Self-Triggered Control to Guarantee Forward Pre-Invariance with Uniformly Positive Inter-Event Times

David Kooi, Mohamed Maghenem, and Ricardo G. Sanfelice

***Abstract*—In this paper, we propose a self-triggered control strategy to guarantee forward pre-invariance of a closed set for a control system modeled by a constrained differential inclusion. Using a (not necessarily periodic) zero-order hold control scheme, this paper addresses two key issues: i) computing the time of the next sampling event, and ii) the assurance of a uniform lower bound on the inter-event times, both while guaranteeing forward invariance. Our results guarantee forward pre-invariance on unbounded sets. Critically, the results impose mild regularity on the right-hand side of the system and on the barrier certificates. Simulations showcase the proposed algorithms and provide comparisons with the literature.**

I. Introduction

Given a continuous-time control system and a control law designed such that the resulting closed-loop system satisfies a prescribed control objective, it is not necessarily the case that a digital implementation, where the input is updated only after some fixed period, will still guarantee that control objective, even if the period is small [1], [2]. Generalizations of such digital implementations, when the control input is not necessarily periodically updated, find motivation in the context of computer-controlled systems [3]. These methods can be decomposed into event-triggered (ET) and self-triggered (ST) control approaches, as described in [4] and [5], respectively; see also [6] for an overview. In ET control, continuous availability of the measurements is typically assumed. Hence, the control input is usually updated whenever the measurements reach a critical region that compromises the control objective. However, continuous availability of the measurements is not possible in some applications. Hence, in ST control strategies [7], the measurements are assumed to be available only when the input needs to be updated. The key questions to answer for ET and ST control systems are as follows: 1) What are the inter-event times that make the control objective satisfied for the resulting closed-loop system? 2) Under what conditions are the resulting inter-event times always larger than a positive constant?

Early results on ST and ET control focus mainly on stability and convergence. These works usually assume the existence of a feedback law that renders the (non-triggered) closed-loop system input to state stable with respect to input perturbations.

D. Kooi and R. G. Sanfelice are with the Department of Electrical and Computer Engineering, University of California, Santa Cruz. Email: dkooi,ricardo@ucsc.edu. M. Maghenem is with CNRS, GIPSA Lab, Grenoble, France mohamed.maghenem@gipsa-lab.grenoble-inp.fr.
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For example, in [8] and [9], ET and ST control algorithms are proposed. In both works, the inter-event times are shown to be larger than a positive constant when the state is not at the origin. In [10], a hybrid system framework is proposed to analyze and design ET control algorithms guaranteeing asymptotic stability of a compact set. More recently, in [11], sufficient conditions to guarantee a uniform lower bound on inter-event times for ET controllers are given; see also [12].

In addition to stability and convergence, safety is one of the common control objectives encountered in applications [13], [14], [15]. Safety is the property that requires the solutions to a system starting from a given set of initial conditions to remain in a desired safe region [16], [17]. Existing results guaranteeing forward invariance using ST and ET control algorithms often assume strong regularity properties on the right-hand side or boundedness of the set to be rendered forward invariant. In [18], a ST control algorithm is proposed while assuming the considered set to be compact. In [19], a ST control algorithm is proposed for linear systems while assuming the considered set to be convex and compact; see also [20], where nonlinear systems with a globally Lipschitz right-hand side are considered. Finally, in [21], an ET control algorithm is proposed for nonlinear systems with a globally bounded right-hand side.

In this paper, we consider a control system modeled by a differential inclusion with an input and a continuous feedback law that renders a given closed set forward invariant for the (non-triggered) closed-loop system. We assume the existence of a barrier function that certifies this invariance property is robust with respect to input perturbations. First, we propose an approach that efficiently determines the next sampling time from a given initial condition. This yields to a sampling sequence that guarantees the invariance property for the resulting ST closed-loop system. Afterwards, we investigate sufficient conditions to guarantee that the inter-event times admit a positive lower bound. Finally, the two results are combined to provide a ST control algorithm that guarantees both the invariance task for the resulting ST closed-loop system and the existence of a strictly positive lower bound on the inter-event times. As opposed to the existing literature, the right-hand side of the system and the chosen barrier function are not assumed to be smooth. Finally, via simulation, we highlight the effectiveness of our approach and compare it to existing works.

The remainder of the paper is organized as follows. Preliminaries are presented in Section II. Problem formulations are given in Section III. The main results are in Section IV. Finally, examples and simulations are in Section V.

Notation Let $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$, and $\mathbb{N}^* :=$

$\{1, 2, \ldots\}$. For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the Euclidean norm of x. For a set $K \subset \mathbb{R}^n$, we use $\mathrm{int}(K)$ to denote its interior, ∂K its boundary, $\mathrm{cl}(K)$ its closure, and $U(K)$ to denote an open neighborhood around K. For a set $O \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O, $K + O := \{x + y : x \in K,\ y \in O\}$, and $KO := \{\langle x, y \rangle : x \in K,\ y \in O\}$. The distance between x and the set K is given by $|x|_K := \inf\{|x - y| : y \in K\}$. By $[x, y]$, we denote the line segment relating x to y, $\langle x, y \rangle := x^\top y$ denotes the scalar product between x and y, and $\langle x, K \rangle := \{x^\top z : z \in K\}$. By $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote a set-valued map associating each element $x \in \mathbb{R}^n$ into a subset $F(x) \subset \mathbb{R}^n$, $\mathrm{dom}\, F$ denotes the domain of definition of F, and $F(K) := \{\eta \in F(x) : x \in K\}$. For a continuously differentiable function $\rho : \mathbb{R}^n \to \mathbb{R}$, $\nabla \rho(x)$ denotes the gradient of ρ evaluated at x. Finally, for a symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ stand for the minimum and maximum eigenvalues of A, respectively.

II. Preliminaries

A. Set-Valued Analysis

A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *locally Lipschitz* if, for each compact set $K \subset \mathbb{R}^n$, there exists $k > 0$ such that, for each $x \in K$ and each $y \in K$, $F(y) \subset F(x) + k|x - y|\mathbb{B}$.

The set-valued map F is said to be *outer semicontinuous* at $x \in \mathbb{R}^m$ if, for all $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^m$ and for all $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ with $x_i \to x$, $y_i \in F(x_i)$ for each i, and $y_i \to y \in \mathbb{R}^n$, we have $y \in F(x)$; see [22, Definition 5.9].

The set-valued map F is said to be *lower semicontinuous* (or, equivalently, *inner semicontinuous*) at $x \in \mathbb{R}^m$ if, for each $\epsilon > 0$ and for each $y_x \in F(x)$, there exists $U(x)$ such that, for each $z \in U(x)$, there exists $y_z \in F(z)$ such that $|y_z - y_x| \leq \epsilon$; see [23, Proposition 2.1].

The set-valued map F is said to be *upper semicontinuous* at $x \in \mathbb{R}^m$ if, for each $\epsilon > 0$, there exists $U(x)$ such that, for each $y \in U(x)$, $F(y) \subset F(x) + \epsilon\mathbb{B}$; see [24, Definition 1.4.1].

The set-valued map F is said to be outer (lower, and upper, respectively) semicontinuous if it is outer (lower, and upper, respectively) for all $x \in \mathbb{R}^m$.

The map F is said to be uniformly upper semicontinuous on $K \subset \mathbb{R}^m$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for each $x \in K$ and for each $y \in x + \delta\mathbb{B}$, $F(y) \subset F(x) + \epsilon\mathbb{B}$.

Finally, a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *locally bounded* if, for each $x \in \mathbb{R}^n$, there exist $U(x)$ and $K > 0$ such that, for each $y \in U(x)$, $|\zeta| \leq K$ for all $\zeta \in F(y)$.

Let $B : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Let Ω be any subset of zero measure in $\mathbb{R}^n$, and let Ω_B be the set of points in $\mathbb{R}^n$ at which B fails to be differentiable. Then, the Clarke generalized gradient at x is defined as

$$\partial_C B(x) := \mathrm{co}\left\{\lim_{i\to\infty} \nabla B(x_i) : x_i \to x,\ x_i \notin \Omega_B,\ x_i \notin \Omega\right\}.$$

Note that, when the function B is differentiable, the Clarke generalized gradient $\partial_C B$ is equivalent to the gradient ∇B. Finally, for a set $K \subset \mathbb{R}^n$, according to [25], the *contingent* cone of K at x is given by

$$T_K(x) := \left\{v \in \mathbb{R}^n : \liminf_{h\to 0^+} \frac{|x + hv|_K}{h} = 0\right\}. \qquad (1)$$

B. Constrained Differential Inclusions

A constrained differential inclusion $\mathcal{H}_f := (C, F)$ is defined as the continuous-time system

$$\mathcal{H}_f : \qquad \dot{x} \in F(x) \quad x \in C \subset \mathbb{R}^n, \qquad (2)$$

with the state $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$, and the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Note that the set C in (2) is not necessarily open and does not necessarily correspond to $\mathbb{R}^n$. Next, we introduce the concept of solution to $\mathcal{H}_f$.

Definition 1: (Solution to $\mathcal{H}_f$) A function $x : \mathrm{dom}\, x \to \mathbb{R}^n$ with $\mathrm{dom}\, x \subset \mathbb{R}_{\geq 0}$ and $t \mapsto x(t)$ locally absolutely continuous is a *solution* to $\mathcal{H}_f$ if

(S1) $x(0) \in \mathrm{cl}(C)$,

(S2) $x(t) \in C$ for all $t \in \mathrm{int}(\mathrm{dom}\, x)$,

(S3) $\frac{dx}{dt}(t) \in F(x(t))$ for almost all $t \in \mathrm{dom}\, x$.

•

A solution x is *complete* if $\mathrm{dom}\, x$ is unbounded. It is *maximal* if there does not exist another solution y such that $\mathrm{dom}\, x$ is a proper subset of $\mathrm{dom}\, y$ and $x(t) = y(t)$ for all $t \in \mathrm{dom}\, x$.[1]

Finally, given $x_o \in \mathrm{cl}(C)$ and $T > 0$, the *reachability map* $R : \mathbb{R}_{\geq 0} \times \mathrm{cl}(C) \rightrightarrows \mathrm{cl}(C)$ for $\mathcal{H}_f$ is given by

$$R(T, x_o) := \{x(t) : x \in \mathcal{S}_{\mathcal{H}_f}(x_o),\ t \in \mathrm{dom}\, x \cap [0, T]\}, \quad (3)$$

where $\mathcal{S}_{\mathcal{H}_f}(x_o)$ is the set of maximal solutions to $\mathcal{H}_f$ starting from x_o.

C. Forward Invariance and Barrier Functions

Consider a system $\mathcal{H}_f := (C, F)$ and a set $X \subset \mathrm{cl}(C)$.

Definition 2 (Forward pre-Invariance): The set X is *forward pre-invariant* for $\mathcal{H}_f$ if, for each initial condition $x_o \in X$ and for each solution $x \in \mathcal{S}_{\mathcal{H}_f}(x_o)$, $x(t) \in X$ for all $t \in \mathrm{dom}\, x$. •

The "pre" in forward pre-invariance is used to accommodate non-complete maximal solutions.

Next, we introduce the notion of barrier function candidate defining a set $X \subset \mathrm{cl}(C)$.

Definition 3 (Barrier Function Candidate): Given $C \subset \mathbb{R}^n$, the function $\rho : C \to \mathbb{R}$ is a *barrier function candidate* defining the set $X \subset \mathrm{cl}(C)$[2] if

$$X = \{x \in \mathrm{cl}(C) : \rho(x) \geq 0\}. \qquad (4)$$

•

For a set X given as in (4), we define

$$X_e := \{x \in \mathbb{R}^n : \rho(x) \geq 0\}. \qquad (5)$$

[1] Note that each complete solution is maximal, but not all maximal solutions are complete.

[2] In some safety related works [26], a barrier function candidate defines the set X according to $X = \{x \in \mathrm{cl}(C) : \rho(x) \leq 0\}$.

III. Problem Formulation

Consider a constrained control system $\mathcal{H}_f^u$ given by

$$\mathcal{H}_f^u: \quad \dot{x} \in F(x,u) \quad x \in C \subset \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{6}$$

Assumption 1: The set-valued map $F : C \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded with convex images. Additionally, the set C is closed. •

Remark 1: We assume the set C to be closed without loss of generality. Indeed, when C is not closed, we can consider its closure, which can add solutions to the system, but the conditions we propose still apply. •

Given a feedback law $\kappa : C \mapsto \mathbb{R}^m$, the resulting closed-loop system is given by

$$\mathcal{H}_f^{cl}: \qquad \dot{x} \in F^{cl}(x) := F(x, \kappa(x)) \qquad x \in C. \tag{7}$$

In a ST control setting, measurements of the state x are available only at sampling instants defined by a sequence $\{t_i\}_{i=0}^{\infty} \subset \mathbb{R}_{\geq 0}$ such that $t_0 = 0$ and $t_{i+1} > t_i$. When the control law κ remains constant between each two samples t_i and t_{i+1}, that is, κ is subject to a zero-order sample and hold, the actual control signal that is applied to the system is given by

$$u(t) = \kappa(x(t_i)) \quad \forall t \in [t_i, t_{i+1}) \quad \forall i \in \mathbb{N}. \tag{8}$$

Next, we define the concept of solutions to $\mathcal{H}_f^u$ in closed loop with (8).

Definition 4 (Solution to ST closed-loop systems): A locally absolutely continuous function $x : \operatorname{dom} x \to C$, $\operatorname{dom} x \subset \mathbb{R}_{\geq 0}$, starting from $x_o \in C$, is a solution to $\mathcal{H}_f^u$ in closed loop with (8) if, in addition to (S2), for all $i \in \mathbb{N}$,

$$\dot{x}(t) \in F(x(t), \kappa(x(t_i))) \text{ for a.a } t \in [t_i, t_{i+1}] \cap \operatorname{dom} x. \tag{9}$$

•

The objective in ST control is to use the state measurements available at the t_i's to deduce the largest possible t_{i+1} such that $t_{i+1} > t_i$ and that the closed loop resulting from controlling $\mathcal{H}_f^u$ using (8) still achieves forward pre-invariance of X.

Problem 1 (Finding the next sampling time): Given the control system $\mathcal{H}_f^u$, a closed set $X \subset C$, and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that X is forward pre-invariant for $\mathcal{H}_f^{cl}$, find a function $T_s : C \mapsto \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for each $x_o \in X$ and for each $t \mapsto x(t)$ solution to

$$\dot{x} \in F(x, \kappa(x_o)) \qquad x \in C \tag{10}$$

starting from $x_o \in X$, we have

$$x(t) \in X \qquad \forall t \in [0, T_s(x_o)] \cap \operatorname{dom} x. \tag{11}$$

•

One solution to Problem 1 is provided in [20], where T_s is proportional to the inverse of the Lipschitz constant of F; see also [18], where X is compact, F is smooth, and T_s is proportional to the inverse of a function upper bounding the decrease rate of a Lyapunov-like function. Our approach is compared to the one in [18] in Section V.

Solving Problem 1 allows us to recursively construct a feasible sampling sequence given by $t_{i+1} = t_i + T_s(x(t_i))$. However, such a sequence is not guaranteed to have the inter-event times $t_{i+1} - t_i$ uniformly larger than a positive constant. The absence of such a guarantee could lead to Zeno behavior. Hence, we formulate the following problem.

Problem 2 (Uniformly nonvanishing inter-event times): Consider the control system $\mathcal{H}_f^u$ in (6) and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that X is forward pre-invariant for $\mathcal{H}_f^{cl}$. Determine conditions guaranteeing the existence of $T_s^* > 0$ and a sampling sequence $\{t_i\}_{i=0}^{\infty}$ solving Problem 1 such that $t_{i+1} - t_i \geq T_s^*$ for all $i \in \mathbb{N}$. •

Existing solutions to Problem 2 require smoothness of the barrier candidate ρ and either boundedness of X [18] or bounded variation of F [20], [21].

IV. Main Results

A. Solutions to Problem 1

In the following result, we propose an answer to Problem 1. The construction of the function T_s at each $x_o \in X$, proposed below, involves an approximation of how fast a solution x starting from x_o can move towards ∂X during a forward propagation interval $[0, \bar{T}]$. This speed is upper bounded by the supremum of the scalar product between the gradient of the barrier function candidate $\rho(\cdot)$ and $F(\cdot, \kappa(x_o))$ on the set that can be reached by the solution x to (10) over the interval $[0, \bar{T}]$. This set is denoted by $\hat{R}(\bar{T}, x_o)$, which is an over-approximation of the reachable set $R(\bar{T}, x_o)$ along the solution to (10).

Theorem 1: Given the control system $\mathcal{H}_f^u$, suppose Assumption 1 holds. Suppose there exist a locally Lipschitz barrier function candidate ρ defining a closed set X as in (4) and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that X is forward pre-invariant for $\mathcal{H}_f^{cl}$. Then, for any $x_o \in C \cap X$ and for any $\bar{T} > 0$, Problem 1 is solved with

$$T_s(x_o) := \begin{cases} \bar{T} & \text{if } M_s(\bar{T}, x_o) \leq 0 \\ \min\left\{\bar{T}, \frac{\rho(x_o)}{M_s(\bar{T}, x_o)}\right\} & \text{otherwise,} \end{cases} \tag{12}$$

where

$$\begin{aligned} M_s(\bar{T}, x_o) := \sup\{&\langle -\gamma, \eta\rangle : \\ &\gamma \in \partial_C \rho(y), \eta \in F(y, \kappa(x_o)) \cap T_C(y), \\ &y \in \hat{R}(\bar{T}, x_o)\}, \end{aligned} \tag{13}$$

and $\hat{R} : \mathbb{R}_{\geq 0} \times C \rightrightarrows C$ is an over-approximation of the reachable set $R(\bar{T}, x_o)$ along the solutions to (10), namely $R(\bar{T}, x_o) \subseteq \hat{R}(\bar{T}, x_o)$. □

Remark 2: The over-estimation $\hat{R}$ of the reachability map R can be always computed because under Assumption 1 and the continuity of κ, $(x, x_o) \mapsto F(x, \kappa(x_o))$ is locally bounded on $C \times C$. However, the tighter this over approximation is, the larger the sampling function T_s will be. Methods to compute $\hat{R}$ are available in [27], [28], [29]. •

B. Solutions to Problem 2

Consider a constrained control system $\mathcal{H}_f^u$ and let $\kappa : C \to \mathbb{R}^m$ be a continuous feedback law rendering the set X, defined using a barrier function candidate ρ as in (4), forward pre-invariant for $\mathcal{H}_f^{cl}$.

Assumption 2: The barrier function candidate ρ is locally Lipschitz. Additionally, given $\kappa : C \to \mathbb{R}^m$, there exist locally Lipschitz functions $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying $\alpha(x) > 0$ for all $x \in \partial X_e$ and $\gamma(x, x) = 0$ such that

$$\begin{aligned} &\langle \zeta, f \rangle \geq \alpha(x) - \gamma(x, \eta) \quad \forall (x, \eta) \in X \times X, \\ &\qquad \forall (\zeta, f) \in \partial_C \rho(x) \times (F(x, \kappa(\eta)) \cap T_C(x)) . \end{aligned} \tag{14}$$

•

We also consider the following additional assumption.

Assumption 3: There exist $T_1 > 0$ and $\beta > 0$ such that, for each solution x to (10) starting from

$$K := \{x \in X : |x|_{\partial X_e} \geq \beta\}, \tag{15}$$

we have $x(t) \in X$ for all $t \in [0, T_1]$. •

Note that Assumption 3 holds for free when the set K is compact.

In the following result, we provide sufficient conditions for the existence of $T_s^* > 0$ that solves Problem 2.

Theorem 2: Given the control system $\mathcal{H}_f^u$, suppose Assumption 1 holds. Suppose there exists a barrier function candidate ρ and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that Assumption 2 holds. Furthermore, with $X \subset C$ as in Definition 3, suppose there exist $\beta > 0$ and $T_1 > 0$ such that Assumption 3 holds. Then, Problem 2 is solved with

$$T_s^* := \min\{T_1, T_2\} \tag{16}$$

provided that

$$T_2 := \min\{T_r(x) : x \in X,\ |x|_{\partial X_e} \leq \beta\} > 0, \tag{17}$$

where, for any $\bar{T} > 0$,

$$T_r(x) := \begin{cases} \bar{T} & \text{if } M_r(\bar{T}, x) \leq 0 \\ \min\left\{\bar{T}, \frac{2\alpha(x)}{M_r(\bar{T},x)}\right\} & \text{otherwise,} \end{cases} \tag{18}$$

$$M_r(\bar{T}, x) := M_\alpha(\bar{T}, x) + M_\gamma(\bar{T}, x), \tag{19}$$

$$\begin{aligned} M_\gamma(\bar{T}, x) :=& \sup\{\langle \gamma_1, \eta \rangle : \gamma_1 \in \partial_C \gamma(y, x), \\ & \eta \in F(y, \kappa(x)) \cap T_C(y), y \in \hat{R}(\bar{T}, x)\}, \end{aligned} \tag{20}$$

$$\begin{aligned} M_\alpha(\bar{T}, x) :=& \sup\{\langle -\gamma_2, \eta \rangle : \gamma_2 \in \partial_C \alpha(y), \\ & \eta \in F(y, \kappa(x)) \cap T_C(y),\ y \in \hat{R}(\bar{T}, x)\}, \end{aligned} \tag{21}$$

and $\hat{R}$ is an over-approximation of the reachable set. □

Note that the sampling function T_r takes advantage of the robustness gained through Assumption 2.

In Lemma 1 below, we provide sufficient conditions to verify Assumption 3 when the set K in (15) is not compact. To do so, we assume the following, for a given ρ and κ.

Assumption 4: Given $\beta > 0$ and $T_1 > 0$ there exists a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ and $\epsilon_1 > 0$ such that

$$V(x) \leq 0 \qquad \forall x \in \{y \in X : \rho(y) = 0\} \tag{22}$$

and, for each $x \in X$, $|x|_{\partial X} \geq \beta$ implies $V(x) \geq \epsilon_1$. Furthermore, there exists a locally Lipschitz function $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, such that

$$\begin{aligned} &\langle \zeta, \eta \rangle \geq \sigma(V(x), V(x_o)) \quad \forall (x, x_o) \in C \times C, \\ &\qquad \forall (\zeta, \eta) \in \partial_C V(x) \times (F(x, \kappa(x_o)) \cap T_C(x)), \end{aligned} \tag{23}$$

$$-\int_0^{\nu_o} \frac{d\nu}{\sigma(\nu, \nu_o)} \in (-\infty, 0] \cup [T_1, +\infty) \quad \forall \nu_o \geq \epsilon_1. \tag{24}$$

•

Remark 3: Note that (24) holds for free when the function σ is linear on its arguments. •

Lemma 1: Given the control system $\mathcal{H}_f^u$, suppose Assumption 1 holds. Suppose there exists a continuous barrier function candidate ρ and a continuous feedback law $\kappa : C \to \mathbb{R}^m$. Furthermore, with $X \subset C$ as in Definition 3, suppose there exists $\beta > 0$ and $T_1 > 0$ such that Assumption 4 holds. Then, Assumption 3 holds with such T_1 and β. □

C. Combining the Solutions to Problems 1 and 2

In the following result, we combine the solutions to Problems 1 and 2, leading to a ST control strategy for the system $\mathcal{H}_f^u$ in (6) that guarantees a strictly positive lower bound on the sampling period. Roughly speaking, when the system's states are deep inside the set X, the next sampling time is computed using Theorem 1. Since this result does not guarantee the existence of a uniform lower bound on inter-event times as the solutions approach ∂X_e, Theorem 2 is used to determine the next sampling time when x is close to ∂X_e.

Theorem 3: Given the control system $\mathcal{H}_f^u$ in (6), suppose Assumption 1 holds. Assume that the solutions to $\mathcal{H}_f^u$ are unique for any piecewise constant input signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Suppose there exist a barrier function candidate ρ and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that Assumption 2 holds. Furthermore, with $X \subset C$ as in Definition 3, suppose there exist $\beta > 0$ and $T_1 > 0$ such that Assumption 3 holds. Then, given $\bar{T} > 0$, the sampling sequence $\{t_i\}_{i=0}^\infty$ designed recursively as

$$t_{i+1} = t_i + \max\{T_s^*, T_r(x(t_i)), T_s(x(t_i))\} \quad \forall i \in \mathbb{N}, \tag{25}$$

where T_s^*, T_r, and T_s are computed according to (16), (18), and (12), respectively, guarantees forward pre-invariance of the set X for the ST closed-loop system in (9). Moreover, the inter-event times are always larger than a positive constant provided that (17) holds. □

Next, we relax Assumption 3 when, for some $\beta > 0$, the following assumption holds.

Assumption 5: The set K in (15) is compact. •

We will also need the following Assumption.

Assumption 6: The set-valued map $x_o \mapsto \hat{R}(\bar{T}, x_o)$, over estimating $x_o \mapsto R(\bar{T}, x_o)$ along the solutions to (10), is outer semicontinuous and locally bounded on X. •

Remark 4: When the map $(x, x_o) \mapsto F(x, \kappa(x_o))$ is outer semicontinuous and locally bounded, we conclude, using [30, Proposition 1], that the map $x_o \mapsto R(\bar{T}, x_o)$ is also outer semicontinuous and locally bounded. Hence, assuming the same regularity properties to hold for $x_o \mapsto \hat{R}(\bar{T}, x_o)$ is not very restrictive. •

Theorem 4: Given the control system $\mathcal{H}_f^u$, suppose that Assumption 1 holds. Assume that the solutions to $\mathcal{H}_f^u$ are unique for any piecewise constant input signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Suppose there exist a barrier function candidate ρ and a continuous feedback law $\kappa : C \to \mathbb{R}^m$, with $X \subset C$ as in Definition 3, such that Assumption 2 holds. Finally, suppose there exists an over-approximation of the reachable set $\hat{R}$ such that Assumption 6 holds. Then, given $\bar{T} > 0$, the sampling sequence $\{t_i\}_{i=0}^{\infty}$ designed recursively as:

$$t_{i+1} = t_i + \max\{T_r(x(t_i)), T_s(x(t_i))\} \quad \forall i \in \mathbb{N}, \tag{26}$$

where T_r and T_s are computed according to (18) and (12), respectively, guarantees forward pre-invariance of the set X for the ST closed-loop system in (9). Moreover, the sampling period is always larger than a positive constant provided that there exists $\beta > 0$ such that Assumption 5 holds and provided that (17) holds. □

Next, we would like to check an inequality similar to (17) only at points in the set $\partial X_e \cap X$. To do so, in addition to Assumptions 2, 5, and 6, for some $\beta^* > 0$ and for $G^* := \{x \in X : |x|_{\partial X_e} \leq \beta^*\}$, we consider the following assumption.

Assumption 7: One of the following is true:
1) The set G^* is compact.
2) The set-valued maps

$$\begin{aligned} &x_o \mapsto \partial_C\gamma(\hat{R}(\bar{T}, x_o), x_o), \quad x_o \mapsto \partial_C\alpha(\hat{R}(\bar{T}, x_o)), \\ &x_o \mapsto F(\hat{R}(\bar{T}, x_o), \kappa(x_o)), \quad x_o \mapsto \alpha(x_o) \end{aligned} \tag{27}$$

are uniformly upper semicontinuous on G^* and bounded on $\partial X_e \cap X$. Furthermore,

$$T_3 := \inf\{\hat{T}_r(z) : z \in \partial X_e \cap X,\ \hat{M}_r(\bar{T}, z) > 0\} > 0, \tag{28}$$
$$\inf\{\alpha(z) : z \in \partial X_e \cap X\} > 0, \tag{29}$$

where $\hat{M}_r(\bar{T}, z) := \sup\{\langle \gamma_1, \eta\rangle : \gamma_1 \in \partial_C\gamma(y, z), \eta \in F(y, \kappa(z)), y \in \hat{R}(\bar{T}, z)\} + \sup\{\langle -\gamma_2, \eta\rangle : \gamma_2 \in \partial_C\alpha(y), \eta \in F(y, \kappa(z)),\ y \in \hat{R}(\bar{T}, z)\}$, $\hat{T}_r(z) := \min\left\{\bar{T}, \frac{2\alpha(z)}{\hat{M}_r(\bar{T}, z)}\right\}$. •

Theorem 5: Given the control system $\mathcal{H}_f^u$, suppose that Assumption 1 holds. Assume that the solutions to $\mathcal{H}_f^u$ are unique for any piece-wise constant input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Suppose there exist a barrier function candidate ρ and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that, with $X \subset C$ as in Definition 3, Assumption 2 holds. Suppose there exists a set-valued map $x_o \mapsto \hat{R}(\bar{T}, x_o)$ such that Assumption 6 holds. Finally, suppose there exists $\beta^* > 0$ such that Assumption 7 holds. Then, the sampling sequence $\{t_i\}_{i=0}^{\infty}$ designed in (26) guarantees forward pre-invariance of X for the ST closed-loop system. Moreover, the inter-event times are always larger than a positive constant provided that Assumption 5 holds for some $\beta \in (0, \beta^*]$. □

V. EXAMPLE

The objective in this example[3] is to compare the inter-event times obtained using Theorem 5 to two other methods. The first one is an event-triggered strategy while the other one is based on the ST strategy proposed in [18, Theorem 4.3]. Consider the control system $\mathcal{H}_f^u = (F, \mathbb{R}^2)$, where

$$F(x, u) := \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{30}$$

$x := (x_1, x_2) \in \mathbb{R}^2$, and $u \in \mathbb{R}$. Furthermore, consider the feedback law $\kappa(x) := Kx := [1 \quad -4]x$. The origin of the closed-loop of $\mathcal{H}_f^u$ using $u = \kappa(x)$, denoted $\mathcal{H}_f^{cl}$, is asymptotically stable. Indeed, using the Lyanpunov function $V(x) := x^\top Px$, with $P := \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix}$, we conclude that $\langle \nabla V(x), Ax + BKx\rangle = -x^\top Qx$ and $Q := \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 1.5 \end{bmatrix}$. Furthermore, we consider the set X given by $X := \{x \in \mathbb{R}^2 : V(x) \leq 0.1\}$. Note that the set X admits the barrier function candidate $\rho(x) := 0.1 - V(x)$. Note that condition (14) is satisfied with $\alpha(x) := x^\top Qx$ and $\gamma(x, \eta) := \frac{1}{2}x^\top PBK(x - \eta)$. Next, the over-approximation $\hat{R}$ from $x_o \in X$ along $[t_o, t_o + \bar{T}]$ is computed as $\hat{R}(\bar{T}, x_o) := R(\bar{T}, x_o) + r\mathbb{B}$, where $R(\bar{T}, x_a) = \{y \in \mathbb{R}^2 : \exists t \in [0, \bar{T}], y = x(t)\}$, and x is the solution to (10) starting from x_o and $r = 0.025$ captures possible errors in the modeling. Note that Assumption 1 is trivially satisfied since F_a is single valued and smooth. Assumption 5 is satisfied since the set X is compact. Also, Assumption 6 is verified using Remark 4. Finally, Assumption 7 is satisfied since Assumption 6 is satisfied and X is compact. When computing $M(x_o, \bar{T})$ for (25), we propose two strategies for selecting the value of $\bar{T}$ as a function of x_o. From now on, we use the notation $T_s(x_o) := T_s(x_o, \bar{T})$ and $T_r(x_o) := T_r(x_o, \bar{T})$.

a. Adapting $\bar{T}$ to the norm of $F(x_o, \kappa(x_o))$: We tested both linear and non-linear relationships between $\bar{T}$ and $|F(x_o, \kappa(x_o))|$. Indeed, consider the map $\bar{T} : \mathbb{R}^n \mapsto [T_{min}, T_{max}]$ given by

$$\bar{T}(x_o) := (T_{max} - T_{min})(1 - F_N(x_o))^{c_s} + T_{min}, \tag{31}$$

where $c_s \in (0, \infty)$, $T_{max} > T_{min} > 0$, and $F_N(x_o) := F(x_o, \kappa(x_o))/\sup\{F(y, \kappa(y)) : y \in X\}$.

b. Evaluating multiple values of $\bar{T}$ over a multiple-step receding horizon: Given $T_{max} > T_{min} > 0$, $N \in \mathbb{N}$, and $\Delta := (T_{max} - T_{min})/N$, we are interested in finding the value of $n \in \{0, 1, ..., N\}$ that maximizes the following value function $J_n := c_h T_n^o + (1 - c_h)T_n^1$, where $c_h \in [0, 1]$, $T_n^o := f(n, x_o)$, $T_n^1 := \max\{f(m, x_1) : m \in \{0, ...N\},\ x_1 = x(T_{min} + n\Delta)\}$, and $f(k, y) := \max\{T_r(T_{min} + k\Delta, y), T_s(T_{min} + k\Delta, y)\}$. The constant c_h adjusts the trade-off between the current sample time T_n^o and the best next sample time T_n^1.

Comparison: All the solutions are simulated from the initial condition $x_o := (-0.1, -0.3)$. Figure 1 shows the evolution of V along a ST closed-loop solution as well as the corresponding inter-event times. Two strategies based on Theorem 5 are

[3] Code at https://github.com/HybridSystemsLab/SelfTriggeredSublevelSet.

simulated. In the first one, the value of $\bar{T}$ is adapted to the norm of $F(x_o, \kappa(x_o))$ as in (31), where the best performance is obtained for $c_s = 150$. Then, the value of $\bar{T}$ is computed following the two-step receding horizon, where the best performance is obtained for $c_h = 0.5$. For these two methods, we took $T_{max} := 2$ and $T_{min} := 0.25$. Furthermore, Figure 1 compares the strategies proposed in this paper to an ET strategy, in which, we update the control input each time the solution reaches ∂X. We also compare to the ST strategy proposed in [18, Theorem 4.3]. The inter-event times obtained using [18, Theorem 4.3] are smaller than those obtained using Theorem 5. Remarkably, according to Table I, our results are comparable to the ET strategy although our strategy does not require continuous availability of the measurements.

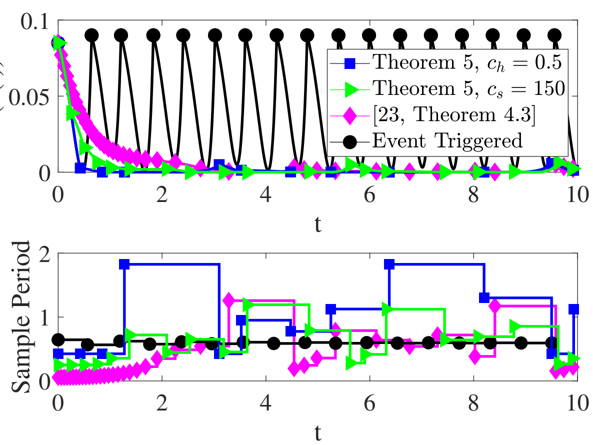

Fig. 1: Closed-loop solutions using various strategies

Theorem 5	Average Period	Minimum Period
Scaled		
$c_s = 1$	0.19	0.04
$c_s = 150$	0.53	0.25
Receding horizon		
$c_h = 1$	0.58	0.425
$c_h = 0.5$	0.92	0.425
[18, Theorem 4.3]	0.26	0.06
Event Triggered	0.59	0.56

TABLE I: Summary of each inter-event properties

VI. Conclusion

In this paper, we present a self-triggered control strategy to guarantee forward pre-invariance of a closed set for a constrained differential inclusion. Sufficient conditions are derived such that the inter-event times are guaranteed to be always larger than a positive constant.

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