Adaptive Safety Using Control Barrier Functions and Hybrid Adaptation

Mohamed Maghenem, Andrew J. Taylor, Aaron D. Ames, and Ricardo G. Sanfelice

***Abstract*—This paper presents a hybrid adaptive law for safety-critical adaptive control of constrained continuous-time systems under the effect of unknown disturbances. The proposed adaptive law features a hybrid update law that, using a hysteresis-type mechanism, appropriately resets the estimate of the disturbance. In contrast to continuous-time adaptation laws for safety-critical control, our hybrid adaptive law relaxes the assumption typically imposed on the unknown disturbances as well as the behavior usually imposed around the boundary of the safe region. We illustrate the benefits of the proposed hybrid law in an adaptive cruise control problem.**

I. Introduction

Safety in the context of dynamical systems can be framed as forward set invariance, i.e., solutions starting from a given set of initial conditions remain in a desired safe region [1], [2], [3]. Control Barrier Functions (CBFs) [4] were recently introduced so as to extend these ideas to nonlinear control systems, and have proven effective in synthesizing control laws that achieve safety [5], [6], [7], including hybrid systems [8], [9]. Control laws synthesized via CBFs rely on an accurate model of the dynamics, and thus may fail to achieve safety in the presence of model uncertainty [10], [11]. The robustness of CBFs to uncertainty has been studied in [12], [13], [14], [15], but often leads to restrictions on the behavior of the system nearby the boundary of the safe set. The integration of CBFs with data-driven learning methods has become increasingly popular [16], [17], [18], but often requires an episodic, offline learning.

In this work, we consider an online, *adaptive* approach for mitigating the impact of model uncertainty on safety guarantees. Adaptive control techniques have shown a great ability to overcome and compensate for the effect of unmodeled disturbances [19], [20]. Building upon the concept of adaptive Control Lyapunov Function (aCLF) for stabilization of nonlinear systems with uncertainty [21], the notion of adaptive Control Barrier Functions (aCBF) has recently been proposed in [22] to address the safety of nonlinear systems in the presence of uncertainty. In [22], an aCBF establishes a *disturbance-to-state* relationship that allows the design of an adaptive law producing an estimate of the disturbance so as to guarantee the required safety property for the closed-loop system. However, the adaptive law proposed in [22] may constrain the solutions to the system to a set that shrinks set over time. The follow-up extension in [23] alleviates this issue through the introduction of data history associated to the system, but restricts the set of initial conditions, which might be conservative. In addition, the results in [23] further require that the estimate of the disturbance remains in a pre-specified, bounded set.

In this paper, motivated by the shortcomings outline above, we present an adaptive law that features hybrid dynamics, in particular, to provide an estimate of the disturbance under weaker conditions than those in previous works. More precisely, we propose an adaptive law that leads to a novel hybrid adaptation system guaranteeing safety that is uniform in the disturbance, hence, robust. This paper makes two main contributions. The first is the formulation of a novel, hybrid dynamics-based method for achieving adaptive safety in the presence of model uncertainty. This method uses the discontinuous nature of hybrid dynamical systems to reset an estimate of the disturbance whenever the system approaches the boundary of the safe region. Unlike the work in [22], [23], this allows the assumption that the disturbance estimate remains uniformly bounded to be avoided. By modeling the plant and the update law as a hybrid dynamical system in the framework of [24], [9], we show that the resulting system is well-posed, which is a property that is instrumental for robustness. Moreover, by augmenting the hybrid adaptation system with a logic variable updated using a hysteresis-type mechanism, we show that its solutions are not Zeno, hence, avoiding the accumulation of events. The second contribution of this paper is a modification of the *disturbance-to-state* relationship defined by an aCBF, wherein safety is only enforced near the boundary of the safe region. This relaxation allows the system to exploit the control input for other purposes, e.g., stabilization, when it is far from the boundary of safe set.

The remainder of the paper is organized as follows. Section II provides some preliminary background material. In Section III, we present the problem of adaptive safety in the context of uncertain constrained continuous-time systems. The aCBF notion is introduced in Section IV, and limitations of the existing continuous-time adaptations are highlighted. In Section VI, we present our main results. An example illustrating the proposed approach is in Section VII.

Notation. Let $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. Given two vectors x and y of the same dimension, m_x denotes the dimension of x, $x^\top$ denotes the transpose of x, $|x|$ denotes the Euclidean norm of x, and $\langle x, y \rangle := x^\top y$ denotes the scalar product of x and y. Given a nonempty set $K \subset \mathbb{R}^{m_x}$, $|x|_K :=$

M. Maghenem is with CNRS, GIPSA Lab, Grenoble, France mohamed.maghenem@gipsa-lab.grenoble-inp.fr.
R. G. Sanfelice is with the Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA ricardo@ucsc.edu. A. J. Taylor and A. D. Ames are with the Department of Computing and Mathematical Sciences, California Institute of Technology, Pasadena, CA 91125, USA ajtaylor@caltech.edu, ames@caltech.edu.

Research partially supported by NSF Grants no. ECS-1710621, CNS-1544396, and CNS-2039054, by AFOSR Grants no. FA9550-19-1-0053, FA9550-19-1-0169, and FA9550-20-1-0238, and by CITRIS and the Banatao Institute at the University of California.

$\inf_{y\in K}|x-y|$ defines the distance between x and the set K, $\Pi_K(x) := \{y \in K : |x-y| = |x|_K\}$ denotes the projection of x on K, $\text{int}(K)$ denotes the interior of K, ∂K denotes its boundary, and $\text{cl}(K)$ denotes its closure. For a nonempty set $O \subset \mathbb{R}^{m_x}$, $K\backslash O$ denotes the subset of elements of K that are not in O. By $\mathbb{B}$, we denote the closed unit ball centered at the origin. For a differentiable map $(x_1, x_2) \mapsto h(x_1, x_2) \in \mathbb{R}$, $\nabla_{x_i} h$ denotes the gradient of h with respect to x_i, $i \in \{1, 2\}$, and ∇h denotes the gradient of h with respect to $x := (x_1, x_2)$. For a symmetric positive definite matrix $\Gamma \in \mathbb{R}^{n\times n}$, $\lambda_{min}(\Gamma)$ and $\lambda_{max}(\Gamma)$ denote the smallest and the largest eigenvalues of Γ, respectively. By $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, we denote a set-valued map associating each element $x \in \mathbb{R}^m$ into a subset $F(x) \subset \mathbb{R}^n$. For a set-valued map $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a set $D \subset \mathbb{R}^m$, $F(D) := \{F(x) : x \in D\}$. Finally, we denote by $\mathcal{S}^m$ the set of symmetric positive definite matrices of dimension $m \times m$.

II. Preliminaries

A. Hybrid Dynamical Systems

Following [24], a hybrid dynamical system $\mathcal{H} = (C, F, D, G)$ is modeled as the combination of constrained differential and difference inclusions given by:

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D, \end{cases} \tag{1}$$

with the state variable $x \in \mathbb{R}^{m_x}$, the flow set $C \subset \mathbb{R}^{m_x}$, the jump set $D \subset \mathbb{R}^{m_x}$, and the flow and jump maps $F : \mathbb{R}^{m_x} \rightrightarrows \mathbb{R}^{m_x}$ and $G : \mathbb{R}^{m_x} \rightrightarrows \mathbb{R}^{m_x}$, respectively.

A hybrid arc ϕ is defined on a hybrid time domain denoted $\text{dom}\,\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$. The hybrid arc ϕ is parametrized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a discrete jump variable $j \in \mathbb{N}$. Its domain of definition $\text{dom}\,\phi$ is such that for each $(T, J) \in \text{dom}\,\phi$, $\text{dom}\,\phi \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\})$ for a sequence $\{t_j\}_{j=0}^{J+1}$, such that $t_{j+1} \geq t_j$, $t_0 = 0$, and $t_{j+1} = T$.

Definition 1 (Concept of solution to $\mathcal{H}$): A hybrid arc $\phi : \text{dom}\,\phi \to \mathbb{R}^{m_x}$ is a *solution* to $\mathcal{H}$ if:

(S1) The hybrid arc ϕ satisfies $\phi(0, 0) \in \text{cl}(C) \cup D$;

(S2) For all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \text{dom}\,\phi\}$ has nonempty interior, the map $t \mapsto \phi(t, j)$ is locally absolutely continuous and satisfies:

$$\begin{aligned} &\phi(t, j) \in C && \text{for all} && t \in \text{int}(I^j), \\ &\dot{\phi}(t, j) \in F(\phi(t, j)) && \text{for almost all} && t \in I^j; \end{aligned} \tag{2}$$

(S3) For all $(t, j) \in \text{dom}\,\phi$ with $(t, j+1) \in \text{dom}\,\phi$, we have:

$$\phi(t, j) \in D, \qquad \phi(t, j+1) \in G(\phi(t, j)). \tag{3}$$

•

A solution ϕ to $\mathcal{H}$ is said to be maximal if there is no solution ψ to $\mathcal{H}$ such that $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \text{dom}\,\phi$ and $\text{dom}\,\phi$ is a proper subset of $\text{dom}\,\psi$. It is said to be complete if $\text{dom}\,\phi$ is unbounded. It is said to be trivial if $\text{dom}\,\phi$ contains only one element. It is said to be continuous if it never jumps. It is said to be eventually discrete if $T := \sup_t \text{dom}\,\phi < \infty$ and $\text{dom}\,\phi \cap (\{T\} \times \mathbb{N})$ contains at least two points. It is said to be Zeno if it is complete and $\sup_t \text{dom}\,\phi < \infty$. The system $\mathcal{H}$ is said to be forward complete if the domain of each maximal solution is unbounded. It is said to be pre-forward complete if each maximal solution is either complete or bounded.

B. Hybrid Basic Conditions

Next, we recall the definition of outer semicontinuity and local boundedness of set-valued maps. Consider a set-valued map $F : K \rightrightarrows \mathbb{R}^n$, where $K \subset \mathbb{R}^m$. The map F is said to be *outer semicontinuous* at $x \in K$ if, for every sequence $\{x_i\}_{i=0}^{\infty} \subset K$ and for every sequence $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ with $\lim_{i\to\infty} x_i = x$, $\lim_{i\to\infty} y_i = y \in \mathbb{R}^n$, and $y_i \in F(x_i)$ for all $i \in \mathbb{N}$, we have $y \in F(x)$; see [24, Definition 5.9]. Moreover, the map F is said to be outer semicontinuous if it is outer semicontinuous for all $x \in K$, respectively. On the other hand, the map F is said to be *locally bounded* if, for any $x \in K$, there exist $U(x)$ an open neighborhood of x and $\beta > 0$ such that $|\zeta| \leq \beta$ for all $\zeta \in F(x')$ and for all $x' \in U(x) \cap K$.

Well-posed [24, Definition 6.2] hybrid systems refer to a class of hybrid systems where the solutions enjoy very useful structural properties [24, Chapter 6]. A hybrid system $\mathcal{H} = (C, F, D, G)$ is well posed if the following conditions, known as the hybrid basic conditions, are satisfied; see [24, Assumption 6.5] and [24, Theorem. 6.8] for more details.

(A1) The sets C and D are closed;

(A2) The flow map $F : C \rightrightarrows \mathbb{R}^{m_x}$ is outer semicontinuous and locally bounded with nonempty images;

(A3) The jump map $G : D \rightrightarrows \mathbb{R}^{m_x}$ is outer semicontinuous and locally bounded with nonempty images.

C. Safety and Uniform Safety

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (1). Furthermore, consider two sets $X_o \subset \text{cl}(C) \cup D$ and $X_u \subset \mathbb{R}^{m_x}$ such that $X_o \cap X_u = \emptyset$.

Definition 2 (Safety): The hybrid system $\mathcal{H}$ is said to be safe with respect to (X_o, X_u) if each solution ϕ starting from X_o satisfies $\phi(t, j) \in \mathbb{R}^{m_x}\backslash X_u$ for all $(t, j) \in \text{dom}\,\phi$. •

Next, we consider a hybrid system $\mathcal{H}_w$ with disturbances given by:

$$\mathcal{H}_w : \begin{cases} \dot{x} \in F_w(x, w) & (x, w) \in C_w \\ x^+ \in G_w(x, w) & (x, w) \in D_w, \end{cases} \tag{4}$$

where $x \in \mathbb{R}^{m_x}$ is the state and $w \in \mathcal{W} \subset \mathbb{R}^{m_w}$ is an unknown disturbance, for example, measurement noise or a modeling uncertainty. The set $\mathcal{W}$ contains the range of the disturbance w, $C_w \subset \mathbb{R}^{m_x} \times \mathcal{W}$ is the flow set, $D_w \subset \mathbb{R}^{m_x} \times \mathcal{W}$ is the jump set, $F_w : C_w \rightrightarrows \mathbb{R}^{m_x}$ is the flow map, and $G_w : D_w \rightrightarrows \mathbb{R}^{m_x}$ is the jump map. See [25, Definition 2.1] for the concept of *solution pairs* (ϕ, w) to hybrid systems $\mathcal{H}_w$ with disturbance.

Definition 3 (Uniform safety): The hybrid system $\mathcal{H}_w$ is said to be safe with respect to (X_o, X_u) uniformly in w if, for each pair (ϕ, w), the solution to $\mathcal{H}_w$ with $\phi(0, 0) \in X_o$, $\phi(t, j) \in \mathbb{R}^{m_x}\backslash X_u$ for all $(t, j) \in \text{dom}\,\phi$. •

III. Problem Formulation

Consider a constrained continuous-time control system with a disturbance w, denoted by $\mathcal{H}_{wu}$, given by

$$\mathcal{H}_{wu} : \dot{z} = f_{wu}(z, w, u) \quad (z, w, u) \in C_{wu}, \tag{5}$$

where $z \in \mathbb{R}^{m_z}$ is the state, $w \in \mathcal{W}$ is an unknown disturbance, $\mathcal{W} \subset \mathbb{R}^{m_w}$ is the set containing the possible values of w, $C_{wu} := \mathbb{R}^{m_z} \times \mathcal{W} \times \mathbb{R}^{m_u}$, $u \in \mathbb{R}^{m_u}$ is the control input, and $f_{wu} : C_{wu} \to \mathbb{R}^{m_z}$ is assumed to be continuous on C_{wu}. Furthermore, we assume that the disturbance $w \in \mathcal{W}$ satisfies the following assumption.

Assumption 1: The disturbance w is constant and there exists a known constant $\bar{w} > 0$ such that $\sup\{|w| : w \in \mathcal{W}\} \leq \bar{w}$. •

Next, we consider assigning the input of $\mathcal{H}_{wu}$ via the dynamic feedback law $u = \kappa(z, \hat{w})$, where $\hat{w}$ can be seen as an estimate of w generated by a hybrid adaptation system $\mathcal{H}_\eta$ given by:

$$\mathcal{H}_\eta : \begin{cases} \dot{\eta} = f_\eta(z, \eta) & (z, \eta) \in C_\eta \\ \eta^+ \in G_\eta(z, \eta) & (z, \eta) \in D_\eta \\ \hat{w} := \varphi(z, \eta), \end{cases} \tag{6}$$

where $\eta \in \mathbb{R}^{m_\eta}$ is the state of $\mathcal{H}_\eta$, $\hat{w} \in \mathbb{R}^{m_w}$ is the output, the sets $C_\eta \subset \mathbb{R}^{m_z} \times \mathbb{R}^{m_\eta}$ and $D_\eta \subset \mathbb{R}^{m_z} \times \mathbb{R}^{m_\eta}$ are the flow and the jump sets, respectively, and the maps $f_\eta : C_\eta \to \mathbb{R}^{m_\eta}$ and $G_\eta : D_\eta \rightrightarrows \mathbb{R}^{m_\eta}$ are the flow and jump maps, respectively. The closed-loop of $\mathcal{H}_{wu}$ using the feedback law $\kappa : \mathbb{R}^{m_z} \times \mathbb{R}^{m_w} \to \mathbb{R}^{m_u}$ and the adaptation $\mathcal{H}_\eta$ is a hybrid system denoted $\mathcal{H}_w$ and given by:

$$\mathcal{H}_w : \begin{cases} \dot{x} = f_w(x, w) & (x, w) \in C_w \\ x^+ \in G_w(x, w) & (x, w) \in D_w, \end{cases}$$

where $x := (z, \eta)$, $C_w := C_\eta \times \mathcal{W}$, $D_w := D_\eta \times \mathcal{W}$, and:

$$\begin{aligned} f_w(x, w) &:= (f_{wu}(z, \kappa(z, \varphi(z, \eta)), w), f_\eta(z, \eta)), \\ G_w(x, w) &:= (z, G_\eta(z, \eta)). \end{aligned}$$

Given two sets $X_o \subset \mathbb{R}^{m_x}$, $m_x := m_z + m_\eta$, and $X_u \subset \mathbb{R}^{m_x} \backslash X_o$, in this paper, we study the following problem.

Problem 1: Design the feedback law κ and the hybrid adaptation system $\mathcal{H}_\eta$ such that the closed-loop system $\mathcal{H}_w$ is safe with respect to (X_o, X_u) uniformly in $w \in \mathcal{W}$. •

Remark 1: Defining the sets (X_o, X_u) in the augmented space $\mathbb{R}^{m_x}$ allows us to consider safety constraints on the state z as well as on the input u. In particular, when the safety requirements involve only the state z, then $(X_o, X_u) := (X_{oz} \times \mathbb{R}^{m_\eta}, X_{uz} \times \mathbb{R}^{m_\eta})$, for some $X_{oz} \subset \mathbb{R}^{m_z}$ and for some $X_{uz} \subset \mathbb{R}^{m_z} \backslash X_{oz}$. •

Note that the hybrid system $\mathcal{H}_w$ flows whenever $\mathcal{H}_\eta$ flows; namely, when $x \in C_\eta$. Furthermore, it jumps whenever $\mathcal{H}_\eta$ jumps; namely, when $x \in D_\eta$. At jumps, z is kept constant since $\mathcal{H}_{wu}$ is a continuous-time plant.

The approach we use to solve Problem 1 is based on an extension, to guarantee safety, of the classical adaptive control technique studied in [21], [19] to ensure only convergence. Furthermore, we generalize the adaptive safety techniques proposed recently in [22] and [23] by introducing a hybrid adaptation algorithm $\mathcal{H}_\eta$ that requires weaker conditions than those in [22], [23].

IV. Adaptive Control Barrier Functions

We start by introducing the notion of adaptive control barrier function candidate. A similar definition is used in [26] to guarantee robust safety properties.

Definition 4 (aCBFc): A function $h_a : \mathbb{R}^{m_z} \times \mathbb{R}^{m_\eta} \to \mathbb{R}$ is an adaptive control barrier function candidate (aCBFc) with respect to (X_o, X_u) if there exists $\epsilon > 0$ such that:

$$h_a(x) < 0 \quad \forall x \in X_u, \quad h_a(x) > \epsilon \quad \forall x \in X_o. \tag{7}$$

•

Remark 2: The second inequality in (7) makes Definition 4 more restrictive than the classical definition of a barrier function candidate, as it is usually assumed to be only non negative on the set of initial conditions X_o; see [3], [4]. •

Given a continuously differentiable aCBFc h_a, with associated constant $\epsilon > 0$ such that (7) holds, and a feedback law κ, we introduce the following *disturbance-to-state* condition:

Assumption 2: There exist a nonempty set $\mathcal{G} \subset \mathcal{S}^{m_w}$ and continuous functions $\gamma : \mathbb{R}^{m_z} \times \mathbb{R}^{m_\eta} \to \mathbb{R}^{m_w}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying:

$$\alpha(s) \geq 0 \qquad \forall s \in (-\epsilon, \epsilon), \tag{8}$$

such that, for all $\Gamma \in \mathcal{G}$ and for all $(x, \hat{w}, w) \in \mathbb{R}^{m_x} \times \mathbb{R}^{m_w} \times \mathcal{W}$:

$$\begin{aligned} \langle \nabla_z h_a(x), f_{wu}(z, \kappa(z, \hat{w}), w) \rangle + \langle \Gamma \gamma(x), \nabla_\eta h_a(x) \rangle \\ \geq \alpha(h_a(x)) + \langle \gamma(x), \hat{w} - w \rangle \end{aligned} \tag{9}$$

•

Definition 5 (aCBF): An aCBFc h_a satisfying Assumption 2 is said to be an adaptive control barrier function (aCBF) with respect (X_o, X_u). •

Remark 3: Note that the notion of aCBF in Assumption 2 is similar to the one used in [23] and is a relaxation of the one proposed in [22]. More precisely, in [22], the function α is assumed to be non-negative on the entire real line $\mathbb{R}$, where in [23], α is assumed to be non-negative only on $(-\epsilon, +\infty)$. •

Remark 4: When the aCBFc h_a is a function of only z and Assumption 2 holds for a singleton set $\mathcal{G} \subset \mathcal{S}^{m_w}$, then it holds for all the elements of $\mathcal{S}^{m_w}$. •

Remark 5: We note that:

$$\begin{aligned} \nabla h_a(x) f_w(x, w) = \langle \nabla_z h_a(x), & f_{wu}(z, \kappa(z, \hat{w}), w) \rangle \\ & + \langle f_\eta(x), \nabla_\eta h_a(x) \rangle. \end{aligned}$$

Hence, when $f_\eta(x) := \Gamma \gamma(x)$, the left-hand side in (9) represents the time derivative of h_a along the solutions to $\mathcal{H}_w$. Condition (9) is standard in the adaptive control literature and allows the design of the adaptation system $\mathcal{H}_\eta$. •

V. LIMITATIONS OF CONTINUOUS-TIME ADAPTATION

Consider the system $\mathcal{H}_{wu}$, a feedback law κ, an aCBFc h_a with respect to (X_o, X_u), and $\epsilon > 0$ such that (7) and Assumption 2 hold. To point out the limitation of using a continuous-time adaptation, we consider the adaptation system:

$$\mathcal{H}_\eta : \begin{cases} \dot{\eta} = \Gamma\gamma(z,\eta) & (z,\eta) \in \mathbb{R}^{m_z} \times \mathbb{R}^{m_w} \\ \hat{w} := \eta, \end{cases} \tag{10}$$

where γ and $\Gamma \in \mathcal{G}$ satisfy Assumption 2.

Solving Problem 1 in this scenario reduces to designing the matrix $\Gamma \in \mathcal{G}$ such that the closed-loop system:

$$\mathcal{H}_w : \begin{bmatrix} \dot{z} \\ \dot{\hat{w}} \end{bmatrix} = f_w(z,\hat{w},w) := \begin{bmatrix} f_{uw}(z,\kappa(z,\hat{w}),w) \\ \Gamma\gamma(z,\hat{w}) \end{bmatrix} \tag{11}$$

with $(z, \hat{w}, w) \in \mathbb{R}^{m_z} \times \mathbb{R}^{m_w} \times \mathcal{W}$ is safe with respect to (X_o, X_u) uniformly in w. Considering a solution $x := (z, \hat{w})$ to $\mathcal{H}_w$ in (11) starting from $(z_o, \hat{w}_o) \in X_o$ yields the following claim:

Claim 1: The solution x never reaches X_u provided that:

$$\begin{aligned} 2\epsilon\lambda_{min}(\Gamma) \geq \max\big\{&|\tilde{w}(0)|^2, \\ &\sup\{|\tilde{w}(t)|^2 : \alpha(h_a(x(t))) < 0,\ t \in \operatorname{dom} x\}\big\}, \end{aligned} \tag{12}$$

where α comes from Assumption 2 and $\tilde{w} := \hat{w} - w$. •

Condition (12) is used in previous works (see [22], [23]) and is the source of the limitation of continuous-time adaptation laws modeled as in (10). This condition implies that $\tilde{w}$ needs to remain uniformly bounded whenever the time derivative of $h(z, \hat{w}, w) := h_a(z, \hat{w}) - \frac{1}{2}\tilde{w}^\top\Gamma^{-1}\tilde{w}$ along the solutions to $\mathcal{H}_w$ is not guaranteed to be positive. Simplified versions of this condition have been employed in the literature. In [22], the function α in Assumption 2 is assumed to be always non-negative. Hence, (12) reduces to $2\epsilon\lambda_{min}(\Gamma) \geq |\tilde{w}(0)|^2$. This assumption imposes a restriction on the initial error $\hat{w}(0) - w$. Furthermore, it constrains the system's input even when the solutions are far from the unsafe region. In [23], it is assumed that, for some $\tilde{v} \in \mathbb{R}^{m_w}$,

$$\sup\{|\tilde{w}_i(t)| : t \in \operatorname{dom} x\} \leq \tilde{v}_i \quad \forall i \in \{1, 2, ..., m_w\}$$

for each solution x to $\mathcal{H}_w$ starting from X_o. Hence, (12) reduces to $2\epsilon\lambda_{min}(\Gamma) \geq |\tilde{v}|^2$. This assumption is also restrictive as it requires the adaptation error to be bounded at all time. This is hard to guarantee in general, in particular, the knowledge of the bound $\tilde{v}$ is required.

In the next section, we propose a class of hybrid adaptation algorithms $\mathcal{H}_\eta$ that avoid such limitations.

VI. MAIN RESULTS

In this paper, we solve Problem 1 by proposing a hybrid adaptation algorithm $\mathcal{H}_\eta$ that appropriately resets the value of $\hat{w}$ in (10) to have $|\tilde{w}| \leq c$, for some $c > 0$, each time a solution reaches the set:

$$D_\eta := \{(z, \hat{w}) \in \mathbb{R}^{m_z} \times \mathbb{R}^{w} : h_a(z, \hat{w}) = \epsilon\}. \tag{13}$$

Under Assumption 1, it is always possible to find such a constant $c > 0$. However, when resetting the value of $\hat{w}$, it is important to guarantee that the considered aCBFc h_a does not become negative. To avoid this scenario, given $\epsilon > 0$ such that (7) holds, we make the following assumption:

Assumption 3: There exist $c > 0$ and $\delta \in (0, \epsilon)$ such that, for each $(z, \hat{w}) \in D_\eta$, we have:

$$G_\eta(z) := \{\bar{w} \in \mathbb{R}^{m_w} : h_a(z, \bar{w}) \geq \epsilon - \delta,\ |\bar{w} - w| \leq c\} \neq \emptyset.$$

•

Note that Assumption 3 holds trivially when the aCBFc h_a is a function of z only. Indeed, resetting $\hat{w}$ in this case will not change the value of h_a.

The hybrid adaptation system $\mathcal{H}_\eta$ is given by:

$$\mathcal{H}_\eta : \begin{cases} \dot{\eta} = \Gamma\gamma(z,\eta) & (z,\eta) \in C_\eta \\ \eta^+ \in G_\eta(z) & (z,\eta) \in D_\eta, \\ \hat{w} = \eta, \end{cases} \tag{14}$$

where D_η is equal to (13) and $C_\eta := (\mathbb{R}^{m_z} \times \mathbb{R}^{m_w}) \backslash D_\eta$.

Theorem 1: Consider the system $\mathcal{H}_{wu}$ in (5) such that Assumption 1 holds and a feedback law $u = \kappa(z, \hat{w})$. Given the initial and unsafe sets $X_o \subset \mathbb{R}^{m_x}$ and $X_u \subset \mathbb{R}^{m_x} \backslash X_o$, suppose there exist a continuously differentiable aCBFc h_a and $\epsilon > 0$ such that (7) and Assumptions 2 and 3 hold. Then, with c and δ coming from Assumption 3, the closed-loop of $\mathcal{H}_{wu}$ using $\mathcal{H}_\eta$ in (14) and the feedback law κ is safe with respect to (X_o, X_u) uniformly in w provided that:

$$\lambda_{min}(\Gamma) \geq c^2/(2(\epsilon - \delta)). \tag{15}$$

□

When using the hybrid adaptation $\mathcal{H}_\eta$ in (14), we do not need α in Assumption 2 to be always positive (as in [22]) or to assume uniform boundedness of $\tilde{w}$ (as in [23]). On the other hand, although $\mathcal{H}_\eta$ in (14) solves Problem 1, it has two drawbacks. The first is that the resulting hybrid system $\mathcal{H}_w$ does not satisfy the hybrid basic conditions (A1)-(A3). In particular, (A1) is not satisfied because the set C_η is not closed. The second is that there is no guarantee that the solutions to $\mathcal{H}_w$ are not eventually discrete or non-Zeno; namely, there may exist solutions to the closed-loop system with jumps that accumulate.

A. Refining the Hybrid Adaptation to Avoid Zeno

In this section, we modify the hybrid adaptation system $\mathcal{H}_\eta$ in (14) to guarantee that $\mathcal{H}_w$ is well posed and that its solutions are not eventually discrete. To do so, we augment the system $\mathcal{H}_\eta$ in (10) by adding a new discrete state variable $q \in \{0, 1\}$. That is, $\eta := (\hat{w}, q)$. This new variable will allow us to enforce a hysteresis-type behavior when triggering the jumps in $\mathcal{H}_\eta$. More precisely, we reset the value of $\hat{w}$ according to the following rules:

1) For some $\epsilon' \in (0, \epsilon)$, each time $h_a(z, \eta) \in [\epsilon', \epsilon]$ and $q = 0$, we switch the value of $\hat{w}$ such that after the jump we have $|\tilde{w}| \leq c$, for some $c > 0$. However, by doing so, it is important to guarantee that such a jump does not render the aCBFc h_a negative. To this end, given $\epsilon > 0$,

$\epsilon' \in (0, \epsilon)$, and $\delta \in (0, \epsilon')$, we consider the following assumption:

Assumption 4: There exists $c > 0$ such that, for each $(z, \hat{w}) \in \mathbb{R}^{m_z} \times \mathbb{R}^{m_w}$ such that $h_a(z, \hat{w}, 0) \in [\epsilon', \epsilon]$, the following set is nonempty:

$$G_{1\hat{w}}(z) := \{\bar{w} \in \mathbb{R}^{m_w} : h_a(z, \bar{w}, 1) \geq \epsilon' - \delta, \\ |\bar{w} - w| \leq c\}.$$

•

2) For some $\epsilon_1 \in (0, \epsilon' - \delta)$, each time $h_a(z, \eta) \leq \epsilon_1$ and $q = 1$, we switch the value of $\hat{w}$ such that after the jump we have $|\tilde{w}| \leq c_1$, for some $c_1 > 0$. Similarly, to guarantee that the aCBFc h_a is not positive after the jump, given $\epsilon_1 \in (0, \epsilon' - \delta)$, we consider the following assumption:

Assumption 5: There exist $c_1 > 0$ and $\delta_1 \in (0, \epsilon_1)$ such that for each $(z, \hat{w}) \in \mathbb{R}^{m_z} \times \mathbb{R}^{m_w}$ such that $h_a(z, \hat{w}, 1) = \epsilon_1$, the following set is nonempty:

$$G_{2\hat{w}}(z) := \{\bar{w} \in \mathbb{R}^{m_w} : h_a(z, \bar{w}, 0) \geq \epsilon_1 - \delta_1, \\ |\bar{w} - w| \leq c_1\}.$$

•

3) Each time a jump is triggered, we update the value of q to $1 - q$.

Remark 6: Note that Assumptions 4 and 5 hold for free when the aCBFc h_a is function of z only. •

As a result, when a jump happens at $h_a(z, \eta) \in [\epsilon', \epsilon]$ (in which case, we have $h_a(z, \eta^+) \geq \epsilon' - \delta > 0$), the next jump can happen only when $h_a(z, \eta) \leq \epsilon_1 < \epsilon' - \delta$. The new hybrid adaptation system $\mathcal{H}_\eta$ is given by

$$\mathcal{H}_\eta : \begin{cases} \dot{\eta} = \begin{bmatrix} \dot{\hat{w}} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \Gamma \gamma(z, \eta) \\ 0 \end{bmatrix} & (z, \eta) \in C_\eta \\ \eta^+ = \begin{bmatrix} \hat{w}^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} G_{\hat{w}}(z, \eta) \\ 1 - q \end{bmatrix} & (z, \eta) \in D_\eta, \end{cases} \tag{16}$$

where $D_\eta := D_1 \cup D_2 \cup D_3$,

$$G_{\hat{w}}(z, \eta) := \begin{cases} G_{1\hat{w}}(z) & \text{if } (z, \eta) \in D_1 \\ G_{2\hat{w}}(z) & \text{if } (z, \eta) \in D_2 \\ G_{2\hat{w}}(\Pi_{\mathbb{R}^{m_z}}(\Pi_{D_2}(z, \eta))) & \text{if } (z, \eta) \in D_3, \end{cases}$$

$$D_1 := \{(z, \eta) \in \mathbb{R}^{m_z} \times \mathbb{R}^{m_w} \times \{0\} : h_a(z, \eta) \in [\epsilon', \epsilon]\},$$
$$D_2 := \{(z, \eta) \in \mathbb{R}^{m_z} \times \mathbb{R}^{m_w} \times \{1\} : h_a(z, \eta) = \epsilon_1\},$$
$$D_3 := \{(z, \eta) \in \mathbb{R}^{m_z} \times \mathbb{R}^{m_w} \times \{1\} : h_a(z, \eta) < \epsilon_1\},$$

and $C_\eta := \text{cl}((\mathbb{R}^{m_z} \times \mathbb{R}^{m_w} \times \{0, 1\}) \backslash D_\eta)$.

Theorem 2: Consider the system $\mathcal{H}_{wu}$ in (5) such that Assumption 1 holds and a feedback law $u = \kappa(z, \hat{w})$. Given the initial and unsafe sets $X_o \subset \mathbb{R}^{m_x}$ and $X_u \subset \mathbb{R}^{m_x} \backslash X_o$, suppose there exist a continuously differentiable aCBFc h_a and $\epsilon > 0$ such that (7) and Assumption 2 hold. Assume that there exist positive constants $(\epsilon', \delta, \epsilon_1)$, with:

$$\epsilon' \in (0, \epsilon), \ \delta \in (0, \epsilon'), \text{ and } \epsilon_1 \in (0, \epsilon' - \delta), \tag{17}$$

such that Assumptions 4 and 5 hold. Then, the closed-loop of $\mathcal{H}_{wu}$ using $\mathcal{H}_\eta$ in (16) and the feedback law κ, denoted $\mathcal{H}_w$, is safe with respect to (X_o, X_u) uniformly in w provided that:

$$\lambda_{min}(\Gamma) \geq c_1^2 / (2(\epsilon_1 - \delta_1)). \tag{18}$$

Moreover, the following properties hold:

1) The conditions (A1)-(A3) are satisfied for $\mathcal{H}_w$.
2) The maximal solutions to $\mathcal{H}_w$ are not eventually discrete. In particular, the size of the interval of flow before jumping from $q = 1$ to $q = 0$ admits a semi-global strictly positive lower bound.

□

Remark 7: [Non-Zenoness of the Hybrid Adaptation] The closed-loop system $\mathcal{H}_w$ using the hybrid adaption $\mathcal{H}_\eta$ in (16) is shown in Theorem 2 to not admit eventually discrete solutions. Furthermore, using [27, Corollary 4.9], we can conclude that, for well-posed hybrid systems not admitting eventually discrete solutions, every bounded solution is non-Zeno. However, as the solutions diverge, they can exhibit a Zeno behavior. To avoid the presence Zeno solutions even when the solutions can diverge, extra assumptions need to be made. Investigating such assumptions is the subject of our future work. •

VII. Example

In this section, we revisit the Adaptive-Cruise-Control (ACC) example studied in [22] (see also [4]). For this, we consider the uncertain nonlinear control system:

$$\dot{z} = \begin{bmatrix} -\frac{1}{m}\Delta(z_1)w + \frac{1}{m}u \\ v_d - z_1 \end{bmatrix} \quad (z, w, u) \in \mathbb{R}^2 \times \mathcal{W} \times \mathbb{R}, \tag{19}$$

where z_1 is the velocity of the vehicle, z_2 the distance between the vehicle and a leading vehicle traveling at a fixed velocity v_d, m is the vehicle's mass, $w \in \mathcal{W} \subset \mathbb{R}^3$ is a vector of unknown parameters such that Assumption 1 holds, and $\Delta(z_1) := [1 \quad z_1 \quad z_1^2]$. The initial and unsafe sets $(X_{oz}, X_{uz}) \subset \mathbb{R}^2 \times \mathbb{R}^2$ are such that, for some $\beta > 0$:

$$\begin{aligned} X_{uz} &:= \{z \in \mathbb{R}^2 : z_2 \leq \beta z_1 - 2\}, \\ X_{oz} &:= \{z \in \mathbb{R}^2 : z_2 \geq \beta z_1 + 2\}. \end{aligned} \tag{20}$$

The objective is to design an adaptive feedback law u to guarantee the following two tasks.

- *Safety:* The z component of the closed-loop solutions starting from X_{oz} never reach X_{uz}.
- *Convergence:* The z component of the closed-loop solutions converges to a target $z_d := (v_d, z_{2d})$ satisfying $z_{2d} > \beta v_d$.

To guarantee safety using Theorem 2, we note that the function $h_a : \mathbb{R}^2 \times \mathbb{R}^{m_\eta} \to \mathbb{R}$ given by $h_a(z, \eta) := z_2 - \beta z_1$ is a valid aCBFc with respect to $(X_o, X_u) := (X_{oz} \times \mathbb{R}^{m_\eta}, X_{uz} \times \mathbb{R}^{m_\eta})$ and (7) holds for any $\epsilon \in (0, 2]$. To find a class of safety inputs $u_s := \kappa_s(z, \hat{w})$ such that Assumption 2 holds with $\mathcal{G} = \mathcal{S}^3$, we take $u_s := \Delta(z_1)\hat{w} + u_{s2}$, where u_{s2} is chosen such that:

$$(\beta/m)u_{s2} \leq -\alpha_s(h_a(x)) - (v_d - z_1) \tag{21}$$

for some continuous function $\alpha_s : \mathbb{R} \to \mathbb{R}$ verifying (8). In which case, (9) holds for any $\Gamma \in \mathcal{S}^3$ and for $\gamma(z,\eta) := (\beta/m)\Delta(z_1)$. Note that, for each positive constants $(\epsilon', \delta, \epsilon_1)$ satisfying (17), Assumptions 4 and 5 hold after Assumption 1 and since h_a is function of z only.

On the other hand, to guarantee the convergence task, we consider the adaptive Control Lyapunov Function candidate (aCLFc) $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by $V_a(z) := (1/2)(z_2 - z_{2d})^2 + (1/2)(z_1 - v_d)^2$. Note that:

$$\langle \nabla_z V_a(z), f_{uw}(z,u,w) \rangle \\ = (z_2 - z_{2d})(v_d - z_1) + (z_1 - v_d)(u - \Delta(z_1)w)/m.$$

To find a class of inputs $u_c := \kappa_c(z, \hat{w}_c)$ that solves the convergence task, we take $u_c := \Delta(z_1)\hat{w}_c + u_{c2}$. This yields:

$$\langle \nabla_z V_a(z),\, f_{uw}(z,u,w) \rangle = (z_2 - z_{2d})(v_d - z_1) + \\ (z_1 - v_d)u_{c2}/m + \Delta(z_1)(z_1 - v_d)(\hat{w}_c - w)/m.$$

To conclude the convergence task via LaSalle's Invariance Principle, we choose $\hat{w}_c$ to be the output of the adaptation system:

$$\mathcal{H}_{\hat{w}_c} : \left\{ \dot{\hat{w}}_c = -\Gamma_c \Delta(z_1)(z_1 - v_d)/m, \qquad \Gamma_c \in \mathcal{S}^3, \right.$$

and u_{c2} to satisfy:

$$(z_1 - v_d)u_{c2}/m \leq -\alpha_c(z_1 - v_d) - (z_2 - z_{2d})(v_d - z_1), \quad (22)$$

for some continuous and positive definite function $\alpha_c : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $\liminf_{s\to\infty} \alpha_c(s) > 0$.

Finally, to solve the safety-plus-convergence task, one can apply any feedback law $u := \kappa(z, \hat{w}, \hat{w}_c)$ such that $u_{s2} := u - \Delta(z_1)\hat{w}$ satisfies (21) and $u_{c2} := u - \Delta(z_1)\hat{w}_c$ satisfies (22), where $\hat{w}$ is the output of the adaptation system $\mathcal{H}_\eta$ in (16) and $\hat{w}_c$ is the output of the adaptation system $\mathcal{H}_{\hat{w}_c}$. Finally, the resulting closed-loop system admits $x := (z, \eta, \hat{w}_c)$ as a state vector. It is safe with respect to $(X_{oz} \times \mathbb{R}^{m_\eta} \times \mathbb{R}^{m_w}, X_{uz} \times \mathbb{R}^{m_\eta} \times \mathbb{R}^{m_w})$, well posed, and its solutions are non Zeno and converge to z_d.

VIII. Conclusion

We presented a novel framework for safety-critical adaptive control of hybrid systems using Control Barrier Functions. We proposed a hybrid update law that enables less conservative behavior than existing methods for safety-critical adaptive control using continuous update laws. We illustrate this method by considering the adaptive cruise control problem. Future work will seek to formalize the unification of data-driven techniques (such as in [23]) and adaptive control techniques through the hybrid framework, and demonstrate the ability of such methods experimentally.

References

[1] M. Nagumo. Über die lage der integralkurven gewöhnlicher differentialgleichungen. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 24:551–559, 1942.

[2] Franco Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

[3] S. Prajna, A. Jadbabaie, and G. J. Pappas. A framework for worst-case and stochastic safety verification using barrier certificates. *IEEE Transactions on Automatic Control*, 52(8):1415–1428, 2007.

[4] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada. Control barrier function based quadratic programs for safety critical systems. *IEEE Transactions on Automatic Control*, 62(8):3861–3876, 2017.

[5] P. Glotfelter, J. Cortés, and M. Egerstedt. Nonsmooth barrier functions with applications to multi-robot systems. *IEEE Control Systems Letters*, 1(2):310–315, 2017.

[6] L. Wang, A. D. Ames, and M. Egerstedt. Safety barrier certificates for collisions-free multirobot systems. *Transactions on Robotics*, 33(3):661–674, 2017.

[7] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada. Control barrier functions: Theory and applications. In *European Control Conference (ECC)*, pages 3420–3431. IEEE, 2019.

[8] M. Maghenem and R. G. Sanfelice. Sufficient conditions for forward invariance and contractivity in hybrid inclusions using barrier functions. *arXiv preprint:1908.03980*, 2019.

[9] R.G. Sanfelice. *Hybrid Feedback Control*. Princeton University Press, New Jersey, 2021.

[10] S. Kolathaya and A. D. Ames. Input-to-state safety with control barrier functions. *Control Systems Letters*, 3(1):108–113, 2018.

[11] A. J. Taylor, A. Singletary, Y. Yue, and A. D. Ames. A control barrier perspective on episodic learning via projection-to-state safety. *IEEE Control Systems Letters*, 5(3):1019–1024, 2021.

[12] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames. Robustness of control barrier functions for safety critical control. *IFAC-PapersOnLine*, 48(27):54–61, 2015.

[13] M. Jankovic. Robust control barrier functions for constrained stabilization of nonlinear systems. *Automatica*, 96:359–367, 2018.

[14] R. Takano and M. Yamakita. Robust constrained stabilization control using control lyapunov and control barrier function in the presence of measurement noises. In *Conference on Control Technology and Applications (CCTA)*, pages 300–305. IEEE, 2018.

[15] T. Gurriet, A. Singletary, J. Reher, L. Ciarletta, E. Feron, and A. Ames. Towards a framework for realizable safety critical control through active set invariance. In *International Conference on Cyber-Physical Systems (ICCPS)*, pages 98–106. IEEE Press, 2018.

[16] A. J. Taylor, A. Singletary, Y. Yue, and A. D. Ames. Learning for safety-critical control with control barrier functions. *Proceedings of Machine Learning Research (PMLR)*, 120:708–717, 2020.

[17] J. Choi, F. Castañeda, C. J. Tomlin, and K. Sreenath. Reinforcement Learning for Safety-Critical Control under Model Uncertainty, using Control Lyapunov Functions and Control Barrier Functions. *arXiv preprint arXiv:2004.07584*, 2020.

[18] M. Cohen and C. Belta. Approximate optimal control for safety-critical systems with control barrier functions. *arXiv preprint arXiv:2008.04122*, 2020.

[19] P. Ioannou and J. Sun. *Robust adaptive control*. Prentice Hall, New Jersey, USA, 1996.

[20] S. Sastry and S. Bodson. *Adaptive control: stability, convergence and robustness*. Courier Dover Publications, 2011.

[21] M. Krstić and P. V. Kokotović. Control Lyapunov functions for adaptive nonlinear stabilization. *Systems & Control Letters*, 26(1):17–23, 1995.

[22] A. J. Taylor and A. D. Ames. Adaptive safety with control barrier functions. In *American Control Conference (ACC)*, pages 1399–1405. IEEE, 2020.

[23] B. T. Lopez, J. E. Slotine, and J. P. How. Robust adaptive control barrier functions: An adaptive and data-driven approach to safety. *IEEE Control Systems Letters*, 5(3):1031–1036, 2021.

[24] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, stability, and robustness*. Princeton University Press, 2012.

[25] J. Chai and R. G. Sanfelice. Forward invariance of sets for hybrid dynamical systems (Part I). *IEEE Transactions on Automatic Control*, 64(6):2426–2441, 2018.

[26] R. Wisniewski and C. Sloth. Converse barrier certificate theorems. *IEEE Transactions on Automatic Control*, 61(5):1356–1361, 2016.

[27] R. Goebel and A. R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573–587, 2006.