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Analysis and Design of Event-triggered Control Algorithms using Hybrid Systems Tools

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Summary

This paper proposes a general framework for analyzing continuous-time systems controlled by event-triggered algorithms. Closed-loop systems resulting from using both static and dynamic output (or state) feedback laws that are implemented via asynchronous event-triggered techniques are modeled as hybrid systems given in terms of hybrid inclusions. Using recently developed tools for robust stability, properties of the proposed models, including stability of compact sets, robustness, and Zeno behavior of solutions are addressed. The framework and results are illustrated in several event-triggered strategies available in the literature, and observations about their key dynamical properties are made.

KEYWORDS:
Event-triggered controls, Hybrid systems, Stability, Robustness

1 | INTRODUCTION

1.1 | Motivation

Classical control theory is rich in mathematical tools for analysis of continuous-time systems modeled as differential equations with inputs as well for the design of continuous-time controllers. With the advent of digital computing, communication, and control, similar tools emerged for analysis and design for discrete-time systems modeled by difference equations. In such systems, at each sampling event, information is measured and the input of the system being controlled is updated. An emulation based approach to the design of controllers for discrete-time systems consists in the design of a continuous controller and subsequent discretization for implementation on a digital processor. For the resulting discrete-time controller to achieve properties that are similar to those of the designed continuous-time controller, a fairly small sampling period is typically required [1]. Small sampling periods – equivalently, high sampling frequencies – are particularly undesired in Networked Control Systems (NCSs) since they may cause network congestion and, consequently, delays in signal transmission. To address these problems, multiple solutions have been proposed in the literature, including the study of multirate systems [2] and aperiodic sampling [3]. The former describes the behavior of NCSs equipped with sensors that have different sampling frequencies; while the latter is mostly concerned with the stability of control systems that are subject to variations in the sampling time. These are rather passive approaches to the network congestion problems facing NCSs. It is well accepted in the community that to actively reduce data transmission in a network, one should employ strategies that allow sensors and actuators to only transmit data over the network when needed. This is the key idea in event-triggered control.

Event-triggered control encompasses a class of controllers in which sampling occurs when certain pre-specified criteria are met. To provide an improvement over the classical discrete-time approach, as indicated in [4], which is one of the earlier articles introducing the event-triggered control paradigm it is mandatory for the event-triggered controller to trigger events only when needed, as infrequent as possible – certainly, Zeno behavior, namely, infinitely many events over a finite time window – should be avoided at all cost. Hence, when designing event-triggered mechanisms, it is essential to understand the properties of the plant and controllers for which guaranteeing a positive lower bound to the time between any two consecutive events, namely, the closed-loop systems are Zeno-free. This property of inter-event times in event-triggered controllers pervades most of the literature on the subject, but we single out the work [5] and in [6] which provides a thorough analysis of the issue. Below we provide an overview of the most widely known event-triggered controller designs. For further information on inter-event times, we refer the reader to [7] and [8].

1.2 | Background

One of the precursory event triggered strategies is referred to as "send-on-delta." It was introduced in [9] as a solution to reduce data transmission over a sensor network. The "send-on-delta" approach samples and transmits data over the network when the difference between the current value of a particular sensor exceeds the value of the last sample by an amount Δ. The analysis in [9] is limited to the efficiency of data transmission.

Another popular event-triggered controller follows the emulation-based approach: design the event-triggered law for a continuous-time plant using an already designed stabilizing control law. In such an approach, the event-triggered mechanism ought to maintain asymptotic stability while only "transmitting" the control input to the plant at the instants when designed "events" happen. Stability analysis of the full closed-loop system with the event-triggered laws in the loop is performed using information of the nominal design, namely, without events. The work in [10] is a hallmark of event-triggered control that is emulation-based. In particular, the controller design in the aforementioned paper relies on the existence of a stabilizing feedback law κ for a continuous-time plant leading to the closed-loop system

$$\dot{x} = f(x, \kappa(x))$$

and a continuously-differentiable function V with the property

$$\langle \nabla V(x), f(x, \kappa(x+e)) \rangle \leq -\alpha(|x|) + \gamma(|e|) \qquad \forall x, e \in \mathbb{R}^n$$

for some continuous functions α, γ that are zero and zero and grow unboundedly with their argument – these are so-called $\mathcal{K}_\infty$ functions. The quantity e is the difference between the current state x and the value of the state x at the last event. Allowing samples to occur only when the event happens, namely, when $\gamma(|e|) \geq \sigma\alpha(|x|)$, the results in [10] establish global asymptotic stability of the origin for the closed-loop system and that solutions starting outside of the origin do not exhibit Zeno behavior. The work in [11] applies the same design principles of [10] to the particular case of linear time-invariant systems, while proving additional properties, including $\mathcal{L}_p$ stability. The same approach finds application on the problem of stabilizing a system over sensor/actuator networks, as shown in [12]. The work in [13] extends the static triggering condition of [14] to a dynamic triggering condition. Notably, in [14], a precise characterization of the inter-sampling time is provided.

With digital implementation of event-triggered control strategies in mind, a periodic implementation of event-triggered control emerged in [15,16,17]. Such implementations periodically monitor the events that trigger controller updates – this is a significant departure from earlier event-triggered control strategies, where the condition triggering the events is monitored continuously. As a result, such approach avoids Zeno solutions of the closed-loop systems, by design. A systematic method to transform a given event-triggered control system into a periodic implementation while preserving stability results using convex over approximation techniques is introduced in [16]. In [17], a design of such implementations relying on a consistent – in the quadratic sense – threshold-based cost function is shown to outperform a traditional periodic control with similar average transmission rate.

The event-triggered strategy introduced in [18] is derived from the work in [19]. It uses a dynamic triggering event that relies on an internal timer and, even though it has stricter requirements than some of the strategies mentioned above, it is shown in [18] that certifies $\mathcal{L}_p$-stability while avoiding Zeno solutions. In [20], algebraic conditions are presented for the stabilization of affine control systems via event-triggered control.

Due to the impulsive nature of event-triggered control, it is natural to analyze and design such control laws using tools for hybrid dynamical systems. The hybrid inclusions introduced in [21] has been used in a few instances for the analysis and design of event-triggered control algorithms in the literature. In [22], a nonlinear continuous-time plant controlled by a Lyapunov-based event-triggered strategy with a single triggering event is modeled and analyzed in the hybrid inclusions framework. Conditions to render an appropriately defined set globally asymptotically stable are proposed therein. The work in [23] employs hybrid inclusions with single-valued flow and jump maps to model, analyze, and design event-triggered control strategies for nonlinear systems. The model therein is flexible enough to capture most strategies in the literature, as well as some proposed by the authors, with a single event-triggering mechanism. The main stability result in [23] is a Lyapunov-like theorem for (uniform) global (pre-)asymptotic stability using a Lyapunov function that strictly decreases along flows and that at jumps, it does not increase.

Conveniently, the work in[23] also provides conditions guaranteeing that the time in between the events is lower bounded for solutions that start away from the origin of the closed-loop system.

Nonlinear control theory provides an almost endless supply of techniques to tackle the challenges posed by nonlinear systems. Event-triggered control adds another "degree of freedom" to the problem because, for each application and each particular controller design, we can devise a plethora of sampling laws. This motivates the widespread interest in event-triggered control research across multiple fields. There have been multiple attempts at unifying event-triggered control (ETC) problem within a general framework (such as the one in[23]) in order to provide a better understanding of their potentials and of the limitations. Undoubtedly, event-triggered control has been found suitable for a number of problems and applications, including the stabilization of affine control systems[24,20]; attitude control[25], quadrotor stabilization[26], model predictive control[27,28], global stabilization of a chain of integrators[29], state estimation[30] and stabilization of strict feedback systems[31]. More importantly, event-triggered control has proved to be most effective for NCSs and adaptive control problems, as evidenced by the multiple contributions in these fields, see, e.g.,[32,33,34,35,36,37,38,39,40,41].

1.3 | Contributions

In this paper, we formulate a general hybrid system model for event-triggered control of continuous-time systems within the hybrid inclusions framework developed in[21]. Such a formulation allows for multiple asynchronous events, explicit perturbations and disturbances, and several tools for analysis and design of the control strategies. In particular, the differential inclusions with constraints are used to describe the continuous dynamics of the plant and of the controller, while difference inclusions with constraints are used to model the discrete changes in the state caused by the triggered events. The main contributions made in this paper include the following:

1. Asynchronous sampling and control update events: the proposed hybrid inclusions model not only allows for static and dynamic output (or state) feedback (potentially set valued) but also allows for two kinds of events in the hybrid closed-loop system: one corresponding to updates of the input to the plant and another corresponding to updates of the measurements of its output. In particular, it allows for local events triggered by part of the state components (hence, covering the setting in[42,43]), which may involve memory states storing the most recent controller and output values. Compared to[23], the proposed model further encompasses multiple event-functions, which, in particular, covers the strategy in[11].

2. Stability analysis using Lyapunov functions that may increase during flows or jumps: the main result for stability analysis in this paper (which is in Theorem 1) fully displays the advantages of using the hybrid inclusions framework in[21]. In particular, this result allows to certify (pre-) asymptotic stability of a closed set with a Lyapunov function that may not decrease both during the continuous evolution and at the events in the hybrid closed-loop system. Unlike the vast majority of previous results in the literature, in particular,[14,11,7,23], this result leads to a novel event-triggered control strategy that exploits the following balancing property: make the Lyapunov function decrease enough at the events so as to compensate for its growth during continuous evolution. Such balancing has great potential in the design of new event-triggered strategies as the continuous regime is where the latest update on the variables starts to lose effectiveness. This new strategy is illustrated in Example 7. In addition, by exploiting regularity properties of the closed-loop system (namely, well-posedness), our main stability analysis result allows us to employ an invariance principle to characterize where solutions converge to when the change of the Lyapunov function along flows or jumps (or both) is weak – namely, it is nonincreasing.

3. Exploit well-posedness to establish uniform lower bounds on inter-event times and robustness, simultaneously: as depicted in Section 1.1, it is typically desired that the event-triggering control algorithm leads to a time in between consecutive events – typically called the inter-event time – that is (uniformly) lower bounded by a positive constant. Though some strategies are known to guarantee such a lower bound from initial conditions away from the origin (or from the set to stabilize), following[44], we remark in an example that, in general, such a property is not robust to arbitrarily small perturbations. This observation is show-cased in Example 12. Motivated by this fact and the desire of guaranteeing a positive uniform inter-event time and robustness simultaneously, we present necessary and sufficient conditions for event-triggered control algorithms that lead to well-posed hybrid closed-loop systems. In addition, results in[21] are exploited to formulate a robust stability result to both generic perturbations as well as perturbations on the condition triggering the events.

In addition, since sequential compactness of solutions is required to be able to apply invariance principles as well as to assure robustness in hybrid systems, our results reveal the properties that the objects defining the event-triggering algorithm should satisfy for sequential compactness to hold. The paper also includes a result that formalizes the properties of the system resulting from performing a temporal regularization of the hybrid closed-loop system, which is useful when Zeno solutions from the origin (or from the set to stabilize) cannot be removed at the design stage. A result on existence and completeness of maximal solutions involving the functions triggering the events is also presented – recall that as a difference to continuous-time systems, existence and uniqueness of solutions to hybrid systems is not guaranteed by Lyapunov inequalities. Furthermore, the proposed model is illustrated by a variety of event-triggered algorithms available in the literature.

1.4 | Organization of the Paper

The remainder of the paper is organized as follows.[1] The general formulation using the hybrid inclusions framework is proposed in Section 2 for closed-loop systems resulting from event-triggered control. Moreover, Section 3 presents the main properties: asymptotic stability (Section 3.1), robustness (Section 3.2), completeness of solutions (Section 3.3), and properties of inter-event times (Section 3.4). Examples are provided throughout the paper to illustrate the results.

Notation:

- $\mathbb{R}_{\geq 0}$ denotes the set $[0,\infty)$; while $\mathbb{R}_{\leq 0}$ denotes the set $(-\infty, 0]$;
- Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, we denote the range of M as $\operatorname{rge} M = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ s.t. } y \in M(x)\}$, the domain of M as $\operatorname{dom} M = \{x \in \mathbb{R}^m : M(x) \neq \emptyset\}$, and the graph of M as $\operatorname{gph} M = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : y \in M(x)\}$.
- The closed unit ball around the origin in $\mathbb{R}^n$ is denoted as $\mathbb{B}$.
- Given a vector x, $|x|$ denotes the 2-norm of x.
- Given $r \in \mathbb{R}$, the r-sublevel set of a function $V : \mathbb{R}^n \to \mathbb{R}$ is $L_V(r) := \{x \in \mathbb{R}^n : V(x) \leq r\}$, and $V^{-1}(r) = \{x \in \mathbb{R}^n : V(x) = r\}$ denotes the r-level set of V.
- We use $|x|_K$ to denote the distance from a point x to a closed set K, i.e., $|x|_K = \inf_{\xi \in K} |x - \xi|$.
- The set of boundary points of a closed set K is denoted by ∂K.
- Given a closed set K, we denote the tangent cone of the set K at a point $x \in K$ as $T_K(x)$.
- Given a map $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a set $K \subset \mathbb{R}^m$, the set $M(K) := \{M(x) : x \in K\} \subset \mathbb{R}^n$ denotes the evaluation of M on K.
- Given sets $S \subset \mathbb{R}^n \times \mathcal{U}$ and $K \subset \mathbb{R}^n$, the projection of S onto $\mathbb{R}^n$ is defined as
$$\Pi(S) := \{x \in \mathbb{R}^n : \exists u \in \mathcal{U} \text{ s.t. } (x,u) \in S\}$$
and the extension of K to $\mathbb{R}^n \times \mathcal{U}$ is defined as
$$\Upsilon(K) := \{(x,u) \in S : x \in K\}.$$
- With $\star \in \{c, d\}$, given sets $S \subset \mathbb{R}^n \times \mathcal{U}_\star$, the set of all possible values $u_\star$ at a point x such that $(x, u_\star) \in S$ is defined as
$$\widetilde{\Psi}_\star(x,S) := \{u_\star \in \mathcal{U}_\star : (x,u_\star) \in S\}.$$
- Given sets $C_u \subset \mathbb{R}^n \times \mathcal{U}_c$ and $D_u \subset \mathbb{R}^n \times \mathcal{U}_d$, the set-valued maps $\Psi_c : \mathbb{R}^n \rightrightarrows \mathcal{U}_c$ and $\Psi_d : \mathbb{R}^n \rightrightarrows \mathcal{U}_d$ are defined for each $x \in \mathbb{R}^n$ as $\Psi_c(x) := \widetilde{\Psi}_c(x, C_u)$ and $\Psi_d(x) := \widetilde{\Psi}_d(x, D_u)$, respectively.
- $V^\circ(z; f)$ denotes Clarke's generalized derivative of V at z in the direction f (see[46]).

[1] Preliminary versions of the results in this paper appeared without proof in the conference article[45]. In additional to the detailed proof steps, when compared to[45], for each section, examples and in-depth explanation of the proposed conditions and results are provided in this paper.

2 | GENERAL FORMULATION OF EVENT-TRIGGERED CONTROL FOR CONTINUOUS-TIME SYSTEMS

In this paper, using hybrid inclusions introduced in [21], we model the closed-loop system obtained from a continuous-time plant controlled by a dynamic controller implemented via event-triggered mechanisms (ETMs). The plant has state $x_p \in \mathcal{X}_p \subset \mathbb{R}^{n_p}$, input $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$, output $y \in \mathcal{Y} \subset \mathbb{R}^{n_y}$, and is given by

$$\begin{aligned} \dot{x}_p &\in F_p(x_p, u), \\ y &\in H_p(x_p). \end{aligned} \tag{1}$$

The dynamic controller has state $x_c \in \mathcal{X}_c \subset \mathbb{R}^{n_c}$ and is given by

$$\begin{aligned} \dot{x}_c &\in F_c(x_c, y), \\ u &\in H_c(x_c, y). \end{aligned} \tag{2}$$

The set-valued maps $F_p : \mathcal{X}_p \times \mathcal{U} \rightrightarrows \mathcal{X}_p$ and $F_c : \mathcal{X}_c \times \mathcal{Y} \rightrightarrows \mathcal{X}_c$ describe the continuous dynamics for the plant and the controller, respectively, while the set-valued maps $H_p : \mathcal{X}_p \rightrightarrows \mathcal{Y}$ and $H_c : \mathcal{X}_c \times \mathcal{Y} \rightrightarrows \mathcal{U}$ assign values for u and y, respectively. For a static (but set-valued) feedback, the controller in (2) reduces to $u \in H_c(y)$. Defining the state $x = (x_p, x_c)$, the closed-loop system (without ETM in the loop) is given by the differential inclusion

$$\dot{x} = \begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} \in \left\{ \zeta \in \begin{bmatrix} F_p(x_p, u) \\ F_c(x_c, y) \end{bmatrix} : \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} H_p(x_p) \\ H_c(x_c, y) \end{bmatrix} \right\}. \tag{3}$$

Next, to prepare for the upcoming results and examples of closed-loop systems with ETMs, we present two existing ETMs in the literature with their plants and controllers modeled as in (1) and (2), respectively. As a result, the closed-loop system without ETMs applied can be expressed in form of (3). These examples are revisited in later sections with their proposed ETMs modeled in the hybrid inclusions framework.

Example 1. (ETM for asymptotic stability via an ISS Lyapunov function and static state-feedback in [14]) For a real-time scheduling problem, [14] develops an ETM for the continuous-time system

$$\dot{x}_p = F_p(x_p, u) \tag{4}$$

that is controlled by the static state-feedback controller

$$u = H_c(x_p). \tag{5}$$

The controller is assumed to render the closed-loop system

$$\dot{x}_p = F_p(x_p, H_c(x_p + e))$$

Input-to-State Stable (ISS) with respect to e. Namely, it is assumed that there exists a continuously differentiable function $\widetilde{V} : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$ and functions $\overline{\alpha}, \underline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x_p|) \leq \widetilde{V}(x_p) \leq \overline{\alpha}(|x_p|) \tag{6a}$$

$$\langle \nabla \widetilde{V}(x_p), F_p(x_p, H_c(x_p + e)) \rangle \leq -\alpha(|x_p|) + \gamma(|e|) \tag{6b}$$

for each $(x_p, e) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$. In later sections, we propose a event-triggered system framework that is applicable to ETM controlled system just like this one. △

Considering the same nonlinear continuous-time plant as in (4), [22] studies a different event-triggered controller design that guarantees Lyapunov stability of the closed-loop system. We present the original formulation as in [22] and a perturbed version of such systems in the next example.

Example 2. (ETM in [22] for static state-feedback) In [22], a memory state, denoted by $\hat{u}$ in our notation, is introduced to "memorize" the previous value of the control law. Given a static state-feedback law in the form of (5), the closed-loop system is given by

$$\dot{x}_p = F_p(x_p, \hat{u}),$$

where $\hat{u}$ is the "held value" if the control law as defined in [22].

Next, we consider the same system and control strategy in [22] but with state perturbations, so as to motivate the usefulness of the proposed differential inclusions model in this paper. The perturbed system of the nonlinear continuous-time plant in (4) is given by

$$\dot{x}_p \in F_p^m(x_p, u) := F_p(x_p, u) + \sigma(x_p)\mathbb{B}, \tag{7}$$

where $\mathbb{B}$ is the unit ball centered at the origin and $\sigma : \mathcal{X}_p \to \mathbb{R}_{\geq 0}$ represents the worst case unmodeled dynamics for the plant. △

Now, we introduce a model for the closed-loop system of the plant in (1) controlled by (2) implemented via event-triggered strategies. When ETMs are in the loop, the closed-loop system has the structure shown in Figure 1. Similar to sample-and-hold control systems, the plant and the controller operate with sampled versions of the output y and of the input u, denoted $\hat{y} \in \mathcal{Y}$ and $\hat{u} \in \mathcal{U}$, respectively. The ETM strategies surpass the traditional periodic sampling ones in the sense that $\hat{y}$ and $\hat{u}$ are updated only when necessary, namely, at *triggering events*. Such events usually describe, but are not limited to, the situation where desired control objectives, such as stability and attractivity, are about to be violated. Consequently, the ETMs optimize the use of resources in data acquisition, communication, among others.

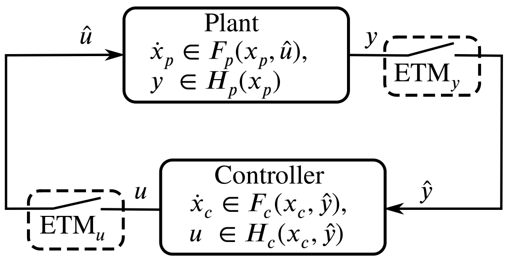

FIGURE 1 Closed-loop system with ETM in the loop.

In addition to the memory variables $\hat{u}$ and $\hat{y}$, our model also incorporates extra variables to fully capture existing strategies in the field. In some ETMs in the literature, e.g.[23,47]., an auxiliary state $\chi \in X \subset \mathbb{R}^{n_\chi}$ is used to capture possible dynamics added in the ETMs. Such a state χ may not be involved in the plant or the controller, but rather play a significant role in triggering events. At corresponding triggering events, the most recent values of the output y from the plant and the input u are assigned to $\hat{y}$ and $\hat{u}$, respectively. Moreover, χ is updated via the general difference inclusion

$$\chi^+ \in G_\chi(x_p, x_c, \hat{y}, \hat{u}, \chi). \tag{8}$$

In between two events, x_p and x_c evolve according to F_p and F_c, respectively, while $\hat{y}$ and $\hat{u}$ are governed by the following general differential inclusions:

$$\begin{aligned} \dot{\hat{y}} &\in \hat{F}_y(x_p, x_c, \hat{y}, \hat{u}, \chi) \\ \dot{\hat{u}} &\in \hat{F}_u(x_p, x_c, \hat{y}, \hat{u}, \chi), \end{aligned}$$

where $\hat{F}_y$ and $\hat{F}_u$ are referred to as *holding functions*. When simply "zero-order hold" is employed for $\hat{y}$ and $\hat{u}$, we have $\hat{F}_y \equiv 0$ and $\hat{F}_u \equiv 0$. Also in between two events, the auxiliary state χ has dynamics

$$\dot{\chi} \in F_\chi(x_p, x_c, \hat{y}, \hat{u}, \chi).$$

The hybrid system model in this paper allows for local triggering events (LTE). These events trigger updates of individual components of the output memory state $\hat{y}$ and the input memory state $\hat{u}$. To this end, the vectors y and $\hat{y}$ are partitioned into N_y subcomponents, while u and $\hat{u}$ are partitioned into N_u subcomponents, i.e.,

$$\begin{aligned} y &= (y_1, y_2, ..., y_{N_y}), & \hat{y} &= (\hat{y}_1, \hat{y}_2, ..., \hat{y}_{N_y}), \\ u &= (u_1, u_2, ..., u_{N_u}), & \hat{u} &= (\hat{u}_1, \hat{u}_2, ..., \hat{u}_{N_u}). \end{aligned}$$

With $i_y \in \{1, 2, ..., N_y\}$ and $i_u \in \{1, 2, ..., N_u\}$, we define triggering event functions updating y_{i_y} and u_{i_u} as, respectively, $\gamma^y_{i_y} : \Xi \to \mathbb{R}^{m_y}$ and $\gamma^u_{i_u} : \Xi \to \mathbb{R}^{m_u}$. The argument of these event functions is given as $\xi := (y, u, \hat{y}, \hat{u}, \chi) \in \Xi$ with $\Xi := \mathcal{Y} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{U} \times X$. When $\gamma^y_{i_y}(\xi) = 0$, namely, when the event performing the update of the i_y−th component of the output is triggered, only the i_y−th component of $\hat{y}$ is updated according to the local output, i.e.,

$$\hat{y}^+_{i_y} = y_{i_y} \text{ and } \hat{y}^+_k = \hat{y}_k \text{ for every } k \in \{1, 2, ..., N_y\}, k \neq i_y.$$

Similarly, when $\gamma^u_{i_u}(\xi) = 0$, namely, when the event performing the update of the i_u−th component of the input is triggered, only the i_u−th component of $\hat{u}$ is updated according to the local input, i.e.,

$$\hat{u}^+_{i_u} = u_{i_u} \text{ and } \hat{u}^+_k = \hat{u}_k \text{ for every } k \in \{1, 2, ..., N_u\}, k \neq i_u.$$

In our proposed hybrid system model, we assume that when $\gamma^y_{i_y}(\xi) \geq 0$ (or $\gamma^u_{i_u}(\xi) \geq 0$) the update of each corresponding component y_{i_y} (or u_{i_u}, respectively) is triggered.[2]

As promoted earlier in Section 1, the closed-loop system resulting from the mechanism described above is modeled as hybrid system in the hybrid inclusions framework introduced in[21]. To make this paper self-contained, we listed the key concepts and properties of such model in Appendix. For the remainder of this section, we formally define the variables, sets and maps of the closed-loop systems with the ETMs in the loop.

Defining the state

$$z = (x_p, x_c, \hat{y}, \hat{u}, \chi) \in \mathcal{Z} := \mathcal{X}_p \times \mathcal{X}_c \times \mathcal{Y} \times \mathcal{U} \times X, \tag{9}$$

the closed-loop hybrid system has a jump set given by

$$D := D_y \cup D_u, \tag{10}$$

where $D_y := \bigcup_{i_y=1}^{N_y} D^y_{i_y}$ and $D_u := \bigcup_{i_u=1}^{N_u} D^u_{i_u}$, with

$$D^y_{i_y} := \{z \in \mathcal{Z} : \gamma^y_{i_y}(\xi) \geq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\},$$
$$D^u_{i_u} := \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) \geq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}.$$

The flow set is given by

$$C := C_y \cap C_u, \tag{11}$$

where $C_y := \bigcap_{i_y=1}^{N_y} C^y_{i_y}$ and $C_u := \bigcap_{i_u=1}^{N_u} C^u_{i_u}$ with

$$C^y_{i_y} := \{z \in \mathcal{Z} : \gamma^y_{i_y}(\xi) \leq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\},$$
$$C^u_{i_u} := \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) \leq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}.$$

Hence, from (11), (10), and the construction of the maps H_p and H_c, we have $C \cup D = \mathcal{Z}$. Then, for each $z \in C$, the flow map is given by

$$F(z) := (F_p(x_p, \hat{u}), F_c(x_c, \hat{y}), \hat{F}_y(z), \hat{F}_u(z), F_\chi(z)) \tag{12}$$

The jump map captures the dynamics at events. The memory states $\hat{y}$ and $\hat{u}$ are updated via local reset functions. More precisely, for every $i_y \in \{1, 2, ..., N_y\}$ and $i_u \in \{1, 2, ..., N_u\}$, we define[3]

$$\begin{aligned} g^y_{i_y}(y, \hat{y}) &:= \begin{cases} (\hat{y}_1, ..., \hat{y}_{i_y-1}, y_{i_y}, \hat{y}_{i_y+1}, ..., \hat{y}_{N_y}) & \text{if } z \in D^y_{i_y} \\ \emptyset & \text{otherwise,} \end{cases} \\ g^u_{i_u}(u, \hat{u}) &:= \begin{cases} (\hat{u}_1, ..., \hat{u}_{i_u-1}, u_{i_u}, \hat{u}_{i_u+1}, ..., \hat{u}_{N_u}) & \text{if } z \in D^u_{i_u} \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned} \tag{13}$$

For every $i_y \in \{1, 2, ..., N_y\}$, when $z \in D^y_{i_y}$, the output memory state $\hat{y}$ is reset via $\hat{y}^+ = g^y_{i_y}(y, \hat{y})$, where only the i_y-th component is updated based on the current value of y, while the other $N_y - 1$ components remain the same as before the jump; when $z \notin D^y_{i_y}$, the map is empty. Similar rules apply to the jump dynamics of the input memory state $\hat{u}$. Hence, the union of these reset functions captures the LTE dynamics. At triggering events, the state x remains unaltered, the auxiliary state χ resets according to (8), and the components of states $\hat{y}$ and $\hat{u}$ are either kept the same or are updated according to (13). Then, following (A2), the

[2]Note that two independent sets of event functions are considered in this model to allow $\hat{y}$ and $\hat{u}$ to be updated via asynchronous events. This general setting reduces to the case of synchronized events by setting $\gamma^y_{i_y}(\xi) = \gamma^u_{i_u}(\xi)$, $N_y = N_u$ and $i_u = i_y$ for every $\xi \in \Xi$.

[3]The maps $g^y_{i_y}$ and $g^u_{i_u}$ are empty outside the respective jump sets to make the forthcoming maps G_y and G_u well-defined, in the sense that there is an update to the measurements and actuation only if the state belongs to the jump set.

jump map is given by

$$G(z) := G_y(z) \cup G_u(z) \tag{14}$$

for each $z \in D := D_u \cup D_y$, where

$$G_y(z) := \left\{ \begin{bmatrix} x \\ \bigcup_{i_y=1}^{N_y} g_{i_y}^y(y, \hat{y}) \\ \hat{u} \\ G_\chi(z) \end{bmatrix} : y \in H_p(x_p) \right\} \qquad \forall z \in D_y,$$

$$G_u(z) := \left\{ \begin{bmatrix} x \\ \hat{y} \\ \bigcup_{i_u=1}^{N_u} g_{i_u}^u(u, \hat{u}) \\ G_\chi(z) \end{bmatrix} : u \in H_c(x_c, \hat{y}) \right\} \qquad \forall z \in D_u.$$

with $x = (x_p, x_c)$. As a result, the full closed-loop system is given by the hybrid inclusions

$$\mathcal{H}_{ET} \begin{cases} \dot{z} \in F(z) & z \in C \\ z^+ \in G(z) & z \in D. \end{cases} \tag{16}$$

The solutions to $\mathcal{H}_{ET}$ as in (16) have continuous and/or discrete behavior depending on the system data (C, F, D, G). Following[21], besides the usual time variable $t \in \mathbb{R}_{\geq 0}$, we consider the number of jumps, $j \in \mathbb{N} := \{0, 1, 2, ...\}$, as an independent variable. Thus, hybrid time is parametrized by (t, j). The domain of a solution to $\mathcal{H}_{ET}$ is given by a hybrid time domain. A hybrid time domain is defined as a subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ can be written as $\cup_{j=0}^{J} \left([t_j, t_{j+1}], j\right)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$. A solution to the hybrid system $\mathcal{H}_{ET}$ is given by a hybrid arc ϕ satisfying the dynamics of the system data (C, F, D, G). A hybrid arc ϕ is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j := \{t : (t, j) \in \operatorname{dom} \phi\}$. For organization purposes, more preliminary definitions and properties used in this paper are limited in the Appendix.

To pave the way for the analysis results in Section 3, we provide conditions guaranteeing that $\mathcal{H}_{ET}$ in (16) to satisfies the hybrid basic conditions.

Lemma 1. The closed-loop hybrid system $\mathcal{H}_{ET}$ in (16), with its data given in (11), (12), (10) and (14), satisfies the hybrid basic conditions in Definition A.2 if

(L1') The set C and D given in (11) and (10) are closed relative to $\mathcal{Z}$;

(L2'.1) $F_p : \mathcal{X}_p \times \mathcal{U} \rightrightarrows \mathcal{X}_p$ is outer semicontinuous and locally bounded relative to the projection of C onto $\mathcal{X}_p \times \mathcal{U}$, and, for each $z \in C$, $F_p(x_p, u)$ is nonempty and convex;

(L2'.2) $F_c : \mathcal{X}_c \times \mathcal{Y} \rightrightarrows \mathcal{X}_c$ is outer semicontinuous and locally bounded relative to the projection of C onto $\mathcal{X}_c \times \mathcal{Y}$, and, for each $z \in C$, $F_c(x_c, y)$ is nonempty and convex;

(L2'.3) $\hat{F}_y : \mathcal{Z} \rightrightarrows \mathcal{Y}$ is outer semicontinuous and locally bounded relative to C, and, for each $z \in C$, $\hat{F}_y(z)$ is nonempty and convex;

(L2'.4) $\hat{F}_u : \mathcal{Z} \rightrightarrows \mathcal{U}$ is outer semicontinuous and locally bounded relative to C, and, for each $z \in C$, $\hat{F}_u(z)$ is nonempty and convex;

(L2'.5) $F_\chi : \mathcal{Z} \rightrightarrows X$ is outer semicontinuous and locally bounded relative to C, and, for each $z \in C$, $F_\chi(z)$ is nonempty and convex;

(L3') The maps H_p, H_c, G_u, G_y and G_χ are outer semicontinuous and locally bounded relative to the projection of C onto $\mathcal{X}_p$, the projection of C onto $\mathcal{X}_c \times \mathcal{Y}$, D_u, D_y, and D, respectively, and $G_\chi(z), G_u(z), G_y(z)$ are nonempty for all $z \in D$.

Remark 1. To satisfy some of the conditions in Lemma 1, we note the following observations. When $\mathcal{Z}$ is closed and the maps H_p and H_c are outer semicontinuous, the sets

$$\{z \in \mathcal{Z} : y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$$

are closed. Moreover, when for each $i_y \in \{1, 2, ..., N_y\}$ and $i_u \in \{1, 2, ..., N_u\}$, $\gamma^y_{i_y}$ and $\gamma^u_{i_u}$ are continuous, the sets

$$\{z : \gamma^u_{i_u}(\xi) \geq 0\} \text{ and } \{z : \gamma^y_{i_y}(\xi) \geq 0\}$$

are closed. Then, since the jump set defined in (10) is restricted by $\mathcal{Z}$, the set D is closed. Similar arguments hold for the flow set. Then, item (L1') in Lemma 1 holds when the following hold:

- The set $\mathcal{Z}$ is closed;
- The maps H_p and H_c are outer semicontinuous; and
- For each $i_y \in \{1, 2, ..., N_y\}$ and $i_u \in \{1, 2, ..., N_u\}$, $\gamma^y_{i_y}$ and $\gamma^u_{i_u}$ are continuous. □

Next, we present event-triggered controlled systems in the literature that fit in the framework $\mathcal{H}_{ET}$ with data (10)-(14).[4] Note that the model given in $\mathcal{H}_{ET}$ can be modified to the case where $\hat{y}$ and $\hat{u}$ are updated simultaneously when $z \in D$, and only one set of triggering event functions is considered. Such a case simply has $N_y = N_u$ and one set of event triggering functions, namely, $\gamma^y(\xi) = \gamma^u(\xi)$, as shown in the examples below.

Example 3. (ETM for output-feedback in[11]) An ETM is designed for a continuous-time LTI plant given by[5]

$$\begin{aligned}\dot{x}_p &= A_p x_p + B_p u\\ y &= C_p x_p,\end{aligned}$$

which is controlled by a dynamic controller given by

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c y\\ u &= C_c x_c + D_c y,\end{aligned}$$

where matrices $A_p, B_p, C_p, A_c, B_c, C_c, D_c$ have appropriate size. The ETM introduced in[11] leads to $N_y = N_u = 1$ and $\gamma^y(\xi) = \gamma^u(\xi) = \min\{|y - \hat{y}|^2 - \sigma_y|y|^2 - \varepsilon_y, |u - \hat{u}|^2 - \sigma_u|u|^2 - \varepsilon_u\}$ with $\xi = (y, u, \hat{y}, \hat{u})$, where $\sigma_y, \sigma_u, \varepsilon_y, \varepsilon_u$ are positive constants to be designed. With $z = (x_p, x_c, \hat{y}, \hat{u}) \in \mathcal{Z} := \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u}$, the closed-loop system is given by

$$\mathcal{H}_{ET}\begin{cases}\dot{z} = F(z) := \begin{bmatrix} A_p & 0 & 0 & B_p\\ 0 & A_c & B_c & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} z & z \in C,\\ z^+ = G(z) := (x_p, x_c, C_p x_p, C_c x_c + D_c C_p x_p) & z \in D,\end{cases}$$

where the flow set C and the jump set D are given as in (11) and (10), respectively.

Note that the formulation in (16) could be exploited to extend the ETM in[11] to the case of asynchronous events for input and output updates. More precisely, instead of updating the memory states $\hat{y}$ and $\hat{u}$ at the same time, two separate event functions can be defined as $\gamma^y(y, \hat{y}) = |y - \hat{y}|^2 - \sigma_y|y|^2 - \varepsilon_y$ and $\gamma^u(u, \hat{u}) = |u - \hat{u}|^2 - \sigma_u|u|^2 - \varepsilon_u$. Hence, whenever $z \in D^y$, only $\hat{y}$ is updated via $\hat{y}^+ = g^y(y, \hat{y}) := C_p x_p$; when $z \in D^u$, only $\hat{u}$ is updated via $\hat{u}^+ = g^u(x_c, \hat{y}) := C_c x_c + D_c \hat{y}$. △

The following examples illustrate $\mathcal{H}_{ET}$ for the state-feedback case with $y = x_p$ and $\hat{y} = \hat{x}_p$.

Example 4. (ETM for state-feedback) In[23], a framework is proposed to control nonlinear continuous-time plants

$$\dot{x}_p = F_p(x_p, u)$$

by dynamic state-feedback controllers of the form

$$\dot{x}_c = F_c(x_c, x_p), \quad u = H_c(x_c, x_p)$$

implemented via ETMs. The model in[23] leads to $N_y = N_u = 1$, state $z = (x_p, x_c, \hat{x}_p, \hat{u}, \chi) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_\chi}$, and a closed-loop system modeled as $\mathcal{H}_{ET}$ with data defined as follows. For each $z \in C$, flow map is given by

$$F(z) := (F_p(x_p, e_u + H_c(x_p, x_c)), F_c(x_c, e_{x_p} + x_p),\ \hat{F}_p(z), \hat{F}_c(z), F_\chi(z))$$

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[4]In the examples to follow, with some abuse of notation, we link the notation in the literature to ours to facilitate comparison.
[5]The unknown disturbances w in[11] are ignored. See Section 3.2 for robustness analysis.

and the jump map is given by

$$G(z) := (x_p, x_c, x_p, H_c(x_p, x_c), G_\chi(z)),$$

where $e_u = \hat{u} - u$ and $e_{x_p} = \hat{x}_p - x_p$. However, the flow set C and jump set D in [23] are only provided specifically for each of the ETMs in the five given examples, among which, all can be written in form (11) and (10); in particular, see Example 1 for the strategy in [23, Section V.C]. △

According to [23, Section V.C], the ETM developed for systems with static state-feedback laws in [14] can be adapted to the framework in [23]. Thus, it also fits $\mathcal{H}_{ET}$.

Example 5. (Revisit Example 1) The ETM proposed in [14] memorizes x_p from the most recent event; hence, $e = \hat{x}_p - x_p$. Such ETM leads to $\mathcal{H}_{ET}$ with $N_y = N_u = 1$ and the event-functions given as

$$\gamma^y(x_p, \hat{x}_p) = \gamma^u(x_p, \hat{x}_p) = \gamma(|\hat{x}_p - x_p|) - \sigma\alpha(|x_p|),$$

where $\sigma \in (0, 1)$. With $z = (x_p, \hat{x}_p) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$, the resulting system is given by the hybrid inclusion

$$\mathcal{H}_{ET} \begin{cases} \dot{z} = F(z) = (F_p(x_p, H_c(\hat{x}_p)), 0) & z \in C \\ z^+ = G(z) = (x_p, x_p) & z \in D. \end{cases} \tag{17}$$

where D and C are given as in (10) and (11), respectively. △

Moreover, our framework given in $\mathcal{H}_{ET}$ is also applicable for the strategy proposed in Example 2.

Example 6. (Revisit Example 2) An event-triggered strategy can be designed for (7) applying a Lyapunov-based strategy similar to the one in [22]. In this direction, we employ a memory state $\hat{u}$ to store the most recent control law, the closed-loop system is given by

$$\dot{x}_p \in F_p(x_p, \hat{u}).$$

The proposed ETM leads to $\mathcal{H}_{ET}$ with $N_y = N_u = 1$ and the event-functions given as

$$\gamma^y(x_p, \hat{u}) = \gamma^u(x_p, \hat{u}) = \langle \nabla V(x_p), F_p^m(x_p, \hat{u}) \rangle + \mu(|x_p|)$$

with $\mu \in \mathcal{K}_\infty$ and $V : \mathcal{X}_p \to \mathbb{R}$ continuously differentiable. Letting $z = (x_p, \hat{u}) \in \mathbb{R}^n \times \mathbb{R}^m$, the resulting closed-loop system is given by

$$\mathcal{H}_{ET} \begin{cases} \dot{z} \in F(z) := (F_p^m(x_p, \hat{u}), 0) & z \in C \\ z^+ = G(z) := (x_p, H_c(x_p)) & z \in D. \end{cases} \tag{18}$$

where D and C are given as in (10) and (11), respectively.

For the perturbed version given in (7), the resulting closed-loop system with the ETM adaptation is given by

$$\mathcal{H}_{ET}^m \begin{cases} \dot{z} \in F^m(z) := (F_p^m(x_p, \hat{u}), 0) & z \in C \\ z^+ = G(z) := (x_p, H_c(x_p)) & z \in D. \end{cases} \tag{19}$$

where D and C are given as in (10) and (11), respectively, with the event-triggered functions given by

$$\gamma^{ym}(x_p, \hat{u}) = \gamma^{um}(x_p, \hat{u}) = \sup_{\zeta \in F_p^m(x_p, \hat{u})} \langle \nabla V(x_p), \zeta \rangle + \mu(|x_p|).$$

In Section 3.1, we provide conditions on V and μ that guarantee the stability of a set for the closed-loop system (19). △

3 | PROPERTIES OF GENERAL FORMULATION

In this section, we study the properties of the closed-loop system $\mathcal{H}_{ET}$ with ETMs modeled as in (16). First, we present conditions to achieve asymptotic stability and convergence for $\mathcal{H}_{ET}$. In Section 3.2, a general robustness property of asymptotically stable compact sets is presented. After that in Section 3.3, conditions to guarantee completeness of all maximal solutions are proposed. Then, in Section 3.4 - 3.6, we address the problem of Zeno behavior of $\mathcal{H}_{ET}$.

3.1 | Stability and Convergence Analysis

The results in [21] for certifying asymptotic stability for general hybrid systems can be employed to design the ETMs in the closed-loop system $\mathcal{H}_{ET}$ in (16). In [21, Chapter 3] and [21, Chapter 7], (uniform) pre-asymptotic stability of a set is defined as the property that, in particular, solutions starting close to $\mathcal{A}$ stay close to it, and maximal solutions that are complete converge to it, uniformly in hybrid time over compact sets; see [21, Definition 3.6]. In [21, Chapter 7], an invariance principle to locate the ω-limit set of maximal and complete solutions is given. The following theorem conveniently summarizes these results.

Theorem 1. Let $\mathcal{Z}$ be given by (9), let $\mathcal{H}_{ET}$ be the hybrid system with data (C, F, D, G) given by (11), (12), (10), and (14), respectively, and let $\mathcal{A} \subset \mathcal{Z}$ be closed. Suppose that there exist a continuous function V that is locally Lipschitz on an open set containing C and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_{3,c} : C \to \mathbb{R}$, $\alpha_{3,d} : D \to \mathbb{R}$ such that

$$\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}}) \qquad \forall z \in \mathcal{Z} \tag{20a}$$

$$V^\circ(z; f) \leq \alpha_{3,c}(z) \qquad \forall z \in C, \forall f \in F(z) \cap T_C(z) \tag{20b}$$

$$V(g) - V(z) \leq \alpha_{3,d}(z) \qquad \forall z \in D,\ \forall g \in G(z). \tag{20c}$$

Then, the following hold:

(a) If $\alpha_{3,c}(z) = \lambda_c V(z)$ and $\alpha_{3,d}(z) = (\exp(\lambda_d) - 1)V(z)$ with $\lambda_c, \lambda_d \in \mathbb{R}$ and there exist $M, \gamma > 0$ such that, for each solution ϕ to $\mathcal{H}_{ET}$,

$$(t, j) \in \operatorname{dom}\phi \ \Rightarrow\ \lambda_c t + \lambda_d j \leq M - \gamma(t + j)$$

then $\mathcal{A}$ is globally (uniformly) pre-asymptotically stable for $\mathcal{H}_{ET}$;

(b) If $\mathcal{H}_{ET}$ satisfies the hybrid basic conditions, $\alpha_{3,c}$ is a negative definite function relative to $\mathcal{A}$ and $\alpha_{3,d}(z) \leq 0$ for each $z \in D$, then the set $\mathcal{A}$ is stable for $\mathcal{H}_{ET}$ and each precompact solution to $\mathcal{H}_{ET}$ approaches[6] the largest weakly invariant subset[7] of

$$V^{-1}(r) \cap ((\mathcal{A} \cap C) \cup (\alpha_{3,d}^{-1}(0) \cap G(\alpha_{3,d}^{-1}(0)))) \tag{21}$$

for some $r \in V(\mathcal{Z})$;

(c) If $\mathcal{H}_{ET}$ satisfies the hybrid basic conditions, $\alpha_{3,d}$ is a negative definite function relative to $\mathcal{A}$ and $\alpha_{3,c}(z) \leq 0$ for each $z \in C$, then the set $\mathcal{A}$ is stable for $\mathcal{H}_{ET}$ and each precompact solution to $\mathcal{H}_{ET}$ approaches the largest weakly invariant subset of

$$V^{-1}(r) \cap (\overline{\alpha_{3,c}^{-1}(0)} \cup (\mathcal{A} \cap D \cap G(\mathcal{A} \cap D))) \tag{22}$$

for some $r \in V(\mathcal{Z})$;

(d) If $\mathcal{H}_{ET}$ satisfies the hybrid basic conditions, $\alpha_{3,d}(z) \leq 0$ for each $z \in D$ and $\alpha_{3,c}(z) \leq 0$ for each $z \in C$, then the set $\mathcal{A}$ is stable for $\mathcal{H}_{ET}$ and each precompact solution to $\mathcal{H}_{ET}$ approaches the largest weakly invariant subset of

$$V^{-1}(r) \cap (\overline{\alpha_{3,c}^{-1}(0)} \cup (\alpha_{3,d}^{-1}(0) \cap G(\alpha_{3,d}^{-1}(0)))) \tag{23}$$

for some $r \in V(\mathcal{Z})$. Additionally, if for each $\xi \in \mathcal{Z}$ with $r := V(\xi) > 0$, there is no complete solution ϕ to $\mathcal{H}_{ET}$ satisfying $\phi(0, 0) = \xi$ and

$$\operatorname{rge}\phi \subset \{z \in \mathcal{Z} : V(z) = r\}, \tag{24}$$

then $\mathcal{A}$ is globally pre-asymptotically stable for $\mathcal{H}_{ET}$.

Proof. For each solution ϕ to $\mathcal{H}_{ET}$ and each $(t, j) \in \operatorname{dom}\phi$, we have

$$\operatorname{dom}\phi \cap ([0, t] \times \{0, 1, \dots, j\}) = \bigcup_{i=0}^{j} \left([t_i, t_{i+1}] \times \{i\}\right),$$

for some sequence $0 = t_0 \leq t_1 \leq \dots \leq t_{j+1} = t$. Thus, for each $i \in \{0, 1, \dots, j\}$, the following holds: $\phi(s, i) \in C$ for all $s \in (t_i, t_{i+1})$. It follows from the assumption that since V is locally Lipschitz on an open set containing C, its Clarke's generalized derivative exists for all $z \in \operatorname{dom} V \cap C$ and

$$\frac{d}{ds}V(\phi(s, i)) \leq V^\circ\left(\phi(s, i); \frac{d}{ds}\phi(s, i)\right) \tag{25}$$

[6] See Definition A.4.
[7] See Definition A.5.

for almost all $s \in [t_i, t_{i+1}]$ (c.f.[46]). From the definition of a solution in Section 2, we have that

$$\frac{d}{ds}\phi(s,i) \in F(\phi(s,i))$$

for almost all $s \in [t_i, t_{i+1}]$. Moreover, for all $s \in (t_i, t_{i+1})$ for which

$$\frac{d}{ds}\phi(s,i)$$

is defined, there exists a sequence $\tau_j > 0$ satisfying $\tau_j \searrow 0$ such that $\phi(s+\tau_j, i) \in C$ for all $j \in \mathbb{N}$. Hence, it follows from Definition A.6 that

$$\frac{d}{ds}\phi(s,i) = \lim_{j\to\infty} \frac{\phi(s+\tau_j, i) - \phi(s,i)}{\tau_j}$$

belongs to the tangent cone to C evaluated at $\phi(s,i)$. From (20b) and (25), we have that

$$\frac{d}{ds}V(\phi(s,i)) \leq \alpha_{3,c}(\phi(s,i))$$

for almost all $s \in [t_i, t_{i+1}]$. Integrating both sides, it follows from (20c) that

$$V(\phi(t,j)) \leq V(\phi(0,0)) + \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \alpha_{3,c}(\phi(s,i))ds + \sum_{i=1}^{j} \alpha_{3,d}(\phi(t_i, i-1)) \tag{26}$$

for each $(t,j) \in \operatorname{dom}\phi$. Next, we prove each of the claims of the theorem.

If the conditions in (a) are satisfied, it follows from (20b), (20c), and (26) that

$$V(\phi(t,j)) \leq \exp(\lambda_c t + \lambda_d j)V(\phi(0,0))$$

for each $(t,j) \in \operatorname{dom}\phi$. Furthermore, since $(t,j) \in \operatorname{dom}\phi \implies \lambda_c t + \lambda_d j \leq M - \gamma(t+j)$ for some positive constants M and γ, then

$$V(\phi(t,j)) \leq \exp(M - \gamma(t+j))V(\phi(0,0)) \tag{27}$$

for each $(t,j) \in \operatorname{dom}\phi$. Uniform global pre-asymptotic stability of $\mathcal{A}$ follows from the previous arguments and from assumption (20a) using the following arguments from[21, Theorem 3.29]. It follows from (20a) and (27) that

$$\alpha_1(|\phi(t,j|_{\mathcal{A}}) \leq V(\phi(t,j) \leq \exp(M-\gamma(t+j))V(\phi(0,0)) \leq \exp(M)\alpha_2(|\phi(0,0|_{\mathcal{A}})$$

for each $(t,j) \in \operatorname{dom}\phi$ and, consequently, $|\phi(t,j)|_{\mathcal{A}} \leq \alpha_1^{-1}(\exp(M)\alpha_2(|\phi(0,0)|_{\mathcal{A}}))$. Since $s \mapsto \alpha_1^{-1}(\exp(M)\alpha_2(s))$ is a class-$\mathcal{K}_\infty$ function and because the bound holds for each maximal solution ϕ to $\mathcal{H}_{ET}$ and every $(t,j) \in \operatorname{dom}\phi$, uniform stability is established. To establish uniform pre-attractivity, pick any $\epsilon, r > 0$ and let $\delta = \alpha_2^{-1}(\exp(-M)\alpha_1(\epsilon))$ so that, if $|\phi(0,0)|_{\mathcal{A}} \leq \delta$, then $|\phi(t,j)|_{\mathcal{A}} \leq \epsilon$ for all $(t,j) \in \operatorname{dom}\phi$ for any solution ϕ to $\mathcal{H}_{ET}$. Let

$$T = 1 + \frac{M - \log(\alpha_1(\delta)) + \log(\alpha_2(r))}{\gamma}.$$

For each solution ϕ to $\mathcal{H}_{ET}$, if $\sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \leq T-1$ then $|\phi(t,j|_{\mathcal{A}} \leq \epsilon$ for all $t+j \geq T$ holds vacuously. Otherwise, there exists $(t',j') \in \operatorname{dom}\phi$ satisfying $T-1 < t'+j' \leq T$ such that $|\phi(t',j')|_{\mathcal{A}} < \delta$. By the choice of δ, $|\phi(t,j)|_{\mathcal{A}} \leq \epsilon$ for all $(t,j) \in \operatorname{dom}\phi$ with $t+j \geq t'+j'$, and, in particular, for all $(t,j) \in \operatorname{dom}\phi$ with $t+j \geq T$, hence uniform pre-attractivity of $\mathcal{A}$ for $\mathcal{H}_{ET}$ has been proven.

To verify the stability of $\mathcal{A}$ for $\mathcal{H}_{ET}$ under the conditions in (b), (c) and (d), note that, due to (20b) and (20c), the growth of V along solutions to $\mathcal{H}_{ET}$ is bounded during flows and during jumps by

$$u_c(z) := \begin{cases} \alpha_{3,c}(z) & \text{if } z \in C \\ -\infty & \text{otherwise} \end{cases} \qquad \forall z \in \mathcal{Z} \tag{28a}$$

$$u_d(z) := \begin{cases} \alpha_{3,d}(z) & \text{if } z \in D \\ -\infty & \text{otherwise} \end{cases} \qquad \forall z \in \mathcal{Z} \tag{28b}$$

respectively. Stability of $\mathcal{A}$ for $\mathcal{H}_{ET}$ then follows from (20a) and from the fact that $u_c(z) \leq 0$ for each $z \in C$ and $u_d(z) \leq 0$ for each $z \in D$ for the conditions in (b), (c) and (d).

Lastly, we prove that, under the conditions in (b), (c) and (d), precompact solutions to $\mathcal{H}_{ET}$ approach the largest weakly invariant subsets of (21), (22) and (23), respectively. It follows from[21, Theorem 8.2] that, under the conditions in (d) and the hybrid basic conditions, each precompact solution to $\mathcal{H}_{ET}$ converges to the largest weakly invariant subset of

$$V^{-1}(r) \cap (\overline{u_c^{-1}(0)} \cup (u_d^{-1}(0) \cap G(u_d^{-1}(0))))$$

for some $r \in V(\mathcal{Z})$, which is equal to (23). If no complete solution ϕ to $\mathcal{H}_{ET}$ remains in a level set of V for all $(t, j) \in \operatorname{dom}\phi$, then $\mathcal{A}$ is pre-attractive, which proves (d). Under the assumptions of (c), we have that $u_d^{-1}(0) = \mathcal{A} \cap D$, thus proving (c) since $G(u_d^{-1}(0)) = G(\mathcal{A} \cap D)$. Under the conditions in (b), we have that $u_c^{-1}(0) = \mathcal{A} \cap C$, thus proving (b).

□

Remark 2. A local version of Theorem 1 also holds by restricting the system to the set of interest. Item a) provides good flexibility in the search for a Lyapunov function as, in particular, covers the case where V grows during flows ($\lambda_c > 0$) but decreases at jumps ($\lambda_d < 0$), which seems natural in event-triggered control as the control input is only updated at events, likely leading to a decrease of V, while in between events V may grow continuously; see Example 7. Furthermore, Theorem 1 pertains to stability and convergence only. The issue of completeness, lower bound on the inter-event times, and robustness are addressed in the upcoming sections. □

Remark 3. Suppose $\mathcal{Z} = \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_\chi}$. When the event functions $\gamma^y_{i_y}$ and $\gamma^u_{i_u}$ are continuously differentiable and are such that, for every $\xi \in \Xi$,

- $\nabla\gamma^y_{i_y}(\xi) \neq \emptyset$ and $\nabla\gamma^u_{i_u}(\xi) \neq \emptyset$, and
- $0 \notin \nabla\gamma^y_{i_y}(\xi)$ and $0 \notin \nabla\gamma^u_{i_u}(\xi)$,

the tangent cone of the set C, which is defined as the sublevel sets of event functions, according to[46, Theorem 2.9.10], can be computed using Clarke's generalized derivatives of their gradients and the flow map in the following special cases. In particular,

1) when $N_y = 1, N_u = 0$, the flow set C is equal to $C_y := \{z \in \mathcal{Z} : \gamma^y(\xi) \leq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$, i.e., it is defined only based on one output event function γ^y. Moreover, the jump map G in (14) reduces to

$$G(z) = \left\{ \begin{bmatrix} x \\ \bigcup_{i_y=1}^{N_y} g^y_{i_y}(y, \hat{y}) \\ \hat{u} \\ G_\chi(z) \end{bmatrix} : y \in H_p(x_p) \right\}.$$

The jump D set reduces to D_y. For every $z \in \partial C := \{z \in \mathcal{Z} : \gamma^y(\xi) = 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$, we have

$$T_C(z) = \{f \in F(z) : \gamma^{y\circ}(\xi; f) \leq 0\}.$$

For every $z \in \operatorname{int} C$, $T_C(z) = \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_\chi}$. Then, given the form of C_y and the fact that γ^y is continuously differentiable, (20b) in Theorem 1 can be replaced by

$$\begin{aligned} V^\circ(z; f) &\leq \alpha_{3,c}(z) \quad \forall z \in \operatorname{int} C, \forall f \in F(z); \\ V^\circ(z; f) &\leq \alpha_{3,c}(z) \quad \forall z \in \partial C, \forall f \in F(z) : \gamma^{y^\circ}(\xi; f) \leq 0 \end{aligned}$$

2) when $N_y = 0, N_u = 1$, we can derive similar results as above with y replaced by u. □

The following example introduces an ETM strategy in which the certificate Lyapunov function can grow during flows, but decreases at jumps. To the best of our knowledge, this strategy is new.

Example 7 (Asymptotically stabilizing ETM with V potentially growing during flows)**.** Let us consider the control system

$$\dot{x}_p = F_p(x_p, u)$$

with state $x_p \in \mathbb{R}^{n_p}$ and input $u \in \mathbb{R}^{n_u}$ such that there exists a positive definite and radially unbounded function $\widetilde{V} : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$ and feedback law $H_c : \mathbb{R}^{n_p} \to \mathbb{R}^{n_u}$. A possible event-triggered implementation of this feedback law is shown in the following hybrid closed-loop system:

$$\begin{bmatrix} \dot{x}_p \\ \dot{\hat{u}} \\ \dot{\chi} \end{bmatrix} = F(x_p, \hat{u}, \chi) := \begin{bmatrix} F_p(x_p, \hat{u}) \\ 0 \\ 1 \end{bmatrix} \qquad (x_p, \hat{u}, \chi) \in C \tag{29a}$$

$$\begin{bmatrix} x_p^+ \\ \hat{u}^+ \\ \chi^+ \end{bmatrix} = G(x_p, \hat{u}, \chi) := \begin{bmatrix} x_p \\ H_c(x_p) \\ 0 \end{bmatrix} \qquad (x_p, \hat{u}, \chi) \in D \tag{29b}$$

with

$$C := \{(x_p, \hat{u}, \chi) \in \mathbb{R}^n \times \mathbb{R}^m \times [0, \bar{\chi}] : \langle \nabla \widetilde{V}(x_p), F_p(x_p, \hat{u}) \rangle \le \eta \widetilde{V}(x_p)\}$$
$$D := \{(x_p, \hat{u}, \chi) \in \mathbb{R}^n \times \mathbb{R}^m \times [\underline{\chi}, \bar{\chi}] : \langle \nabla \widetilde{V}(x_p), F_p(x_p, \hat{u}) \rangle \ge \eta \widetilde{V}(x_p)\}$$

for some constants $\underline{\chi}, \overline{\chi} > 0$ satisfying $\underline{\chi} \le \overline{\chi}$ and $\eta \in \mathbb{R}$, where $\hat{u} \in \mathbb{R}^{n_u}$ denotes a memory variable that stores the value of the input at events and $\chi \in [0, \overline{\chi}]$ is a timer. The input to the system is updated to $H_c(x_p)$ whenever the derivative of the $\widetilde{V}$ exceeds the pre-defined bound $\langle \nabla \widetilde{V}(x_p), f(x_p, \hat{u}) \rangle \le \eta \widetilde{V}(x_p)$ or when χ is in $[\underline{\chi}, \overline{\chi}]$.

To analyze the stability properties of the closed-loop system, let us consider the function

$$V(x_p, \hat{u}, \chi) = \exp(\sigma \chi) \widetilde{V}(x_p) \qquad \forall (x_p, \hat{u}, \chi) \in \mathbb{R}^n \times \mathbb{R}^m \times [0, \overline{\chi}]$$

with $\sigma > 0$. After some straightforward computations, we have that

$$\langle \nabla V(x_p, \hat{u}, \chi), F(x_p, \hat{u}, \chi) \rangle \le (\sigma + \eta) V(x_p, \hat{u}, \chi) \qquad \forall (x_p, \hat{u}, \chi) \in C$$
$$V(x_p, H_c(x_p), 0) - V(x_p, \hat{u}, \chi) \le (\exp(-\sigma \underline{\chi}) - 1) V(x_p, \hat{u}, \chi) \qquad \forall (x_p, \hat{u}, \chi) \in D.$$

It follows from[48, Lemma 4.3] and from the assumptions on $\widetilde{V}$ that there exist $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ such that

$$\tilde{\alpha}_1(|x_p|) \le \widetilde{V}(x_p) \le \tilde{\alpha}_2(|x_p|).$$

Letting

$$\mathcal{A} := \{(x_p, \hat{u}, \chi) \in \mathbb{R}^n \times \mathbb{R}^m \times [0, \overline{\chi}] : x_p = 0\},$$

condition (20a) is satisfied with $\alpha_1 = \tilde{\alpha}_1$ and $\alpha_2 = \exp(\sigma \overline{\chi}) \tilde{\alpha}_2$. Condition (20b) is satisfied with

$$\alpha_{3,c}(x_p, \hat{u}, \chi) = \lambda_c V(x_p, \hat{u}, \chi)$$

and $\lambda_c = \sigma + \eta$. Condition (20c) is satisfied with

$$\alpha_{3,d}(x_p, \hat{u}, \chi) = (\exp(\lambda_d) - 1) V(x_p, \hat{u}, \chi)$$

and $\lambda_d = -\sigma \underline{\chi} < 0$. It follows from the construction of (29) that the maximum time between jumps is $\overline{\chi}$. Hence, for each maximal solution ϕ to (29) and each $(t, j) \in \operatorname{dom} \phi$, the number of jumps j up to the continuous time t is not lower than $\frac{t}{\overline{\chi}} - 1$, i.e., the following holds for each solution ϕ to (29):

$$(t, j) \in \operatorname{dom} \phi \;\Rightarrow\; j \ge \frac{t}{\overline{\chi}} - 1, \tag{30}$$

thus, choosing $p, q > 1$ satisfying $p^{-1} + q^{-1} = 1$, it follows from (30) that

$$\begin{aligned} \lambda_d j + \lambda_c t &= \frac{\lambda_d}{p} j + \frac{\lambda_d}{q} j + \lambda_c t \\ &\le -\frac{\lambda_d}{q} + \max\left\{ \frac{\lambda_d}{p}, \frac{\lambda_d}{q\overline{\chi}} + \lambda_c \right\} (t + j) \end{aligned}$$

for all $(t, j) \in \operatorname{dom} \phi$, because $\lambda_d < 0$. Hence, $(t, j) \in \operatorname{dom} \phi$ implies that $\lambda_d j + \lambda_c t \le M - \gamma(t + j)$ for $M = -\lambda_d / q > 0$ and $\gamma = -\max\left\{ \frac{\lambda_d}{p}, \frac{\lambda_d}{q\overline{\chi}} + \lambda_c \right\}$. Replacing $\lambda_d = -\sigma \overline{\chi}$ and $\lambda_c = \sigma + \eta$ in the expressions for M and γ, we see that $M > 0$ and $\gamma > 0$ for some $q > 1$ if

$$\eta < \sigma \left(\frac{\underline{\chi}}{\overline{\chi}} - 1 \right).$$

If the latter condition is satisfied, we conclude from Theorem 1.a) that the closed set $\mathcal{A}$ is globally pre-asymptotically stable for (29). Note that the aforementioned conditions do not insist on the decrease of V during flows, since these are balanced by strict decreases during jumps. Also, notice a solution $\phi = (x_p, \hat{u}, \chi)$ to (29) is not complete if at least one of the following conditions is verified: there exists $\xi \in C \backslash D$ such that $F(\xi) \cap T_C(\xi) = \emptyset$, in which case there is no flowing solution from ξ; there exists $(t, j) \in \operatorname{dom} \phi$ such that $\phi(t, j) \in \mathbb{R}^n \times \mathbb{R}^m \times [0, \underline{\chi})$ and $\langle \nabla \widetilde{V}(x_p(t, j)), F_p(x_p(t, j), \hat{u}(t, j)) \rangle > \eta \widetilde{V}(x_p(t, j))$ since, in this case, $\phi(t, j) \notin C \cup D$; the solution has finite continuous escape time, in the sense that there exists $T < \infty$ such that $|\phi(t, J)| \to \infty$ as $t \to T$ with $J := \sup_j \operatorname{dom} \phi < \infty$. However, for every complete solution to (29) that converges to $\mathcal{A}$, we have that $\hat{u}$ converges to $H_c(0)$, because $\hat{u}^+ = H_c(x_p)$ at jumps and there is a jump whenever $\chi = \overline{\chi}$.

In the particular case that $\eta = -\sigma$, we have that $\langle \nabla V(x_p, \hat{u}, \chi), F(x_p, \hat{u}, \chi) \rangle \le 0$ for each $(x_p, \hat{u}, \chi) \in C$, hence Theorem 1.c is also applicable. It follows from this result that $\mathcal{A}$ is stable and each precompact solution converges to the largest weakly invariant subset of

$$V^{-1}(r) \cap (\{(x_p, \hat{u}, \chi) \in C : \langle \nabla V(x_p, \hat{u}, \chi), F(x_p, \hat{u}, \chi) \rangle = 0\} \cup \mathcal{A})$$

for some $r \ge 0$. Since every complete solution to (29) has persistent jumps there is no complete solution to (29) satisfying $\operatorname{rge} \phi \subset V^{-1}(r)$ with $r > 0$, hence complete solutions to (29) must approach $\mathcal{A}$, proving that $\mathcal{A}$ is globally pre-asymptotically stable for (29). △

Theorem 1.*a* provides a stronger stability result than classical event-triggered control designs since

1) it provides conditions for the *global uniform pre-asymptotic stability* of a *closed* set $\mathcal{A}$ rather than global asymptotic stability of a setpoint or a compact set;
2) we consider Clarke's generalized derivative rather than the regular time-derivative;
3) we use the framework of hybrid inclusions, which can be more broadly applied.

To further emphasize the differences to existing event-triggered controllers, we provide the example below.

Example 8. An alternative closed-loop system to (29) is presented in[22] is as follows: using a positive definite and radially unbounded function $\widetilde{V} : \mathbb{R}^{n_p} \to \mathbb{R}_{\ge 0}$ and a feedback law $H_c : \mathbb{R}^{n_p} \to \mathbb{R}^{n_u}$, we define

$$\begin{bmatrix} \dot{x}_p \\ \dot{\hat{u}} \end{bmatrix} = F(x_p, u) := \begin{bmatrix} F_p(x_p, \hat{u}) \\ 0 \end{bmatrix} \qquad (x_p, \hat{u}) \in C \tag{31a}$$

$$\begin{bmatrix} x_p^+ \\ \hat{u}^+ \end{bmatrix} = G(x_p, \hat{u}) := \begin{bmatrix} x_p \\ H_c(x_p) \end{bmatrix} \qquad (x_p, \hat{u}) \in D \tag{31b}$$

with

$$C := \{(x_p, \hat{u}) \in \mathbb{R}^n \times \mathbb{R}^m : \langle \nabla \widetilde{V}(x_p), F_p(x_p, \hat{u}) \rangle \le \eta \widetilde{V}(x_p)\}$$
$$D := \{(x_p, \hat{u}) \in \mathbb{R}^n \times \mathbb{R}^m : \langle \nabla \widetilde{V}(x_p), F_p(x_p, \hat{u}) \rangle \ge \eta \widetilde{V}(x_p)\}.$$

In this case, $\widetilde{V}$ satisfies the conditions of (b) of Theorem 1 for $\mathcal{A} := \{(x_p, \hat{u}) \in \mathbb{R}^n \times \mathbb{R}^m : x_p = 0\}$ if $\eta < 0$, implying that $\mathcal{A}$ is stable for (31) and that precompact solutions to (31) converge to the largest weakly invariant subset of

$$\widetilde{V}^{-1}(r) \cap ((\mathcal{A} \cap C) \cup (D \cap G(D))) \tag{32}$$

for some $r \ge 0$. If $\eta = 0$, then the conditions of Theorem 1.d) are verified for $\mathcal{A}$ and each precompact solution to the closed-loop system converges to the largest weakly invariant subset of

$$\widetilde{V}^{-1}(r) \cap (\overline{\alpha_{3,c}^{-1}(0)} \cup (D \cap G(D))) \tag{33}$$

for some $r \ge 0$, where

$$\alpha_{3,c}(x_p, \hat{u}) = \begin{cases} \langle \nabla \widetilde{V}(x_p), F_p(x_p, \hat{u}) \rangle & \text{if } (x_p, \hat{u}) \in C \\ -\infty & \text{otherwise} \end{cases}$$

for each $(x_p, \hat{u}) \in \mathbb{R}^n \times \mathbb{R}^m$. Neither condition (33) nor (32) are enough to assert pre-asymptotic stability of $\mathcal{A}$. In order to do that, more assumptions are required such as the one in (24). Note that it may be the case that $\langle \nabla V(x_p), F_p(x_p, \hat{u}) \rangle = 0$ for some $(x_p, \hat{u}) \in \{(x_p, \hat{u}) \in \mathcal{A} : \hat{u} \neq 0\}$, in which case $\hat{u}$ does not necessarily converge to 0, even if $H_c(0) = 0$, because, unlike Example 7, there is no persistence of jumps. △

Next, we revisit some of the examples of Section 2 and study their stability properties in light of Theorem 1.

Example 9. (Stability conditions for[14]) Considering the hybrid system model in Example 5, let $\mathcal{A} := \{(x_p, \hat{x}_p) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} : x_p = 0\}$. It is possible to verify that, under (6b), conditions (20) are satisfied for (17) with

$$\alpha_{3,c}(x_p, \hat{x}_p) = -(1-\sigma)\alpha(|x_p|) \qquad \forall (x_p, \hat{x}_p) \in C \tag{34a}$$
$$\alpha_{3,d}(x_p, \hat{x}_p) = 0 \qquad \forall (x_p, \hat{x}_p) \in D, \tag{34b}$$

where $0 < \sigma < 1$. Hence we can apply Theorem 1.b) in order to conclude that $\mathcal{A}$ is stable for (17) and each precompact solution converges to the largest weakly invariant set in $V^{-1}(r) \cap ((\mathcal{A} \cap C) \cup (\alpha_{3,d}^{-1}(0) \cap G(\alpha_{3,d}^{-1}(0))))$ for some $r \in V(\mathcal{Z})$. Noting that $\mathcal{A} \cap C = \{0\}$ and $\alpha_{3,d}^{-1}(0) \cap G(\alpha_{3,d}^{-1}(0)) = D \cap G(D) = \{0\}$, we have that each precompact solution to (17) converges to $\mathcal{A}$. Global asymptotic stability of $\mathcal{A}$ for (17) follows from the fact that every maximal solution to (17) is precompact due to (6a) (see Section 3.3). △

Example 10. (Stability conditions for[22]) Considering the hybrid system model in Example 6, in order to globally asymptotically stabilize the closed set

$$\mathcal{A} := \{(x_p, \hat{u}) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_u} : x_p = 0\} \tag{35}$$

for (19), we follow a controller design that is similar to the one in[22]. Given $\mu_1, \mu_2, \mu_3 \in \mathcal{K}_\infty$ and a static-feedback controller $H_c : \mathbb{R}^{n_p} \to \mathbb{R}^{n_u}$ such that $H_c(0) = 0$, we assume that

$$\begin{aligned} &\mu_1(|x_p|) \le V(x_p) \le \mu_2(|x_p|) \\ &\sup_{\zeta \in F_p^m(x_p, H_c(x_p))} \langle \nabla V(x_p), \zeta \rangle \le -\mu_3(|x_p|) \end{aligned} \tag{36}$$

for each $x_p \in \mathbb{R}^{n_p}$. It follows that (20) is satisfied with $\alpha_1 \equiv \mu_1$, $\alpha_2 \equiv \mu_2$, $\alpha_{3,c}(x_p, \hat{u}) = -\mu(|x_p|)$ for all $(x_p, \hat{u}) \in C$ and $\alpha_{3,d}(x_p, \hat{u}) = 0$ for all $(x_p, \hat{u}) \in D$, hence the conditions of Theorem 1.b) are met and, consequently, $\mathcal{A}$ is stable for (19). Furthermore, each precompact solution to (19) converges to the largest weakly invariant subset of

$$V^{-1}(r) \cap ((\mathcal{A} \cap C) \cup (D \cup G(D))) \tag{37}$$

for some $r \ge 0$. In order to prove global asymptotic stability of $\mathcal{A}$ for (19), it remains to be shown that every solution to (19) is precompact and that the largest weakly invariant subset of (37) is $\mathcal{A}$ but, to do this, we need further assumptions on the hybrid system data.

For each $(x_p, \hat{u}) \in D \cap G(D)$, we have that

$$\sup_{\zeta \in F_p^m(x_p, H_c(x_p))} \langle \nabla V(x_p), \zeta \rangle \le -\mu_3(|x_p|) \tag{38a}$$
$$\sup_{\zeta \in F_p^m(x_p, \hat{u})} \langle \nabla V(x_p), \zeta \rangle \ge -\mu(|x_p|). \tag{38b}$$

Assuming that $\mu(s) < \mu_3(s)$ *for all* $s > 0$, the conditions (38) hold only if $x_p = 0$, hence $D \cap G(D) = \mathcal{A}$ and, consequently, we conclude that each precompact solution to (19) converges to $\mathcal{A}$.

Note that, unlike Example 9, it is not necessarily the case that $\hat{u}$ converges to 0, thus it is necessary to inspect solutions to (19) in order to conclude that they are bounded. In this direction, note that each sublevel set of V is forward invariant for (19), hence $x_p(t, j)$ is bounded for every solution $(t, j) \mapsto (x_p, \hat{u})(t, j)$ to the closed-loop hybrid system. Since the memory variable $\hat{u}$ remains constant during flows and is updated to x_p at jumps, it follows that $\hat{u}(t, j)$ is also bounded for every solution $(t, j) \mapsto (x_p, \hat{u})(t, j)$, thus solutions to (19) are bounded. The results in this example are summarized by the next proposition.

Proposition 1. Given the compact set $\mathcal{A}$ in (35), if there exist $\mu, \mu_1, \mu_2, \mu_3 \in \mathcal{K}_\infty$ and $H_c : \mathbb{R}^{n_p} \to \mathbb{R}^{n_u}$ such that (36) holds and $\mu(s) < \mu_3(s)$ for each $s > 0$, then $\mathcal{A}$ is globally pre-asymptotically stable for the hybrid closed-loop system (18). △

3.2 | Robustness Analysis

When the closed-loop system $\mathcal{H}_{ET}$ given as in (16) satisfies the properties in Lemma 1, $\mathcal{H}_{ET}$ is nominally well-posed[21, Definition 6.2]. Moreover, given a compact set that is (uniformly) pre-asymptotically stable for such $\mathcal{H}_{ET}$, the stability property is robust to small perturbations. In particular, perturbations on y, u, and the plant dynamics leads to

$$\dot{x} \in \left\{ \zeta \in \begin{bmatrix} F_p(x_p, u + d_u) + d_p \\ F_c(x_c, y + d_y) \end{bmatrix} : \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} H_p(x_p) \\ H_c(x_c, y + d_y) \end{bmatrix} \right\},$$

where d_y, d_u correspond to the noise and d_p captures unmodeled dynamics.

In addition to the general noise to $\mathcal{H}_{ET}$ given by $\widetilde{d}_1$ and $\widetilde{d}_2$, we also consider the effect of noise on event functions. In particular, we consider perturbed events that are given by

$$\gamma^y_{i_y}(\xi) \leq \rho_y \text{ and } \gamma^u_{i_u}(\xi) \leq \rho_u, \tag{39}$$

for triggering updates in $\hat{y}$ and $\hat{u}$, respectively, with $i_y \in \{1, 2, ..., N_y\}$, $i_u \in \{1, 2, ..., N_u\}$ and $\rho_y, \rho_u > 0$. Similar to the nominal event functions, i.e., $\gamma^y_{i_y}(\xi) = 0$ and $\gamma^u_{i_u}(\xi) = 0$ given in Section 2, the perturbed events in (39) are used to construct the flow sets and jump sets. Consequently, the closed-loop system with ETMs has the same structure as in (11) and (10), however, with slight different definitions. For the ease of reading, we denote these perturbed flow and jump set as $\widetilde{C}$ and $\widetilde{D}$. Given $\rho_y, \rho_u > 0$, the perturbed jump set is given by $\widetilde{D} = \widetilde{D}_y \cup \widetilde{D}_u$, where

$$\widetilde{D}_y := \bigcup_{i_y=1}^{N_y} \widetilde{D}^y_{i_y}, \quad \widetilde{D}_u := \bigcup_{i_u=1}^{N_u} \widetilde{D}^u_{i_u},$$

and for each $i_y \in \{1, 2, ..., N_y\}$, $\widetilde{D}^y_{i_y} := \{z \in \mathcal{Z} : \gamma^y_{i_y}(\xi) \geq -\rho_y\}$, and for each $i_u \in \{1, 2, ..., N_u\}$, $\widetilde{D}^u_{i_u} := \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) \geq -\rho_u\}$; hence, the perturbed flow set is given by

$$\widetilde{C}_y := \bigcap_{i_y=1}^{N_y} \widetilde{C}^y_{i_y}, \quad \widetilde{C}_u := \bigcap_{i_u=1}^{N_u} \widetilde{C}^u_{i_u},$$

and, for each $i_y \in \{1, 2, ..., N_y\}$, $\widetilde{C}^y_{i_y} := \{z \in \mathcal{Z} : \gamma^y_{i_y}(\xi) \leq \rho_y\}$, and, for each $i_u \in \{1, 2, ..., N_u\}$, $\widetilde{C}^u_{i_u} := \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) \leq \rho_u\}$.

Hence, with $\widetilde{d}_1 := (0, 0, d_y, d_u, 0) \in \mathcal{Z}$, $\widetilde{d}_2 := (d_p, 0, 0, 0, 0)$, the closed-loop system $\mathcal{H}_{ET}$ given as in (16) with such perturbations, which is denoted by $\widetilde{\mathcal{H}}_{ET}$, has state $z = (x_p, x_c, \hat{y}, \hat{u}, \chi)$ and dynamics[8]

$$\widetilde{\mathcal{H}}_{ET} \begin{cases} \dot{z} \in F(z + \widetilde{d}_1) + \widetilde{d}_2 & z + \widetilde{d}_1 \in \widetilde{C} \\ z^+ \in G(z + \widetilde{d}_1) & z + \widetilde{d}_1 \in \widetilde{D}. \end{cases}$$

The following result establishes that pre-asymptotic stability is robust to small measurement noise, unmodeled dynamics and event function perturbations.

Theorem 2. Suppose $\mathcal{H}_{ET}$ satisfies the hybrid basic conditions in Lemma 1 and there exists a compact set $\mathcal{A} \subset \mathcal{Z}$ that is pre-asymptotically stable for $\mathcal{H}_{ET}$ with basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}}$ (see[21, Definition 7.3]). Then, there exists $\beta \in \mathcal{KL}$ such that, for each $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}^p_{\mathcal{A}}$, there exists $\delta > 0$ and $\rho^* > 0$, such that for any two measurable functions $\widetilde{d}_1, \widetilde{d}_2 : \mathbb{R}_{\geq 0} \mapsto \delta\mathbb{B}$ and each $\rho_u, \rho_y \in (0, \rho^*]$, every solution $\widetilde{\phi} \in \mathcal{S}_{\widetilde{\mathcal{H}}_{ET}}(K)$ satisfies

$$|\widetilde{\phi}(t, j)|_{\mathcal{A}} \leq \beta(|\widetilde{\phi}(0, 0)|_{\mathcal{A}}, t + j) + \varepsilon \quad \forall (t, j) \in \operatorname{dom} \widetilde{\phi}. \tag{40}$$

Proof. Since for every $t \geq 0$, $\widetilde{d}_1(t), \widetilde{d}_2(t) \in \delta\mathbb{B}$, the perturbed hybrid system $\widetilde{\mathcal{H}}_{ET}$ can be rewritten as

$$\begin{cases} \dot{z} \in F_\delta(z) & z \in C_\delta \\ z^+ \in G_\delta(z) & z \in D_\delta \end{cases}$$

where

$$\begin{aligned} F_\delta(z) &:= \overline{\text{co}} F((z + \delta\mathbb{B}) \cap \widetilde{C}) + \delta\mathbb{B} \\ G_\delta(z) &:= \left\{ \eta \ : \ \eta \in g + \delta\mathbb{B}, g \in G((z + \delta\mathbb{B}) \cap \widetilde{D}) \right\} \\ C_\delta &:= \left\{ z \ : \ (z + \delta\mathbb{B}) \cap \widetilde{C} \neq \emptyset \right\} \\ D_\delta &:= \left\{ z \ : \ (z + \delta\mathbb{B}) \cap \widetilde{D} \neq \emptyset \right\}. \end{aligned}$$

This hybrid system is an outer perturbation of $\mathcal{H}_{ET}$ and satisfies the (C1), (C2), (C3), and (C4) in[49]. Then, by application of[21, Theorem 7.12], the locally pre-asymptotic stability and compactness of $\mathcal{A}$ imply the set $\mathcal{B}^p_{\mathcal{A}}$ is open and that $\mathcal{A}$ is $\mathcal{KL}$ pre-asymptotically stable on $\mathcal{B}^p_{\mathcal{A}}$. Then, applying[21, Lemma 7.20], $\mathcal{A}$ is semiglobally practically robustly $\mathcal{KL}$ pre-asymptotically stable on $\mathcal{B}^p_{\mathcal{A}}$. More preciously, according to[21, Definition 7.18], item (b), with $\omega(z) := |z|_{\mathcal{A}}$ defined for every $z \in \mathcal{Z}$, for every compact

[8]Perturbations on C and D, in particular, in the ETM, is also allowed.

set $K \subset \mathcal{B}^p_{\mathcal{A}}$, there exists $\beta \in \mathcal{KL}$ such that, for each $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}^p_{\mathcal{A}}$, there exists $\delta > 0$ such that every $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}_{ET}}$ satisfies (40). $\square$

3.3 | Completeness of Maximal Solutions

The stability properties presented in Section 3.1 leads to every maximal solution to the system to be bounded but not necessarily complete. In this section, we propose conditions to guarantee completeness of every maximal solution to $\mathcal{H}_{ET}$ in (16). To this end, we apply Proposition A.1.

Proposition 2. Suppose the hybrid system $\mathcal{H}_{ET}$ in (16) with system data given as in (10)-(14) satisfies the hybrid basic conditions. Then, there exists a nontrivial solution to $\mathcal{H}_{ET}$ from every initial point in $C \cup D = \mathcal{Z}$ if

(VC') For every $z \in \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) < 0, \gamma^y_{i_y}(\xi) < 0, \forall i_u \in \{1, 2, ..., N_u\}, \forall i_y \in \{1, 2, ..., N_y\}, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$, $F(z) \cap T_C(z) \neq \emptyset$.

Moreover, every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$ is complete if

(b') case (b) in Proposition A.1 does not hold for every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$, and;

(c') $G_\chi(D) \subset X$.

Proof. By definition of C and D given in (11) and (10), every $z \in \{\zeta \in \mathcal{Z} : \gamma^u_{i_u}(\xi) = 0, \gamma^y_{i_y}(\xi) = 0, i_u \in \{1, 2, ..., N_u\}, i_y \in \{1, 2, ..., N_y\}, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$ is also such that $z \in D$. Then, assumption (VC') implies that for every $z \in C \setminus D$, $F(z) \cap T_C(z) \neq \emptyset$. Hence, (VC) in Proposition A.1 holds for every $z \in C \setminus D$, and there exists a nontrivial solution to $\mathcal{H}_{ET}$ from every initial point in $C \cup D$.

Then, each of the nontrivial $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$ satisfies one of the three cases in Proposition A.1. Case (b) does not hold by assumption (b'). Moreover, when (c') holds, the jump map G maps $z \in \mathcal{Z}$ to set $\mathcal{Z}$. Since $C \cup D = \mathcal{Z}$, we have that assumption (c') implies case *(c)* does not apply for every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$. Thus, every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$ is complete. $\square$

Remark 4. When $\mathcal{Z} = \mathbb{R}^n$, the set $C \setminus D$ is open. Since for every $z \in \text{int}\,(C \setminus D)$, $F(z) \subset T_C(z) = \mathbb{R}^n$, condition (VC') holds trivially. In principle, condition (b') is a solution-dependent property, which can be guaranteed when either C is compact, F is bounded on C, or F is Marchaud on C. All maximal solutions to the closed-loop in (17) in Example 1 are complete when $F_p(x_p, H_c(\hat{x}_p))$ is locally Lipschitz. For the general hybrid system $\mathcal{H}_{ET}$ as in (16) and a closed set $\mathcal{A}$ that satisfies Theorem 1, suppose condition (VC') and (c') in Proposition 2 hold and there exists $V, \alpha_1, \alpha_2, \lambda_c$ and λ_d as described in item a) of Theorem 1 and that V is radially unbounded. Then, every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$ is complete. Moreover, the set $\mathcal{A}$ is globally asymptotic stable for $\mathcal{H}_{ET}$. $\square$

We illustrate Proposition 2 by revisiting Example 1 below.

Example 11 (Example 1 re-revisited)**.** As stated in Example 1, the event-function is given as $\gamma^y(x_p, \hat{x}_p) = \gamma^u(x_p, \hat{x}_p) = \gamma(|\hat{x}_p - x_p|) - \sigma\alpha(|x_p|)$, hence, we have the flow set C given by

$$C := \{z \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} : \gamma(|\hat{x}_p - x_p|) - \sigma\alpha(|x_p|) \leq 0\}.$$

By definition of tangent cone, for every $z \in \{z \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} : \gamma(|\hat{x}_p - x_p|) - \sigma\alpha(|x_p|) < 0\}$, $T_C(z) = \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$. Hence, (VC') in Proposition 2 holds. Moreover, (b') in Proposition 2 holds since F_p is Lipschitz continuous as stated in[14, Theorem III.1]. Thus, every maximal solution to hybrid system (17) is complete, as (c') holds trivially. $\triangle$

3.4 | Lower Bound on Inter-Event Times by Design

In this section, we present conditions on the system data to guarantee a uniform positive lower bound on inter-event time for all solutions to $\mathcal{H}_{ET}$ as in (16). By guaranteeing such a bound, the jumps do not happen arbitrarily close in time. Moreover, the proposed conditions ensure a lower bound on the time between events for systems with small perturbations, for which we impose the hybrid basic conditions on the system of interest. As the following example shows, when such a lower bound is not guaranteed, a “vanishing" perturbation leads to Zeno solutions.

Example 12. (Example 1 revisited) Consider the ETM presented in[14] applied to a dynamical system with state $x_p \in \mathbb{R}$ given by

$$\dot{x}_p = F_p(x_p, u) := u, \quad u = H_c(x_p) = -x_p. \tag{41}$$

Then, the closed-loop system is given as in (17) with $z = (x_p, \hat{x}_p)$, $F(z) := [-\hat{x}_p \ 0]^\top$ and $G(z) := [x_p \ x_p]^\top$. According to[14], we pick triggering event

$$\gamma^u(x_p, \hat{x}_p) = \gamma^y(x_p, \hat{x}_p) = |\hat{x}_p - x_p| - \sigma|x_p|.$$

Suppose u is affected by a disturbance d_u. Then, the resulting perturbed system has flow map defined as $\widetilde{F}(z) := [-\hat{x}_p + d_u \ 0]^\top$ for every $z \in C := \{z \in \mathbb{R}^2 : |\hat{x}_p - x_p| - \sigma|x_p| \leq 0\}$, the jump map remains the same as in (17), and the jump set is given as $z \in D := \{z \in \mathbb{R}^2 : |\hat{x}_p - x_p| - \sigma|x_p| \geq 0\}$. Figure 2 illustrates solution to the system under influence of the vanishing

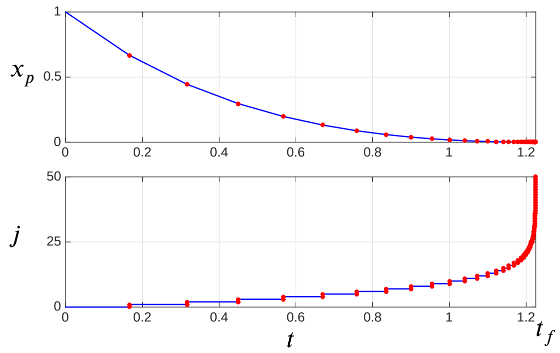

FIGURE 2 Simulation of closed-loop system in Example 1 with vanishing perturbation d_u on u.

perturbation $d_u = -\hat{x}_p|\hat{x}_p|^{b-1}$ with $\sigma = b = 1/2$ and initial conditions $\hat{x}_p(0,0) = x_p(0,0) = 1$. The resulting solution induces Zeno behavior with accumulation point in the time domain given by $t_f \approx 1.22$.[9] To see this, let x_i for each $i \in \mathbb{N}$ denote the state at events. It follows from the definition of the jump logic that $|x_{i+1} - x_i| = \sigma|x_{i+1}|$ for each $i \in \mathbb{N}$ and, by induction, $x_i = x_0/(1+\sigma)^i$ for each $i \in \mathbb{N}$, where $x_0 = x_p(0) = 1$. On the other hand, for each $i \in \mathbb{N}$, we have a constant input equal to $-x_i - x_i|x_i|^{-1/2}$, thus the inter-event time is

$$t_{i+1} - t_i = \frac{\sigma x_i/(1+\sigma)}{x_i + x_i|x_i|^{-1/2}}$$

for each $i \in \mathbb{N}$ and, recalling the assumption that $x_0 = 1$, the total elapsed time is determined by the convergent series

$$\sum_{i=0}^{\infty} t_{i+1} - t_i = \sum_{i=0}^{\infty} \frac{\sigma/(1+\sigma)}{1 + (1+\sigma)^{i/2}}.$$

$\triangle$

By construction of (16), jumps are triggered by either "event type y," i.e., $\gamma^y_{i_y}(\xi) = 0$ with $i_y \in \{1, 2, ..., N_y\}$, or "event type u," i.e., $\gamma^u_{i_u}(\xi) = 0$ with $i_u \in \{1, 2, ..., N_u\}$. Since these events are asynchronous, it suffices to guarantee a uniform positive lower bound for each type of event. To this end, for a given solution $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$, let $E := \bigcup_{j=0}^{J-1} (t_{j+1}, j)$ be the set of all points in dom ϕ at which a jump occurs ($J := \sup_j \operatorname{dom} \phi$ can be finite or infinite). Moreover, we denote the collection of points in dom ϕ at which a jump is triggered by "event type y" as E_y, while E_u denotes the collection of points in dom ϕ at which a jump is triggered by "event type u." Note that $E_u \cup E_y = E$. Then, given a solution $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$, the minimum inter-event time for "event type y" is given by

$$\Delta t_y = \inf\{t'' - t' : (t', j'), (t'', j'') \in E_y, j' < j''\}. \tag{42}$$

[9] Code at github.com/HybridSystemsLab/EventTriggerScalarZeno

Similarly, the minimum inter-event time for "event type u" is given by

$$\Delta t_u = \inf\{t''-t' : (t',j'),(t'',j'') \in E_u, j'<j''\}. \tag{43}$$

Following[50, Lemma 2.7], we provide a necessary and sufficient condition for the existence of a positive uniform lower bound on inter-event time.

Proposition 3. (positive lower bound on inter-event times) Suppose $\mathcal{H}_{ET}$ satisfies the hybrid basic conditions and that every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$ is precompact. Then, for every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$

1. there exists $\lambda_y > 0$ such that Δt_y given as in (42) satisfies $\Delta t_y \geq \lambda_y$ if and only if
$$D_y \cap G(D_y) = \emptyset;$$
2. there exists $\lambda_u > 0$ such that Δt_u given as in (43) satisfies $\Delta t_u \geq \lambda_u$ if and only if
$$D_u \cap G(D_u) = \emptyset;$$

Proof. We show the detailed proof for claim *1)* below, and the proof for claim *2)* follows the same steps. Necessity($\Leftarrow$): (resembles the proof for[50, Lemma 2.7].) Consider any solution $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$. Since ϕ is precompact, i.e., ϕ is complete and bounded, $F(\operatorname{rge}\phi)$ is bounded. By assumption (A2), F is locally bounded, thus, there exists $\delta > 0$ such that $|\dot{\phi}(t,j)| < \delta$ for all $(t,j) \in \operatorname{dom}\phi$. Let E_y be the collection of points in $\operatorname{dom}\phi$ at which a jump is triggered by event type y. Then the set $\overline{\phi(E_y)} \subset \mathcal{Z}$ is compact.[10] In addition, by construction of D_y, the set D_y is closed and $\overline{\phi(E_y)} \subset D_y$ by definition of E_y. Moreover, $G(\overline{\phi(E_y)}) \subset G(D_y)$ is closed since jump map G is outer semicontinuous by assumption (A3). Then, by assumption $D_y \cap G(D_y) = \emptyset$, we have $\overline{\phi(E_y)} \cap G(\overline{\phi(E_y)}) = \emptyset$, and the distance ε between $\overline{\phi(E_y)}$ and $G(\overline{\phi(E_y)})$ is such that $\varepsilon > 0$. More precisely, $|\phi(t_{j_y+1}, j_y) - \phi(t_{j_y^*+1}, j_y^*)| \geq \varepsilon$, for every $(t_{j_y+1}, j_y), (t_{j_y^*+1}, j_y^*) \in E_y$ such that $j_y^* = \min\{j \in (j_y, J-1) : (t,j) \in E_y\}$. Hence, for every Δt_y given as in (42), we have $\Delta t_y \geq \lambda_y = \frac{\varepsilon}{\delta}$.

Sufficiency($\Rightarrow$): Proof by transposition. Suppose $D_y \cap G(D_y) \neq \emptyset$, let $D' = D_y \cap G(D_y)$. By definition of solutions to $\mathcal{H}_{ET}$, there exists a solution $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}(D')$ with $(0,0),(0,1) \in E_y$. Thus, the corresponding Δt_y given as in (42) is $0 - 0 = 0$. Hence, there does not exists $\lambda_y > 0$ such that every Δt_y given as in (42) satisfies $\Delta t_y \geq \lambda_y$. ☐

Remark 5. Furthermore, when solutions are bounded for system in (16), for every solution from a compact sets of initial conditions, there exists a unique $\lambda_y > 0$ such that Δt_y given as in (42) satisfies $\Delta t_y \geq \lambda_y$; and there exists a unique $\lambda_u > 0$ such that Δt_u given as in (43) satisfies $\Delta t_u \geq \lambda_u$. ☐

Example 13 (Revisit Example 3)**.** If σ_y, ϵ_y, σ_u and ϵ_u in Example 3 are constants greater than zero, then $\gamma^y(\xi^+) = \gamma^u(\xi^+)$ is less than 0 for each $\xi^+ := (u, y, \hat{u}^+, \hat{y}^+)$, because $\hat{u}^+ = u$ and $\hat{y}^+ = y$ for all $z^+ \in G(z)$ and $z \in D$. Since γ^y and γ^u must be greater or equal to 0 for the state to belong to D, then we conclude that $G(D) \cap D = \emptyset$, thus verifying the conditions in Proposition 3. △

Note that some assumptions on system data and solutions in Proposition 3 are not "necessary" in the sense that without these assumptions,[11] the necessary and sufficient condition for the existence of the lower bound is still valid. We impose these conditions because they guarantee nominal well-posedness, which, as seen in Section 3.2, is crucial in the robustness and stability analysis for hybrid systems.

3.5 | Lower Bound on Inter-Event Time via Temporal Regularization

The conditions in Proposition 3 guarantee a lower bound on inter-event times. When those conditions are not enforced at the design stage, the closed-loop system may have Zeno solutions from initial conditions in $\mathcal{A}$ or from nearby it. A way to guarantee such a lower bound is to *temporally regularize* the closed-loop system by adding a timer to each ETM with dynamics that allow events to occur only after a particular positive amount of time has elapsed after every respective event. To this end, let τ be a timer with positive threshold $T \in [0, T^*)$, where T^* is a fixed positive parameter.[12] The augmented version of the closed-loop

[10] The set $\phi(E_y) := \{\phi(t,j) : (t,j) \in E_y\}$.
[11] The convexity of $F(x)$ required by (A2) in Definition A.2 is one such assumption.
[12] The threshold could be function of the augmented state.

system $\mathcal{H}_{ET} = (C, F, D, G)$ in (16) is denoted $\widetilde{\mathcal{H}}_{ET}$, has state $\widetilde{z} = (z, \tau) \in \mathcal{Z} \times \mathbb{R}_{\geq 0}$, and dynamics

$$\dot{\widetilde{z}} \in F(z) \times \rho(\tau) \qquad \widetilde{z} \in \left(C \times \mathbb{R}_{\geq 0}\right) \cup (\mathcal{Z} \times [0, T]) \tag{44a}$$

$$\widetilde{z}^{+} \in G(z) \times \{0\} \qquad \widetilde{z} \in D \times [T, \infty) \tag{44b}$$

where ρ is designed to have τ converge to $[0, T^*]$. A particular choice is $\rho(\tau) = 1$ for each $\tau \in [0, T^*)$, $\rho(\tau) = [0, 1]$ for $\tau = T^*$, and $\rho(\tau) = -\tau + T^*$ for each $\tau > T^*$. Note that when $T = 0$ the z component of $\widetilde{\mathcal{H}}_{ET}$ matches that of $\mathcal{H}_{ET}$. We have the following result.

Theorem 3. Suppose the set $\mathcal{A}$ is compact and pre-asymptotically stable for the closed-loop system $\mathcal{H}_{ET}$ in (16) with basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}}$. Then, the set $\mathcal{A} \times [0, T^*]$ has the following semiglobal practical (in the parameter T) stability property: there exists a class-$\mathcal{KL}$ function $\widetilde{\beta}$ such that for each compact set $K_z \times K_\tau \subset \mathcal{B}^p_{\mathcal{A}} \times \mathbb{R}_{\geq 0}$ and each $\varepsilon > 0$ there exists $\overline{T} \in (0, T^*)$ such that for each $T \in (0, \overline{T}]$, every solution $\widetilde{\phi}$ to $\widetilde{\mathcal{H}}_{ET}$ with $\widetilde{\phi}(0,0) \in K_z \times K_\tau$ satisfies

$$|\widetilde{\phi}(t,j)|_{\mathcal{A}\times[0,T^*]} \leq \widetilde{\beta}(|\widetilde{\phi}(0,0)|_{\mathcal{A}\times[0,T^*]}, t + j) + \varepsilon \;\; \forall (t,j) \in \operatorname{dom} \widetilde{\phi}.$$

Proof. Denote the data of $\widetilde{\mathcal{H}}_{ET}$ as $C_T := \left(C \times \mathbb{R}_{\geq 0}\right) \cup (\mathcal{Z} \times [0, T])$, $F_T(\widetilde{z}) = F(z) \times \rho(\tau)$, $D_T := D \times [T, \infty)$ and $G_T(\widetilde{z}) := G(z) \times \{0\}$. Next, we build an augmentation of $\mathcal{H}_{ET}$, denoted $\mathcal{H}'_{ET}$, with one extra one-dimensional state τ for which the compact set $\mathcal{A} \times \{0\}$ is pre-asymptotically stable. Its state is the same as that of $\widetilde{\mathcal{H}}_{ET}$, so we denote it as $\widetilde{z} = (z, \tau)$. The data of $\mathcal{H}'_{ET}$ is defined as

$$C' := \left(C \times \mathbb{R}_{\geq 0}\right) \cup (\mathcal{Z} \times \{0\}) \qquad D' := D \times [0, \infty)$$

$$F'(\widetilde{z}) := F(z) \times \rho'(\tau), \quad \rho'(\tau) := \begin{cases} [0,1] & \text{if } \tau = T^* \\ 1 & \text{if } \tau \in [0, T^*) \\ -\tau + T^* & \text{if } \tau > T^* \end{cases} \qquad G'(\widetilde{z}) := G(z) \times \{0\}$$

Since $\mathcal{A}$ is compact and pre-asymptotically stable for $\mathcal{H}_{ET}$, and $\mathcal{H}_{ET}$ is well-posed, we have that there exists a class-$\mathcal{KL}$ function β such that every maximal solution ϕ_z to $\mathcal{H}_{ET}$ with $\phi_z(0,0)$ in the basin of attraction $\mathcal{B}_{\mathcal{A}}$ satisfies

$$|\phi_z(t,j)|_{\mathcal{A}} \leq \beta(|\phi_z(0,0)|_{\mathcal{A}}, t + j) \qquad \forall (t,j) \in \operatorname{dom} \phi_z$$

Let $\widetilde{\phi} = (\widetilde{\phi}_z, \widetilde{\phi}_\tau)$ be a maximal solution to $\mathcal{H}'_{ET}$ with $\widetilde{\phi}_z(0,0) \in \mathcal{B}_{\mathcal{A}}$. We consider the following two cases:

1) If $\widetilde{\phi}_\tau(0,0) = 0$ then, by the construction of D', jumps are possible only if $\widetilde{\phi}_z(0,0) \in D$. So $\widetilde{\phi}$ has a jump at $(t,j) = (0,0)$ only in that case; namely, only if a solution to $\mathcal{H}_{ET}$ from $\phi_z(0,0) = \widetilde{\phi}_z(0,0)$ would jump. Note that jumps of $\mathcal{H}'_{ET}$ reset τ to zero, so if a jump occurs at $(t,j) = (0,0)$ then $\widetilde{\phi}_\tau(0,1) = 0$. By the construction of C', flows from $\widetilde{\phi}_\tau(0,0) = 0$ are only possible if $\widetilde{\phi}_z$ can flow within C; in fact, if that was not the case, then flows would occur within $\mathcal{Z} \times \{0\}$, in which case $\widetilde{\phi}_\tau$ should remain at zero, which is impossible since $\rho'(\widetilde{\phi}_\tau(0,0)) = 1$. So $\widetilde{\phi}$ flows from $(t,j) = (0,0)$ only if a solution to $\mathcal{H}_{ET}$ from $\phi_z(0,0) = \widetilde{\phi}_z(0,0)$ would be able to flow. Note that flows of $\mathcal{H}'_{ET}$ only allow τ to evolve in $[0, T^*]$; in fact, from $\widetilde{\phi}_\tau(0,0) = 0$, $\widetilde{\phi}_\tau$ grows linearly in time t to T^*, where it stops. Then, for solutions from $\widetilde{\phi}_\tau(0,0) = 0$ the z component is not affected by the addition of the timer τ. Then, we have that such solutions satisfy

$$|\widetilde{\phi}(t,j)|_{\mathcal{A}\times[0,T^*]} \leq \beta(|\widetilde{\phi}_z(0,0)|_{\mathcal{A}}, t + j) \qquad \forall (t,j) \in \operatorname{dom} \widetilde{\phi} \tag{45}$$

2) Consider the case $\widetilde{\phi}_\tau(0,0) \in [0, T^*]$. As in 1), by the construction of D', jumps are possible only if $\widetilde{\phi}_z(0,0) \in D$. Therefore, the solution jumps only if a solution to $\mathcal{H}_{ET}$ from $\phi_z(0,0) = \widetilde{\phi}_z(0,0)$ would jump, in which case $\widetilde{\phi}_\tau(0,1) = 0$ and the same argument in 1) applies. If flow is possible, then those are allowed by C', in which case, flow must occur within $C \times \mathbb{R}_{\geq 0}$. By construction of ρ', $\widetilde{\phi}_\tau$ remains in $[0, T^*]$ for all future hybrid time. Then, the bound in (45) holds.

3) Finally, consider the case $\widetilde{\phi}_\tau(0,0) > T^*$. The analysis for initial jumps follow similarly. The difference is during flows. If flow is possible, then those are allowed by C', in which case, flow must occur within $C \times \mathbb{R}_{\geq 0}$. Now, by construction of ρ', $\widetilde{\phi}_\tau$ evolves accordingly to $\dot{\tau} = -\tau + T^*$. If flow occurs for all future hybrid time, then

$$|\widetilde{\phi}_\tau(t,0)|_{[0,T^*]} \leq |\widetilde{\phi}_\tau(0,0)|_{[0,T^*]} \exp(-t) \qquad \forall (t,0) \in \operatorname{dom} \widetilde{\phi}$$

If flow does not occur for all future hybrid time, then by definition of G', $\widetilde{\phi}_z(t, j+1) = 0$, and the argument in 1) applies.

Combining the above arguments we obtain a $\mathcal{KL}$-bound for $\mathcal{H}'_{ET}$ establishing pre-asymptotic stability of $\mathcal{A} \times [0, T^*]$.

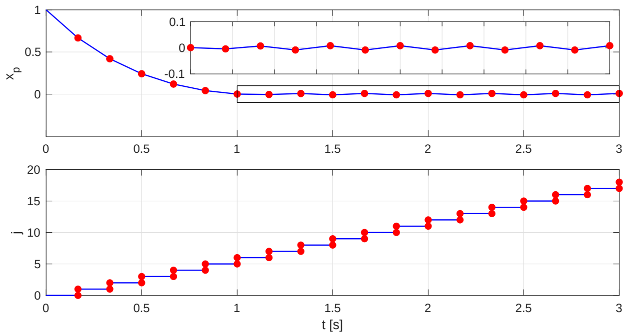

FIGURE 3 Simulation results of the system (46) in the presence of a disturbance that induces Zeno solutions for the non-regularized system as in Figure 2. It is possible to verify that the state of the system converges to a neighborhood of the origin and then oscillates around it due to the disturbance.

Now, the data of $\widetilde{\mathcal{H}}_{ET}$ is such that the sequence of sets C_i and D_i, which correspond to C_T and D_T with $T = \frac{1}{i}$, satisfy

$$\limsup_{i\to\infty} C_i = \limsup_{i\to\infty} \left(C \times \mathbb{R}_{\geq 0}\right) \cup (\mathcal{Z} \times [0, 1/i]) = \bigcap_i \left(C \times \mathbb{R}_{\geq 0}\right) \cup (\mathcal{Z} \times [0, 1/i]) = \left(C \times \mathbb{R}_{\geq 0}\right) \cup (\mathcal{Z} \times \{0\}) = C'$$

$$\limsup_{i\to\infty} D_i = \limsup_{i\to\infty} D \times [1/i, \infty) = \bigcup_i D \times [1/i, \infty) = D \times [0, \infty) = D'$$

since C_i and D_i are monotone; see[51, Exercise 4.3, page 111]; and F_i and G_i, which correspond to F_T and G_T with $T = \frac{1}{i}$, satisfy

$$F_0(\widetilde{z}) \subset F(z) \times \rho'(\tau) = F'(\widetilde{z})$$
$$G_0(\widetilde{z}) = G(z) \times \{0\} = G'(\widetilde{z})$$

for each $\widetilde{z} \in \mathbb{R}^n$, where F_0 and G_0 are the outer graphical limits of F_i and G_i. Since $\widetilde{\mathcal{H}}_{ET}$ satisfies the hybrid basic conditions, with the above properties of the data of $\mathcal{H}'_{ET}$,[52, Theorem 6.6] concludes the proof. □

Below, we revisit Example 12 in order to demonstrate an application of temporal regularization that removes Zeno solutions in the presence of vanishing disturbances.

Example 14 (Example 12 revisited)**.** In this example, we consider the dynamical system and feedback law (41), but we modify the event-triggered control that was presented in Example 12 according to (44). It is possible to verify that the intersampling time for the closed-loop system in Example 12 is $T^* = \sigma/(1+\sigma)$, we may set the threshold T in (44) to a nonnegative number smaller than T^* without compromising global asymptotic stability. To further illustrate the temporal regularization method, we reproduce below the closed-loop system (44) for the event-triggered control of Example 12:

$$\begin{bmatrix} \dot{x}_p \\ \dot{\hat{x}}_p \\ \dot{\tau} \end{bmatrix} = F(x_p, \hat{x}_p, \tau) \begin{bmatrix} -\hat{x}_p \\ 0 \\ \rho(\tau) \end{bmatrix}$$
$$(x_p, \hat{x}_p, \tau) \in C := \{(x_p, \hat{x}_p, \tau) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} : |\hat{x}_p - x_p| \leq \sigma |x_p| \text{ or } \tau \in [0, T]\} \quad \text{(46a)}$$

$$\begin{bmatrix} x_p^+ \\ \hat{x}_p^+ \\ \tau^+ \end{bmatrix} = G(x_p, \hat{x}_p, \tau) \begin{bmatrix} x_p \\ x_p \\ 0 \end{bmatrix}$$
$$(x_p, \hat{x}_p, \tau) \in D := \{(x_p, \hat{x}_p, \tau) \in \mathbb{R}^2 \times [T, +\infty) : |\hat{x}_p - x_p| \geq \sigma |x_p|\}. \quad \text{(46b)}$$

This construction satisfies the conditions of Proposition 3, hence the effects of vanishing noise that was demonstrated in Example 12 does not happen.

3.6 | Zeno Stability

Though not recommended due to the reasons illustrated in Example 12, if Zeno solutions from $\mathcal{A}$ are acceptable, one might be interested in determining if solutions starting nearby $\mathcal{A}$ are also Zeno. The following result, which is a restatement of[53, Proposition 4.5], provides a set of conditions for Zeno solutions to exist from nearby $\mathcal{A}$. Below, given a set $\mathcal{X}$ and a solution ϕ,

$$\mathcal{T}_{\mathcal{X}}(\phi) := \sup\{t \;:\; \exists j, (t,j) \in \operatorname{dom}\phi, \phi(t,j) \in \mathcal{X}\ \}$$

Proposition 4. Suppose that the closed-loop system $\mathcal{H}_{ET} = (C, F, D, G)$ satisfies the hybrid basic conditions and that $\mathcal{A}$ is compact. Then, $\mathcal{A}$ is pre-asymptotically stable, there exists $\varepsilon > 0$ such that every maximal solution ϕ to $\mathcal{H}_{ET}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, \varepsilon]$ is Zeno, and $\mathcal{T}_{\mathcal{Z}}(\phi) = \mathcal{T}_{\mathcal{Z}\setminus\mathcal{A}}(\phi)$ if and only if

1. There exists $\epsilon > 0$ such that every maximal solution ϕ to $\mathcal{H}_{ET}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, \epsilon]$ has $\mathcal{T}_{\mathcal{Z}\setminus\mathcal{A}}(\phi) \in (0, \infty)$;
2. There is no absolutely continuous function $z_c : [0, \epsilon] \to \mathcal{Z}$, $\epsilon > 0$, such that $z_c(0) \in \mathcal{A}$, $z_c(\epsilon) \notin \mathcal{A}$, and

$$\dot{z}_c(t) \in -F(z_c(t)) \;\; z_c(t) \in C \qquad \text{for almost all } t \in [0, \epsilon]$$

Note solutions to the proposed ETM in[14] are Zeno when they start from the set $\mathcal{A}$. However, solutions start near the set $\mathcal{A}$ are not Zeno as shown therein and in[23].

4 | CONCLUSION

In this paper, we propose a framework that can be used to model several existing event-triggered strategies, which are designed to reduce computation times when compared to continuous feedback. The proposed model and tools derived from the theory of hybrid systems in[21]. Moreover, our results provide better understanding of the requirements for uniformly lower bounded inter-event times and robustness; extend existing event-triggered control strategies and lead to new ETMs that can further relax the computational requirements of event-triggered control. In particular, the proposed Lyapunov-based conditions allowed the development of an event-triggered control strategy for which a Lyapunov function increases during the continuous regime – due to information obtained at the previous event getting “stale” – but decreases enough when new information is obtained at events (see Example 7). We find that this discovery has the potential to lead to a very promising direction of research in event-triggered control. Conditions explicitly involving the event-triggering functions are provided to guarantee completeness of solutions and an uniform positive lower bound on inter-event times.

Another promising direction of research is whether existing event-triggering conditions can be redesigned to increase the lower bound on the inter-event times at points in the state space that are far from the origin (or the set to stabilize) and for points nearby it, such restriction is relaxed. In general, the number of computation events (or computation time) depend on the form of γ_u and γ_y functions. The result in Section 3.6 provides an initial observation on that as it provides conditions under which the behavior of solutions nearby the set to be stabilized are Zeno. A follow up consists of designing a (potentially hybrid) algorithm that at points far from that set, the solutions have a large enough uniform lower bound on the inter-event times.

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References

1. Franklin GF, Powell JD, Workman ML. *Digital Control of Dynamic Systems*. Addison-Wesley series in electrical and computer engineering:Addison-Wesley . 1998.

2. Berg MC, Amit N, Powell JD. Multirate Digital Control System Design. *IEEE Transactions on Automatic Control* 1988; 33(12): 1139–1150. doi: 10.1109/9.14436

3. Fujioka H. Stability analysis of systems with aperiodic sample-and-hold devices. *Automatica* 2009; 45(3): 771–775. doi: 10.1016/j.automatica.2008.10.017

4. Åarzén KE. A simple event-based PID controller. *IFAC Proceedings Volumes* 1999; 32(2): 8687–8692.

5. Borgers DP, Heemels WPM. Event-separation properties of event-triggered control systems. *IEEE Transactions on Automatic Control* 2014; 59(10): 2644–2656. doi: 10.1109/TAC.2014.2325272

6. Postoyan R, Heemels W, Sanfelice R. Inter-Event Times Analysis for Planar Linear Event-Triggered Controlled Systems (I). In: ; 2019.

7. Heemels W, Johansson KH, Tabuada P. An introduction to event-triggered and self-triggered control. In: Proceedings of the 51st IEEE Conference on Decision and Control (CDC); 2012: 3270–3285.

8. Zhang XM, Han QL, Zhang BL. An Overview and Deep Investigation on Sampled-Data-Based Event-Triggered Control and Filtering for Networked Systems. *IEEE Transactions on Industrial Informatics* 2017; 13(1): 4–16. doi: 10.1109/TII.2016.2607150

9. Miskowicz M. Send-on-delta concept: An event-based data reporting strategy. *Sensors* 2006; 6(1): 49–63. doi: 10.3390/s6010049

10. Tabuada P. Event-Triggered Real-Time Scheduling of Stabilizing Control Tasks. *IEEE Transactions on Automatic Control* 2007; 52(9): 1680–1685. doi: 10.1109/TAC.2007.904277

11. Donkers M, Heemels W. Output-based event-triggered control with guaranteed L_∞-gain and improved event-triggering. In: Proceedings of the 49th IEEE Conference on Decision and Control (CDC); 2010: 3246–3251.

12. Mazo M, Tabuada P. On event-triggered and self-triggered control over sensor/actuator networks. In: Proceedings of the 47th IEEE Conference on Decision and Control; 2008: 435–440

13. Girard A. Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control* 2015; 60(7): 1992–1997.

14. Tabuada P. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control* 2007; 52(9): 1680–1685.

15. Heemels WPM, Donkers MCF, Teel AR. Periodic Event-Triggered Control for Linear Systems. *Transactions on Automatic Control* 2013; 58(4): 847–861.

16. Borgers DP, Postoyan R, Anta A, Tabuada P, Nešić D, Heemels WP. Periodic event-triggered control of nonlinear systems using overapproximation techniques. *Automatica* 2018; 94: 81–87. doi: 10.1016/j.automatica.2018.04.019

17. Asadi Khashooei B, Antunes DJ, Heemels WPM. A Consistent Threshold-Based Policy for Event-Triggered Control. *IEEE Control Systems Letters* 2018; 2(3): 447–452. doi: 10.1109/LCSYS.2018.2840970

18. Dolk VS, Borgers DP, Heemels WPM. Dynamic event-triggered control: Tradeoffs between transmission intervals and performance. *Proceedings of the 53th IEEE Conference on Decision and Control*; 2014: 2764–2769. doi: 10.1109/CDC.2014.7039813

19. Nesic D, Teel AR, Carnevale D. Explicit Computation of the Sampling Period in Emulation of Controllers for Nonlinear Sampled-Data Systems. *IEEE Transactions on Automatic Control* 2009; 54(3): 619–624.

20. Theodosis D, Tsinias J. Sufficient lie algebraic conditions for sampled-data feedback stabilization. In: Proceedings of the 54th IEEE Conference on Decision and Control (CDC); 2015: 6490–6495.

21. Goebel R, Sanfelice RG, Teel AR. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. New Jersey: Princeton University Press . 2012

22. Seuret A, Prieur C. Event-triggered sampling algorithms based on a Lyapunov function. In: Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference; 2011: 6128–6133.

23. Postoyan R, Tabuada P, Nešić D, Anta A. A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control* 2015; 60(4): 982–996.

24. Marchand N, Durand S, Guerrero-Castellanos J. A general formula for event-based stabilization of nonlinear systems. *IEEE Transactions on Automatic Control* 2013; 58(5): 1332–1337.

25. Boisseau B, Durand S, Martinez-Molina J, Raharijaona T, Marchand N. Attitude control of a gyroscope actuator using event-based discrete-time approach. In: IEEE International Conference on Event-based Control, Communication, and Signal Processing (EBCCSP); 2015: 1–6.

26. Guerrero-Castellanos J, Téllez-Guzmán J, Durand S, Marchand N, Alvarez-Muñoz J, Gonzalez-Diaz VR. Attitude stabilization of a quadrotor by means of event-triggered nonlinear control. *Journal of Intelligent & Robotic Systems* 2014; 73(1-4): 123–135.

27. Li H, Shi Y. Event-triggered robust model predictive control of continuous-time nonlinear systems. *Automatica* 2014; 50(5): 1507–1513. doi: 10.1016/j.automatica.2014.03.015

28. Hashimoto K, Adachi S, Dimarogonas DV. Event-triggered intermittent sampling for nonlinear model predictive control. *Automatica* 2017; 81(March): 148–155. doi: 10.1016/j.automatica.2017.03.028

29. Xie Y, Lin Z. Global Stabilization of a Chain of Integrators by a Switching Event-Triggered Bounded Control. In: Proceedings of the 2017 American Control Conference; 2017.

30. Battistelli G, Chisci L, Selvi D. A distributed Kalman filter with event-triggered communication and guaranteed stability. *Automatica* 2018; 93: 75–82. doi: 10.1016/j.automatica.2018.03.005

31. Li YX, Yang GH. Model-Based Adaptive Event-Triggered Control of Strict-Feedback Nonlinear Systems. *IEEE Transactions on Neural Networks and Learning Systems* 2018; 29(4): 1033–1045. doi: 10.1109/TNNLS.2017.2650238

32. Wang J, Mounier H, Cela A, Niculescu SI. Event driven intelligent PID controllers with applications to motion control. In: IFAC Proceedings Volumes; 2011: 10080–10085.

33. Frazzoli E, Johansson KH. Distributed Event-Triggered Control for Multi-Agent Systems. *IEEE Transactions on Automatic Control* 2012; 57(5): 1291 – 1297. doi: 10.1109/TAC.2011.2174666

34. Molin A, Hirche S. On the optimality of certainty equivalence for event-triggered control systems. *IEEE Transactions on Automatic Control* 2013; 58(2): 470–474. doi: 10.1109/TAC.2012.2206719

35. Yue D, Tian E, Han QL. A delay system method for designing event-triggered controllers of networked control systems. *IEEE Transactions on Automatic Control* 2013; 58(2): 475–481. doi: 10.1109/TAC.2012.2206694

36. Yu J, Karaman S, Rus D. Persistent Monitoring of Events with Stochastic Arrivals at Multiple Stations. *IEEE Transactions on Robotics* 2013; 31(3): 521 – 535.

37. Vamvoudakis KG. Event-triggered optimal adaptive control algorithm for continuous-time nonlinear systems. *IEEE/CAA Journal of Automatica Sinica* 2014; 1(3): 282–293. doi: 10.1109/JAS.2014.7004686

38. Almeida J, Silvestre C, Pascoal AM. Synchronization of multi-agent systems using event-triggered and self-triggered broadcasts. *IEEE Transactions on Automatic Control* 2017; 69(9): 4741 – 4746. doi: 10.1109/TAC.2017.2671029

39. Poveda JI, Teel AR. A Robust Event-Triggered Approach for Fast Sampled-Data Extremization and Learning. *IEEE Transactions on Automatic Control* 2017; 62(10): 4949–4964. doi: 10.1109/TAC.2017.2674519

40. Solowjow F, Baumann D, Garcke J, Trimpe S. Event-Triggered Learning for Resource-Efficient Networked Control. In: Proceedings of the 2018 American Control Conference; 2018: 6506–6512

41. Yang R, Zhang H, Feng G, Yan H, Wang Z. Robust cooperative output regulation of multi-agent systems via adaptive event-triggered control. *Automatica* 2019; 102: 129–136. doi: 10.1016/j.automatica.2019.01.001

42. Liu T, Jiang Z. Event-based control of nonlinear systems with partial state and output feedback. *Automatica* 2015; 53: 10–22.

43. De Persis C, Sailer R, Wirth F. Parsimonious event-triggered distributed control: A Zeno free approach. *Automatica* 2013; 49(7): 2116–2124.

44. Borgers DP, Heemels W. Event-separation properties of event-triggered control systems. *IEEE Transactions on Automatic Control* 2014; 59(10): 2644–2656.

45. Chai J, Casau P, Sanfelice RG. Analysis and Design of Event-triggered Control Algorithms using Hybrid Systems Tools. In: Proceedings of the 56th IEEE Conference on Decision and Control. ; 2017: 6057-6062.

46. Clarke FH. *Optimization and Nonsmooth Analysis*. Philadelphia: SIAM . 1987.

47. Lunze J, Lehmann D. A state-feedback approach to event-based control. *Automatica* 2010; 46(1): 211–215.

48. Khalil H. *Nonlinear Systems*. Prentice Hall. 3rd ed. 2002.

49. Goebel R, Teel AR. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica* 2006; 42(4): 573–587.

50. Sanfelice RG, Goebel R, Teel AR. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control* 2007; 52(12): 2282–2297. doi: http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=04395187

51. Rockafellar R, Wets RJB. *Variational Analysis*. Berlin Heidelberg: Springer . 1998.

52. Goebel R, Teel A. Solutions to Hybrid Inclusions Via Set and Graphical Convergence with Stability Theory Applications. *Automatica* 2006; 42(4): 573-587.

53. Goebel R, Teel AR. Lyapunov characterization of Zeno behavior in hybrid systems. In: IEEE. Proceedings of the 47th IEEE Conference on Decision and Control; 2008: 2752–2757.

□

APPENDIX

A PRELIMINARIES ON HYBRID SYSTEMS

We classify the solutions to hybrid systems as follows.

Definition A.1. A solution ϕ to $\mathcal{H}_{ET}$ is said to be

- nontrivial if $\operatorname{dom}\phi$ has at least two points;
- complete if $\operatorname{dom}\phi$ is unbounded;
- precompact if it is complete and bounded;
- Zeno if it is complete and $\sup\{t : (t, j) \in \operatorname{dom}\phi\} < \infty$;
- maximal if there does not exist another solution ϕ' such that ϕ is a truncation of ϕ' to some proper subset of $\operatorname{dom}\phi'$.

We use $\mathcal{S}_{\mathcal{H}_{ET}}$ to denote the set of maximal solutions to the hybrid system $\mathcal{H}_{ET}$. See [21, Chapter 2] for more details about solutions to hybrid systems. □

The following regularity conditions on the system data for a (closed-loop) hybrid system $\mathcal{H}_{ET}$ are used in our results. They guarantee robustness of stability of compact sets with respect to perturbations; see[21, Chapter 6] for details.

Definition A.2. (Hybrid Basic Conditions) A hybrid system $\mathcal{H}_{ET}$ with state $z \in \mathcal{Z}$ is said to satisfy the hybrid basic conditions if its data (C, F, D, G) is such that

(L1) $C \subset \mathcal{Z}$ and $D \subset \mathcal{Z}$ are closed sets relative to $\mathcal{Z}$;

(L2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C,[13] and $F(z)$ is nonempty and convex for all $z \in C$;

(L3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, and $G(z)$ is nonempty for all $z \in D$. □

For hybrid systems $\mathcal{H}_{ET} = (C, F, D, G)$,[21, Definition 3.6] introduces the following stability notion.

Definition A.3. (Stability) A closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *stable* for $\mathcal{H}_{ET}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution χ with $|\chi(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|\chi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} \chi$;
- *locally pre-attractive* for $\mathcal{H}_{ET}$ if there exists $\mu > 0$ such that every solution χ to $\mathcal{H}_{ET}$ with $|\chi(0,0)|_{\mathcal{A}} \leq \mu$ is such that $(t, j) \mapsto |x(t,j)|_{\mathcal{A}}$ is bounded and, if χ is complete, then also

$$\lim_{t+j\to\infty} |\chi(t,j)|_{\mathcal{A}} = 0; \tag{A1}$$

- *pre-asymptotically stable* for $\mathcal{H}_{ET}$ if it is both stable and pre-attractive for $\mathcal{H}_{ET}$.

If $\mathcal{A}$ is pre-asymptotically stable for $\mathcal{H}_{ET}$ and every solution to $\mathcal{H}_{ET}$ is complete then one may drop the prefix "pre-" and say that $\mathcal{A}$ is asymptotically stable. In addition, if every solution to $\mathcal{H}_{ET}$ is bounded and if complete it satisfies (A1), then $\mathcal{A}$ is globally asymptotically stable for $\mathcal{A}$.

The following Definition is extracted from the result on ω-limit sets to hybrid trajectories in[50, Lemma 3.3].

Definition A.4. A hybrid trajectory ϕ approaches the set $S \in \mathbb{R}^n$ if for all $\varepsilon > 0$ there exists $(\bar{t}, \bar{j}) \in \operatorname{dom} \phi$ such that for all (t, j) satisfying $(t, j) \succcurlyeq (\bar{t}, \bar{j}), (t, j) \in \operatorname{dom} \phi, \phi(t, j) \in S + \varepsilon\mathbb{B}$.

To keep this paper self-contained, we quote the definition of the largest weakly invariant set as follows directly from[21, Definition 6.19].

Definition A.5 (Weak invariance)**.** Given a hybrid system $\mathcal{H}_{ET}$, a set $S \subset \mathbb{R}^n$ is said to be

- weakly forward invariant if for every $\xi \in S$ there exists at least one complete $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}(\xi)$ with $\operatorname{rge} \phi \subset S$;
- weakly backward invariant if for every $\xi \in S$, every $\tau > 0$, there exists at least one $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}(S)$ such that for some $(t^*, j^*) \in \operatorname{dom} \phi$, $t^* + j^* \geq \tau$, it is the case that $\phi(t^*, j^*) = \xi$ and $\phi(t, j) \in S$ for all $(t, j) \in \operatorname{dom} \phi$ with $t + j \leq t^* + j^*$;
- weakly invariant if it is both weakly forward invariant and weakly back- ward invariant.

The following result is important in the construction of the jump logic associated with the event-triggered strategy that is presented in this paper. In other words, Lemma A.1 guarantees that the union operation defined in (A2) preserves outer semicontinuity and local boundedness.

Lemma A.1. Given closed subsets D_1 and D_2 of $\mathbb{R}^n$ and set-valued maps $G_1 : D_1 \rightrightarrows \mathbb{R}^n$ and $G_2 : D_2 \rightrightarrows \mathbb{R}^n$ that are outer semicontinuous and locally bounded relative to D_1 and D_2, respectively, the set $D := D_1 \cup D_2$ is closed and the set-valued map $G : D \rightrightarrows \mathbb{R}^n$, given by

$$G(x) := G_1(x) \cup G_2(x) = \begin{cases} G_1(x) & \text{if } x \in D_1 \backslash D_2 \\ G_2(x) & \text{if } x \in D_2 \backslash D_1 \\ G_1(x) \cup G_2(x) & \text{if } x \in D_1 \cap D_2 \end{cases} \tag{A2}$$

[13] We say that a set-valued mapping $M : \mathbb{R}^m \to \mathbb{R}^n$ is outer semicontinuous (osc) at $x \in \mathbb{R}^m$ if for every sequence of points x_i convergent to x and any convergent sequence of points $y_i \in M(x_i)$, one has $y \in M(x)$, where $\lim_{i\to\infty} y_i = y$. The mapping M is outer semicontinuous if it is outer semicontinuous at each $x \in \mathbb{R}^m$. Given a set $S \subset \mathbb{R}^m$, $M : \mathbb{R}^m \to \mathbb{R}^n$ is outer semicontinuous relative to S if the set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $M(x)$ for $x \in S$ and $\emptyset$ for $x \notin S$ is outer semicontinuous at each $x \in S$. Moreover, M is locally bounded at $x \in \mathbb{R}^m$ if there exists a neighborhood U_x of x such that $M(U_x) \subset \mathbb{R}^n$ is bounded. The mapping M is locally bounded if it is locally bounded at each $x \in \mathbb{R}^m$. Given a set $S \subset \mathbb{R}^m$, the mapping M is locally bounded relative to S if the set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $M(x)$ for $x \in S$ and $\emptyset$ for $x \notin S$ is locally bounded at each $x \in S$.

for each $x \in D$, is outer semicontinuous and locally bounded relative to D.

Proof. To see that the equality (A2) holds, note that, for each $x \in D_1 \backslash D_2$, we have $G_2(x) = \emptyset$, thus $G_1(x) \cup G_2(x) = G_1(x)$ for each $x \in D_1 \backslash D_2$. Similarly, we conclude that $G_1(x) \cup G_2(x) = G_2(x)$ for each $x \in D_2 \backslash D_1$.

The set D corresponds to the union of two closed sets, hence it is closed. The set $D_1 \backslash D_2$ is open relative to D_1, because D_2 is closed. Therefore, for each sequence x_i convergent to $x \in D_1 \backslash D_2$, there exists $i^\star \geq 0$ such that $x_i \in D_1 \backslash D_2$ for all $i \geq i^\star$. Consequently, for each sequence $y_i \in G(x_i)$ convergent to y, we have that $y_i \in G_1(x_i)$ for all $i \geq i^\star$. It follows from outer semicontinuity of G_1 relative to D_1 that $y \in G_1(x) = G(x)$. Similarly, we can prove that for each sequence x_i convergent to $x \in D_2 \backslash D_1$ and for each sequence $y_i \in G(x_i)$ convergent to y, we have that $y \in G_2(x) = G(x)$. For each sequence x_i convergent to $x \in D_1 \cap D_2$ and for each sequence $y_i \in G(x_i)$ convergent to y, we have that $y_i \in G_1(x_i)$ or $y_i \in G_2(x_i)$ for all $i \in \mathbb{N}$. Hence, it follows from outer semicontinuity of G_1 and G_2 relative to D_1 and D_2, respectively, that $y \in G_1(x)$ or $y \in G_2(x)$, thus $y \in G_1(x) \cup G_2(x)$. Since $x \in D_1 \cap D_2$ due to closedness of $D_1 \cap D_2$, we have that $y \in G(x)$. We conclude that G is outer semicontinuous relative to D.

To prove local boundedness of G relative to D, we split the proof into three cases: $x \in D_1 \backslash D_2$, $x \in D_2 \backslash D_1$ and $x \in D_1 \cap D_2$. If $x \in D_1 \backslash D_2$, by local boundedness of G_1 relative to D_1, there exists a neighborhood U of x such that $G_1(U)$ is bounded. Since $G_1(U') = G(U') \subset G_1(U)$ for $U' = U \cap (D_1 \backslash D_2)$, it follows that for every $x \in D_1 \backslash D_2$ there exists a neighborhood U' of x such $G(U')$ is bounded. Similarly, it is possible to show that, for each $x \in D_2 \backslash D_1$, there exists a neighborhood U' of x such $G(U')$ is bounded. It follows from local boundedness of G_1 and G_2 relative to D_1 and D_2, respectively, that, for each $x \in D_1 \cap D_2$, there exist neighborhoods U_1 and U_2 of x such that $G_1(U_1)$ and $G_2(U_2)$ are bounded. For each neighborhood U' of $x \in D_1 \cup D_2$, we have that

$$G(U') = G_1(U') \cup G_2(U') \tag{A3}$$

because G_1 and G_2 are empty outside D_1 and D_2, respectively. Since U_1 and U_2 are neighborhoods of x, there exists $\epsilon > 0$ such that $x + \epsilon\mathbb{B} := \{v \in \mathbb{R}^n : |x - v| < \epsilon\}$ satisfies $(x + \epsilon\mathbb{B}) \cap D_1 \subset U_1$ and $(x + \epsilon\mathbb{B}) \cap D_2 \subset U_2$. Letting $U' := (x + \epsilon\mathbb{B}) \cap (D_1 \cup D_2)$, it follows from (A3) that $G(U') \subset G_1(U_1) \cup G_2(U_2)$ which is bounded. We conclude that G is locally bounded relative to D. ☐

We also recall the concept of tangent cone to a set K (see, e.g.,[21, Definition 5.12]), which is also known as the Bouligand tangent cone or contingent cone.

Definition A.6. (Tangent Cone) The tangent cone to a closed set $K \subset \mathbb{R}^n$ at a point $z \in \mathbb{R}^n$, denoted as $T_K(z)$, is the set of all vectors $\omega \in \mathbb{R}^n$ for which there exist sequences $z_i \in K$, $\tau_i > 0$ with $z_i \to z$, $\tau_i \searrow 0$ and

$$\omega = \lim_{i\to\infty} \frac{z_i - z}{\tau_i}.$$

To formulate our results, we need the following conditions guaranteeing existence of solutions to hybrid system, see[21, Proposition 6.10].

Proposition A.1. Let $\mathcal{H}_{ET} = (C, F, D, G)$ satisfy the hybrid basic conditions. Let $\xi \in C \cup D$. If $\xi \in D$ or

(VC) there exist a neighborhood U of ξ such that for every $z \in U \cap C$, $F(z) \cap T_C(z) \neq \emptyset$,

then there exists a nontrivial solution ϕ to $\mathcal{H}_{ET}$ with $\phi(0,0) = \xi$. If (VC) holds for every $\xi \in C \setminus D$, then there exists a nontrivial solution to $\mathcal{H}_{ET}$ from every point in $C \cup D$, and every $\phi \in \mathcal{S}_{\mathcal{H}_{ET}}$ satisfies exactly one of the following:

(a) ϕ is complete;

(b) ϕ is not complete and "ends with flow": for $(T, J) = \sup \operatorname{dom} \phi$, the interval I^J has nonempty interior and $t \mapsto \phi(t, J)$ is a maximal solution to $\dot{z} \in F(z)$, in fact
$\lim_{t\to T} |\phi(t, J)| = \infty$;

(c) ϕ is not complete and "ends with jump": for $(T, J) = \sup \operatorname{dom} \phi$, one has $\phi(T, J) \notin C \cup D$.

Furthermore, if $G(D) \subset C \cup D$, then (c) does not occur.

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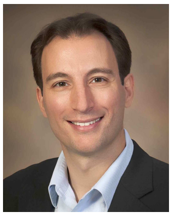

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