Computation of Controlled Invariants for Nonlinear Systems: Application to Safe Neural Networks Approximation and Control ⋆

Adnane Saoud * **Ricardo G. Sanfelice** *

* *Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA (e-mail: asaoud, ricardo@ucsc.edu).*

Abstract: In this paper, we consider the problem of computing multidimensional interval controlled invariants for nonlinear input-affine systems. We first present sufficient conditions for an interval to be controlled invariant. Then, we introduce the concept of local framers, based on which we present a sound algorithm to compute interval controlled invariants. Finally, we show how the proposed framework makes it possible to provide safety guarantees when using deep neural networks, either as a model or a controller of nonlinear systems. Illustrative examples are provided showing the merits of the proposed approach and its scalability properties.

Keywords: Invariance, nonlinear systems, framers, neural networks.

1. INTRODUCTION

Controlled invariance plays an important role in control theory (Blanchini and Miani [2008]). The notion of controlled invariance reflects the ability to control the system so that all trajectories initialized in a set remain there for all future time. Formulation and definitions of the concept of controlled invariance of a set are summarized in (Aubin [2009], Blanchini and Miani [2008]) for continuous-time and discrete-time systems, and in (Chai and Sanfelice [2020]) for hybrid systems. For continuous-time systems, different approaches have been proposed in the literature to compute controlled invariants. In (Blanchini and Miani [2008]) controlled invariants are obtained as sublevel sets of Lyapunov-like functions. Controlled invariants for polynomial systems have been investigated using linear programming in (Korda et al. [2014]). For general nonlinear systems, polytopics controlled invariants are computed in (Cannon et al. [2003]) by embedding the nonlinear dynamics into linear ones. Other approaches have been proposed recently using symbolic control techniques (Saoud [2019], Tabuada [2009]).

In this paper, we are interested in the study of multidimensional interval (or simply interval) controlled invariants for a class of nonlinear input-affine systems. To the best of the authors knowledge, only monotone systems have been considered when dealing with interval controlled invariants. The authors in (Abate et al. [2009]) are dealing with monotone autonomous multi-affine systems and in (Meyer et al. [2016]) the authors present an approach for the computation of robust controlled invariants for monotone systems with inputs. In this paper, a new approach to compute interval controlled invariants that is applicable to a class of nonlinear input-affine systems is proposed. First, we present sufficient conditions for an interval to be controlled invariant. Then, we introduce the concept of local framer and present a sound algorithm to compute interval controlled invariants. Finally, we show how the proposed approach can be used to ensure invariance when using deep neural networks for the purpose of approximation and control of continuous-time nonlinear input-affine systems. To validate the practicality of the proposed we consider the control problem of a Kuramoto oscillator and an agent moving in a 2-D plane.

The remainder of this paper is organized as follows. In Section 2 we introduce the class of systems considered and sufficient conditions for an interval to be controlled invariant. In Section 3, we present applications of the proposed framework to safe deep neural networks approximation and control. Finally, Section 4 presents numerical results validating the merits of the proposed approach. Due to space constraints, the proofs are omitted and will be published elsewhere.

Notation: The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{\leq 0}$ denote the set of positive integer, real, non negative real, and nonpositive real numbers, respectively. Given vectors $x^a, x^b \in \mathbb{R}^n$, $x^a \leq x^b$ stands for $x_i^a \leq x_i^b$ for all $i \in \{1, 2, \dots, n\}$ and $x^a < x^b$ stands for $x_i^a < x_i^b$ for all $i \in \{1, 2, \dots, n\}$. Using this partial order, we define a multidimensional interval set as follows: for $\underline{x}, \overline{x} \in \mathbb{R}^n$, with $\underline{x} \leq \overline{x}$, $[\underline{x}; \overline{x}] = \{x \in \mathbb{R}^n \mid x \leq \overline{x},\ x \geq \underline{x}\}$ and $(\underline{x}; \overline{x}) = \{x \in \mathbb{R}^n \mid x < \overline{x},\ x > \underline{x}\}$. Given a map $f : \mathbb{R}^n \to \mathbb{R}^m$ and a compact

⋆ A. Saoud and R. G. Sanfelice are with Department of Electrical and Computer Engineering, University of California, Santa Cruz. Email: asaoud,ricardo@ucsc.edu. This research has been partially supported by NSF Grants no. ECS-1710621, CNS-1544396, and CNS-2039054, by AFOSR Grants no. FA9550-19-1-0053, FA9550-19-1-0169, and FA9550-20-1-0238, and by CITRIS and the Banatao Institute at the University of California.

set $A \subset \mathbb{R}^n$, $f(A) = \{y \in \mathbb{R}^m \mid \exists x \in A \text{ and } f(x) = y\}$ is the reachable set from the set A under the map f. Given two sets $A, B \subset \mathbb{R}^n$, the set $A + B$ represents the Minkowski sum of the sets A and B, which is defined as $A + B = \{a + b \mid a \in A,\ b \in B\}$. For $x \in \mathbb{R}$, $z = (z_1, z_2, \ldots, z_{n-1}) \in \mathbb{R}^{n-1}$, and $i \in \{1, 2, \ldots, n\}$, we use $\lceil x, z \rceil_i$ to denote the unique element $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ such that $y_i = x$, $y_j = z_j$ for all $j \in \{1, 2, \ldots, i-1\}$ and $y_j = z_{j-1}$ for all $j \in \{i+1, i+2, \ldots, n\}$, i.e. $\lceil x, z \rceil_i = (z_1, z_2, \ldots, z_{i-1}, x, z_i, \ldots, z_{n-1}) \in \mathbb{R}^n$.

2. CONTROLLED INVARIANTS

2.1 Nonlinear input-affine systems

Consider a nonlinear input-affine system Σ defined by

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathcal{X} \subset \mathbb{R}^n$ is the state and $u \in \mathcal{U} = [\underline{\mathcal{U}}; \overline{\mathcal{U}}] \subset \mathbb{R}^m$ is the control input. The functions f and g are locally Lipschitz, where $\underline{\mathcal{U}}, \overline{\mathcal{U}} \in \mathbb{R}^m$. The trajectories of (1) are denoted by $\Phi(., x_0, u)$, where $\Phi(t, x_0, u)$ is the state reached at time $t \in \mathbb{R}_{\geq 0}$ from the initial state x_0 under the control input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. When the control inputs of system (1) are generated by a state-feedback controller $\kappa : \mathbb{R}^n \to \mathcal{U}$, the dynamics of the closed-loop system are given by

$$\dot{x} = f(x) + g(x)\kappa(x) \tag{2}$$

and its trajectories from x_0 are denoted by $\Phi_\kappa(., x_0)$. We first have the following standard assumption on the controller κ ensuring uniqueness and completeness of maximal trajectories of the closed-loop system.

Assumption 1. The controller $\kappa : \mathcal{X} \to \mathcal{U}$ is such that from any initial condition $x_0 \in \mathcal{X}$, the system (2) admits a unique solution $\Phi_\kappa(., x_0)$ originating from x_0 defined for all $t \geq 0$. •

The following local control property is employed in the main results of this section.

Assumption 2. System (1) can be written as follows: for each $i \in \{1, 2, \ldots, n\}$

$$\dot{x}_i = f_i(x) + g_i(x)u_i, \tag{3}$$

with $x = (x_1, x_2, \ldots, x_n) \in \mathcal{X} \subset \mathbb{R}^n$ and $u = (u_1, u_2, \ldots, u_n) \in \mathcal{U} \subset \mathbb{R}^m$, where $u_i \in \mathcal{U}_i = [\underline{\mathcal{U}_i}; \overline{\mathcal{U}_i}]$, and $\underline{\mathcal{U}_i}$ (respectively, $\overline{\mathcal{U}_i}$) is the i-th component of $\underline{\mathcal{U}}$ (respectively, $\overline{\mathcal{U}}$). •

Intuitively, the local control property in Assumption 2 means that every component of the control input directly influences only a single component of the state in (1). This property can be more exhaustively written as follows. Expressing g as

$$g(x) = \begin{bmatrix} g_{11}(x) & g_{12}(x) & \ldots & g_{1m}(x) \\ g_{21}(x) & g_{22}(x) & \ldots & g_{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}(x) & g_{n2}(x) & \ldots & g_{nm}(x) \end{bmatrix}$$

it can be seen that if there exists a column of the matrix $g(x)$ with two nonzero elements in the i-th column, u_i will affect two state components of $x = (x_1, x_2, \ldots, x_n)$. Similarly, if the column g_{*i} is zero, then the system does not depend on u_i. Hence, system (1) satisfies the local control property in Assumption 2 if and only if each column of the matrix g has only one nonzero element.

For the system in (3), $u_i \in \mathcal{U}_i$, $i \in \{1, 2, \ldots, n\}$, represents all input components with a direct influence on x_i. With some abuse of notation, we rewrite the dynamics of each x_i, $i \in \{1, 2, \ldots, n\}$, in (3) as follows:

$$\dot{x}_i = f_i(x_i, z_i) + g_i(x_i, z_i)u_i \tag{4}$$

with $z_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$, $i \in \{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, n\}$, we define the linear map $\pi_{i,1} : \mathbb{R}^n \to \mathbb{R}$ and $\pi_{i,2} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that for all $x = (x_1, x_2, \ldots, x_n)$, $\pi_{i,1}(x) = x_i$ and $\pi_{i,2}(x) = z_i$. In the following, we will denote $\mathcal{X}_i = \pi_{i,1}(\mathcal{X})$ and $\mathcal{Z}_i = \pi_{i,2}(\mathcal{X})$.

Remark 1. Let us mention that in some cases, even if Assumption 2 is not satisfied by the original system, one can find a change of coordinates ensuring Assumption 2 for the transformed system.

2.2 Sufficient conditions for controlled invariance

In this section we provide a result for the computation of controlled invariants for the considered class of systems. First, we recall the concept of controlled invariant (Blanchini and Miani [2008]). In simple words, a controlled invariant set is a set for which there exists a controller such that if the state of the system is initialized in this set then its solutions remain there for all time.

Definition 1. The set $K \subseteq \mathcal{X}$ is said to be a controlled invariant for the system Σ in (1) if there exists a controller $\kappa : K \to \mathcal{U}$ such that, for all $x_0 \in K$, each solution $\Phi_\kappa(., x_0) : \mathbb{R}_{\geq 0} \to \mathcal{X}$ satisfies $\Phi_\kappa(t, x_0) \in K$ for all $t \in \mathbb{R}_{\geq 0}$. When this property holds, κ is said to be an invariance controller for the system Σ and the set K.

The following result provides a characterization of controlled invariance of a set K given by an interval. The invariance controller is designed by exploiting the sign of the nonlinear functions f and g in the boundary of K.

Theorem 1. Suppose that the nonlinear system Σ defined in (1) satisfies Assumption 2. Let $K \subseteq \mathcal{X}$ be an interval defined as $K = [\underline{x}; \overline{x}]$, where $\underline{x} < \overline{x}$. The set K is a controlled invariant for the system Σ if, for each $i \in \{1, 2, \ldots, n\}$, there exist $\underline{u}_i, \overline{u}_i \in \mathcal{U}_i$ such that

$$f_i(\underline{x}_i, [\underline{z}_i; \overline{z}_i]) + g_i(\underline{x}_i, [\underline{z}_i; \overline{z}_i])\overline{u}_i \subset \mathbb{R}_{\geq 0} \tag{5}$$

$$f_i(\overline{x}_i, [\underline{z}_i; \overline{z}_i]) + g_i(\overline{x}_i, [\underline{z}_i; \overline{z}_i])\underline{u}_i \subset \mathbb{R}_{\leq 0} \tag{6}$$

where f_i and g_i are given as in (4).

A graphical representation of the conditions in Theorem 1 is provided in Figure 1.

Remark 2. At times, we may have that, for some $i \in \{1, 2, \ldots, n\}$, $g_i(x) = 0$ for all $x \in \mathcal{X}$. In such a case, conditions (5) and (6) reduce to, respectively, $f_i(\underline{x}_i, [\underline{z}_i; \overline{z}_i]) \subset \mathbb{R}_{\geq 0}$ and $f_i(\overline{x}_i, [\underline{z}_i; \overline{z}_i]) \subset \mathbb{R}_{\leq 0}$, whose satisfaction solely depends on the properties of f alone.

Theorem 1 provides theoretical guidelines to compute interval controlled invariants. The main difficulty is to provide an efficient algorithmic procedure to check conditions (5) and (6). Indeed, the difficulty lies in the fact that the sets $f_i(\underline{x}_i, [\underline{z}_i; \overline{z}_i])$, $g_i(\underline{x}_i, [\underline{z}_i; \overline{z}_i])$, $f_i(\overline{x}_i, [\underline{z}_i; \overline{z}_i])$,

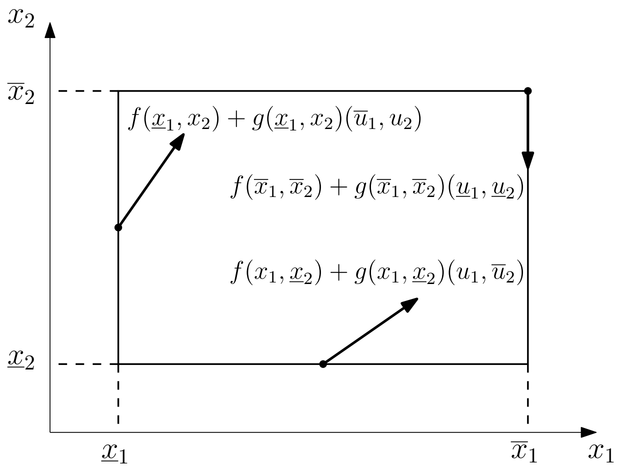


Fig. 1. A graphical representation of the conditions in Theorem 1 for a two dimensional system. The vector fields are pointing toward the interior of K or tangentially to K at points in the boundary of the interval K.

and $g_i(\overline{x}_i, [\underline{z}_i; \overline{z}_i])$ are difficult to compute numerically for general nonlinear functions f and g.

In the following, we introduce the concept of local framers, and show how this concept makes it possible to check the satisfaction of conditions (5) and (6).

2.3 Local framers and controlled invariance

The concept of framer [1] has been extensively used for the construction of interval observers (Moisan et al. [2007]) since they allow to provide bounds for the unknown state.

Definition 2. Given a set $\mathcal{X} \subset \mathbb{R}^n$ and a function $f : \mathcal{X} \to \mathbb{R}^m$, a local framer of f is given by a family of collections of pair of maps $\underline{f}_i, \overline{f}_i : \mathcal{X}_i \times \mathcal{Z}_i \times \mathcal{Z}_i \to \mathbb{R}$, $i \in \{1, 2, \ldots, n\}$, such that for all $x_i \in \mathcal{X}_i$ and for all $\underline{z}_i, \overline{z}_i \in \mathcal{Z}_i$, with $\underline{z}_i \leq \overline{z}_i$, we have

$$f_i(x_i, [\underline{z}_i; \overline{z}_i]) \subseteq [\underline{f}_i(x_i, \underline{z}_i, \overline{z}_i); \overline{f}_i(x_i, \underline{z}_i, \overline{z}_i)] \tag{7}$$

where $f_i : \mathcal{X} \to \mathbb{R}$ is the i-th component of the function f.

Next, we show how the existence of a local framer allows to compute interval controlled invariants for the considered class of systems. The issue of computing local framers is discussed Remark 3.

Proposition 1. Suppose that the system Σ defined in (1) satisfies Assumption 2. Let $\underline{f}_i$, $\overline{f}_i$ (respectively, $\underline{g}_i$, $\overline{g}_i$), $i \in \{1, 2, \ldots, n\}$, be a local framer of f (respectively, g). Let $K \subseteq \mathcal{X}$ be an interval defined as $K = [\underline{x}; \overline{x}]$, where $\underline{x} < \overline{x}$. The set K is a controlled invariant for the system Σ if, for each $i \in \{1, 2, \ldots, n\}$, there exist $\underline{u}_i, \overline{u}_i \in \mathcal{U}_i$ such that

$$\begin{cases} \underline{f}_i(x_i, \underline{z}_i, \overline{z}_i) + \underline{g}_i(x_i, \underline{z}_i, \overline{z}_i)\overline{u}_i \geq 0 & \text{if } \overline{u}_i \geq 0 \\ \underline{f}_i(x_i, \underline{z}_i, \overline{z}_i) + \overline{g}_i(x_i, \underline{z}_i, \overline{z}_i)\overline{u}_i \geq 0 & \text{if } \overline{u}_i < 0 \end{cases} \tag{8}$$

$$\begin{cases} \overline{f}_i(x_i, \underline{z}_i, \overline{z}_i) + \overline{g}_i(x_i, \underline{z}_i, \overline{z}_i)\underline{u}_i \leq 0 & \text{if } \underline{u}_i \geq 0 \\ \overline{f}_i(x_i, \underline{z}_i, \overline{z}_i) + \underline{g}_i(x_i, \underline{z}_i, \overline{z}_i)\underline{u}_i \leq 0 & \text{if } \underline{u}_i < 0 \end{cases} \tag{9}$$

[1] The same concept has been used in interval arithmetic under the name of interval enclosure function (Jaulin et al. [2001])

The result in Proposition 1 leads to the following algorithm.

Algorithm 1: Controlled invariance

Input: Functions f and g of a system Σ as in (1) and an interval $K = [\underline{x}; \overline{x}]$ with dimension n.
Output: $True$, if K is a controlled invariant for Σ.
begin
for $i = 1 : n$
  Compute local framers:
    $\underline{f}_i(x_i, \underline{z}_i, \overline{z}_i)$, $\underline{g}_i(x_i, \underline{z}_i, \overline{z}_i)$, $\overline{f}_i(x_i, \underline{z}_i, \overline{z}_i)$,
    $\overline{g}_i(x_i, \underline{z}_i, \overline{z}_i)$
  Set $b_i = 1$ if conditions (8) and (9) of Proposition 1 are satisfied, otherwise set $b_i = 0$
end for
$b = \bigwedge_{i=1}^{n} b_i$
return True if $b = 1$.

Remark 3. Algorithm 1 is linear with respect to the state space dimension. Indeed, the computational complexity is given by $\mathcal{O}((\alpha + \beta)n)$, where n represents the state-space dimension, α represents the complexity to compute the local framer, and β represents the complexity to check (8)-(9) in Algorithm 1.

The following result characterizes the class of possible invariance controllers enforcing conditions (8) and (9) of Proposition 1.

Corollary 1. Suppose that the system Σ defined in (1) satisfies Assumption 2. Let $\underline{f}_i$, $\overline{f}_i$, $i \in \{1, 2, \ldots, n\}$, be a local framer of f. Similarly, let $\underline{g}_i$, $\overline{g}_i$, $i \in \{1, 2, \ldots, n\}$, be a local framer of g. Let $K \subseteq \mathcal{X}$ be an interval defined as $K = [\underline{x}; \overline{x}]$, where $\underline{x} < \overline{x}$. If the set K is a controlled invariant for the system Σ, then any continuous function $\kappa : \mathcal{X} \to \mathcal{U}$ satisfying Assumption 1 with $\kappa(x) = (\kappa_1(x), \kappa_2(x), \ldots, \kappa_n(x))$ and $\kappa_i : \mathcal{X} \to \mathcal{U}_i$, $i \in \{1, 2, \ldots, n\}$, satisfying

$$\kappa_i(x) \in \begin{cases} \{u_i \in \mathcal{U}_i \mid \text{ condition (8) holds }\} & \text{if } x_i = \underline{x}_i \\ \mathcal{U}_i & \text{if } x_i \in (\underline{x}_i; \overline{x}_i) \\ \{u_i \in \mathcal{U}_i \mid \text{ condition (9) holds }\} & \text{if } x_i = \overline{x}_i \end{cases}$$

is an invariance controller for the system Σ and the set K.

Remark 4. The continuity of the controller κ in Corollary 1 can be dropped, at the cost of potentially more restrictive conditions. Indeed, in view of ([Aubin, 2009, Theorem 5.2.1]), conditions (8) and (9) do not need to be satisfied only at the boundary of the set of interest K, but also on the external part of a sufficiently small neighborhood of the set K.

Remark 5. (On the computation of local framers). The construction of local framers can be done for general nonlinear functions using tools from interval arithmetic. If a function $f : \mathbb{R}^n \to \mathbb{R}^n$ can be expressed as a finite composition of the basic operators $\{+, -, \times, /\}$ and elementary functions (e.g., $sin, cos, exp, log, \ldots$), a local framer for f can be constructed using the natural inclusion approach in ([Jaulin et al., 2001, Section 2.4]) by replacing each variable and each operator or function by their interval counterparts.

3. APPLICATIONS

In this section, we show how the proposed approach can be used to ensure safety when using deep neural networks for the purpose of approximation and control for the considered class of systems.

3.1 Safe deep neural networks approximation of continuous time dynamical systems

It is well known that deep neural networks (DNNs) can be used as universal approximators of static functions (Hornik [1991]). Indeed, any continuous function can be approximated over a compact set up to a desired level of accuracy by the selection of suitable activation functions and an adequate number of hidden layer neurons. Motivated by this property, DNNs have been extensively used in the control community as models for complex systems, where the unknown dynamics are generally learned using a DNN to be used for control purposes, ([Lewis et al., 2020, Section 6-2], Bansal et al. [2016]). Since DNNs approximation results are valid as long as the state of the system belongs to the compact set on which the data have been collected, one need to ensure, a priori, that the trajectories of the closed-loop controlled system will always remain in that compact set. In this section, we first start by presenting some preliminaries on DNNs, then we formulate the safe DNNs approximation problem and finally propose a solution using the tools presented in the previous section.

Preliminaries on deep neural networks: In this paper we assume that the DNN is a feedforward neural network. A feedforward neural network has $N^0 = \bar{n}$ input neurons, L hidden layers, with N^l, $l \in \{1, 2, \ldots, L\}$, neurons per hidden layer, and one output layer with $N^{L+1} = \bar{m}$ output neurons. The neural network $\mathcal{N} : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{m}}$ can be formally defined for $\xi \in \mathbb{R}^{\bar{n}}$ as follows:

$$\mathcal{N}(\xi) = a^{L+1} \circ f^{L+1} \circ a^L \circ f^L \circ \ldots \circ a^1 \circ f^1(\xi) \qquad (10)$$

where, for each $l \in \{1, 2, \ldots, L+1\}$, a^l is the activation function and each f^l is an affine transformation of the output of the previous layer given by

$$f^l(x) = W^l x + b^l \qquad (11)$$

for each $l \in \{1, 2, \ldots, L+1\}$, the matrix W^l and the offset vector b^l have the following size:

$$W^l \in \begin{cases} \mathbb{R}^{N^1 \times \bar{n}} & \text{if } l = 1 \\ \mathbb{R}^{N^l \times N^{l-1}} & \text{if } l \in \{2, 3, \ldots, L\} \\ \mathbb{R}^{\bar{m} \times N^L} & \text{if } l = L+1 \end{cases} \qquad (12)$$

$$b^l \in \begin{cases} \mathbb{R}^{N^l} & \text{if } l \in \{1, 2, \ldots, L\} \\ \mathbb{R}^{\bar{m}} & \text{if } l = L+1 \end{cases} \qquad (13)$$

The activation function of the output layer, a^{L+1}, is the identity map from $\mathbb{R}^{\bar{m}}$ to $\mathbb{R}^{\bar{m}}$.

Problem statement: Given the system Σ defined in (1) with the functions f and g modeled as DNNs $\mathcal{N}_f$ and $\mathcal{N}_g$, where the DNNs $\mathcal{N}_f$ and $\mathcal{N}_g$ have been learned using the data collected on a compact interval $K \subseteq \mathbb{R}^n$, provide a collection of controllers [2] $\kappa : K \to \mathcal{U}$, ensuring that the set K is an invariant for the closed-loop system $\Sigma_{\mathcal{N}}$ defined by

$$\dot{x} = \mathcal{N}_f(x) + \mathcal{N}_g(x)\kappa(x) \qquad (14)$$

Solution strategy: To provide a solution to our problem, we will follow the approach proposed in Section 2. The main ingredient used to compute controlled invariants are the local framers.

In the following, we will briefly describe how the approach proposed in the toolbox ReluVal (Wang et al. [2018]) can be adapted to efficiently compute accurate local framers for DNNs. Let us mention that ReluVal works only with ReLU [3] activation functions. The approach of ReluVal is made of three main ingredients:

- Symbolic interval propagation: it consists of a layer-by-layer application of the natural inclusion approach to the linear map (11). Indeed, if we consider the layer l of the DNN, the local framer of the linear map f^l in (11) is given for $i \in \{1, 2, \ldots, n\}$, $x_i^{l-1} \in \mathcal{X}_i$ and $\underline{z}_i^{l-1}, \overline{z}_i^{l-1} \in \mathcal{Z}_i$ as follows:
$$\begin{aligned} \underline{f}_i^l(x_i^{l-1}, \underline{z}_i^{l-1}, \overline{z}_i^{l-1}) = {} & W_+^l \lceil x_i^{l-1}, \underline{z}_i^{l-1} \rceil_i + \\ & W_-^l \lceil x_i^{l-1}, \overline{z}_i^{l-1} \rceil_i + b^l \\ \overline{f}_i^l(x_i^{l-1}, \underline{z}_i^{l-1}, \overline{z}_i^{l-1}) = {} & W_+^l \lceil x_i^{l-1}, \overline{z}_i^{l-1} \rceil_i + \\ & W_-^l \lceil x_i^{l-1}, \underline{z}_i^{l-1} \rceil_i + b^l \end{aligned} \qquad (15)$$
where for the matrix $A \in \mathbb{R}^{n \times m}$ the matrices A_+ and A_- are defined for $i, j \in \{1, 2, \ldots, n\}$ as $(A_+)_{ij} = \max\{A_{ij}, 0\}$, and $(A_-)_{ij} = \min\{A_{ij}, 0\}$.
- Concretization: the ReLU function is then applied to the symbolic equations in (15) and we distinguish three cases: if $\underline{f}_i^l(x_i^{l-1}, \underline{z}_i^{l-1}, \overline{z}_i^{l-1}) > 0$, we keep the symbolic dependency on the input variables. If $\overline{f}_i^l(x_i^{l-1}, \underline{z}_i^{l-1}, \overline{z}_i^{l-1}) \leq 0$ we concertize to 0. Finally, if $\underline{f}_i^l(x_i^{l-1}, \underline{z}_i^{l-1}, \overline{z}_i^{l-1}) \leq 0$ and $\overline{f}_i^l(x_i^{l-1}, \underline{z}_i^{l-1}, \overline{z}_i^{l-1}) \geq 0$ then while passing the local framers through the ReLU, we can no longer keep the symbolic representation. Therefore, we concretize the values of the local framers.
- Refinement: to improve the precision of the local framer, an iterative refinement is performed, through the use of bisections.

Once the local framers are computed, we rely on the result of Corollary 1 to provide a collection of controllers, ensuring that the set of interest K is an invariant for the closed-loop system.

3.2 Safety verification of deep neural network controllers for continuous-time dynamical systems

DNNs have been extensively used as controllers in the control community when dealing with complex and challenging dynamical systems (Koopman and Wagner [2017], Sallab et al. [2017]). Motivated by the lack of formal guarantees when using DNNs (as discussed in the previous subsection), in this section, we present an approach to verify invariance properties of dynamical systems under control of continuous-time DNN controllers with ReLU activation functions.

[2] We are providing here a collection of controllers ensuring the safety of the closed-loop system, among these controllers, one can choose one allowing to achieve a higher level specification such as stability or more complex logic specification (Tabuada [2009]).

[3] For $x \in \mathbb{R}$, $Relu(x) = \max\{0, x\}$.

Problem statement: Given the system Σ defined in (1), an interval $K \subseteq \mathbb{R}^n$, a DNN continuous-time controller $\mathcal{N} : X \to U$, which has already been trained, verify whether the set K is an invariant of the closed-loop system given by

$$\dot{x} = f(x) + g(x)\mathcal{N}(x) \tag{16}$$

Solution strategy: To provide a solution to this control problem, we follow the approach proposed in Section 2 and use local framers both for the system dynamics and the controller. The following corollary of Proposition 1 summarizes the main idea.

Corollary 2. Suppose that the system Σ defined in (1) satisfies Assumption 2. Let $\underline{f}_i$, $\overline{f}_i$ (respectively, $\underline{g}_i$, $\overline{g}_i$), $i \in \{1, 2, \ldots, n\}$, be a local framer of f (respectively, g). Let $K \subseteq \mathcal{X}$ be an interval defined as $K = [\underline{x}; \overline{x}]$, where $\underline{x} < \overline{x}$. Let $\mathcal{N}$ be the already trained DNN controller and let $\underline{\mathcal{N}}_i, \overline{\mathcal{N}}_i$, $i \in \{1, 2, \ldots, n\}$, be a local framer for $\mathcal{N}$ (which can be constructed, as discussed in the previous subsection). The set K is a controlled invariant for the closed-loop controlled system in (16) if the following conditions are satisfied: [4]

$$\underline{f}_i + \min\{\underline{g}_i\underline{\mathcal{N}}_i, \underline{g}_i\overline{\mathcal{N}}_i, \overline{g}_i\underline{\mathcal{N}}_i, \overline{g}_i\overline{\mathcal{N}}_i\} \geq 0$$
$$\overline{f}_i + \max\{\underline{g}_i\underline{\mathcal{N}}_i, \underline{g}_i\overline{\mathcal{N}}_i, \overline{g}_i\underline{\mathcal{N}}_i, \overline{g}_i\overline{\mathcal{N}}_i\} \leq 0$$

4. EXAMPLES

4.1 Kuramoto oscillator

The Kuramoto oscillator has many applications ranging from automated vehicle coordination to power networks (Dorfler and Bullo [2012]). The fully interconnected Kuramoto model consists of n phase oscillators with state θ_i, $i \in \{1, 2, \ldots, n\}$, evolving according to

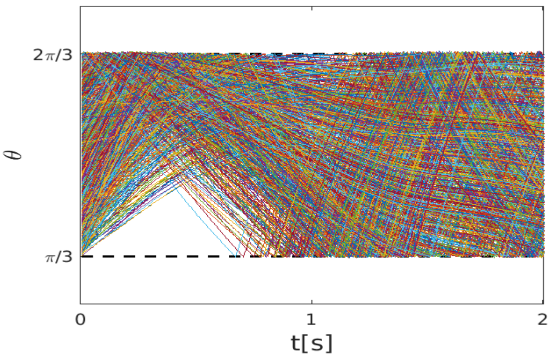
$$\dot{\theta} = f(\theta) + g(\theta)u \tag{17}$$

with $f(\theta) = w + \frac{K}{n}\phi(\theta)$ and $g(\theta) = 1$, where $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathcal{X} = [0; 2\pi]^n$ and $w = (w_1, w_2, \ldots, w_n)$ is the natural frequency of the oscillators. In the simulations, w_i, $i \in \{1, 2, \ldots, n\}$ is randomly chosen in the set $[0, \pi]$. The map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is given by

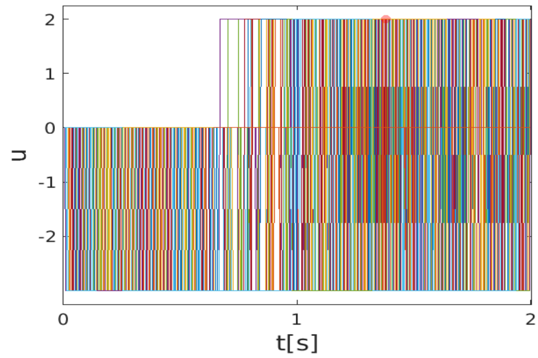
$$\phi(\theta) = \left(\sum_{j=1}^{n} \sin(\theta_j - \theta_1), \sum_{j=1}^{n} \sin(\theta_j - \theta_2) \ldots, \sum_{j=1}^{n} \sin(\theta_j - \theta_n) \right).$$

The constant $K = 2$ is the coupling strength, and control input $u = (u_1, u_2, \ldots, u_n)$ with $u_i \in [-3.5; 2]$, $i \in \{1, 2, \ldots, n\}$. The proposed model satisfies Assumption 2 with f_i and g_i, $i \in \{1, 2, \ldots, n\}$, given by $f_i(\theta) = w_i + \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$ and $g_i(\theta) = 1$.

The objective is to construct a controller allowing to keep the trajectories of the closed loop system in the safe set $K = [\frac{\pi}{3}; \frac{2\pi}{3}]^n$. Safety of Kuramoto oscillators has been addressed recently in (Jagtap et al. [2020]), where the authors use compositional barrier certificates, which can be constructed only under a small-gain like condition. In this paper we do not rely on such type of conditions.



(a) Evolution of the state of the Kuramoto oscillator with $n = 1000$ using a bang-bang controller



(b) Evolution of the input of the Kuramoto oscillator with $n = 1000$ using a bang-bang controller

To check that the set of interest K is a controlled invariant, we use Algorithm 1. The computation of the local framer have been conducted using the natural inclusion approach ([Jaulin et al., 2001, Section 2.4]). Then, we select a controller realization from the set of all possible controllers described in Corollary 1. For simulations [5], we use a bang-bang control where we simply keep the previous value of u. Figure 2a shows the evolution of the state variables. For each oscillator, the initial condition is chosen randomly in the interval $\theta_i(0) = [\frac{\pi}{3}; \frac{2\pi}{3}]$, $i \in \{1, 2, \ldots, n\}$, and one can see that the overall safety objective is satisfied. On Figure 2b we represent the input signal generated by the bang-bang controller. To check the controlled invariance using Algorithm 1, the computation time is less than 2 minutes for 1000 oscillator, which shows the practical scalability of the proposed approach. The numerical implementations have been done in MATLAB, Processor 2.8 GHz Intel Core i7, Memory 8 GB 1333 MHz DDR3.

4.2 Safe deep neural network approximation

We consider the nonlinear system Σ described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x) + g(x)u := \begin{bmatrix} x_2 \sin(k_1 x_1) \\ x_1 \cos(k_2 x_2^2) \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system represents an agent moving in the 2-D plane with position $x = (x_1, x_2)$. The authors in (Hou et al. [2009]) assume that the dynamics of Σ are unknown and use a single hidden layer neural network to learn the model. In this paper, we approximate the system using a DNN

[4] The dependence of the functions $\underline{f}_i, \overline{f}_i$, $\underline{g}_i, \overline{g}_i$ and $\underline{\mathcal{N}}_i, \overline{\mathcal{N}}_i$ on $x_i, \underline{z}_i$ and $\overline{z}_i$) is omitted to improve readability.

[5] Files for this simulation can be found at the following address: `https://github.com/HybridSystemsLab/InvarianceKuramoto`

model, i.e, the function f is approximated by a DNN $\mathcal{N}_f$ [6]. The data is collected on the state space of interest $\mathcal{X} = [-2;2] \times [-2;2]$, where the input space is given by $\mathcal{U} = [-2;2] \times [-2;2]$. In view of Problem 3.1.2, the objective is to ensure the safety of the DNN approximation result. More precisely, the objective is to provide a collection of state-feedback controllers $\kappa : \mathcal{X} \to \mathcal{U}$ ensuring that the set of interest $\mathcal{X}$, on which the data has been collected is an invariant for the closed loop system

$$\dot{x} = \mathcal{N}_f(x) + \kappa(x). \tag{18}$$

For the simulations [7], the function f is approximated by a 2 hidden layers NN ($L = 2$), The number of neurons each hidden layer is 4 ($N^1 = N^2 = 8$). The ReLU activation function is used in the NN hidden layers with a randomly initialized input weights. The training has been done using the Matlab Deep Learning toolbox. We used a feedforward DNN with Levenberg-Marquardt training algorithm and Mean Squared Error as a loss function. Once the approximative DNN $\mathcal{N}_f$ is obtained, we follow the approach proposed in Section 3.1.3. We first use the ReluVal toolbox (Wang et al. [2018]) to compute the local framers. Then, we rely on Algorithm 1 and Corollary 1 to provide a collection of controllers, ensuring that the set of interest $\mathcal{X}$ is an invariant for the closed-loop system in (18). The collection of possible controllers is given by $\kappa : \mathcal{X} \to \mathcal{U}$ with $\kappa(x) = (\kappa_1(x), \kappa_2(x))$, where $\kappa_i : \mathcal{X} \to \mathcal{U}_i$, $i \in \{1, 2\}$ satisfy

$$\kappa_1(x) \in \begin{cases} [1.9; 2] & \text{if } x_1 = -2 \\ [-2; 2] & \text{if } x_1 \in (-2; 2) \\ [-2; -1.9] & \text{if } x_1 = 2 \end{cases}$$

$$\kappa_2(x) \in \begin{cases} [0.6; 2] & \text{if } x_2 = -2 \\ [-2; 2] & \text{if } x_2 \in (-2; 2) \\ [-2; -0.6] & \text{if } x_2 = 2 \end{cases}$$

5. CONCLUSION

In this paper, we have presented an approach to the computation of interval controlled invariants for nonlinear input-affine systems using the concept of local framers. Moreover, we have shown how the proposed approach makes it possible to ensure safety properties when using DNN representations for the purpose of modelling and control. Illustrative examples are presented showing the merits of the proposed approach and its scalability properties. In future works, we will address the question of the computation of tight local framers, which will make it possible to construct a complete algorithm. Moreover, we will generalize the approach to deal with other set structures beyond intervals used in this paper.

REFERENCES

Abate, A., Tiwari, A., and Sastry, S. (2009). Box invariance in biologically-inspired dynamical systems. *Automatica*, 45(7), 1601–1610.

Aubin, J.P. (2009). *Viability theory*. Springer Science & Business Media.

Bansal, S., Akametalu, A.K., Jiang, F.J., Laine, F., and Tomlin, C.J. (2016). Learning quadrotor dynamics using neural network for flight control. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, 4653–4660.

Blanchini, F. and Miani, S. (2008). *Set-theoretic methods in control*. Springer.

Cannon, M., Deshmukh, V., and Kouvaritakis, B. (2003). Nonlinear model predictive control with polytopic invariant sets. *Automatica*, 39(8), 1487–1494.

Chai, J. and Sanfelice, R.G. (2020). Forward invariance of sets for hybrid dynamical systems (part ii). *IEEE Transactions on Automatic Control*.

Dorfler, F. and Bullo, F. (2012). Synchronization and transient stability in power networks and nonuniform kuramoto oscillators. *SIAM Journal on Control and Optimization*, 50(3), 1616–1642.

Hornik, K. (1991). Approximation capabilities of multilayer feedforward networks. *Neural networks*, 4(2), 251–257.

Hou, Z.G., Cheng, L., and Tan, M. (2009). Decentralized robust adaptive control for the multiagent system consensus problem using neural networks. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 39(3), 636–647.

Jagtap, P., Swikir, A., and Zamani, M. (2020). Compositional construction of control barrier functions for interconnected control systems. In *Proceedings of the 23rd International Conference on Hybrid Systems: Computation and Control*, 1–11.

Jaulin, L., Kieffer, M., Didrit, O., and Walter, E. (2001). Interval analysis. In *Applied interval analysis*, 11–43. Springer.

Koopman, P. and Wagner, M. (2017). Autonomous vehicle safety: An interdisciplinary challenge. *IEEE Intelligent Transportation Systems Magazine*, 9(1), 90–96.

Korda, M., Henrion, D., and Jones, C.N. (2014). Convex computation of the maximum controlled invariant set for polynomial control systems. *SIAM Journal on Control and Optimization*, 52(5), 2944–2969.

Lewis, F., Jagannathan, S., and Yesildirak, A. (2020). *Neural network control of robot manipulators and nonlinear systems*. CRC press.

Meyer, P.J., Girard, A., and Witrant, E. (2016). Robust controlled invariance for monotone systems: application to ventilation regulation in buildings. *Automatica*, 70, 14–20.

Moisan, M., Bernard, O., and Gouzé, J.L. (2007). Near optimal interval observers bundle for uncertain bioreactors. In *2007 European Control Conference (ECC)*, 5115–5122. IEEE.

Sallab, A.E., Abdou, M., Perot, E., and Yogamani, S. (2017). Deep reinforcement learning framework for autonomous driving. *Electronic Imaging*, 2017(19), 70–76.

Saoud, A. (2019). *Compositional and Efficient Controller Synthesis for Cyber-Physical Systems*. Ph.D. thesis, Université Paris Saclay.

Tabuada, P. (2009). *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media.

Wang, S., Pei, K., Whitehouse, J., Yang, J., and Jana, S. (2018). Formal security analysis of neural networks using symbolic intervals. In *27th {USENIX} Security Symposium ({USENIX} Security 18)*, 1599–1614.

[6] The function g is simply given by $g(x_1, x_2) = [1; 1]$.

[7] Files for this simulation can be found at the following address: `https://github.com/HybridSystemsLab/InvarianceNN`