A Local Hybrid Observer for a Class of Hybrid Dynamical Systems with Linear Maps and Unknown Jump Times

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Abstract— **We propose a local observer design for hybrid systems with linear flow, jump, and output maps, whose jump times are not known/detected. Assuming the solutions of interest admit a dwell-time, evolve in a compact set, and the pair of flow/output maps is observable allows us to use a sufficiently fast linear observer during flow and trigger the observer jumps when its estimate reaches the jump set. However, since, as we show, using the plant output around the jump times is actually counterproductive, we propose to "disconnect" the correction term of the observer around the jump times and let the estimate flow in open-loop with the plant flow map. Local attractivity of an appropriate zero-error set is then shown for the obtained observer and illustrated in simulations.**

I. Introduction

The problem of designing observers for general hybrid systems presenting both continuous-time behavior and discrete-time behavior is still largely unsolved, mainly due to the fact that the plant jump times, that is, the times at which discrete events occur in the plant solution generally depend on its initial condition, which is unknown in the context of observer design. When the plant jump times are known or can be detected, it is natural to design an observer that is synchronized with the plant, i.e., whose jumps are triggered at the same time as those of the plant. Such an approach has been pursued under assumptions on the time elapsed between successive jumps (reverse/average dwell-time for instance) in a large variety of contexts, including impulsive (possibly switched) systems [1], [13], [20], sampled-data systems [16], [7], [19], and general hybrid systems [6], [17], [5], among others. Because the observer jumps at the same time as the plant, both observer and plant solutions are defined on the same (hybrid) time domain, which facilitates the analysis of the estimation error and the design of an observer.

Unfortunately, exact synchronization between the plant and the observer is usually difficult to achieve in practice, due to noisy/delayed jump detection. Robustness with respect to delays in triggering the observer jumps has been studied in [5], but only practical stability outside the delay intervals may be expected. Besides, in other contexts, it may even be impossible to detect the jumps of the plant via the measurements. Motivated by these shortcomings, we investigate in this paper the possibility of achieving – at least local – asymptotic stability of the estimation error without relying on the detection of the plant jumps.

When the observer jumps are not triggered at the same time as those of the plant, the mismatch of time domains between the plant and the observer solutions makes the formulation of observability and, in turn, observer design very challenging [3]. In the particular context of switched systems, numerous results are available for the design of observers able to estimate the switching signal; see [2], [15], [21] among many others. On the other hand, very few observer results exist for general hybrid systems [10] when the plant jump times are unknown. Exceptions are [14], [11], where the existence of a change of coordinates transforming the jump map into the identity map is studied, thus allowing the use of a continuous-time observer in those new coordinates. Also in [8], an observer with non synchronized jumps is designed for billiard-type systems, but the knowledge of the plant jump times is still needed to trigger the observer jumps.

In this paper, we consider a general class of hybrid systems [10] with linear flow, jump, and output maps, whose solutions of interest admit a dwell-time, evolve in a compact set and whose pair of flow/output maps is observable. Our goal is to design an observer that does not require the knowledge or detection of the plant jump times. While it is tempting to use a sufficiently fast linear observer during flow and trigger the jumps when its estimate reaches the jump set, we remark that using the plant output around the jump times is counterproductive. Indeed, unlike standard output disturbances like noise or delays whose nominal behavior is to be small or absent, arbitrarily small asynchronism of the plant and observer jump times typically leads to large errors due to discontinuity in the solutions at the jumps, even in the ideal context where the output is noise-free. Hence, under some appropriate assumptions on the behavior of solutions around the jump set, we propose to "disconnect" the correction term of the observer around the jump times and let the estimate flow in open-loop with the plant flow map until it naturally reaches the jump set. Local attractivity of an appropriate zero-error set is then shown for the obtained observer and the performance compared to a more standard synchronous observer with delayed jump detection in an example.

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A. Notation and Preliminaries

We denote $\mathbb{R}$ (resp. $\mathbb{N}$) the set of real numbers (resp. integers) and $\mathbb{R}_{\geq 0} := [0, +\infty)$. For $x \in \mathbb{R}^n$ and $\mathcal{A} \subseteq \mathbb{R}^n$, $|x|_{\mathcal{A}}$ denotes the distance from x to $\mathcal{A}$. For a matrix P, $\mathrm{eig}(P)$ denotes the set of its eigenvalues, and $\underline{\lambda}(P)$ (resp. $\overline{\lambda}(P)$) stands for its smallest (resp. largest) singular value. For a set $\mathcal{S} \subset \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n \times n}$, $M\mathcal{S}$ denotes the set $\{Mx : x \in \mathcal{S}\}$. We consider hybrid dynamical systems as in [10], whose solutions are defined on *hybrid time-domains*. A subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time-domain* if $E = \bigcup_{j=0}^{j_m - 1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_{j_m}$, and it is a hybrid time domain if for any $(t_m, j_m) \in E$, $E \cap [0, t_m] \times \{0, \ldots, j_m\}$ is a compact hybrid time domain. For a solution $(t, j) \mapsto x(t, j)$ (see [10, Definition 2.6]), we denote $\mathrm{dom}\, x$ its domain, $\mathrm{dom}_t\, x$ (resp.$\mathrm{dom}_j\, x$) its projection on the ordinary time (resp. jump) component. We say that x is *t-complete* if $\mathrm{dom}_t\, x$ is unbounded and that it has a dwell-time $\tau_m > 0$ if it flows at least τ_m units of time in between consecutive jumps.

II. Problem statement

A. Framework

We consider a hybrid plant of the form [10]

$$\mathcal{H} \left\{ \begin{array}{ll} \dot{x} = A_c\, x & x \in C \\ x^+ = A_d\, x & x \in D \end{array} \right. \quad , \quad y = H\, x \tag{1}$$

with state $x \in \mathbb{R}^{d_x}$ and output $y \in \mathbb{R}^{d_y}$, matrices $A_c, A_d \in \mathbb{R}^{d_x \times d_x}$, $H \in \mathbb{R}^{d_y \times d_x}$, and flow and jump sets C and D. For this class of hybrid systems, we are interested in estimating the state of $\mathcal{H}$ when its solutions are initialized in a subset $\mathcal{X}_0 \subset C \cup D$. We denote $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ the set of maximal solutions of $\mathcal{H}$ with initial condition in $\mathcal{X}_0$.

As in [6], [17], if the plant jump times were known or detected, one could implement an observer for (1) of the form

$$\hat{\mathcal{H}} \left\{ \begin{array}{ll} \dot{\hat{x}} = A_c \hat{x} - L_c(H\hat{x} - y) & \text{when } \mathcal{H} \text{ flows} \\ \hat{x}^+ = A_d \hat{x} - L_d(H\hat{x} - y) & \text{when } \mathcal{H} \text{ jumps} \end{array} \right. \tag{2}$$

that is synchronized with the plant, for some gains $L_c, L_d \in \mathbb{R}^{d_x \times d_y}$ to be chosen such that $\hat{x}$ asymptotically reconstructs the plant state x. The advantage of such a setting is that the dynamics of the extended state $(x, \hat{x})$ are easy to write, which facilitates the analysis of the estimation error.

Unfortunately, as mentioned above, exact synchronization between the plant and the observer is difficult to achieve in practice and we investigate here the possibility of building an – at least local – observer whose jumps are triggered based on its own estimate of the plant state, rather than an exogenous signal.

The following assumption describes the class of hybrid systems considered in this study.

Assumption 2.1: Given $\mathcal{H} = (C, A_c, D, A_d)$ and $\mathcal{X}_0 \subset C \cup D$, there exist $\tau_m > 0$ and a compact subset $\mathcal{X}$ of $C \cup D$ such that any solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$

- is t-complete with dwell-time τ_m; and
- remains in $\mathcal{X}$ at all times.

In addition, the output matrix H is such that the pair (A_c, H) observable.

The uniform dwell-time assumption enables our design to rely on an observer of the flow dynamics that can be made arbitrarily fast. Under well-posedness, the existence of such a dwell-time is guaranteed if $g(D) \cap D = \emptyset$ using [18, Lemma 2.7] and the fact that all the solutions from $\mathcal{X}_0$ evolve in the compact set $\mathcal{X}$.

B. Arbitrarily Fast Linear Observer for the Flow

Since the pair (A_c, H) is observable, it admits a linear observer, whose eigenvalues can be assigned arbitrarily fast. For that, we define a change of coordinates $\mathcal{V} \in \mathbb{R}^{d_x \times d_x}$ transforming (A_c, H) into a block-diagonal observable form, namely such that

$$\mathcal{V} A_c \mathcal{V}^{-1} = \boldsymbol{A} + \boldsymbol{D}\boldsymbol{H} \quad , \quad H\mathcal{V}^{-1} = \boldsymbol{H}$$

with

$$\boldsymbol{A} := \mathrm{blkdiag}(A_1, \ldots, A_{d_y}) \ , \ \boldsymbol{D} := \mathrm{blkdiag}(D_1, \ldots, D_{d_y})$$
$$\boldsymbol{H} := \mathrm{blkdiag}(H_1, \ldots, H_{d_y})$$

,

$$A_i = \begin{pmatrix} 0 & 0 & & \ldots & 0 \\ 1 & 0 & & & \\ \vdots & \ddots & \ddots & & \\ 0 & & 1 & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{d_i \times d_i}$$

$$H_i = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{1 \times d_i} \ ,$$

$D_i \in \mathbb{R}^{d_i \times 1}$, and d_i integers such that $\sum_{i=1}^{d_y} d_i = d_y$. Consider vectors K_i such that $A_i - K_i H_i$ is Hurwitz, and for a positive scalar ℓ, define $\mathcal{L}_i(\ell) := \mathrm{diag}(\ell^{d_i - 1}, \ldots, \ell, 1)$. Then, a linear high-gain observer during flow can be designed as

$$\dot{\hat{x}} = F_\ell(\hat{x}, y)$$

with

$$F_\ell(\hat{x}, y) := A_c\hat{x} - \mathcal{V}^{-1}(\boldsymbol{D} + \ell\boldsymbol{\mathcal{L}}(\ell)\boldsymbol{K})(H\hat{x} - y) \tag{3}$$

where

$$\boldsymbol{K} := \mathrm{blkdiag}(K_1, \ldots, K_{d_y}) \ , \ \boldsymbol{\mathcal{L}} := \mathrm{blkdiag}(\mathcal{L}_1, \ldots, \mathcal{L}_{d_y}).$$

We thus have $\mathrm{eig}(A_c - L_c H_c) = \ell\, \mathrm{eig}(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{H})$, so that ℓ is a high-gain parameter enabling to accelerate the convergence of the observer.

It is shown in [5] that when the jump times of the plant are known or immediately detected, a possible observer consists of $\hat{\mathcal{H}}$ defined in (2), with L_c defined as in (3) during flow and $L_d = 0$. Indeed, for ℓ sufficiently large compared to A_d and the dwell-time τ_m, one can show that the (exponential) decrease of the estimation error during flow wins over its (polynomial) increase at jumps and the estimation error thus asymptotically converges to zero.

Still relying on the dwell-time and the available high-gain observer of the flow dynamics, the construction of a local

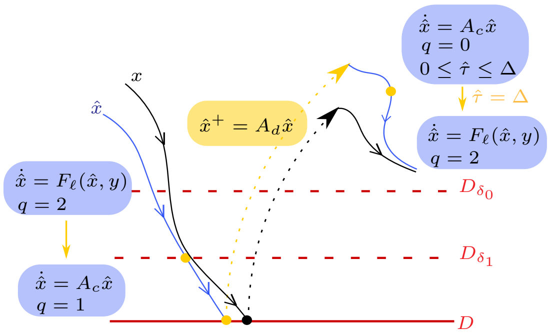

Fig. 1. Sketch of hybrid mechanism in observer (6), with plant trajectory in black, observer trajectory in blue/yellow, with blue (resp. yellow) representing observer flow (resp. jumps), in the case where $\hat{x}$ remains in $\mathrm{cl}(C \cup D)$ and $\Pi(\hat{x}) = \hat{x}$.

hybrid observer which does not require the detection of the plant jumps is presented in Section III. A sketch of the proof of asymptotic convergence is then provided in Section IV and the performance of the observer illustrated in Section V.

III. Local Hybrid Observer

A. Open-Loop Estimation around Jump Times

A first idea would be to use the observer (3) during flow and simply trigger the jumps of the observer when $\hat{x} \in D$, with the jump map $\hat{x}^+ = A_d\,\hat{x}$. Indeed, if the estimation error sufficiently decreases during flow, one can expect that the observer jumps will occur close in time to those of the plant and somehow the observer will synchronize and converge. However, around the observer jump times, because the plant typically jumps slightly sooner or later than the observer, the input y feeding the observer flow map might actually constitute a disturbance and hinder the observer convergence. More precisely, assume that $\hat{x}$ and x are both close to D and x jumps first. Then, the input y after the jump could steer $\hat{x}$ away from D to catch up with x through flow and $\hat{x}$ could miss its jump. The same reasoning holds in the reverse case where $\hat{x}$ jumps slightly ahead of x, and where the use of y would force $\hat{x}$ to track the value of x before the jump instead of simply waiting for x to catch up.

This issue is dealt with in [8] by making $\hat{x}$ follow a mirrored image of x with respect to D during the jump time mismatches. But this is done in a very particular setting, where $g \circ g$ is the identity, and more importantly, it requires the knowledge of the plant jump times in order to decide whether $\hat{x}$ should follow x or its mirrored image.

In this paper, since the plant jump times are unknown, we propose to "disconnect" the correction term of the observer around the jump set D. More precisely, we propose a hybrid mechanism that lets $\hat{x}$ flow in "open-loop" according to A_c until it naturally reaches D, and only reconnects the correction term a short while Δ later, in a way that ensures the plant has also jumped in the meantime. This process is illustrated in Figure 1. For this to work, we assume that i) the plant eventually reaches D when entering a certain neighborhood of D and flowing with A_c (see (P1) below), ii) the plant necessarily jumps from D (see (P2)), and iii) the plant takes at least $0 < \tau_m^0 < \tau_m$ units of time to reach that neighborhood again (see (P3)). Similar conditions are used in [9] in the context of trajectory tracking. More precisely, consider a projection map $\Pi : \mathbb{R}^{d_x} \to \mathrm{cl}(C \cup D)$ for which there exists $a_p \geq 1$ such that

$$|x - \Pi(\hat{x})| \leq a_p|x - \hat{x}| \qquad \forall (x, \hat{x}) \in \mathcal{X} \times \mathbb{R}^{d_x} \ . \tag{4}$$

In particular, (4) implies that $\Pi = \mathrm{Id}$ on $\mathcal{X}$. Denoting, for $\delta > 0$,

$$D_\delta = \{x \in \mathrm{cl}(C \cup D) \ : \ |x|_D \leq \delta\} \ ,$$

we make the following assumption.

Assumption 3.1: Given $\mathcal{H} = (C, A_c, D, A_d)$, $\mathcal{X}$ defined in Assumption 2.1, there exists $\delta_0 > 0$ such that:

(P1) for any $x \in D_{\delta_0}$, there exists $\tau_D \geq 0$ such that
- $\exp(A_c\tau_D)x \in D$
- $\exp(A_c t)x \in D_{\delta_0} \setminus D$ for all $t \in [0, \tau_D)$.

In addition, the map $\mathfrak{T} : D_{\delta_0} \to \mathbb{R}_{\geq 0}$, which associates τ_D to each $x \in D_{\delta_0}$, is locally Lipschitz.

(P2) No flow in D_{δ_0} is possible for $\dot{x} = A_c x$ starting from D.

(P3) $A_d D \cap D_{\delta_0} = \emptyset$ and there exists $\tau_m^0 > 0$ such that solutions of $\dot{x} = A_c x$ starting from $A_d D$ are defined over the interval $[0, \tau_m^0]$ with $\Pi(x) \notin D_{\delta_0}$.

Sufficient conditions on the data A_c, C and D ensuring the Lipschitzness of the *time-to-impact* function $\mathfrak{T}$ in (P1) are given in a more general context in [12] and references therein. Actually, when there exists a continuously differentiable function $\varpi : \mathbb{R}^{d_x} \to \mathbb{R}$ such that the map $\mathfrak{T}$ is characterized at each $x \in D_{\delta_0}$ by $\varpi\big(\exp(A_c\mathfrak{T}(x))x\big) = 0$, the continuous differentiability of $\mathfrak{T}$ is guaranteed by the implicit function theorem under the *transversality condition*

$$\frac{\partial \varpi}{\partial x}(x)A_c x \neq 0 \qquad \forall x \in D \ . \tag{5}$$

Note that (5) also ensures that no flow is possible in D, namely (P2) holds.

B. Hybrid Observer Construction

Suppose Assumption 3.1 holds with δ_0 and τ_m^0. Pick $0 < \delta_1 < \delta_0$ and $0 < \Delta < \frac{\tau_m^0}{2}$. In order to implement the observation strategy explained in the previous section, we define our hybrid observer with three states $(\hat{x}, \tau, q)$, where τ is a timer and $q \in \{0, 1, 2\}$ describes the "operating mode" of the observer. When $q = 2$, $\hat{x}$ flows with the linear observer given by (3). When $\Pi(\hat{x})$ reaches D_{δ_1}, the observer jumps to mode $q = 1$ with $\hat{x}$ reset to $\Pi(\hat{x})$. During this mode, $\hat{x}$ flows with A_c until it reaches D, which we know will happen in finite time thanks to (P1). At this point, $\hat{x}$ is reset to $A_d\hat{x}$, the mode changes to $q = 0$, and the timer τ is launched. During this phase, we let $\hat{x}$ flow again with A_c during Δ units of time. When the timer expires, the observer jumps back to mode $q = 2$.

The proposed observation strategy is captured by the following hybrid observer, denoted $\hat{\mathcal{H}}$:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\tau} \\ \dot{q} \end{pmatrix} = \begin{cases} \begin{pmatrix} F_\ell(\hat{x}, y) \\ 0 \\ 0 \end{pmatrix} & \text{if } (\hat{x}, \tau, q) \in C_2 \\ \begin{pmatrix} A_c\, \hat{x} \\ 0 \\ 0 \end{pmatrix} & \text{if } (\hat{x}, \tau, q) \in C_1 \\ \begin{pmatrix} A_c\, \hat{x} \\ 1 \\ 0 \end{pmatrix} & \text{if } (\hat{x}, \tau, q) \in C_0 \end{cases} \tag{6a}$$

$$\begin{pmatrix} \hat{x}^+ \\ \tau^+ \\ q^+ \end{pmatrix} = \begin{cases} \begin{pmatrix} \Pi(\hat{x}) \\ 0 \\ 1 \end{pmatrix} & \text{if } (\hat{x}, \tau, q) \in D_2 \\ \begin{pmatrix} A_d\, \hat{x} \\ 0 \\ 0 \end{pmatrix} & \text{if } (\hat{x}, \tau, q) \in D_1 \\ \begin{pmatrix} \hat{x} \\ 0 \\ 2 \end{pmatrix} & \text{if } (\hat{x}, \tau, q) \in D_0 \end{cases} \tag{6b}$$

with F_ℓ defined in (3), the (disjoint) flow sets defined by

$$\begin{aligned} C_2 &= \left\{\hat{x} \in \mathbb{R}^{d_x} \;:\; \Pi(\hat{x}) \in \mathrm{cl}(\mathbb{R}^{d_x} \setminus D_{\delta_1})\right\} \times \{0\} \times \{2\} \\ C_1 &= D_{\delta_0} \times \{0\} \times \{1\} \\ C_0 &= \mathbb{R}^{d_x} \times [0, \Delta] \times \{0\} \end{aligned}$$

and (disjoint) jump sets by

$$\begin{aligned} D_2 &= \left\{\hat{x} \in \mathbb{R}^{d_x} \;:\; \Pi(\hat{x}) \in D_{\delta_1}\right\} \times \{0\} \times \{2\} \\ D_1 &= D \times \{0\} \times \{1\} \\ D_0 &= \left\{\hat{x} \in \mathbb{R}^{d_x} \;:\; \Pi(\hat{x}) \in \mathrm{cl}(\mathbb{R}^{d_x} \setminus D_{\delta_0})\right\} \times \{\Delta\} \times \{0\} \end{aligned}$$

Of course, the plant $\mathcal{H}$ evolves in parallel with the observer, with jumps that are not necessarily synchronized with those of the observer. However, as long as the estimation error $\hat{x} - x$ is sufficiently small, the following hold:

a) When the observer flows in mode $q = 2$, $|\,\Pi(\hat{x})|_D \geq \delta_1$ so $x \notin D$ and the plant is also flowing, with y evolving continuously;
b) When the observer enters mode $q = 1$, $|\,\Pi(\hat{x})|_D = \delta_1$, so $x \in D_{\delta_0}$ and from (P1)-(P2), x jumps in a near future, some time during the phase where $q \in \{1, 0\}$;
c) Once x has jumped, the observer has time to finish the phase $q \in \{1, 0\}$ and flow again in mode $q = 2$ with the high-gain observer, before x reenters D_{δ_0} according to (P3) and the fact that $\Delta < \tau_m^0/2$.

The latter item ensures the estimation error has time to decrease with output-injection before another open-loop sequence starts.

C. Main result

Let us define $\mathcal{A} = \mathcal{A}_{\mathrm{eq}} \cup \mathcal{A}_1 \cup \mathcal{A}_0$, where

$$\begin{aligned} \mathcal{A}_{\mathrm{eq}} &= \{(x, \hat{x}, q) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_x} \times \{0, 1, 2\} \;:\; \hat{x} = x\} \\ \mathcal{A}_1 &= \{(x, \hat{x}, q) \in A_d D \times D \times \{1\} \;:\; x = A_d \hat{x}\} \\ \mathcal{A}_0 &= \{(x, \hat{x}, q) \in D \times A_d D \times \{0\} \;:\; \hat{x} = A_d x\} \end{aligned}$$

The set $\mathcal{A}_{\mathrm{eq}}$ corresponds to a zero estimation error, while the sets $\mathcal{A}_1$ and $\mathcal{A}_0$ correspond to $\hat{x}$ being one jump right ahead or behind of x. Including $\mathcal{A}_1$ and $\mathcal{A}_0$ cannot be avoided in an asymptotic analysis of a hybrid observer, since such errors are inevitable arbitrarily close to the jump times (*peaking phenomenon*), unless exact synchronization of the plant and observer jump times is achieved.

The following theorem shows that for ℓ sufficiently large, $\mathcal{A}$ is locally attractive for the interconnection of $\mathcal{H}$ and $\hat{\mathcal{H}}$. Because the plant and observer solutions are not defined on the same (hybrid) time domain, we use the notion of j-reparametrization introduced in [4]. More precisely, given a hybrid arc x, x^{r} is a full j-reparametrization of x if there exists a map $\rho : \mathbb{N} \to \mathbb{N}$ verifying $\rho(0) = 0$, $\rho(j+1) - \rho(j) \in \{0, 1\}$, and such that $x^{\mathrm{r}}(t, j) = x(t, \rho(j))$ for all $(t, j) \in \operatorname{dom} x^{\mathrm{r}}$ with $\operatorname{dom} x = \bigcup_{(t,j) \in \operatorname{dom} x^{\mathrm{r}}} (t, \rho(j))$.

Theorem 3.2: Suppose Assumptions 2.1 and 3.1 hold. Then, there exists $\ell^* > 0$ such that for all $\ell \geq \ell^*$, there exist $\epsilon_\ell > 0$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$, any maximal solution $\phi := (\hat{x}, \tau, q)$ to $\hat{\mathcal{H}}$ defined by (6) *with input $y = Hx$ and initialized in $C_2 \cup (D_{\delta_1} \times \{0\} \times \{1\})$ such that*

$$|(x, \hat{x}, q)(0, 0)|_{\mathcal{A}} < \epsilon_\ell \tag{7}$$

is t-complete and there exist full j-reparametrizations x^{r} and ϕ^{r} of x and ϕ, respectively, such that $\operatorname{dom} x^{\mathrm{r}} = \operatorname{dom} \phi^{\mathrm{r}}$ and

$$\lim_{t+j \to \infty} |(x^{\mathrm{r}}, \hat{x}^{\mathrm{r}}, q^{\mathrm{r}})(t, j)|_{\mathcal{A}} = 0\ . \tag{8}$$

In other words, by definition of $\mathcal{A}$, $\hat{x}$ asymptotically converges to x (modeled by $\mathcal{A}_{\mathrm{eq}}$), except around the jump times where $\hat{x}$ may be a jump ahead/behind x (modeled by $\mathcal{A}_1$ and $\mathcal{A}_0$). However, thanks to $\mathfrak{T}$ being Lipschitz, the length of those time mismatches asymptotically goes to zero.

The analysis of the estimation error heavily relies on items a)-b)-c) described above and thus necessitates a sufficiently small initial error, guaranteeing that $\hat{x}$ is only one jump ahead/behind x. One may proceed with initialization as follows. If we believe that at the initial time, x is not about to jump or has not just jumped (i.e., $x(0,0)$ is not close to either D or $A_d D$), one may initialize $(\hat{x}, \tau, q)$ to $q(0,0) = 2$, $\tau(0,0) = 0$ and $\hat{x}(0,0) \notin D_{\delta_1}$ such that the estimation error $\hat{x}(0,0) - x(0,0)$ is sufficiently small to satisfy (7). On the other hand, if $x(0,0) \in D_{\delta_0}$ or is close to $A_d D$, one should initialize $(\hat{x}, \tau, q)$ to $q(0,0) = 1$, $\tau(0,0) = 0$ and $\hat{x}(0,0) \in D_{\delta_1}$ such that either $\hat{x}(0,0) - x(0,0)$ or $A_d \hat{x}(0,0) - x(0,0)$ is sufficiently small according to (7).

IV. Sketch of proof of Theorem 3.2

Consider a positive definite matrix $P \in \mathbb{R}^{d_x \times d_x}$ such that

$$(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{H})^\top P + P(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{H}) \leq -\lambda P$$

for some $\lambda > 0$. Then, the Lyapunov function

$$V_\ell(x, \hat{x}) = (x - \hat{x})^\top \mathcal{V}^\top \boldsymbol{\mathcal{L}}(\ell)^{-1} P \boldsymbol{\mathcal{L}}(\ell)^{-1} \mathcal{V}\,(x - \hat{x})$$

verifies for all $(x, \hat{x}) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$

$$\underline{c}(\ell)|\hat{x} - x|^2 \leq V_\ell(x, z) \leq \overline{c}(\ell)|\hat{x} - x|^2 \tag{9a}$$

$$\langle \nabla V_\ell(x,\hat{x}), \mathcal{F}_\ell(x,\hat{x})\rangle \leq -\ell\lambda\, V_\ell(x,\hat{x}) \tag{9b}$$

with $\underline{c}(\ell) = \frac{\underline{\lambda}(\mathcal{V}^\top P \mathcal{V})}{\ell^{2(d-1)}}$, $\overline{c}(\ell) = \overline{\lambda}(\mathcal{V}^\top P\mathcal{V})$, $d = \max d_i$, $\mathcal{F}_\ell(x,\hat{x}) = (f(x), F_\ell(\hat{x}, Hx))$ and F_ℓ defined in (3).

Consider $\varepsilon < \min\{\delta_1, \delta_0 - \delta_1\}$. For all $(x,\hat{x}) \in \mathbb{R}^{d_x}\times\mathbb{R}^{d_x}$,

(a) If $x\in D$ and $|x-\hat{x}|\leq\varepsilon$, then $|\hat{x}|_D < \delta_1$.
(b) If $|\hat{x}|_D \leq \delta_1$ and $|x-\hat{x}|\leq \varepsilon$, then $|x|_D\leq\delta_0$.

Let $v_\ell := \underline{c}(\ell)\left(\frac{\varepsilon}{a_p}\right)^2$. Then, from (9a), (4), items (a)-(b) hold when $V_\ell(x,\hat{x})\leq v_\ell$ also for $\hat{x}$ replaced by $\Pi(\hat{x})$.

A. t-Completeness of Observer Solutions

Assume $q(0,0) = 2$. When $(\hat{x},\tau,q)\in C_2$, since $\Pi(\hat{x})\in \mathrm{cl}(C\cup D)$, we have $|\Pi(\hat{x})|_D \geq \delta_1$, so as long as $V(x,\hat{x})\leq v_\ell$, by item (a), $x\notin D$, and both the plant and the observer flow with y continuous. If at some point $|\Pi(\hat{x})|_D = \delta_1$, a jump is possible in the observer from D_2, in which case $\hat{x}^+ = \Pi(\hat{x})\in D_{\delta_1}\setminus D$ and $q^+ = 1$, so that $(\hat{x}^+,\tau^+,q^+)\in C_1\setminus D_1$ and the observer can only flow. Then, while the observer flows in C_1, it is in open-loop. Since $\hat{x}$ starts from inside D_{δ_0} and flows with A_c, we know by (P1) that $\hat{x}$ remains in D_{δ_0} and reaches D in finite-time. Besides, no jump can happen in the observer before $\hat{x}$ has reached D by definition of D_1. When $\hat{x}$ reaches D, using (P2) and $(\hat{x},\tau,q)\in D_1$, the observer jumps with $\hat{x}^+ = A_d\hat{x}$ and $q^+ = 0$. From there, $(\hat{x},\tau,q)\in C_0\setminus D_0$, with $\tau = 0$ and $\hat{x}\in A_dD$, and the observer should flow as long as $\tau\leq\Delta$, i.e., during Δ units of time. Since $\Delta < \tau_m^0$, $\hat{x}$ can indeed flow with A_c during that time and with $\Pi(\hat{x})\notin D_{\delta_0}$ according to (P3). Thus, when τ reaches Δ, we have $\Pi(\hat{x})\notin D_{\delta_0}$, i.e., $(\hat{x},\tau,q)\in D_0$ and since no flow is possible in C_0 when $\tau = \Delta$, a jump occurs with $\tau^+ = 0$ and $q^+ = 2$. Since $\hat{x}^+ = \hat{x}$ and $D_{\delta_1}\subset D_{\delta_0}$, $(\hat{x},\tau^+,q^+)\in C_2$ and we are back to where the argument started.

On the other hand, if $q(0,0) = 1$, by assumption $\hat{x}(0,0)\in D_{\delta_1}$ so the same reasoning holds, starting from the third item. Therefore, as long as $V_\ell(x,z)\leq v_\ell$ during the phases with $q = 2$, solutions are t-complete, alternating between modes $2\to 1\to 0\to 2$.

B. Evolution of Estimation Error through each Cycle

When $q = 2$, from (9b), V_ℓ decreases exponentially at rate $\ell\lambda$. Let $0 < v_1 < v_\ell$. Starting from $V_\ell(x,\hat{x})\leq v_1$ with $q = 2$, we follow the estimation error through a succession of modes $q = 1$ and $q = 0$, until q switches back to 2. By exploiting the Lipschitzness of Π and $\mathfrak{T}$, one shows that i) the time mismatch Δ_τ between the plant and observer jumps is bounded by $a_\tau a_p\sqrt{\frac{v_1}{\underline{c}(\ell)}}$ and ii) if $\Delta_\tau < \Delta$, V_ℓ grows by less than $a\frac{\overline{c}(\ell)}{\underline{c}(\ell)}$, with $a > 0$ independent from ℓ. Then, throughout the following phase with $q = 2$, V_ℓ exponentially decreases again. Still using Assumption 3.1, one shows that this phase lasts $\tau_m' \geq \tau_m^0 - 2\Delta > 0$. Therefore, after a full cycle, V_ℓ decreases by at least $\mu_\ell := a e^{-\ell\lambda\tau_m'}\frac{\overline{c}(\ell)}{\underline{c}(\ell)}$.

C. Iterating Cycles

Exploiting exponential growth over polynomial growth, let us pick ℓ sufficiently large such that $\mu_\ell < 1$ and v_1 sufficiently small to have $v_1 < \frac{\underline{c}(\ell)}{a_p^2}\min\left\{\frac{\Delta^2}{a_\tau^2}, \frac{1}{a}\frac{\underline{c}(\ell)}{\overline{c}(\ell)}\varepsilon^2\right\}$. Then, choosing the initial error sufficiently small ensures that $V_\ell < v_1$ before each transition $q = 2\to 1$, and $V_\ell < v_\ell$ at all times. Hence, ϕ is t-complete and, denoting v_k the value of V_ℓ before each transition $q = 2\to 1$, we have $v_k\leq \mu_\ell^{k-1}v_1$. The rest of the proof is purely technical and shows (8) via appropriate j-reparametrizations.

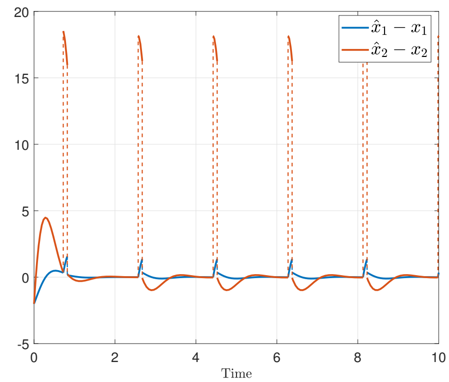

Fig. 2. Estimation error for plant (10) with observer (2) where $F = F_\ell$ with $\ell = 5$ and $K = (1,1)^\top$, $x(0,0) = (4,-2)$, $\hat{x}(0,0) = (6,0)$, and delay in the jump detection of 0.1 units of time.

V. Example

Consider a bouncing ball modeled by $\mathcal{H}$ with state $(x_1,x_2)\in\mathbb{R}^2$ and

$$\begin{aligned}
&A_c = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\, B_c = \left(\begin{smallmatrix}0\\-\mathfrak{g}\end{smallmatrix}\right),\, A_d = \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right),\, H = (1,0)\\
&C = \{(x_1,x_2)\in\mathbb{R}^2 \,:\, x_1\geq 0\}\\
&D = \{(x_1,x_2)\in\mathbb{R}^2 \,:\, x_1 = 0\,,\, x_2\leq 0\}
\end{aligned} \tag{10}$$

where x_1 is the position of the ball, x_2 its velocity and the flow map is given by $\dot{x} = A_cx + B_c$ instead of[1] $\dot{x} = A_cx$ in (1). Solutions initialized in a compact subset $\mathcal{X}_0\subset\mathbb{R}^2\setminus\{0,0\}$ are bounded and have a (uniform) dwell-time. Besides, the pair (A_c,H) is observable so that Assumption 2.1 holds. Since the jump times can be detected from the output $y = x_1$ going through 0, it is proposed in [6] to use an observer of the type (2), with L_c given by (3), $L_d = 0$, and jumps triggered at the same time as those of the plant. However, slight delays in the jump detection prevent the estimate convergence, as illustrated in Figure 2.

Instead, we would like to implement observer (6), which automatically synchronizes its jumps with those of the plant. Unfortunately, (P3) of Assumption 3.1 does not hold directly with D defined in (10) because $(0,0)\in D$ allows a discrete solution. But from the definition of $\mathcal{X}_0$, we know there exists $m > 0$ such that the plant solution remains outside of the open ball $\mathbb{B}_m$. Therefore, the solutions of interest are solution to $\mathcal{H}$ with D replaced by

$$D_m := D\setminus\mathbb{B}_m = \{(x_1,x_2)\in\mathbb{R}^2 \,:\, x_1 = 0\,,\, x_2\leq -m\}$$

[1] A constant term added to the flow/jump maps of $\mathcal{H}$ does not change the analysis as long as it is also added in the observer dynamics (2) and (6).

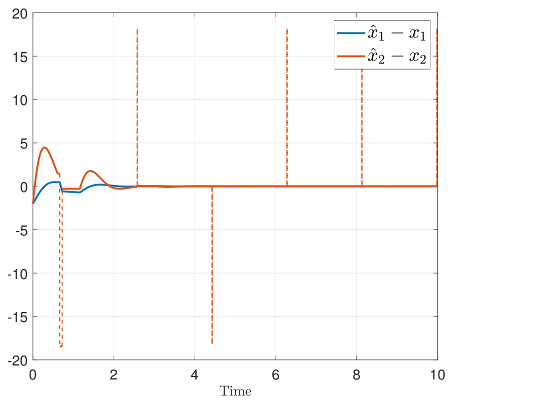

Fig. 3. Estimation error for plant (10) with observer (6) where $F = F_\ell$ with $\ell = 5$ and $K = (1, 1)^\top$, $\delta_0 = 1$, $\delta_1 = 0.5$, $\Delta = 0.5$, and initial conditions $x(0, 0) = (4, -2)$, $\hat{x}(0, 0) = (6, 0)$, and $q(0, 0) = 2$.

and (P3) now holds with D_m for $\delta_0 < m$ and τ_m^0 smaller than the minimal time needed for a solution to flow from $A_d D_m$ to D_{m,δ_0}. Since the solutions remain in $C \cup D_m$, this minimal time is achieved for solutions flowing from $(0, m)$ to either (δ_0, x_2) with $x_2 < \delta_0 - m$, or $(x_1, \delta_0 - m)$ with $0 < x_1 < \delta_0$. Besides, the lipschitzness of $\mathfrak{T}$ in (P1) is proved by observing that for $x \in D_{m,\delta_0}$, $\mathfrak{T}$ is characterized by $\varpi(\Psi(x, \mathfrak{T}(x))) = 0$, where $\varpi(x) = x_1$ on $\mathbb{R}^2$ and $\Psi(x, \tau)$ denotes the solution of $\dot{x} = A_c x + B_c$ at time τ initialized at x. Since $\frac{\mathrm{d}\varpi}{\mathrm{d}x}(x)(A_c x + B_c) < 0$ for $x \in D_m$, the map $\mathfrak{T}$ is indeed continuously differentiable by the Implicit Function Theorem. Therefore, (P1) holds. Finally, (P2) clearly holds since no flow is possible from D_m into $C \cup D_m$, and thus Assumption 3.1 holds. On the other hand, $\mathrm{cl}(C \cup D_m) = C$ being closed and convex, the map Π can be chosen as the orthogonal projection on C, which verifies (4) with $a_p = 1$.

Figure 3 shows the results of a simulation of observer (6) with the same initial conditions and same gain L_c as above. The estimation error asymptotically converges to 0, except at the jump times where $\hat{x}$ is either one jump ahead or behind x. We actually recover similar performance as in [11], where an (invertible) *gluing* function is computed to transform the hybrid dynamics into a continuous-time system where a continuous-time observer can be designed. The design of [11] has the advantage of being global, but there is no general method to build such a gluing function. On the other hand, the design of this paper is local but systematic. Indeed, unlike in [11], any other observable pair (A_c, H) and any jump matrix A_d could have been considered, as long as Assumptions 2.1 and 3.1 hold.

VI. Conclusion

We have proposed a local observer for linear hybrid dynamical systems whose jump times are unknown and whose pair of flow/output maps is observable. The observer relies on a sufficiently fast linear observer of the flow and jumps triggered based on the plant state estimate, in a way that "disconnects" the correction term around the jump times. Compared to designs in [6], [17], [5] where the observer jumps are synchronized with those of the plant, this novel observer avoids the problems of delayed/noisy detection of the plant jump times. Unfortunately, the convergence is only local, but further work includes combining both methodologies to ensure both globality and asymptotic convergence.

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