A Robust Hybrid Finite Time Parameter Estimator With Relaxed Persistence of Excitation Condition

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Abstract— **In this paper, we consider the problem of estimating a constant (or piecewise constant) parameter of a linear regression model. Using a hybrid systems framework, a hybrid algorithm is proposed allowing the estimate to converge to the exact value of the unknown parameter in predetermined finite time. Interestingly, we show that for the case of a constant parameter, the convergence property of the hybrid algorithm holds while only requiring the regressor to be exciting on a finite interval. For the case of a piecewise constant parameter, the classical persistency of excitation condition is required to guarantee the convergence. Robustness of the proposed algorithm with respect to measurements noise is analysed. Finally, illustrative examples are provided showing the merits of the proposed approach.**

I. INTRODUCTION

Accurate estimation of model parameters of a system is critical in most applications. Different algorithms have been proposed in the adaptive control community to tackle this problem [1], [2]. A common approach consists of exploiting information about the structure of the system and a collection of available system signals (called regressors) to compute online estimates of the systems parameters. An optimality criterion is defined and the behaviour of the parameters estimates can be rigorously analysed through the use of Lyapunov theory.

In static linear regression models [2], the relationship between the regressors and the output is linear. For such models, the estimation of parameters is based on the classical gradient descent algorithm [1], [2], which requires a persistence of excitation condition [1], [2]. Different approaches have been proposed in the literature to relax the excitation requirement while ensuring asymptotic convergence of the estimator to the exact value [3], [4], [5]. Regarding convergence within finite time intervals, the authors in [6] (respectively [7]) proposed algorithms for finite time (respectively, fixed time) estimation of parameters, while the regressor can converge to zero asymptotically or in finite time.

Motivated by the results on finite time observers [8], this paper presents a hybrid estimator with predetermined time for convergence of the estimates to piecewise constant unknown parameters. We first study the stability and finite time convergence properties of our hybrid estimator for the case of a constant unknown parameter. Interestingly, we show that the regressor needs to be exciting only over a bounded interval (given here as the time interval before the first jump). Then we show how this result can be generalized to piecewise constant unknown parameters, while precisely specifying the intervals on which the regressor needs to be exciting. Finally, we use tools developed in hybrid systems theory to provide robustness of the proposed estimator.

In spirit, our approach is closely related to the one in [9]. Both results provide finite time estimators using a hybrid framework. In comparison to our work, the approach in [9] differs in three directions. First, they are dealing with dynamical systems, while our work deals with an algebraic input-output model. Second, we are using a different estimation algorithm, based on the use of two coupled estimators. Finally, while the results in [9] rely on a persistency of excitation condition to ensure that their hybrid system is well defined and to guarantee completeness of solutions, we only need the regressor to be exciting on a finite time interval. Another related work is [10] which proposes an interval excitation condition to ensure the convergence of an MRAC system. Their result is based on dynamic regressor extension and mixing (DREM), which consists of decomposing the global estimation problem into a series of scalar problems using dynamic operators. In this paper, we are dealing with the original parameter estimation problem without any prior transformation. Moreover, we provide robustness results using tools from the hybrid systems theory. Other finite-time approaches have been proposed for the design of observers [11], [12] where the system is given by a dynamical model. The main difference between our work and these results is that rather than bounding the evolution of the estimation error using the norm of the dynamics, we leverage persistency of excitation type conditions which are less conservative.

The remainder of this paper is organized as follows. Section II introduces the classical linear regression system and the hybrid systems framework. Section III illustrates the construction of the hybrid estimator for a simple scalar case. In Section IV, we show the finite time convergence of the proposed estimator with respect to constant and piecewise constant unknown parameters. In Section V, robustness of the proposed hybrid estimator to generic noise is investigated. Finally, Section VI presents numerical results highlighting the robustness properties of the proposed estimator. Due to space constraints, the proofs are omitted and will be published elsewhere.

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II. Preliminaries

A. Notations

The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$ denote the set of positive integers, real and non negative real numbers, respectively. The identity matrix of appropriate dimension is denoted by I. The Euclidean norm of vectors is denoted $|.|$. For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, $\operatorname{dom} f$ denotes the domain of definition of f. A continuous function α is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$, and $\alpha(r)$ goes to infinity as r tends to infinity. Given a point $y \in \mathbb{R}^n$ and a non-empty set $\mathcal{A} \subseteq \mathbb{R}^n$, $|y|_\mathcal{A} := \inf_{x \in \mathbb{R}^n} |x - y|$. For two functions $f : X \to Y$ and $g : Y \to Z$, and a point $x \in X$, $f \circ g(x) = f(g(x))$.

B. Linear regression

A general linear regression model is given by

$$y(t) = \theta^{*\top} \phi(t) \tag{1}$$

where $t \mapsto y(t) \in \mathbb{R}$ represents the known scalar output, $\theta^* \in \mathbb{R}^n$ is the unknown constant parameter to identify, and $t \mapsto \phi(t) \in \mathbb{R}^n$ is the known regressor. Since θ^* is unknown, an estimator of the form $\hat{y}(t) = \theta^\top(t)\phi(t)$ can be constructed, where $t \mapsto \hat{y}(t) \in \mathbb{R}$ is the estimated output and $t \mapsto \theta(t) \in \mathbb{R}^n$ is the estimate of the unknown parameter θ^*. The error between the true and estimated outputs is defined as

$$e(t) := \hat{y}(t) - y(t) = \tilde{\theta}^\top(t)\phi(t) \tag{2}$$

where $t \mapsto \tilde{\theta}(t) := \theta(t) - \theta^*$ is the parameter estimation error. The typical objective is to design a law that adjusts the parameter estimate θ, based on the knowledge of e and ϕ, in order to ensure that $\tilde{\theta}$ and e converge towards zero. For this purpose, a continuous gradient descent-like algorithm can be designed with the following cost function:

$$J(e) := \frac{1}{2} e^2 \tag{3}$$

Using this cost function, the classical continuous-time gradient like algorithm [1], [2] is defined as follows:

$$\dot{\theta} = -\gamma \nabla_\theta J(e(t)) = -\gamma \phi(t)(\phi^\top(t)\theta - y(t)) \tag{4}$$

where $\gamma > 0$ is the constant adaptation rate. It can be shown using Barbalat Lemma [2] that this algorithm makes the estimation error $t \mapsto e(t)$ goes to zero as time t goes to infinity. To show the convergence of θ to θ^*, a persistency of excitation condition [1], [2] is required in general (see Definition 3 for a formal definition of persistency of excitation).

Our objective is to construct a hybrid parameter estimator allowing for θ to converge to θ^* in finite time with a relaxed persistence of excitation condition. For this reason, some preliminary tools on hybrid systems [13] are introduced in the next section.

C. Preliminaries on hybrid systems

In this paper, a hybrid system $\mathcal{H}$ has data (C, f, D, g) and is defined by

$$\begin{aligned} \dot{z} &= f(z, u) \quad z \in C, \\ z^+ &= g(z, u) \quad z \in D, \end{aligned} \tag{5}$$

where $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, f is the flow map capturing the continuous dynamics, and C defines the flow set on which f is effective. The map g is called the jump map and models the discrete change of z, while D defines the jump set, from which jumps are allowed. Given an input u, a solution to $\mathcal{H}$ is given by the pair (z, u), which is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where z is a hybrid arc, t is the ordinary time keeping track of the flows, and j is the jump index counting the number of jumps. When the system has no input or its input is identically zero, its solution is given by z. The domain $\operatorname{dom}(z, u) = \operatorname{dom} z = \operatorname{dom} u \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N}$ of a solution pair (z, u) to $\mathcal{H}$ is a hybrid time domain, in the sense that for every $(T, J) \in \operatorname{dom} z$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ with $t_0 = 0$ such that

$$\operatorname{dom} z \cap ([0, T] \times \{0, 1, \ldots, J\}) = \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times j).$$

Two hybrid arcs are said to be (τ, ε)-close if they satisfy the following property

Definition 1: Given $\tau, \varepsilon > 0$, two hybrid arcs z_1 and z_2 to $\mathcal{H}$ are (τ, ε)-close if

- for all $(t, j) \in \operatorname{dom} z_1$ with $t + j \leq \tau$ there exists s such that $(s, j) \in \operatorname{dom} z_2$, $|t - s| \leq \varepsilon$, and $|z_1(t, j) - z_2(s, j)| \leq \varepsilon$;
- for all $(t, j) \in \operatorname{dom} z_2$ with $t + j \leq \tau$ there exists s such that $(s, j) \in \operatorname{dom} z_1$, $|t - s| \leq \varepsilon$, and $|z_2(t, j) - z_1(s, j)| \leq \varepsilon$

III. Motivational example: scalar case

In this section, we explain the construction of the proposed hybrid parameter estimator on a scalar example. Consider the linear regression model defined in (1) with $n = 1$. The objective is to design an algorithm ensuring the convergence of the parameter θ to θ^* in finite time with a relaxed persistence of excitation. The proposed algorithm consists of two update laws with positive parameters, γ_1 and γ_2, and a jump time δ and is captured by the following hybrid system $\mathcal{H}$. Given the known signals $\phi, y \in \mathbb{R}$. When the timer $\tau_a \in [0, \delta)$, the system flows according to

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\tau}_a \\ \dot{\tau}_b \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\gamma_1 \phi(\tau_b)(\phi^\top(\tau_b)\theta_1 - y(\tau_b)) \\ -\gamma_2 \phi(\tau_b)(\phi^\top(\tau_b)\theta_2 - y(\tau_b)) \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{6}$$

and when $\tau_a = \delta$ and $q = 0$, we reset the state according to

$$\begin{bmatrix} \theta_1^+ \\ \theta_2^+ \\ \tau_a^+ \\ \tau_b^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} R(\theta_1, \theta_2) \\ R(\theta_1, \theta_2) \\ 0 \\ \tau_b \\ q + 1 \end{bmatrix} \tag{7}$$

The state of the hybrid system $\mathcal{H}$ is $z = (\theta_1, \theta_2, \tau_a, \tau_b, q) \in \mathcal{X} := \mathbb{R}^2 \times [0, \delta] \times \mathbb{R}_{\geq 0} \times \mathbb{N}$. The signals $\phi, y \in \mathbb{R}$ are known and used as inputs of the hybrid model. The state components θ_1 and θ_2 represent the estimates with update laws based on the positive adaptation rates γ_1 and γ_2, respectively. The state component τ_a is a timer used to trigger the jump at $\delta > 0$. The state components τ_b and q make it possible to convert the time-varying system into a time-invariant one.

The functional R is defined as $R(\theta_1, \theta_2) = K_1\theta_1 + K_2\theta_2$ where the gains K_1 and K_2 are given as

$$K_1 = \left(-\exp\left(-\gamma_2 \int_0^\delta \phi^2(t)dt\right)\right) \times \left(\exp\left(-\gamma_1 \int_0^\delta \phi^2(t)dt\right) - \exp\left(-\gamma_2 \int_0^\delta \phi^2(t)dt\right)\right)^{-1}$$
$$K_2 = 1 - K_1$$

From (6)-(7), the resulting flow set is $C := \mathcal{X}$ and the jump set is $D := \{z \in \mathcal{X} : \tau_a = \delta, q = 0\}$. With this construction, the idea is to trigger the jump at $\tau_a = \delta$. The logic variable q ensure that the jump, defined by the reset map in (7), occurs only one time, when $q = 0$.

Let us mention that if $\gamma_1 \neq \gamma_2$ and $\int_0^\delta \phi^2(t)dt \neq 0$, then $\left(\exp\left(-\gamma_1 \int_0^\delta \phi^2(t)dt\right) - \exp\left(-\gamma_2 \int_0^\delta \phi^2(t)dt\right)\right)^{-1}$ is invertible, and the gains K_1 and K_2 are well defined. The following result shows how the proposed hybrid parameter estimator allows to identify θ^* in finite time.

Proposition 1: Given $\delta, \gamma_1, \gamma_2 > 0$ such that $\gamma_1 \neq \gamma_2$, if the regressor ϕ is such that $\int_0^\delta \phi^2(t)dt \neq 0$, then for any solution z to the hybrid system $\mathcal{H}$ in (6)-(7) with initial condition $z(0,0) \in \mathcal{X}_0 = \{z \in \mathcal{X} : \theta_1 = \theta_2, \tau_a = \tau_b = q = 0\}$, the states θ_1 and θ_2 converge to θ^* in finite time δ and one jump, i.e, $\theta_1(t,j) = \theta_2(t,j) = \theta^*$ for all $(t,j) \in \operatorname{dom} z$ and $t + j \geq \delta + 1$.

The basic idea of the hybrid estimator (6)-(7) is as follows: the estimates θ_1 and θ_2 at the time instant $(\delta, 0)$ are given by

$$\theta_1(\delta, 0) = \exp\left(-\gamma_1 \int_0^\delta \phi^2(s)ds\right)\theta_1(0,0) + \theta^*$$
$$\theta_2(\delta, 0) = \exp\left(-\gamma_2 \int_0^\delta \phi^2(s)ds\right)\theta_2(0,0) + \theta^*.$$

The idea is to select the same initial values $\theta_1(0,0)$ and $\theta_2(0,0)$ and design K_1 and K_2 such that $\theta_i(\delta, 1) = R(\theta_1, \theta_2) = K_1\theta_1(\delta, 0) + K_2\theta_2(\delta, 0) = \theta^*$ for each $i \in \{1, 2\}$.

Remark 1: It is noteworthy that for the scalar case we are able to ensure the finite time convergence of the parameter estimates θ_1 and θ_2 to the true value θ^* in finite time without requiring the regressor to be persistently exciting. Indeed, the regressor signal needs only to satisfy $\int_0^\delta \phi^2(t)dt \neq 0$. The result of Proposition 1 motivates the use of the concept of excitation over a finite interval introduced in the following section for the non-scalar case.

IV. Hybrid finite time convergent algorithm

Following the scalar case presented in Section III, this section presents a hybrid parameter estimator for finite-time convergence with respect to constant and piecewise constant unknown parameter θ^* for the general case of (1).

A. Excitation conditions

We start by recalling from [2] the notion of excitation for the regressor signal.

Definition 2: Given $\sigma \geq 0$ and $\mu > 0$, a signal $t \mapsto \phi(t) \in \mathbb{R}^n$ is exciting over the finite interval $[\sigma, \sigma + \mu]$ if there exists $\eta > 0$ such that

$$\int_\sigma^{\sigma+\mu} \phi(t)\phi^\top(t)dt \geq \eta I. \tag{8}$$

Definition 3: A signal $t \mapsto \phi(t) \in \mathbb{R}^n$ is μ-persistently exciting if there exist $\mu > 0$ and $\eta > 0$ such that for all $\sigma \geq 0$

$$\int_\sigma^{\sigma+\mu} \phi(t)\phi^\top(t)dt \geq \eta I. \tag{9}$$

In addition, a signal $t \mapsto \phi(t) \in \mathbb{R}^n$ is said to be persistently exciting if there exists $\mu > 0$ such that $t \mapsto \phi(t) \in \mathbb{R}^n$ is μ-persistently exciting.

B. Constant unknown parameter

The finite time adaptation law to estimate θ^* is formalized as a hybrid system and defined as follows:

$$\begin{aligned} \dot{z} &= f(z, u) \quad z \in C, \\ z^+ &= g(z, u) \quad z \in D, \end{aligned} \tag{10}$$

Its state is $z = (\theta_1, \theta_2, \tau_a, \tau_b, q) \in \mathcal{X} = \mathbb{R}^n \times \mathbb{R}^n \times [0, \delta] \times \mathbb{R}_{\geq 0} \times \mathbb{N}$. The input of the hybrid model is given by $u = (\phi, y)$ where ϕ and y are known signals. The state components θ_1 and θ_2 represent the estimates with update laws based on the adaptation rates γ_1 and γ_2, respectively. The state component τ_a is a timer used to trigger the jump at $\delta > 0$. The state components τ_b and q are included to make it possible to convert the time-varying system for constant parameter θ^*, as well as the forthcoming system for the case when the parameter θ^* changes over time, into a time-invariant one. The flow and jump maps are given by

$$f(z,u) = \begin{bmatrix} -\gamma_1\phi(\tau_b)(\phi^\top(\tau_b)\theta_1 - y(\tau_b)) \\ -\gamma_2\phi(\tau_b)(\phi^\top(\tau_b)\theta_2 - y(\tau_b)) \\ 1 \\ 1 \\ 0 \end{bmatrix} \forall z \in C := \mathcal{X}$$

$$g(z,u) = \begin{bmatrix} R(\theta_1, \theta_2) \\ R(\theta_1, \theta_2) \\ 0 \\ \tau_b \\ 1 + q \end{bmatrix} \forall z \in D := \{z \in \mathcal{X} : \tau_a = \delta, q = 0\}$$

where $R(\theta_1, \theta_2) = K_1\theta_1 + K_2\theta_2$ and the gains K_1 and K_2 are given by the functionals

$$\begin{aligned} K_1 &= -\Phi_2(\delta, 0)\left(\Phi_1(\delta, 0) - \Phi_2(\delta, 0)\right)^{-1} \\ K_2 &= I - K_1 \end{aligned} \tag{11}$$

where for each $i \in \{1,2\}$, Φ_i is the state transition matrix of the time-varying system

$$\dot{\tilde{\theta}}_i = -\gamma_i \phi(t)\phi^\top(t)\tilde{\theta}_i.$$

The logic variable q ensures that the jump occurs only one time, when $q = 0$.

Let us first provide conditions for the functionals K_1 and K_2 to be well defined.

Proposition 2: Given $\delta, \gamma_1, \gamma_2 > 0$ such that $\gamma_1 \neq \gamma_2$, if the regressor $t \mapsto \phi(t)$ is exciting over the finite interval $[0, \delta]$, with $\int_0^\delta \phi(u)\phi^\top(u)du \geq \eta I$ for some $\eta > 0$, there exists $\phi_M \geq 0$ such that $|\phi(s)| \leq \phi_M$ for all $s \in [0, \delta]$ and if the constants $\delta, \gamma_1, \gamma_2$ satisfy

$$\phi_M^2 \gamma_2 \delta \in (0,1) \tag{12}$$

$$\left(1 - \frac{2\eta\gamma_1}{(1+\phi_M^2\gamma_1\delta)^2}\right)\left(1 + \frac{2\gamma_2\phi_M^2\delta}{(1-\phi_M^2\gamma_2\delta)^2}\right) \in (0,1) \tag{13}$$

then the functionals K_1 and K_2 in (11) are well defined, in particular, $\Phi_1(\delta, 0) - \Phi_2(\delta, 0)$ is invertible.

Remark 2: The function defined by

$$\delta \mapsto \left(1 - \frac{2\eta\gamma_1}{(1+\phi_M^2\gamma_1\delta)^2}\right)\left(1 + \frac{2\gamma_2\phi_M^2\delta}{(1-\phi_M^2\gamma_2\delta)^2}\right)$$

evaluated at $\delta = 0$ gives $1 - 2\eta\gamma_1$. Hence, conditions (12)-(13) are always satisfied by appropriately choosing the parameters δ, γ_1, and γ_2 for a given η and ϕ_M.

Next, we show convergence of the estimates θ_1 and θ_2 to θ^*. To this end, we define the following set:

$$\mathcal{A} = \{z \in \mathcal{X} : \theta_1 = \theta_2 = \theta^*\} \tag{14}$$

Theorem 1: Given $\delta, \gamma_1, \gamma_2 > 0$ such that $\gamma_1 \neq \gamma_2$, and assume the unknown parameter $\theta^* \in \mathbb{R}^n$ is constant. If the conditions in Proposition 2 hold, then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that any solution z to the hybrid system $\mathcal{H}$ in (10) from $z(0,0) \in \mathcal{X}_0 = \{z \in \mathcal{X} : \theta_1 = \theta_2, \tau_a = \tau_b = q = 0\}$ satisfies

$$|z(t,j)|_\mathcal{A} \leq \alpha_1^{-1} \circ \alpha_2(|z(0,0)|_\mathcal{A})$$

for all $(t,j) \in \operatorname{dom} z$, and

$$\theta_1(t,j) = \theta_2(t,j) = \theta^*$$

for all $(t,j) \in \operatorname{dom} z$ satisfying $t \geq \delta$ and $j \geq 1$.

Remark 3: It is noteworthy that for the particular case when the parameter θ^* is constant, we are able to ensure the finite time convergence of the parameter error to zero in finite time, without requiring the regressor ϕ to be persistently exciting as in [1]; see Definition 3. Let us also mention that our result is less conservative than the approach proposed in [5]. Indeed, in [5] the regressor does not need to be persistently exciting, but it is required to not be square integrable[1] ($\phi \notin \mathcal{L}_2$), while in Theorem 1, we only require the input signal to be exciting on the interval $[0, \delta]$; see Definition 2. Finally, let us point out that if there exists $a > 0$ such that ϕ is exciting over the interval $[0, a]$, then one can choose $\delta \geq a$, which is a less conservative condition than the conditions on the regressor ϕ proposed in [6].

[1]A vector signal $\phi \in \mathbb{R}^n$ is said to be square intergrable, denoted $\phi \in \mathcal{L}_2$, if $\int_0^{+\infty} |\phi(t)|^2 dt < \infty$.

C. Piecewise constant unknown parameter

When the unknown parameter θ^* is a piecewise constant function, it is also possible to estimate it in finite time. However, one jump is not enough. Therefore, recursive jumps are embedded in our hybrid parameter estimator. For this purpose, we propose a hybrid finite time convergent adaptation law as in (10)

$$\begin{aligned} \dot{z} &= f(z,u) \quad z \in C, \\ z^+ &= g(z,u) \quad z \in D, \end{aligned} \tag{15}$$

with state $z = (\theta_1, \theta_2, \tau_a, \tau_b, q) \in \mathcal{X} = \mathbb{R}^n \times \mathbb{R}^n \times [0,\delta] \times \mathbb{R}_{\geq 0} \times \mathbb{N}$ and data (C, f, D, g) as in (10), but with $D := \{z \in \mathcal{X} : \tau_a = \delta\}$ and with a map R given by $R(\theta_1, \theta_2, q) = K_1(q)\theta_1 + K_2(q)\theta_2$ where

$$\begin{aligned} K_1(q) &= -\Phi_2((q+1)\delta, q\delta) \times \\ &\qquad (\Phi_1((q+1)\delta, q\delta) - \Phi_2((q+1)\delta, q\delta))^{-1} \\ K_2(q) &= I - K_1(q) \end{aligned} \tag{16}$$

In the case of piecewise constant unknown parameter, the jump set D and the functionals K_1 and K_2 differ from the ones in (10) to account for successive jumps. As in the previous section, for K_1 and K_2 to be well defined, we have the following result

Proposition 3: Given $\delta, \gamma_1, \gamma_2 > 0$ such that $\gamma_1 \neq \gamma_2$, if there exists $\mu > 0$ such that $\mu \leq \delta$ and the regressor $t \mapsto \phi(t)$ is μ-persistently exciting, with $\int_\sigma^{\sigma+\delta} \phi(u)\phi^\top(u)du \geq \eta I$ for some $\eta > 0$ and for all $\sigma \geq 0$, there exists $\phi_M \geq 0$ such that $|\phi(s)| \leq \phi_M$ for all $s \in \mathbb{R}_{\geq 0}$ and if the constants $\delta, \gamma_1, \gamma_2$ satisfy conditions (12)-(13), then the gains K_1 and K_2 in (16) are well defined.

The following result establishes the main convergence property induced by the hybrid system $\mathcal{H}$ in (15) when unknown parameter θ^* is piecewise constant.

Theorem 2: Given $\delta, \gamma_1, \gamma_2 > 0$ such that $\gamma_1 \neq \gamma_2$, and assume the unknown parameter $\theta^* : [0, +\infty) \to \mathbb{R}^n$ is piecewise constant, where the time instants at which the parameter changes values are defined by a finite or infinite sequence $\{d_0, d_1, d_2, \ldots\}$ satisfying $0 \leq d_k < d_{k+1}$ for all $k \in \mathbb{N}$ and $\cup_{k=0}^{+\infty}[d_k, d_{k+1}) = [0, +\infty)$. If the conditions in Proposition 3 hold and if the parameter δ is chosen such that

$$0 < 2\mu \leq 2\delta < \min_{k \in \mathbb{N}}\{d_{k+1} - d_k\} \tag{17}$$

then for any solution z to the hybrid system $\mathcal{H}$ in (15) from $z(0,0) \in \mathcal{X}_0 = \{z \in \mathcal{X} : \theta_1 = \theta_2, \tau_a = \tau_b = q = 0\}$, the following property is satisfied: for each $j \in \mathbb{N}_{\geq 1}$ there exists an interval with nonempty interior $I'_j \subseteq I_j \cup I_{j+1}$ such that $\theta_1(t,j) = \theta_2(t,j) = \theta^*$ for all $t \in I'_j$.

The previous result shows that the parameter θ^* is exactly estimated, after finite time since the last time it changed. Indeed, the intervals I'_j imply that whenever the parameter changes its value, the proposed estimator converges to the exact value θ^* no later than 2δ. Let us also mention that similarly to Theorem 1, the same bounds on the solutions can be established.

V. Robustness to measurement noise

In this section, we analyse the robustness of the proposed hybrid parameter estimator with respect to bounded time-varying measurement noise. For the sake of readability, we focus on constant unknown parameters. However, the robustness results can be generalized using the same approach to deal with piecewise constant unknown parameters.

The linear regression model with measurement noise on y is expressed as

$$y(t) = \theta^{*\top}\phi(t) + w(t) \tag{18}$$

where $\theta^* \in \mathbb{R}^n$ is a constant parameter and $t \mapsto w(t) \in \mathbb{R}$ represents measurement noise. The estimator $\hat{y}$ of the real output y is defined as $\hat{y}(t) = \hat{\theta}^\top(t)\phi(t)$ and the error between the true and estimated outputs is given by $e(t) = \hat{y}(t) - y(t) = \tilde{\theta}^\top(t)\phi(t) - w(t)$. Using the cost function $J(e) = \frac{1}{2}e^2$, the classical continuous-time gradient like algorithm is defined as follows:

$$\dot{\tilde{\theta}}(t) = -\gamma\phi(t)\phi^\top(t)\tilde{\theta}(t) + \gamma\phi(t)w(t). \tag{19}$$

where $\gamma > 0$ is a constant adaptation rate.

Starting from the noise free hybrid estimator defined in (10), our hybrid parameter estimator under the effect of $t \mapsto w(t)$ is then defined as a hybrid system $\tilde{\mathcal{H}}$ with data $(C, \tilde{f}, D, \tilde{g})$ and described as follows:

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}(\tilde{z}) \quad \tilde{z} \in C, \\ \tilde{z}^+ &= \tilde{g}(\tilde{z}) \quad \tilde{z} \in D, \end{aligned} \tag{20}$$

the flow and jump maps are given by $\tilde{f}(\tilde{z}) = f(\tilde{z}) + (\gamma_1\phi(\tau_b), \gamma_2\phi(\tau_b), 0, 0, 0)w(\tau_b)$ and $\tilde{g}(\tilde{z}) = g(\tilde{z})$, where f and g are as in (10). The flow and jump sets are given by $C := \mathcal{X}$, and $D := \{z \in \mathcal{X} : \tau_a = \delta, q = 0\}$. To analyse the effect of measurement noise, we rely on the robustness tools developed in the hybrid systems framework [13]. Consider the noise-free hybrid system $\mathcal{H}$ in (15) and the noisy hybrid system $\tilde{\mathcal{H}}$ in (20). We have the following result showing closeness of the trajectories of the noisy hybrid system $\tilde{\mathcal{H}}$ and the noise-free hybrid system $\mathcal{H}$. We rely on the notion of (τ, ε)-closeness of trajectories.

Proposition 4: Given $\delta, \gamma_1, \gamma_2 > 0$ such that $\gamma_1 \neq \gamma_2$, and assume the unknown parameter $\theta^* \in \mathbb{R}^n$ is constant. Let $\mathcal{K} \subseteq \mathbb{R}^{2n}$ be a compact set and let $\tau, \varepsilon > 0$. If the conditions in Proposition 2 hold, then we have the existence of $\bar{w} > 0$ such that if $|w(t)| \leq \bar{w}$, for all $t \in \mathbb{R}_{\geq 0}$, the following holds: for every solution $\tilde{z}$ to $\tilde{\mathcal{H}}$ with an initial condition $\tilde{z}(0,0) \in \mathcal{K} \times [0,\delta] \times \mathbb{R}_{\geq 0} \times \mathbb{N} \cap \mathcal{X}_0$, with $\mathcal{X}_0 = \{z \in \mathcal{X} : \theta_1 = \theta_2, \tau_a = \tau_b = q = 0\}$, there exists a solution z to $\mathcal{H}$ with initial condition $z(0,0) \in \mathcal{K} \times [0,\delta] \times \mathbb{R}_{\geq 0} \times \mathbb{N} \cap \mathcal{X}_0$ such that $\tilde{z}$ and z are (τ, ε) close.

Next we show that under a persistency of excitation condition the noisy system $\tilde{\mathcal{H}}$ is input to state stable (ISS) for any essentially bounded measurement noise w. The question of establishing ISS properties under relaxed excitation conditions is left as future research.

Proposition 5: Given $\delta, \gamma_1, \gamma_2 > 0$ such that $\gamma_1 \neq \gamma_2$, and assume the unknown parameter $\theta^* \in \mathbb{R}^n$ is constant. If the regressor $t \mapsto \phi(t)$ is δ-persistently exciting, with $\int_\sigma^{\sigma+\delta} \phi(u)\phi^\top(u)du \geq \eta I$ for some $\eta > 0$ and for all $\sigma \geq 0$, there exists $\phi_M \geq 0$ such that $|\phi(s)| \leq \phi_M$ for all $s \in \mathbb{R}_{\geq 0}$ and if the constants $\delta, \gamma_1, \gamma_2$ satisfy conditions (12)-(13), then any solution $\tilde{z}$ to $\tilde{\mathcal{H}}$ from $z(0,0) \in \mathcal{X}_0 = \{\tilde{z} \in \mathcal{X} : \theta_1 = \theta_2, \tau_a = \tau_b = q = 0\}$ satisfies

$$\begin{aligned} |\tilde{z}(t,j)|_{\mathcal{A}} &\leq \rho(j)(\beta(|\tilde{z}(0,0)|_{\mathcal{A}}, t) + \alpha_1(|w|_\infty)) \\ &\quad + (1-\rho(j))(\alpha_2(|w|_\infty) + \alpha_1(|w|_\infty)) \end{aligned} \tag{21}$$

with $\rho(0) = 1$ and $\rho(j) = 0$ for $j \in \mathbb{N}_{>0}$, $\beta(s,t) = \exp(\delta)\exp(-t)s$, $\alpha_1(s) = \max\{\gamma_1, \gamma_2\}\phi_M\frac{\kappa}{\lambda}s$ and $\alpha_2(s) = \left((1-\kappa_1\kappa_2)^{-1}(\gamma_1+\gamma_2) + \gamma_2\right)\phi_M\frac{\kappa}{\lambda}s$ with $\kappa_1 = \sqrt{1 - \frac{2\eta\gamma_1}{(1+\phi_M^2\gamma_1\delta)^2}}$ and $\kappa_2 = \sqrt{1 + \frac{2\gamma_2\phi_M^2\delta}{(1-\phi_M^2\gamma_2\delta)^2}}$.

Remark 4: It can be seen from (21) that while the hybrid estimator makes it possible to estimate the unknown parameter at the first jump $(t,j) = (\delta, 1)$ in the noise-free case, the estimation error at the first jump in the noisy case is bounded by $(1-\kappa_1\kappa_2)^{-1}(\gamma_1+\gamma_2)\phi_M|w|_\infty\frac{\kappa}{\lambda} + \gamma_2\phi_M|w|_\infty\frac{\kappa}{\lambda}$. Hence, the estimation error in the noisy case is larger when the term $(1-\kappa_1\kappa_2)$ is smaller. Then, to make the estimation error smaller at the second jump, the idea is to select the parameters δ, γ_1 and γ_2, to make the term $\kappa_1\kappa_2$ closer to 0 while ensuring the satisfaction of conditions (12)-(13).

VI. Examples

A. Constant parameter with excitation on finite time interval

Consider the linear regression model in (1) with $\theta^* = (1,1)$ and $\phi(t) = [\phi_1(t), \phi_2(t)]^\top$, where $\phi_2(t) = \exp(-0.6t)$ and ϕ_1 is given by $\phi_1(t)$ for $t \in [0,2]$ and $\phi_1(t) = 0$ if $t > 2$. It is clear that the regressor ϕ is not persistently exciting, so the classical gradient descent algorithm presented in (4) cannot be applied. Moreover, we have that ϕ is square integrable, so the result presented in [5] cannot be applied either. Finally, the system does not satisfy the conditions in algorithms 1 and 2 in [6]. It can also be seen that ϕ is only exciting over the time interval $[0,2]$. The parameters for the simulations are given by $\gamma_1 = 0.05$, $\gamma_2 = 0.5$ and $\delta = 1$ and the initial conditions are chosen from $\mathcal{X}_0$ with $\theta_1(0,0) = \theta_2(0,0) = (7,5)$. One can check that conditions (12)-(13) are satisfied.

The simulation results[2] are shown in Figure 1. The solid lines represent the estimation error of both parameters in the noise free case, $w(t) = 0$, and the dashed lines represent a noise given by $w(t) = 6\sin(10t)$. We can see that in the noise free case, the parameter error converges to zero in 1 second. In the noisy case, the proposed hybrid algorithm generate robust bounded trajectories.

[2]Files for this simulation can be found at the following address: https://github.com/HybridSystemsLab/FinitetimeConstantparameter

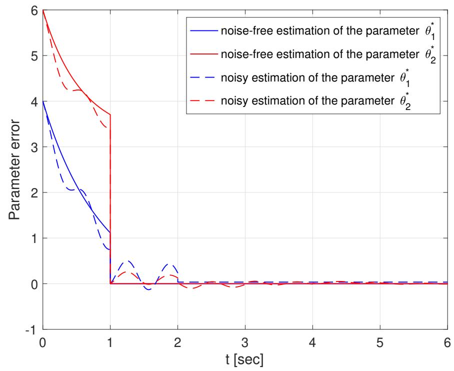

Fig. 1: Evolution of the parameter estimation error with and without noise for unknown constant parameters

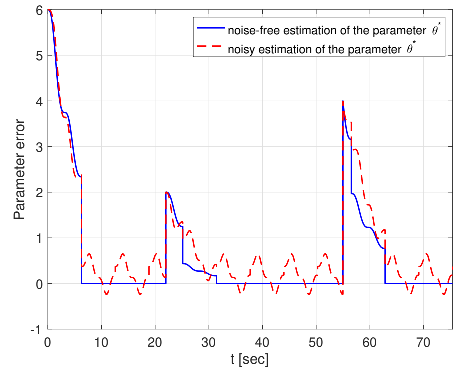

Fig. 2: Evolution of the parameter estimation error with and without noise for unknown piecewise-constant parameters

B. Piecewise constant parameter with persistency of excitation

We apply the proposed hybrid algorithm for the case of a piecewise-constant unknown parameter. Consider the linear regression model in (1) with $\theta^* = 3$ on the time interval $[0, 7\pi]$, $\theta^* = 1$ on the time interval $(7\pi, \frac{35\pi}{2}]$, and $\theta^* = -3$ on the time interval $(\frac{35\pi}{2}, +\infty)$. The regressor is given by $\phi(t) = \sin(t)$. One can check that ϕ is 2π-exciting. Then, in view of Theorem 2, the period δ for the hybrid estimator needs to satisfy $\delta \geq 2\pi$. For the sake of simplicity, we choose $\delta = 2\pi$ which is consistent with the requirement given in (17), since $\min_{k\in\{1,2,3\}}\{d_{k+1} - d_k\} = 24\pi - \frac{35\pi}{2} = \frac{13\pi}{2}$. Following the construction in Theorem 2, the sequence $\{d_k\}$ is given by $d_0 = 0$, $d_1 = 7\pi$, $d_2 = \frac{35\pi}{2}$, $d_3 = +\infty$. The parameters for the simulations are given by $\gamma_1 = 0.3$, and $\gamma_2 = 0.7$ and the initial conditions are chosen from $\mathcal{X}_0$ with $\theta_1(0,0) = 9$, $\theta_2(0,0) = 9$. The simulation results[3] are shown in Figure 2 (we are only showing the trajectory of $\tilde{\theta}_1$). The solid blue line represents the noise-free case, $w(t) = 0$, and the dashed red line represents a noise given by $w(t) = 2\sin(2t)$. Let us first describe the evolution of the trajectory in the noise free case. The first convergence of the parameter error to zero occurs at $\delta = 2\pi = 6.28$. The parameter θ^* change its value at $t = 7\pi$ and the second convergence of the parameter error to zero occurs at $t = 31.42$. It can be seen that the difference between the second convergence to zero and the first change of the parameter is less than $2\delta = 4\pi$, $(31.42 - 7\pi = 9.42 \leq 2\delta)$. Finally, the parameter θ^* changes its value at $t = \frac{35\pi}{2}$ and the third convergence of the parameter error to zero occurs at $t = 62.83$, and one can again check that the difference between the third convergence to zero and the second change of the parameter is less than 2δ. In the noisy case, the hybrid algorithm generates robust bounded trajectories.

VII. Conclusion

In this paper, a robust hybrid finite time parameter estimator is proposed. We have shown that the proposed hybrid estimator allows for finite time convergence, while only requiring the regressor to be exciting on a finite interval. Moreover, robustness with respect to time-varying measurements noise is analysed using tools from hybrid systems theory. Finally, numerical examples are provided, showing the practicality of the proposed approach. In future work we will develop finite-time parameter estimators for the classical MRAC model.

[3]Files for this simulation can be found at the following address: https://github.com/HybridSystemsLab/FinitetimePiecewiseconstantparameter

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