A Duality Approach to Set Invariance and Safety for Nonlinear Systems

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Abstract— **This paper proposes a duality approach to guarantee set invariance for nonlinear dynamical systems. Building from the so-called mirror descent algorithm from the optimization literature, we develop a new version of the given nonlinear system such that the desired set is forward (pre-)invariant. Such new version of the model is constructed using duality between the given system — called the *primal system* — and a new system — called the *dual system*. By appropriately mapping the dual system back to the original space, the resulting system — called the *modified primal system* — has the desired set forward pre-invariant. The power of the approach is illustrated in several applications pertaining to constrained optimization and feedback control under constraints. Academic examples are provided to illustrate the approach and utility of the results.**

I. Introduction

Forward invariance of a set for a dynamical system is the property that solutions from the set remain in the set for all time. Approaches to ensure such property that do not require to explicitly compute the solutions to the system are essential for systems that are nonlinear, in particular, when safety is the property to be certified. One of the first results guaranteeing forward invariance of a set without computing the solutions to the system is credited to Nagumo; see [1]. In this article, sufficient conditions for forward invariance of a set are given in terms of the contingent cone and the right-hand side of the system. Due to the imposed regularity properties, these conditions only need to be checked on the boundary of the set to render forward invariant. The Nagumo Theorem has been extended to many settings, in particular, to differential inclusions in [2], to impulse differential inclusions in [3], and to hybrid inclusions in [4].

An alternative approach to those involving contingent (or tangent) cone conditions just mentioned consists of employing energy-like functions – much like Lyapunov functions – and check that their variation along the solution is nonincreasing around the set to render forward invariant. Such functions, typically called barrier functions, give rise to infinitesimal conditions that depend on some form of gradient of the function and the right-hand side of the system. Sufficient conditions for forward invariance of a set, as well as safety, using barrier functions for continuous-time systems appeared in [5] and for hybrid systems in [6], to just list a few related references. While barrier functions are powerful as they play the same role for forward invariance and safety as Lyapunov functions play for asymptotic stability, they are also typically hard to find to certify forward invariance of a set.

In this paper, motivated by difficulties in finding barrier functions, we develop a method to certify forward invariance of a set that, rather than requiring checking conditions involving contingent cone or barrier functions, exploits ideas from constrained optimization. Specifically, our approach builds from the so-called *mirror descent algorithm* for optimization formulated in [7] (see also [8]) to solve the constrained optimization problem

$$\begin{aligned} \min \quad & g(x) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned}$$

via duality. To solve this problem using the mirror descent algorithm, one constructs a dual map and applies the gradient descent method in dual space, rather than in primal space, and then one maps the result back to the primal space. Our idea is to exploit this approach to certify forward (pre-)invariance of a set as follows. Given a dynamical system

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n, \tag{1}$$

which we refer to as the *primal system* on the *primal space* $\mathbb{R}^n$, and a set $\mathcal{X} \subset \mathbb{R}^n$ to render forward invariant, we construct a strongly convex function ψ with domain $\mathcal{X}$ given by

$$\psi(x) = \phi(x) + \delta_{\mathcal{X}}(x) \quad \forall x \in \mathbb{R}^n. \tag{2}$$

Let ψ^* be the convex conjugate of ψ. Then the gradient of ψ^*, namely, $\nabla\psi^*$, which we call the *dual map*, is a map that provides a bridge from the dual space to the primal space. The dual map enables us to construct a *dual system*, which, conveniently, is unconstrained, as well as a new version of the primal system called the *modified primal system*, which under appropriate conditions, has the set $\mathcal{X}$ forward pre-invariant. More precisely, we show that when ϕ in (2) is lower semicontinuous, strongly convex, $\mathcal{X}$ is closed and convex, and the right-hand side in (1) is locally Lipschitz, we have that solutions to the dual system map to solutions to the modified primal system that remain in $\mathcal{X}$. Furthermore, when the dual map is differentiable, we can further show that the set $\mathcal{X}$ is forward (pre-)invariant for the modified primal system. We illustrate the power of the proposed method in several applications and academic examples.

The rest of the paper is organized as follows. In Section II, we recap concepts from optimization and dynamical systems. In Section III, the main results about duality for set invariance are introduced. Applications and examples are presented in Section IV. Due to space constraints, proofs will be published elsewhere.

Notation. Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}_{\geq 0} = [0, \infty)$. For $x, y \in \mathbb{R}^n$, $\langle x, y \rangle$ denotes the inner product between x and y. For $A, B \subset \mathbb{R}^n$, $A \setminus B$ denotes the subset of elements of A that are not in B. The set $\overline{\mathbb{R}}$ indicates the extended real line, i.e., $\overline{\mathbb{R}} = [-\infty, \infty]$. For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\operatorname{dom} f$ represents the domain of f, $\operatorname{gph} f$ indicates graph of f, and $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ denotes the gradient of f. The set S^n_{++} denotes the set of all $n \times n$, symmetric and positive definite matrices. The 2-norm is denoted by $|x|$, and $|x|_K := \inf_{y\in K} |x - y|$ is the distance from x to the nonempty set K. We denote the closed unit ball centered at the origin as $\mathbb{B}$. The interior, the boundary, and the closure of the set S are denoted as $operatorname int(S)$, ∂S, and $\operatorname{cl}(S)$, respectively. I denotes the identity matrix with appropriate dimension.

II. Preliminaries

Here, we present basic definitions and concepts, which are needed in our approach.

A. Basic Definitions and Results

The following definitions are based on [9], [10] and [8].

Definition 1. *(Strong convexity) A proper function $f : E \to \overline{\mathbb{R}}$ is* strongly convex *if there exists a constant $\sigma > 0$ such that*

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)-\frac{1}{2}\sigma\lambda(1-\lambda)|x-y|^2$$

for all $x, y \in \operatorname{dom} f$ and all $\lambda \in (0, 1)$.

Definition 2. *(Convex conjugate) For any convex function $f : E \to \overline{\mathbb{R}}$, the* convex conjugate function *$f^* : E^* \to \overline{\mathbb{R}}$ is defined by*

$$f^*(z) = \sup_{x\in \operatorname{dom} f} \{\langle z, x\rangle - f(x)\} \quad \forall z \in \operatorname{dom} f^* \tag{3}$$

Note that $\operatorname{dom} f^* \subset E^*$, where E^* is dual space of E. The domain of f^* consists of points $z \in E^*$ such that f^* is finite, i.e., $\operatorname{dom} f^* = \{z \in E^* : f^*(z) < \infty\}$.

Based on [9], a set-valued map $F : X \rightrightarrows Z$ maps points in X to subsets of Z, is represented by the double arrow $\rightrightarrows$. The graph of F is defined as

$$\operatorname{gph} F = \{(x, z) \in X \times Z : z \in F(x)\}.$$

Definition 3. *(Lower semicontinuous) The function $f : E \to \overline{\mathbb{R}}$ is* lower semicontinuous (lsc) *at $\bar{x} \in E$ if*

$$\liminf_{x\to\bar{x}} f(x) = f(\bar{x}).$$

The function f is lower semicontinuous on *E if it is lower semicontinuous for every $x \in E$.*

Proposition 1. *[9, Proposition 11.3] For any proper, lsc, convex function f, ∂f, subgradient of f, and ∂f^* are inverses of each other, i.e.,*

$$\bar{z} \in \partial f(\bar{x}) \Leftrightarrow \bar{x} \in \partial f^*(\bar{z}) \Leftrightarrow f(\bar{x}) + f^*(\bar{z}) = \langle \bar{x}, \bar{z}\rangle$$

whereas $f(x)+f^*(z) \geq \langle x, z\rangle$ for all x, z. Hence, the graph of ∂f is closed and

$$\partial f(\bar{x}) = \operatorname*{arg\,max}_{z\in \operatorname{dom} f^*} \{\langle \bar{x}, z\rangle - f^*(z)\}, \tag{4}$$

$$\partial f^*(\bar{z}) = \operatorname*{arg\,max}_{x\in \operatorname{dom} f} \{\langle \bar{z}, x\rangle - f(x)\}. \tag{5}$$

□

The following theorem states properties of strong convex functions and its dual ones.

Theorem 1. *[10, Theorem 4.2.1] Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with constant $\sigma > 0$ on $\mathbb{R}^n$. Then, $\operatorname{dom} f^* = \mathbb{R}^n$ and ∇f^* is Lipschitz continuous with Lipschitz constant $\frac{1}{\sigma}$ on $\mathbb{R}^n$.*

Definition 4. *(Coercive, [9, Definition 3.25]) A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be* coercive *if it is bounded below on bounded sets and $\liminf_{|x|\to+\infty} \frac{f(x)}{|x|} = +\infty$.*

Remark 1. *Any proper, lsc, strongly convex function is coercive. Indeed, this property follows from the fact that any strongly convex function f can be written as $\hat{f} + \frac{\mu}{2}|\cdot|^2$ for some proper, lsc, convex function $\hat{f}$ and $\mu \in \mathbb{R}_{>0}$. Since $\hat{f}$ is convex and it has a global underestimator hyperplane [11], Definition 4 implies that f is coercive.*

Corollary 1. *[10, Corollary 4.2.10] Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, twice differentiable, and coercive. Assume that $\nabla^2 f(x)$ is a positive definite matrix for all $x \in \mathbb{R}^n$. Then f^* has the same properties and*

$$\nabla^2 f^*(s) = [\nabla^2 f(\nabla f^{-1}(s))]^{-1} \quad \text{for all } s \in \mathbb{R}^n. \tag{6}$$

□

Definition 5. *(Normal Cone) Given a closed convex set $\mathcal{X} \subset \mathbb{R}^n$, the* normal cone *of $\mathcal{X}$ is given by*

$$N_{\mathcal{X}}(x) = \begin{cases} \{\hat{f} \in \mathbb{R}^n : \hat{f}^\top(y-x) \leq 0 \quad \forall y \in \mathcal{X}\} & if \quad x \in \mathcal{X} \\ \emptyset & if \quad x \notin \mathcal{X}. \end{cases}$$

Definition 6. *Given $f : \mathbb{R}^n \to \mathbb{R}^n$, the* range *of f is given by*

$$\operatorname{rge} f = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^n \; y = f(x)\}.$$

Definition 7. *(Convex Indicator Function) Given a convex set $\mathcal{X} \subset \mathbb{R}^n$, the* convex indicator *of $\mathcal{X}$, denoted $\delta_{\mathcal{X}}$, is defined as*

$$\delta_{\mathcal{X}}(x) = \begin{cases} 0 & if \quad x \in \mathcal{X}, \\ +\infty & otherwise. \end{cases} \tag{7}$$

B. Autonomous Dynamical Systems

Consider a differential equation of the form

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n \tag{8}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. A locally absolutely continuous function, $t \mapsto x(t)$ is a (nontrivial) solution to (8), starting from $x_0 \in \mathbb{R}^n$, if $\operatorname{dom} x = [0, t_{max})$ for $t_{max} \in \mathbb{R}_{>0} \cup +\infty$, and $\frac{dx}{dt}(t) = f(x(t))$ for almost all $t \in \operatorname{dom} x$. Based on [12], a solution x is *complete* if its domain is unbounded, and it is *maximal* if there does not exist another solution y such that $\operatorname{dom} x$ is a proper subset of $\operatorname{dom} y$ and $x(t) = y(t)$ for all $t \in \operatorname{dom} x$.

Definition 8. *(Forward pre-invariance) The set $K \subset \mathbb{R}^n$ is said to be* forward pre-invariant *for* (8) *if for every $x_0 \in K$, every solution $t \mapsto x(t)$ from x_0 satisfies $x(t) \in K$ for all $t \in \operatorname{dom} x$.*

When K is *forward pre-invariant* for (8) and every maximal solution from K is complete, we say that K is forward invariant for (8).

III. Duality for Set Invariance

A. Outline of approach

Given system (8), a set $\mathcal{X} \subset \mathbb{R}^n$, and a function ψ, suppose ψ is a strongly convex function with domain $\mathcal{X}$ and convex conjugate ψ^*. We define the *dual system* (with respect to (8)) as

$$\dot{z} = f(\nabla\psi^*(z)) \tag{9}$$

where $\nabla\psi^* : \mathbb{R}^n \to \mathcal{X}$ is the gradient of ψ^*, which is a map from the dual space to the primal space, which we refer to as the *dual map*. Based on Theorem 1, if ψ is proper, lsc, and strongly convex, then ψ^* is defined on $\mathbb{R}^n$, and $\nabla\psi^*$ is a Lipschitz continuous function and is given by

$$\nabla\psi^*(z) = \underset{x\in\operatorname{dom}\psi}{\arg\max}\{\langle z, x\rangle - \psi(x)\} \tag{10}$$

as illustrated in Proposition 1. Therefore, if f is locally Lipschitz, the dual system (9) has a unique solution $t \mapsto z(t)$ for each $z_0 \in \mathbb{R}^n$. Accordingly, this solution can be mapped to the primal space $\mathbb{R}^n$, namely, to the state space of (8) which plays the role of *primal system* as follows:

$$x(t) = \nabla\psi^*(z(t)) \quad \forall t \in \operatorname{dom} z. \tag{11}$$

As we show in Lemma 1, the range of the dual map is $\mathcal{X}$. Then, for each $x_0 \in \mathcal{X}$, there exist z_0 such that $x_0 = \nabla\psi^*(z_0)$. The resulting solution $t \mapsto z(t)$ to the dual system (9) from z_0 and $t \mapsto x(t)$ from (11) define a solution to a system of the form

$$\begin{aligned}\dot{s} &= \langle \nabla^2\psi^*(z), f(\nabla\psi^*(z))\rangle \\ &= \langle \nabla^2\psi^*(z), f(s)\rangle.\end{aligned} \tag{12}$$

We refer to this system as the *modified primal system.* In the following, we show that every solution $t \mapsto x(t)$ to the modified primal system from $\mathcal{X}$ stays in $\mathcal{X}$, implying that $\mathcal{X}$ is forward pre-invariant.

B. Assumptions and Supporting Results

Lemma 1. *Suppose $\psi : \mathbb{R}^n \to \mathbb{R}$ is a proper, lsc, and strongly convex function with domain equal to $\mathcal{X} \subset \mathbb{R}^n$, and ψ^* is the convex conjugate of ψ. Then, the range of $\nabla\psi^*$ is $\mathcal{X}$.* □

Given a convex set $\mathcal{X}$, a function ψ with domain $\mathcal{X}$ can be written as in (2). The following Assumption is required for the function ψ in (2) to satisfy the assumptions of Proposition 1 and Theorem 1.

Assumption 1. *Given $\mathcal{X} \subset \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$*

- *$\mathcal{X}$ is a nonempty, closed, and convex set, and*
- *ϕ is proper, lsc, strongly convex, and twice continuously differentiable on $\mathbb{R}^n$.*

Remark 2. *Consider ψ defined in* (2) *such that $\mathcal{X}$ and ϕ satisfy Assumption 1. Then, follows directly that ψ is proper, lsc, and strongly convex.*

The function ψ in (2) is differentiable in the interior of $\mathcal{X}$ and its subdifferential mapping, which is a map from primal space to the dual space, is given as follows.

Lemma 2. *Suppose $\mathcal{X}$ and ϕ satisfy Assumption 1. Let ψ be defined as in* (2)*, where $\delta_{\mathcal{X}}$ is the convex indicator function of $\mathcal{X}$. Then, the subdifferential mapping $\partial\psi : \mathcal{X} \rightrightarrows \mathbb{R}^n$ of ψ is given by*

$$\partial\psi(x) = \begin{cases}\nabla\phi(x) & \text{if} \quad x \in \mathit{int}(\mathcal{X}),\\ \nabla\phi(x) + N_{\mathcal{X}}(x) & \text{if} \quad x \in \partial\mathcal{X},\\ \emptyset & \text{if} \quad x \notin \mathcal{X}.\end{cases} \tag{13}$$

□

The conjugate function of ψ in (2) is differentiable and its gradient is a map from dual space to the primal space. Using Lemma 1, we show that its range is equal to $\mathcal{X}$.

Lemma 3. *Suppose $\mathcal{X}$ and ϕ satisfy Assumption 1. Let ψ be defined as in* (2) *for each $x \in \mathbb{R}^n$, where $\delta_{\mathcal{X}}$ is the convex indicator function of $\mathcal{X}$, and let ψ^* be the convex conjugate of ψ, then the range of $\nabla\psi^*$ is equal to $\mathcal{X}$.* □

Remark 3. *Suppose ϕ satisfies Assumption 1. Remark 1 and Corollary 1 imply that ϕ^* is twice differentiable. Using $\partial\psi(x) = \nabla\phi(x)$ for each $x \in \operatorname{int}(\mathcal{X})$, and the fact that*

$$\nabla^2\psi^*(\nabla\phi(x)) = \nabla^2\phi^*(\nabla\phi(x)) \quad \forall x \in \operatorname{int}(\mathcal{X}),$$

we conclude that $\nabla^2\psi^*$ exists at least on $\operatorname{int}(\mathcal{X})$. Furthermore, since $\nabla\psi^*$ is Lipschitz continuous, Rademacher's Theorem [13, Theorem 3.1.2] implies that $\nabla^2\psi^*$ exists almost everywhere.

Based on Remark 3, we introduce the following assumption.

Assumption 2. *The Hessian of the convex conjugate function ψ^*, where ψ is defined in* (2)*, is piecewise continuous on $\mathbb{R}^n$.*

The following lemma is exploited to define the modified primal system as function of s only.

Lemma 4. *Consider ψ defined as in* (2) *for each $x \in \mathbb{R}^n$, where $\mathcal{X}$ and ϕ satisfy Assumptions 1 and 2. Then, for each $x \in \mathcal{X}$ and each $z \in \partial\psi(x)$, $\nabla^2\psi^*(z) = \nabla^2\psi^*(\nabla\phi(x))$.* □

C. Construction of the modified primal system

In light of Lemma 4, using ψ in (2) and $\nabla\psi^*$ in (10), we have

$$\nabla\psi^*(\nabla\phi(x)) = \begin{cases}\nabla\phi^*(\nabla\phi(x)) & \text{if} \quad x \in \operatorname{int}(\mathcal{X}),\\ g(\nabla\phi(x)) & \text{if} \quad x \in \mathbb{R}^n \setminus \operatorname{int}(\mathcal{X}).\end{cases} \tag{14}$$

Therefore, $x \mapsto \nabla\psi^*(\nabla\phi(x))$ is a piecewise function. Using Corollary 1, we conclude that $\nabla^2\phi^*$ exists and is continuous.

Moreover, Assumption 2 implies that g is continuously differentiable. Then, following (12), the modified primal system is defined on $\mathbb{R}^n$ as

$$\begin{aligned}\dot{s} &= \langle \nabla^2 \psi^*(\nabla\phi(s)), f(s)\rangle \\ &= \begin{cases} \langle \nabla^2 \phi^*(\nabla\phi(s)), f(s)\rangle & \text{if} \quad s \in \operatorname{int}(\mathcal{X}), \\ \langle \nabla g(\nabla\phi(s)), f(s)\rangle & \text{if} \quad s \in \mathbb{R}^n \setminus \operatorname{int}(\mathcal{X}). \end{cases}\end{aligned} \tag{15}$$

D. Main result

Next, we show that $\mathcal{X}$ is forward pre-invariant for the modified primal system (15).

Theorem 2. *Consider*

- *$f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz continuous, and*
- *ψ defined in (2), where $\mathcal{X} \subset \mathbb{R}^n$ and ϕ satisfy Assumptions 1 and 2.*

Suppose $t \mapsto z(t)$ is a solution to the dual system $\dot{z} = f(\nabla\psi^*(z))$ from z_0. Then, $t \mapsto s(t)$ defined as $s(t) = \nabla\psi^*(z(t))$ for each $t \in \operatorname{dom} z$ is a solution to the modified primal system in (15) from $s_0 = \nabla\psi^*(z_0)$. Furthermore, the set $\mathcal{X}$ is forward pre-invariant for the modified primal system in (15).

In the following section, we illustrate our results with applications.

IV. Applications

A. Forward Invariance via Modification of f

Consider system (8) and $\mathcal{X} \subset \mathbb{R}^n$. We want to find $\tilde{f}$ that is as close as possible to f such that $\mathcal{X}$ is forward pre-invariant for $\dot{x} = \tilde{f}(x)$. To do that, we consider ϕ quadratic. We have the following result.

Proposition 2. *Suppose*

- *$f : \mathbb{R}^n \to \mathbb{R}^n$ in (8) is a locally Lipschitz continuous function,*
- *ψ is defined in (2), where $\mathcal{X} \subset \mathbb{R}^n$ and ϕ satisfy Assumptions 1 and 2, and ϕ is the quadratic function, $\phi(x) := \frac{1}{2}x^\top Px$ with $P \in S^n_{++}$.*

Then, the modified primal system is given by

$$\dot{s} = \tilde{f}(s) = \begin{cases} f(s) & \text{if} \quad s \in \operatorname{int}(\mathcal{X}) \\ \langle \nabla g(Ps), Pf(s)\rangle & \text{if} \quad s \in \mathbb{R}^n \setminus \operatorname{int}(\mathcal{X}), \end{cases} \tag{16}$$

and the following hold:

- *$\mathcal{X}$ is forward pre-invariant for the modified primal system in (16).*
- *Suppose $t \mapsto s(t)$ and $t \mapsto x(t)$ are solutions to $\dot{s} = \tilde{f}(s)$ and to $\dot{x} = f(x)$, respectively, both from $s_0 \in \mathcal{X}$. Then, $s(t) = x(t)$ for all $t \in \operatorname{dom} x$ such that $x(t) \in \mathcal{X}$.*

□

In the following example, we illustrate how to use the proposed method to include constraints in the heavy ball algorithm for optimization (see, e.g. [14]).

Example 1. *Given the dynamical system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_2 - b\nabla L(x_1) \end{cases} \tag{17}$$

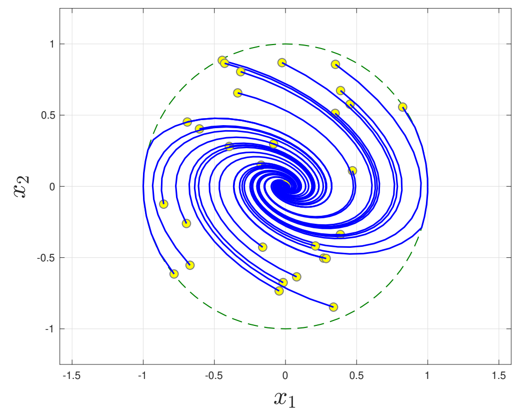

Fig. 1. State trajectories for the system in Example 1, from thirty initial points in $\mathcal{X} = \{x \in \mathbb{R}^2 | \, |x| \leq 1\}$, with $\phi(x) = \frac{1}{2}x^\top Px$.

where $L(x_1) = \frac{1}{2}x_1^2$ and constants a, b are positive and given the set $\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, |x| \leq 1\}$, we want to find $\tilde{f}$ such that the set $\mathcal{X}$ is forward invariant. The system in (17) corresponds to the heavy ball algorithm which typically does not handle constraints. We show how to handle constraints using the approach proposed in this paper.

Let $\phi(x) := \frac{1}{2}x^\top Px$ with $P \in S^n_{++}$, We use ψ in (2) and (10) to obtain the dual map

$$\nabla\psi^*(z) = \begin{cases} P^{-1}z & \text{if} \quad |P^{-1}z| < 1, \\ \dfrac{P^{-1}z}{|P^{-1}z|} & \text{if} \quad |P^{-1}z| \geq 1. \end{cases} \tag{18}$$

Because P^{-1} is positive definite, $P^{-1}z$ is zero only when $z = 0$ which does not happen when $P^{-1}z \geq 1$.

Next, the Hessian of ψ is given by

$$\nabla^2\psi^*(z) = \begin{cases} P^{-1} & \text{if} |P^{-1}z| < 1, \\ \left(\dfrac{I}{|P^{-1}z|} - \dfrac{(P^{-1}z)(P^{-1}z)^\top}{|P^{-1}z|^3}\right)P^{-1} & \text{if} |P^{-1}z| \geq 1, \end{cases}$$

and since $\nabla\phi(x) = Px$, we have

$$\nabla^2\psi^*(\nabla\phi(x)) = \begin{cases} P^{-1} & \text{if} \quad |x| < 1, \\ \left(\dfrac{I}{|x|} - \dfrac{xx^\top}{|x|^3}\right)P^{-1} & \text{if} \quad |x| \geq 1 \end{cases} \tag{19}$$

Therefore, the right-hand side of the modified primal system is

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if} \quad |s| < 1, \\ \left(\dfrac{I}{|s|} - \dfrac{ss^\top}{|s|^3}\right)f(s) & \text{if} \quad |s| \geq 1 \end{cases} \tag{20}$$

Since the right-hand side of the system in (17) is Lipschitz, Proposition 2 implies that $\mathcal{X}$ is forward pre-invariant for $\dot{s} = \tilde{f}(s)$. The trajectories resulting for 30 random initial points in the unit disc are shown in Figure 1, where $P = \begin{pmatrix} 5.2 & 0.7 \\ 0.7 & 8 \end{pmatrix}$, and $a = b = 0.5$. □

B. Controlled Forward Invariance

Consider a nonlinear control system given by

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p \tag{21}$$

and $\mathcal{X} \subset \mathbb{R}^n$. The following result allows us to design a control law $x \mapsto \kappa(x)$ such that $\mathcal{X}$ is forward pre-invariant for $\dot{x} = f(x, \kappa(x))$.

Proposition 3. *Given system (21), where f is locally Lipschitz in both arguments, suppose ψ is defined in (2) for each $x \in \mathbb{R}^n$, where $\mathcal{X} \subset \mathbb{R}^n$ and ϕ satisfy Assumptions 1 and 2. If there exists a control law $\kappa : \mathbb{R}^n \to \mathbb{R}^p$ that is piecewise locally Lipschitz on $\mathbb{R}^n$ and a function $f_z : \mathbb{R}^n \to \mathbb{R}^n$ such that $\nabla^2\psi^*(\nabla\phi(s))f_z(s) = f(s, \kappa(s))$, then $\mathcal{X}$ is forward pre-invariant for $\dot{x} = f(x, \kappa(x))$.* □

The following example illustrates Proposition 3 in a simple control system to highlight the proposed approach.

Example 2. *Given $\dot{x} = u$ such that $x \in \mathbb{R}^2$, and $u \in \mathbb{R}^2$, and $\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, x^\top P_1 x \le 1\}$, where P_1 is symmetric positive definite matrix, we want to find a control law κ to make the set $\mathcal{X}$ forward pre-invariant. We define $\phi(x) := \frac{1}{2}x^\top P x$, where $P = \alpha P_1$ and $\alpha > 0$ for simplicity. Let $f_z(x) := \lambda(x)^{-1}v(x)$ where $\lambda(x)$ and $v(x)$ are the largest eigenvalue and corresponding eigenvector of $\nabla^2\psi^*(\nabla\phi(x))$, respectively. Defining the control law $\kappa(x) = v(x)$, for each $x \in \mathbb{R}^2$, we have*

$$\begin{aligned}\dot{x} &= \nabla^2\psi^*(\nabla\phi(x))f_z(x)\\ &= \nabla^2\psi^*(\nabla\phi(x))\lambda(x)^{-1}v(x)\\ &= \kappa(x).\end{aligned}$$

Then, using Proposition 3, we conclude that the set $\mathcal{X}$ is forward pre-invariant for the system $\dot{x} = \kappa(x)$. Next, we synthesize κ.

Using that $\nabla^2\psi^*(z) = \nabla^2\phi^*(z) = P^{-1}$ in the interior of $\mathcal{X}$, we conclude that the eigenvectors corresponding to $\nabla^2\psi^*(\nabla\phi(x))$ are constant. As a consequence the control law can be arbitrary in the interior of $\mathcal{X}$. However, at other points, the control law is defined as the eigenvector corresponding to the largest eigenvalue of the Hessian of ψ^*. Therefore, the control law is defined as follows

$$\kappa(x) = \begin{cases}\kappa_0(x) & \text{if } x^\top P_1 x < 1,\\ \beta v(x) & \text{if } x^\top P_1 x \ge 1\end{cases} \tag{22}$$

where $\beta \in \mathbb{R}_{>0}$ and $\kappa_0 : \mathbb{R}^n \to \mathbb{R}^n$ is any arbitrary locally Lipschitz function. Then, κ is piecewise locally Liptschitz on $\mathbb{R}^n$ and meets the requirement of Proposition 3.

To specify $x \mapsto v(x)$, we use ψ in (2) and (10) to obtain the dual map

$$\nabla\psi^*(z) = \begin{cases}P^{-1}z & \text{if} \quad z^\top P^{-1}z < \alpha,\\ \sqrt{\alpha}\dfrac{P^{-1}z}{\sqrt{z^\top P^{-1}z}} & \text{if} \quad z^\top P_1^{-1}z \ge \alpha.\end{cases} \tag{23}$$

Next, using $\nabla\phi(x) = Px$, we have

$$\nabla^2\psi^*(\nabla\phi(x)) = \begin{cases}P^{-1} & \text{if} \quad x^\top P_1 x < 1,\\ \sqrt{\alpha}\dfrac{x^\top PxP^{-1} - xx^\top}{(x^\top Px)^{\frac{3}{2}}} & \text{if} \quad x^\top P_1 x \ge 1.\end{cases} \tag{24}$$

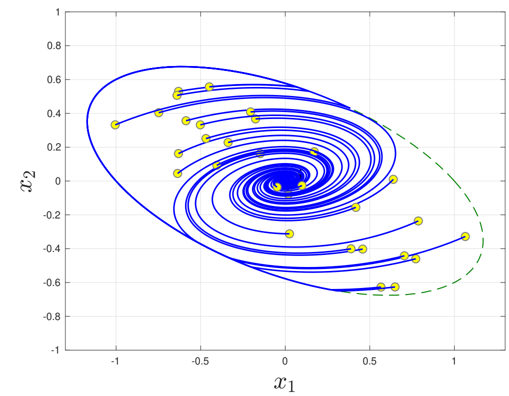

Fig. 2. The trajectory results of Example 2, for thirty initial points in the ellipse.

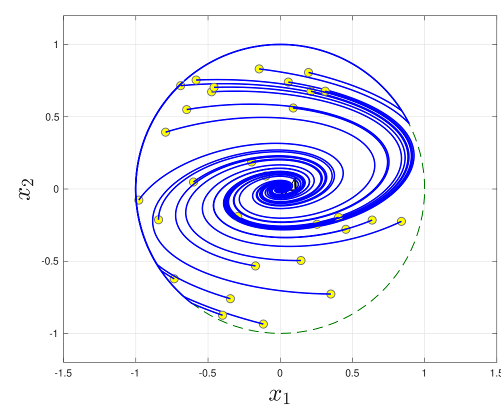

Fig. 3. State trajectories for the system in Example 2, for thirty initial points in the unit ball.

Let $P^{-1} = \begin{pmatrix} q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}$. The largest eigenvalue of $\nabla^2\psi^*(\nabla\phi(x))$ is

$$\lambda = \frac{\sqrt{\alpha}}{2}\left[\frac{q_{11}+q_{22}}{(x^\top Px)^{\frac{1}{2}}} - \frac{x_1^2+x_2^2}{(x^\top Px)^{\frac{3}{2}}} + \sqrt{\left(\frac{q_{11}-q_{22}}{(x^\top Px)^{\frac{1}{2}}} - \frac{x_1^2-x_2^2}{(x^\top Px)^{\frac{3}{2}}}\right)^2 + 4\left(\frac{q_{12}}{\alpha(x^\top Px)^{\frac{1}{2}}} - \frac{x_1x_2}{(x^\top Px)^{\frac{3}{2}}}\right)^2}\,\right].$$

The corresponding eigenvector is given by

$$v(x) = \left(\sqrt{\alpha}\left(\frac{q_{12}}{(x^\top Px)^{\frac{1}{2}}} - \frac{x_1x_2}{(x^\top Px)^{\frac{3}{2}}}\right), \lambda - \sqrt{\alpha}\left(\frac{q_{11}}{(x^\top Px)^{\frac{1}{2}}} + \frac{x_1^2}{(x^\top Px)^{\frac{3}{2}}}\right)\right)^\top$$

Suppose κ_0 in (22) is given by $\kappa_0(x) := (-x_1+4x_2, -x_1-x_2^3)^\top$. Since κ_0 and v are continuous, the control law κ satisfies the regularity condition of Proposition 3.

Let $P_1 = \begin{pmatrix} 1 & 0.9\\ 0.9 & 3\end{pmatrix}$, $\alpha = 1$ and $\beta = 1$, the trajectories for thirty random initial points in the ellipse defined by $\mathcal{X}$ are illustrated in Figure 2. The result when $\mathcal{X}$ is the unit ball, namely, when $P_1 = I$, with the same κ_0, $\alpha = 2$, and and $\beta = 1$ is depicted in Figure 3.

□

In the next example, we illustrate how to ensure forward invariance of a set and asymptotically stability of the origin.

Example 3. *Given the system*

$$\dot{x} = u \qquad x \in \mathbb{R}^2,\ u \in \mathbb{R}^2, \tag{25}$$

the set $\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, x_2^2 + x_1 - 1 \leq 0\}$, *and the system*

$$\dot{x} = f(x, u_1) = \begin{bmatrix} -x_1 - x_2 \\ x_1^3 + x_2 u_1 \end{bmatrix} \qquad x \in \mathbb{R}^2,\ u_1 \in \mathbb{R}, \tag{26}$$

we want to design a control law κ *for the system in* (25) *to make the set* $\mathcal{X}$ *forward pre-invariant and such that the solutions to the resulting closed-loop system match with those of system* (26) *in the interior of the set* $\mathcal{X}$ *and the origin is asymptotically stable.*

First, we design u_1 *such that the origin is asymptotically stable for* (26). *Using control Lyapunov function* $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, *the control law is given by* $\kappa_1(x) = -1$ *for each* $x \in \mathbb{R}^2$. *Next,* κ *can be defined by*

$$\kappa(x) = \begin{cases} f(x, \kappa_1(x)) & \text{if} \quad x \in \mathit{int}(\mathcal{X}) \\ \alpha v(x) & \text{otherwise} \end{cases} \tag{27}$$

where $v(x)$ *is the eigenvector of the Hessian matrix* $\nabla^2\psi^*(\nabla\phi(x))$ *and* $\alpha \in \mathbb{R}_{>0}$. *We take* $\phi(x) = \frac{1}{2}x^\top P x$ *where* $P \in S_{++}^n$, *then* $\psi(x) = \phi(x) + \delta_{\mathcal{X}}(x)$ *and the dual map* $\nabla\psi^*$ *is given as follows:*

$$\begin{aligned} \nabla\psi^*(z) = \arg\min \quad & \left\{\frac{1}{2}x^\top P x - z^\top x\right\} \\ \text{s.t.} \quad & x_1 + x_2^2 - 1 \leq 0 \end{aligned} \tag{28}$$

This can be written as a quadratically constrained quadratic program with one constraint as follows:

$$\begin{aligned} \nabla\psi^*(z) = \arg\min \quad & \left\{\frac{1}{2}x^\top P x - z^\top x\right\} \\ \text{s.t.} \quad & \frac{1}{2}x^\top P_1 x + q_1^\top x + r \leq 0 \end{aligned} \tag{29}$$

where $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $q_1 = (1, 0)^\top$, *and* $r = -1$. *The convex optimization problem* (29) *can be solved using the KKT conditions, which reduce to*

$$\begin{cases} (P + \lambda^* P_1)x^* + (\lambda^* q - z) = 0 \\ \frac{1}{2}x^{*\top} P_1 x^* + q_1^\top x^* + r \leq 0 \\ \lambda^* \geq 0 \\ \lambda^* \left(\frac{1}{2}x^{*\top} P_1 x^* + q_1^\top x^* + r\right) = 0 \end{cases} \tag{30}$$

Then, the Hessian $\nabla^2\psi^*(\nabla\phi(x))$ *is calculated and the resulting control law is equal to the eigenvector corresponding the largest eigenvalue of the Hessian.*

Let $P = \begin{pmatrix} 9 & 1 \\ 1 & 4 \end{pmatrix}$. *The trajectories to the system in* (25), *with* $u = \kappa$ *in* (27) *for thirty random initial points in* $\mathcal{X}$ *are illustrated in Figure 4. Thus, we designed a control law to ensure safety and stability for the system* (25) *and* (26). □

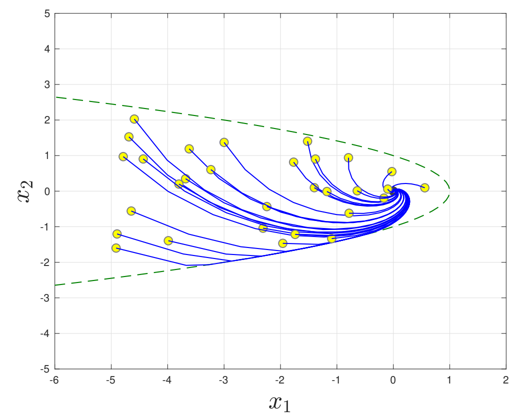

Fig. 4. State trajectories for the system in Example 3, from thirty initial points inside the set defined by the rotated parabola.

V. Conclusion

This paper proposes an approach to certify set invariance for nonlinear systems. Given a nonlinear (primal) system and a set, we employ duality to build a dual system and a modified primal system for which the desired set is forward (pre-) invariant. Several applications are provided to illustrate the proposed approach, in particular, for constrained optimization as well as feedback control under constraints. In future work, we will extend the ideas to differential inclusions and hybrid systems.

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