *such that* $\gamma(t, j) = 0$, $u(t, j) = 0$.

d) For every $(t, j) \in \operatorname{dom}(\xi, q)$ *such that* $\gamma(t, j) = 0$, $\beta(t, j)\tan\theta(t, j) = \alpha(t, j)$.

Remark 1. The proof of Proposition 1 uses the fact that $\mathcal{H}_v$ can be associated with a hybrid system $\mathcal{H}_v^*$ given in the framework in [12] and that every solution to $\mathcal{H}_v$ solving ($\star$) is also a solution to $\mathcal{H}_v^*$ (agreeing with the concept of solution in [12] [2]). This property follows by construction of $\mathcal{H}_v^*$. Hybrid systems in [12]

[2] In [12], solutions to hybrid systems are given on compact time intervals by absolutely continuous functions ξ_j on $[t_j, t_{j+1}]$ such that, for each $j \in \{1, 2, \ldots, \nu\}$ (with finite $\nu \in \mathbb{N}$) and for finite sequences of logic states $\{q_j\}$ and control inputs $\{u_j\}$, satisfy the flow condition $\dot{\xi}_j = f_{q_j}(\xi_j(t), u_j(t))$ for almost all $t \in [t_j, t_{j+1}]$ and the jump condition $(\xi_j(t_j), \xi_{j+1}(t_j)) \in \mathcal{S}_{q_j, q_{j+1}}$ for each t_j, where t_j denotes the jump time (which is assumed to belong to the interior of the compact time interval where solutions are defined) and $\mathcal{S}_{q_j, q_{j+1}}$ is the switching set at the j-th jump (see [12, Definition 3] for more details). Hence, passing from a solution ζ on a bounded hybrid time domain $\operatorname{dom}\zeta$ with jumps at different (t_j, j)'s, first jump at $(t_1, 0)$ with $t_1 > t_0$, and last jump at

and [15] have a continuous state ξ with flows governed by $\dot{\xi} = f_q(\xi, u)$ when ξ belongs to a smooth manifold $\mathcal{M}_q$, where $q \in Q$ is a discrete state (which remains constant during flows). Jumps from mode q to mode q' satisfy: 1) the switching condition $(\xi, \xi') \in \mathcal{S}_{q,q'}$, where ξ is the continuous state before the jump, ξ' is the continuous state after the jump, and $\mathcal{S}_{q,q'}$ is the switching set; and 2) a temporal constraint enforcing that the jump time for the current mode is in the set $J_q \subset \mathbb{R}$. To obtain $\mathcal{H}_v^*$, the sets C_q in $\mathcal{H}_v$ are replaced by smooth manifolds $\mathcal{M}_q$, $C_q \subset \mathcal{M}_q$, while the jump set and the jump map are replaced by the switching condition given by

$$\mathcal{S}_{1,2} = \mathcal{S}_{2,1} = \hat{\mathcal{S}} := \left\{ (\xi, \xi) \;\middle|\; y = 0, \xi \in \mathbb{R}^3 \right\} \ ,$$

and $J_1 = J_2 = \mathbb{R}$. Then, the properties of the adjoint pair guaranteed by [12, Theorem 1] automatically imply item a) in Proposition 1 (see [12, Definition 9]). The condition for optimality at switches for the adjoint state λ implies that only the second component of λ, i.e. β, has a jump while the other two components are continuous (see Remark 2). This implies item b) in Proposition 1. The Hamiltonian maximization condition guaranteed to hold by [12, Theorem 1] implies that

$$H_{q(t,j)}(\xi(t,j), \lambda(t,j), \lambda_\circ, u(t,j)) = \max_{w \in U} H_{q(t,j)}(\xi(t,j), \lambda(t,j), \lambda_\circ, w)$$

for almost every $t \in [t_j, t_{j+1}]$, $(t,j) \in \operatorname{dom} \lambda$ (see [12, Definition 10]). It follows that the control law in item c) in Proposition 1 maximizes H_q. By integrating the adjoint state λ when $u = 0$, Proposition 1.d follows automatically. ■

Remark 2. [12, Theorem 1] implies that at jumps, the optimal solution, optimal control, and adjoint pair satisfy the switching condition $(-\lambda(t,j), \lambda(t,j+1)) \in K_j^\perp$ for each j for which there exists $t \in [0,T]$ such that $(t,j), (t,j+1) \in \operatorname{dom} \lambda$, where $K_j^\perp$ is the polar of the Boltyanskii approximating cone to $\mathcal{S}_{q(t,j),q(t,j+1)} (= \hat{\mathcal{S}})$. The set $\hat{\mathcal{S}}$ is such that $K_j^\perp$ is given by $\left\{ w \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \langle w, v \rangle \leq 0 \ \forall v \in \hat{\mathcal{S}} \right\}$ since the Boltyanskii approximating cone to $\hat{\mathcal{S}}$ is the set itself. Then, since by definition of $\hat{\mathcal{S}}$ the second and fourth components of v in $K_j^\perp$ are zero, $(-\lambda(t,j), \lambda(t,j+1)) \in K_j^\perp$ if and only if $\alpha(t,j) = \alpha(t,j+1)$, $\gamma(t,j) = \gamma(t,j+1)$, which implies that only β can have a jump. This property can also be obtained using the optimality principles in [14]. ■

4.1 Optimality of paths

The properties of the adjoint pair $(\lambda, \lambda_\circ)$ and the control input u in Proposition 1 can be related to properties of the continuous component ξ of the solutions to $(\star)$. These characterize the optimal paths from given initial and terminal constraints, as the following theorem states.

$(t_J, J-1)$ with $t_J < T$, where $T := \sup \{ t \in \mathbb{R}_{\geq 0} \mid \exists j \in \mathbb{N} \text{ such that} (t,j) \in \operatorname{dom} \zeta \}$ and $J := \sup \{ j \in \mathbb{N} \mid \exists t \in \mathbb{R}_{\geq 0} \text{ such that} (t,j) \in \operatorname{dom} \zeta \}$), to a solution as in [12, Definition 3] is straightforward.

Theorem 1 (optimality conditions of solutions to ($\star$)). *Each optimal solution (ξ, q) to ($\star$) with optimal control u, minimum transfer time T, and $J-1$ number of jumps is such that:*

a) The continuous component ξ is a smooth concatenation of finitely many pieces from the set $\{\mathcal{C}^+, \mathcal{C}^-, \mathcal{L}\}$.
b) The input component u is piecewise constant with finitely many pieces taking value in $\{-1, 0, 1\}$.
c) Each piece of the continuous component ξ contained in C_q, $q \in Q$, is Dubins optimal between the first and last point of such piece, i.e., it is given as in (2).
d) For each $(t, j) \in \operatorname{dom}(\xi, q)$ for which $(x(t, j), y(t, j), \theta(t, j)) \in D_{q(t,j)}$, the solution has a jump and:
  d.1) If the path before the jump is $\mathcal{C}$ then the path after the jump is $\mathcal{C}$.
  d.2) If the path before the jump is $\mathcal{L}$ then the path after the jump is $\mathcal{L}$ and $\theta(t, j)$ is zero or any multiple of π.

Remark 3. The proof of Theorem 1 uses Proposition 1 and the fact that, since the jump condition in $\mathcal{H}_v$ is time independent (that is, $J_1 = J_2 = \mathbb{R}$), the Hamiltonian value condition guaranteed to hold by [12, Theorem 1] implies that there exists $h^* \in \mathbb{R}$ such that

$$h^* = H_{q(t,j)}(\xi(t, j), \lambda(t, j), \lambda_\circ, u(t, j))$$

for almost every $t \in [t_j, t_{j+1}]$, $(t, j) \in \operatorname{dom} \lambda$ (see [12, Definition 13]). ∎

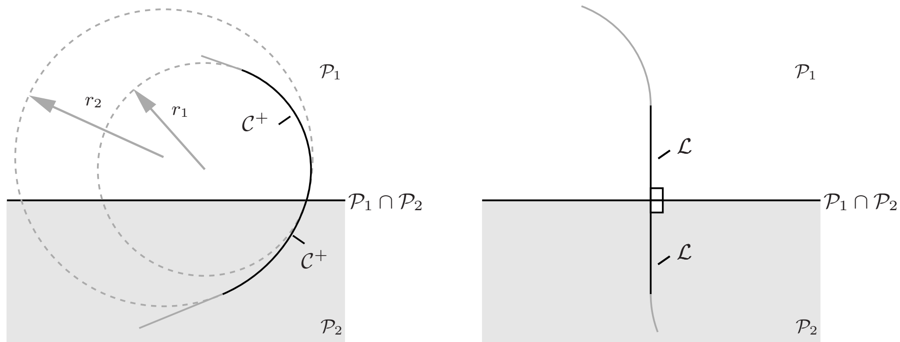
Figure 2 depicts optimal paths around the boundary of the patches. Item d.1) in Theorem 1 implies that optimal paths that cross the boundary are of the same type at each side of it. More precisely, if before crossing the boundary, the optimal path is of type $\mathcal{C}$ ($\mathcal{C}^+$ or $\mathcal{C}^-$), then the optimal path after crossing the boundary is also of type $\mathcal{C}$ ($\mathcal{C}^+$ or $\mathcal{C}^-$, respectively). Figure 2(a) depicts an optimal path of type $\mathcal{C}^+$. Statement d.2) in Theorem 1 implies that $\mathcal{L}$-type paths at the boundary are optimal only if they are orthogonal to the boundary. Figure 2(b) depicts this situation.

Using Theorem 1, it is possible to determine optimal families of paths for a class of solutions to ($\star$). The following statements follow directly from Dubins' result and Theorem 1.

Corollary 1 (optimal paths w/one jump). *Every optimal solution (ξ, q) to ($\star$) with only one jump is such that the continuous component ξ is a smooth concatenation of $\mathcal{C}, \mathcal{L}$ paths pieces and is given by one of the following four types of paths*

$$\mathcal{C}_1\mathcal{L}_1\mathcal{C}_2\mathcal{L}_2\mathcal{C}_3,\ \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathcal{L}_1\mathcal{C}_4,\ \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathcal{C}_4\mathcal{C}_5,\ \mathcal{C}_1\mathcal{L}_1\mathcal{C}_2\mathcal{C}_3\mathcal{C}_4\ , \tag{5}$$

in addition to any such path obtained when some of the path pieces (but not all) have zero length. Furthermore, if the path piece intersecting the boundary is of type $\mathcal{L}$, then the continuous component ξ describes a path of type $\mathcal{C}_1\mathcal{L}_1\mathcal{C}_2$ (or any such path obtained when $\mathcal{C}_1$ and/or $\mathcal{C}_2$ have zero length).



(a) $\mathcal{C}^+$-type of path at the boundary. Path pieces $\mathcal{C}^+$ in patch $\mathcal{P}_1$ with radius $r_1 = v_1$ and in patch $\mathcal{P}_2$ with radius $r_2 = v_2$, $v_2 > v_1$.

(b) $\mathcal{L}$-type of path at the boundary. The angle between the path and the boundary in each patch is $\pi/2$.

Fig. 2. Optimal paths nearby the boundary: paths of types $\mathcal{C}^+$ and $\mathcal{L}$ satisfying the necessary conditions in Theorem 1.

A consequence of Theorem 1 that is useful when computing optimal paths is the following.

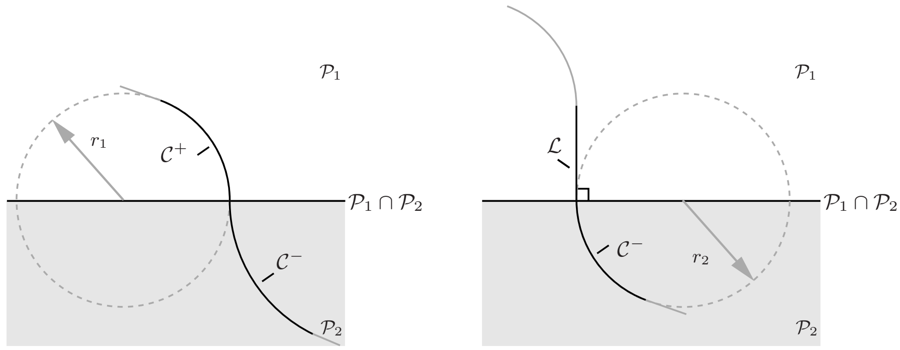
Corollary 2 (nonoptimal paths). *For the optimal control problem ($\star$), solutions to $\mathcal{H}_v$ satisfying (C1)-(C3) with the continuous component ξ describing paths that change at the boundary are nonoptimal, that is, paths that before and after the boundary are given by $\mathcal{C}^+$ and $\mathcal{L}$, $\mathcal{C}^-$ and $\mathcal{L}$, $\mathcal{L}$ and $\mathcal{C}^+$, $\mathcal{L}$ and $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{C}^-$, or $\mathcal{C}^-$ and $\mathcal{C}^+$, respectively, are nonoptimal.*

Figure 3 depicts two of the path types that Corollary 2 determines to be nonoptimal.

4.2 Refraction law at boundary

The optimal control law given in Proposition 1.c and the properties of the component γ of the adjoint state λ given in Proposition 1.b imply that the control law is constant at jumps of $\mathcal{H}_v$ (note that u is piecewise continuous for each fixed j with discontinuities at (t, j)'s where the path type changes). While θ remains constant at the boundary, the initial and final angles (and their variations) of the paths intersecting the boundary satisfy the following algebraic condition involving the patch velocities v_1 and v_2.

Theorem 2 (refraction law for ($\star$)). *Let (ξ, q) be an optimal solution to ($\star$). Let θ_1 and θ_2 denote the initial and final angle, respectively, of a path piece intersecting the boundary $\mathcal{P}_1 \cap \mathcal{P}_2$, as show in Figure 4. Let $\Delta\theta_1, \Delta\theta_2 \in \mathbb{R}$ be given by $\Delta\theta_1 := \theta^* - \theta_1$, $\Delta\theta_2 := \theta_2 - \theta^*$, where θ^* is the angle between the path*



(a) Nonoptimal $\mathcal{C}^+/\mathcal{C}^-$-type path at the boundary. Path piece $\mathcal{C}^+$ in patch $\mathcal{P}_1$ with radius $r_1 = v_1$ and path piece $\mathcal{C}^-$ in patch $\mathcal{P}_2$ with radius $r_2 = v_2$, $v_2 > v_1$.

(b) Nonoptimal $\mathcal{L}/\mathcal{C}^-$-type path at the boundary. Path piece $\mathcal{C}^-$ in patch $\mathcal{P}_2$ with radius $r_2 = v_2$.

Fig. 3. Nonoptimal paths at the boundary: paths of type $\mathcal{C}^+/\mathcal{C}^-$ and $\mathcal{L}/\mathcal{C}^-$ changing at the boundary and hence, not satisfying the necessary conditions for optimality in Theorem 1.

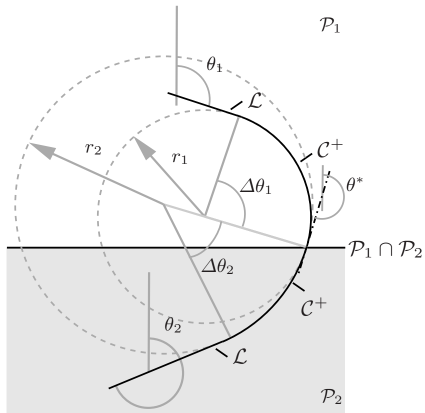
and the boundary $\mathcal{P}_1 \cap \mathcal{P}_2$ at their intersection (with respect to the vertical axis). If the path piece intersecting $\mathcal{P}_1 \cap \mathcal{P}_2$ is of type $\mathcal{C}$, then $v_1, v_2, \theta_1, \theta_2, \Delta\theta_1$ and $\Delta\theta_2$ satisfy

$$\frac{v_1}{v_2} = \frac{1 + \cot\theta_2 \cot\left(\frac{\Delta\theta_1 - \Delta\theta_2}{2} + \frac{\theta_1+\theta_2}{2}\right)}{1 + \cot\theta_1 \cot\left(\frac{\Delta\theta_1 - \Delta\theta_2}{2} + \frac{\theta_1+\theta_2}{2}\right)} , \tag{6}$$

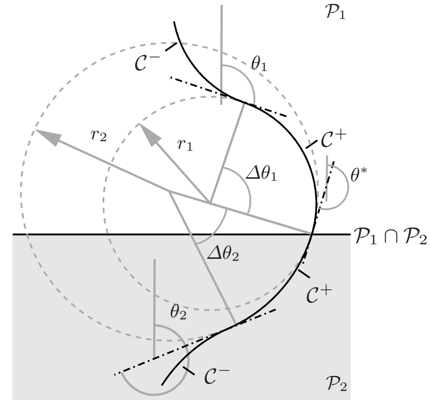
and if the path piece intersecting $\mathcal{P}_1 \cap \mathcal{P}_2$ is of type $\mathcal{L}$, then θ_1 and θ_2 are equal to π.

Remark 4. Equation (6) in Theorem 2 implies that for a path of type $\mathcal{C}$ intersecting $\mathcal{P}_1 \cap \mathcal{P}_2$ to be optimal, $\theta_1, \theta_2, \Delta\theta_1$ and $\Delta\theta_2$ shown in Figure 4 must satisfy (6). When the path intersecting $\mathcal{P}_1 \cap \mathcal{P}_2$ is of type $\mathcal{L}$, by Corollary 1, the path $\mathcal{L}$ is orthogonal to $\mathcal{P}_1 \cap \mathcal{P}_2$ and consequently, there is no "refraction" at the boundary. This is depicted in Figure 2(b). The proof of Theorem 2 follows from the properties of the optimal solution and adjoint state at jumps stated in Theorem 1 and Proposition 1.d. ■

Equation (6) can be interpreted as a refraction law at the boundary of the two patches for the angles (and their variations) θ_1, θ_2 (and $\Delta\theta_1, \Delta\theta_2$). This parallels Snell's law of refraction in optics, which states a relationship between the angles of rays of light when passing through the boundary of two isotropic media with different refraction coefficients. More precisely, given two media with



(a) Refraction for $\mathcal{LCL}$-type of path nearby the boundary. The $\mathcal{L}$ path pieces define the angles θ_1, θ_2 and their variations $\Delta\theta_1, \Delta\theta_2$.



(b) Refraction for $\mathcal{CCC}$-type of path nearby the boundary. The tangents (plotted with .– lines) at the point of path change define the angles θ_1, θ_2 and their variations $\Delta\theta_1, \Delta\theta_2$.

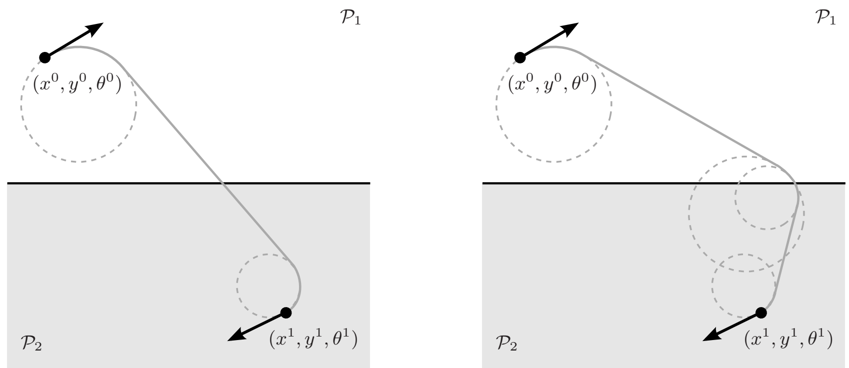
Fig. 4. Refraction law for paths at the boundary. The initial and final angles of optimal paths intersecting the boundary given by θ_1 and θ_2, respectively, and their variations $(\Delta\theta_1, \Delta\theta_2)$ satisfy equation (6), which is a generalization of Snell's law of refraction.

different refraction indexes v_1 and v_2, Snell's law of refraction states that

$$\frac{v_1}{v_2} = \frac{\sin\theta_1}{\sin\theta_2}\ , \tag{7}$$

where θ_1 is the angle of incidence and θ_2 is the angle of refraction. This law can be derived by solving a minimum-time problem between two points, one in each medium. Moreover, the dynamics of the rays of light can be associated to the differential equations $\dot{x} = v_i$, where v_i is the velocity in the i-th medium, $i = 1, 2$. Theorem 2 generalizes Snell's law to the case when the dynamics of the rays of light are given by (1). In fact, (6) reduces to (7) when $\Delta\theta_1 = \theta_1$ and $\Delta\theta_2 = \theta_2$. In the context of autonomous vehicles, (6) consists of a generalization of the refraction law for optimal steering of a point-mass vehicle, as in [6, 7], to the Dubins vehicle case.

To further illustrate our results, consider $v_1 = 2v_2 > 0$, (x^0, y^0, θ^0), and (x^1, y^1, θ^1) as depicted in Figure 5. A path corresponding to a solution to $\mathcal{H}_v$ matching the initial and terminal constraints is shown in Figure 5(a). Since the $\mathcal{L}$-type path piece smoothly connecting the $\mathcal{C}$-type paths at (x^0, y^0, θ^0) and (x^1, y^1, θ^1) does not intersect the boundary $\mathcal{P}_1 \cap \mathcal{P}_2$ orthogonally, Theorem 1.d implies that it is nonoptimal (see also Corollary 1). Note that this path is not taking advantage of the fact that in patch $\mathcal{P}_1$, the vehicle can travel twice faster



(a) (Nonoptimal) path of type $\mathcal{CLC}$.

(b) Path of type $\mathcal{CLCLC}$.

Fig. 5. Optimal control of Dubins vehicle on patches with velocities $v_1 = 2v_2$. The path depicted in (a) is nonoptimal since its $\mathcal{L}$-type piece is not orthogonal to the boundary $\mathcal{P}_1 \cap \mathcal{P}_2$ (it is also nonoptimal since it does not exploit the fact that the maximum velocity in patch $\mathcal{P}_1$ is twice faster than in patch $\mathcal{P}_2$). The path depicted in (b) is a candidate for optimality as it satisfies the conditions in Theorem 1 and Corollary 1.

than in patch $\mathcal{P}_2$. Paths candidate for being optimal are like the one depicted in Figure 5(b) as it satisfies the conditions in Theorem 1 and Corollary 1.

5 Conclusions

We have derived necessary conditions for the optimality of paths with bounded maximum curvature. To establish our results, we formulated the problem as a hybrid optimal control problem and used optimality principles from the literature. Our results provide verifiable conditions for optimality of paths. These include conditions both in the interior of the patches and at their common boundary, as well as a refraction law for the angles which generalizes Snell's law of refraction in optics to the current setting. Applications of our results include optimal motion planning tasks for autonomous vehicles with Dubins vehicle dynamics.

6 Acknowledgments

This research has been partially supported by ARO through grant W911NF-07-1-0499, and by NSF through grant 0715025. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the supporting organizations.

References

1. Dubins, L.E.: On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. American Journal of Mathematics **79** (1957) 497–516
2. Pontryagin, L.S., Boltyanskij, V.G., Gamkrelidze, R.V., Mishchenko, E.F.: The mathematical theory of optimal processes. Wiley (1962)
3. Boissonnat, J.D., Cérézo, A., Leblond, J.: Shortest paths of bounded curvature in the plane. Journal of Intelligent and Robotic Systems **11** (1994) 5–20
4. Reeds, J.A., Shepp, L.A.: Optimal paths for a car that goes both forwards and backwards. Pacific Journal of Mathematics **145** (1990) 367–393
5. Sussmann, H.J., Tang, G.: Shortest paths for the Reeds-Shepp car a worked out example of the use of geometric techniques in nonlinear optimal control. Technical report, Rutgers Center for Systems and Control Technical Report (1991)
6. Alexander, R.S., Rowe, N.C.: Path planning by optimal-path-map construction for homogeneous-cost two-dimensional regions. Proc. IEEE International Conference on Robotics and Automation (1990) 1924–1929
7. Rowe, N.C., Alexander, R.S.: Finding optimal-path maps for path planning across weighted regions. The International Journal of Robotics Research **19** (2000) 83–95
8. Shkel, A.M., Lumelsky, V.: Classification of the Dubins' set. Robotics and Autonomous Systems **34** (2001) 179–202
9. Balkcom, D.J., Mason, M.T.: Time optimal trajectories for bounded velocity differential drive vehicles. The International J. Robotics Research **21** (2002) 199–217
10. Chitsaz, H., LaValle, S.M., Balkcom, D.J., Mason, M.T.: Minimum wheel-rotation paths for differential-drive mobile robots. Proceedings IEEE International Conference on Robotics and Automation (2006)
11. Sussmann, H.J.: Some recent results on the maximum principle of optimal control theory. Systems and Control in the Twenty-First Century (1997) 351–372
12. Sussmann, H.J.: A maximum principle for hybrid optimal control problems. In: Proc. 38th IEEE Conference on Decision and Control. (1999) 425–430
13. Garavello, M., Piccoli, B.: Hybrid necessary principle. SIAM J. Control Optim. **43**(5) (2005) 1867–1887
14. Shaikh, M.S., Caines, P.E.: On the hybrid optimal control problem: Theory and algorithms. IEEE Transactions on Automatic Control **52** (2007) 1587–1603
15. Piccoli, B.: Necessary conditions for hybrid optimization. In: Proc. 38th IEEE Conference on Decision and Control. (1999) 410–415
16. D'Apice, C., Garavello, M., Manzo, R., Piccoli, B.: Hybrid optimal control: case study of a car with gears. International Journal of Control **76** (2003) 1272–1284
17. Goebel, R., Hespanha, J., Teel, A., Cai, C., Sanfelice, R.: Hybrid systems: generalized solutions and robust stability. In: Proc. 6th IFAC Symposium in Nonlinear Control Systems. (2004) 1–12
18. Goebel, R., Teel, A.: Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. Automatica **42**(4) (2006) 573–587
19. Sanfelice, R., Goebel, R., Teel, A.: Generalized solutions to hybrid dynamical systems. To appear in ESAIM: Control, Optimisation and Calculus of Variations (2008)
20. Sussmann, H.J.: A nonsmooth hybrid maximum principle. In: Stability and Stabilization of Nonlinear Systems, Lecture Notes in Control and Information Sciences. Springer (1999) 325–354