Robust Finite-Time Parameter Estimation for Linear Dynamical Systems

Ryan S. Johnson, Adnane Saoud, and Ricardo G. Sanfelice

Abstract— **We consider the problem of estimating a constant or piecewise constant vector of unknown parameters for a linear dynamical system. Using a hybrid systems framework, a hybrid algorithm that achieves finite-time convergence of the parameter estimate to the true value is proposed. Sufficient conditions that guarantee convergence of the parameter estimate are provided. Robustness of the proposed algorithm with respect to measurements noise is analyzed, and examples are provided showing the merits of the proposed approach.**

I. Introduction

Accurate estimation of a system's unknown parameters is critical in many engineering applications [1]. One such application is the classical model-reference adaptive control problem, which has been studied since the 1960s [2] and has experienced a recent resurgence with the advent of machine learning applications [3]. This estimation algorithm computes online an estimate of the unknown parameters by exploiting the available input signals and information about the structure of the system [4], [5]. In the case of linear time-varying systems, analyzing the convergence rate of the parameter estimate can be translated into showing exponential stability of the origin [1], [5]. A persistence of excitation condition is necessary and sufficient for exponential stability of linear time-varying systems [6].

Motivated by the recent results on finite-time parameter estimation for linear regression models [7], this paper presents a hybrid estimator for linear dynamical systems that guarantees convergence of the parameter estimate to the true value in finite time. In Section III-A, we show that, for the case of a constant unknown parameter, the parameter estimate converges to the true value after one jump. Then, in Section III-B, we generalize this result to piecewise constant unknown parameters by allowing the algorithm to jump multiple times. Robustness of the proposed algorithm to measurement noise is discussed in Section IV, and simulation results are presented in Section V.

Our approach is related to the ones in [7], [8] – both results provide a finite-time estimator using a hybrid system framework. In comparison to our work, the approach in [8] is different in two aspects. First, we use a different estimation algorithm based on the use of two coupled estimators. Second, while the results in [8] rely on a persistence of excitation condition to ensure that their hybrid system is well defined and to guarantee completeness of solutions, we impose a condition related to the invertibility of the solution components and show, using a numerical example, that our algorithm is capable of converging in finite time when the regressor is exciting over only a finite time interval. Moreover, our work differs from the one in [7] in three aspects. First, the authors of [7] deal with a linear regression model while we deal with a dynamical model. Second, the construction of the update law is different from the one in [7]. Finally, the analysis of the convergence properties of the system is more complicated for dynamical models than for linear regression models. Indeed, while the authors in [7] rely on a quadratic Lyapunov function, our analysis requires a more involved Lyapunov function so as to account for the coupling between the state and parameter estimators. Due to space constraints, the proofs of some results are sketched or omitted and will be published elsewhere.

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II. Preliminaries

A. Notation

We denote the set of real, nonnegative, positive, and natural numbers (including zero) as $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, and $\mathbb{N}$, respectively. The matrix I denotes the identity matrix of appropriate dimension. The set of symmetric positive definite matrices of dimension $n \times n$ is dentoed S^n_{++}. The Euclidean norm of vectors and the induced matrix norm is denoted $|\cdot|$, and the infinity norm is denoted $|\cdot|_\infty$. The distance of a point x to a nonempty set S is denoted $|x|_S = \inf_{y \in S} |y - x|$. Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the domain of M is the set $\operatorname{dom} M = \{x \in \mathbb{R}^m \ : \ M(x) \neq \varnothing\}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$ function (denoted $\alpha \in \mathcal{K}_\infty$) if α is zero at zero, strictly increasing, and unbounded.

B. Parameter estimation

A classical model-reference adaptive (closed-loop) control system is given by

$$\dot{x} = A_0 x + B(t)\theta \tag{1}$$

where $x \in \mathbb{R}^n$ is the known state vector, $t \mapsto B(t) \in \mathbb{R}^{n \times p}$ is the known regresson matrix, the matrix $A_0 \in \mathbb{R}^{n \times n}$ is known, and $\theta \in \mathbb{R}^p$ is an unknown vector of parameters.

The unknown parameter θ can be estimated using an update law of the form [1]

$$\begin{aligned}\dot{\hat{\theta}} &= \gamma B^\top(t)P(x-\hat{x})\\ \dot{\hat{x}} &= A_0x + B(t)\hat{\theta} - A(x-\hat{x})\end{aligned} \tag{2}$$

where $\hat{\theta} \in \mathbb{R}^p$ is the estimate of the unknown parameter θ, $\hat{x}$ is the estimate of the state x, $A \in \mathbb{R}^{n\times n}$ is a user-selected Hurwitz matrix, $P = P^\top \in \mathbb{R}^{n\times n}$ is a positive definite matrix that solves $A^\top P + PA = -Q$, where $Q = Q^\top \in \mathbb{R}^{n\times n}$ is a user-selected positive definite matrix, and $\gamma > 0$ is a design parameter that modifies the convergence rate.

Denote the parameter estimation error as $\tilde{\theta} := \theta - \hat{\theta}$ and the state estimation error as $\tilde{x} := x - \hat{x}$. Then, the error dynamics can be written as

$$\dot{\tilde{x}} = A\tilde{x} + B(t)\tilde{\theta}, \qquad \dot{\tilde{\theta}} = -\gamma B^\top(t)P\tilde{x}. \tag{3}$$

The convergence properties of (3) are typically analyzed using a quadratic Lyapunov function of the form $V(\tilde{x},\tilde{\theta}) = \tilde{x}^\top P\tilde{x} + \gamma^{-1}\tilde{\theta}^\top\tilde{\theta}$ whose time derivative satisfies $\dot{V}(\tilde{x},\tilde{\theta}) \le -\underline{\mu}(Q)|\tilde{x}|^2$. It can be shown using Barbalat's Lemma [4] that $t \mapsto \tilde{x}(t)$ converges to zero as time t goes to infinity. In order to show convergence of $\hat{\theta}$ to θ as t goes to infinity, a persistence of excitation condition [1], [5], [9] is required (see Definition 2.2 below for a formal definition of persistence of excitation).

The objective of this paper is to estimate the unknown parameter vector θ in finite time using hybrid systems tools. For this reason, we review some preliminaries on hybrid systems in the following section.

C. Preliminaries on hybrid systems

In this paper, a hybrid system $\mathcal{H}$ is defined as in [10] by (C, F, D, G) as

$$\mathcal{H} = \begin{cases} \dot{\xi} = F(\xi, u) & (\xi, u) \in C \\ \xi^+ = G(\xi, u) & (\xi, u) \in D \end{cases} \tag{4}$$

where $\xi \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the flow map defining a differential equation capturing the continuous dynamics, and $C \subset \mathbb{R}^n$ defines the flow set on which flows are permitted. The mapping $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the jump map defining the law resetting ξ at jumps, and $D \subset \mathbb{R}^n$ is the jump set on which jumps are permitted.

A solution ξ to $\mathcal{H}$ is a hybrid arc that is parameterized by $(t, j) \in \mathbb{R}_{\ge 0} \times \mathbb{N}$, where t is the elapsed ordinary time and j is the number of jumps that have occurred. The domain of ξ, denoted $\operatorname{dom}\xi \subset \mathbb{R}_{\ge 0} \times \mathbb{N}$, is a *hybrid time domain*, in the sense that for every $(T, J) \in \operatorname{dom}\xi$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ with $t_0 = 0$ such that $\operatorname{dom}\xi \cap ([0,T] \times \{0,1,\ldots,J\}) = \bigcup_{j=0}^{J}([t_j, t_{j+1}], \{j\})$. A solution ξ to $\mathcal{H}$ is called *maximal* if it cannot be extended further. A solution is called *complete* if its domain is unbounded.

D. Excitation conditions

We employ the following notions of excitation for time-varying signals [4].

Definition 2.1: Given $t \ge 0$ and $\eta_1 > 0$, a signal $t \mapsto B(t) \in \mathbb{R}^{n\times p}$ is *exciting over the finite interval* $[t, t+\eta_1]$ if there exist constants $\eta_2, b_M > 0$ such that

$$\int_t^{t+\eta_1} B^\top(s)\, B(s)\, ds \ge \eta_2 I \tag{5}$$

and ess sup $\{|B(t)|_\infty, |\dot{B}(t)|_\infty : t \ge 0\} \le b_M$.

Definition 2.2: Given $\eta_1 > 0$, a signal $t \mapsto B(t) \in \mathbb{R}^{n\times p}$ is *persistently exciting* if there exist constants $\eta_2, b_M > 0$ such that, for all $t \ge 0$,

$$\int_t^{t+\eta_1} B^\top(s)\, B(s)\, ds \ge \eta_2 I \tag{6}$$

and ess sup $\{|B(t)|_\infty, |\dot{B}(t)|_\infty : t \ge 0\} \le b_M$.

III. Hybrid Finite-Time Convergent Algorithm

In this section, we present a hybrid parameter estimator for finite-time convergence with respect to constant and piecewise constant unknown parameters. We first focus on the case of constant unknown parameters.

Recall the update laws for $\hat{x}$ and $\hat{\theta}$ in (2) and denote $z := (x, \theta)$ and $\Gamma := (A, \gamma, Q)$. Then, given $t \mapsto B(t)$, we express the dynamics of $\hat{z} := (\hat{x}, \hat{\theta})$ in a compact form as

$$\dot{\hat{z}} = h(x, \hat{z}, \Gamma, B(t)) := \begin{bmatrix} A_0x + B(t)\hat{\theta} - A(x-\hat{x}) \\ \gamma B^\top(t)P(x-\hat{x}) \end{bmatrix}. \tag{7}$$

Next, we express the error dynamics in (3) as

$$\dot{\tilde{z}} = \Psi(\Gamma, B(t))\tilde{z} \tag{8}$$

where $\tilde{z} := (\tilde{x}, \tilde{\theta})$ and the functional Ψ is given by

$$\Psi(\Gamma, B(t)) := \begin{bmatrix} A & B(t) \\ -\gamma B^\top(t)P & 0 \end{bmatrix} \tag{9}$$

where Γ denotes explicitly the dependence of Ψ on (A, γ, Q). The continuous evolution of $\tilde{z}$ between times t_0 and t with $0 \le t_0 < t$ is given by $\tilde{z}(t) = \Omega(t, t_0)\tilde{z}(t_0)$ where, given Γ and $t \mapsto B(t)$, Ω is the state transition matrix for (8).

A. Constant unknown parameter

We extend the finite-time parameter estimation approach in [7] to classes of continuous-time systems whose solutions satisfy (1). We begin by explaining the intuition behind the algorithm before providing a formal statement of the result. The algorithm is expressed as a hybrid system, denoted $\mathcal{H}$, and operates as follows. Given $\Gamma_1 := (A_1, \gamma_1, Q_1)$ and $\Gamma_2 := (A_2, \gamma_2, Q_2)$ where $A_1, A_2 \in \mathbb{R}^{n\times n}$, $\gamma_1, \gamma_2 > 0$, and $Q_1, Q_2 \in S^n_{++}$ are design parameters, denote the state of $\mathcal{H}$ as $\xi = (x, \hat{z}_1, \hat{z}_2, \Phi_1, \Phi_2, q)$, where

- x is the plant state, with dynamics given in (1);
- $\hat{z}_1, \hat{z}_2 \in \mathbb{R}^{n+p}$ are estimates of z, with dynamics

$$\dot{\hat{z}}_1 = h(x, \hat{z}_1, \Gamma_1, B(t)), \quad \dot{\hat{z}}_2 = h(x, \hat{z}_2, \Gamma_2, B(t))$$

where h is given in (7). The dynamics of the errors $\tilde{z}_1 := z - \hat{z}_1$ and $\tilde{z}_2 := z - \hat{z}_2$ are given by

$$\dot{\tilde{z}}_1 = \Psi(\Gamma_1, B(t))\tilde{z}_1, \quad \dot{\tilde{z}}_2 = \Psi(\Gamma_2, B(t))\tilde{z}_2 \tag{10}$$

where Ψ is given in (9);

- $\Phi_1, \Phi_2 \in \mathbb{R}^{(n+p)\times(n+p)}$ have dynamics

$$\dot{\Phi}_1 = \Psi(\Gamma_1, B(t))\Phi_1, \quad \dot{\Phi}_2 = \Psi(\Gamma_2, B(t))\Phi_2$$

where Ψ is given in (9). Hence, when initialized as the identity matrix, Φ_1 and Φ_2 are equivalent to the the state transition matrices Ω_1 and Ω_2 for, respectively, the systems in (10);

- $q \in \{0, 1\}$ is a logic variable.

Next, let $(t, j) \mapsto \xi(t, j)$ be a solution to $\mathcal{H}$ – hence, defined on a hybrid time domain – and consider the initial interval of flow I^0, where $I^j := \{t : (t, j) \in \operatorname{dom}\xi\}$, with initial conditions $\Phi_1(0, 0) = \Phi_2(0, 0) = I$, and $\hat{z}_1(0, 0) = \hat{z}_2(0, 0)$ arbitrary. At any time $t \in I^0$, the solutions components Φ_1 and Φ_2 satisfy

$$\tilde{z}_1(t, 0) = \Phi_1(t, 0)\tilde{z}_1(0, 0), \quad \tilde{z}_2(t, 0) = \Phi_2(t, 0)\tilde{z}_2(0, 0).$$

Then, if there exists a positive time $t_1 \in I^0$ such that the matrix $\Phi_1(t_1, 0) - \Phi_2(t_1, 0)$ is invertible, resetting $\hat{z}_1$ and $\hat{z}_2$ to the value of the function $R(\xi) := K_1(\xi)\hat{z}_1 + K_2(\xi)\hat{z}_2$, where

$$K_1(\xi) := -\Phi_2\big(\Phi_1 - \Phi_2\big)^{-1}, \quad K_2(\xi) := I - K_1(\xi) \tag{11}$$

leads to, for each $i \in \{1, 2\}$,

$$\begin{aligned}
\hat{z}_i(t_1,1) &= R(\xi(t_1,0)) = K_1\hat{z}_1(t_1,0) + K_2\hat{z}_2(t_1,0)\\
&= K_1(\tilde{z}_1(t_1,0) + z(t_1,0)) + K_2(\tilde{z}_2(t_1,0) + z(t_1,0))\\
&= K_1\tilde{z}_1(t_1,0) + K_2\tilde{z}_2(t_1,0) + (K_1 + K_2)z(t_1,0)\\
&= K_1(\tilde{z}_1(t_1,0) - \tilde{z}_2(t_1,0)) + \tilde{z}_2(t_1,0) + z(t_1,0)\\
&= K_1\big(\Phi_1(t_1,0) - \Phi_2(t_1,0)\big)\tilde{z}_2(0,0)\\
&\quad + \Phi_2(t_1,0)\tilde{z}_2(0,0) + z(t_1,0)\\
&= \Phi_2(t_1,0)\big(-\tilde{z}_2(0,0) + \tilde{z}_2(0,0)\big) + z(t_1,0) = z(t_1,0)
\end{aligned} \tag{12}$$

where the argument of K_1 and K_2 is omitted for readability. Hence, we have finite-time convergence of $\hat{z}_1$ and $\hat{z}_2$ to z.

To ensure the existence of a jump time $(t_1, 0)$, we choose A_1 and A_2 such that $\hat{z}_1$ is convergent but $\hat{z}_2$ is divergent. To avoid the solution components $\hat{z}_2$ and Φ_2 from growing unbounded, for all $(t, j) \in \operatorname{dom}\xi$ satisfying $t \geq t_1$ and $j \geq 1$, we assign $\hat{z}_2$ the dynamics of $\hat{z}_1$ and Φ_2 the dynamics of Φ_1. After the jump, we have $\hat{z}_2(t_1, j) = \hat{z}_1(t_1, j)$ and $\Phi_2(t_1, j) = \Phi_1(t_1, j)$, and thus $\hat{z}_2(t, j) = \hat{z}_1(t, j)$ and $\Phi_2(t, j) = \Phi_1(t, j)$ for all $t \geq t_1$ and $j \geq 1$.

We implement the estimation scheme outlined above as a hybrid algorithm, denoted $\mathcal{H}$, whose jump map computes $\hat{z}_1$ and $\hat{z}_2$ as in the first line of (12). The hybrid system $\mathcal{H}$ has state $\xi := (x, \hat{z}_1, \hat{z}_2, \Phi_1, \Phi_2, q) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^{n+p} \times \mathbb{R}^{n+p} \times \mathbb{R}^{(n+p)\times(n+p)} \times \mathbb{R}^{(n+p)\times(n+p)} \times \{0, 1\}$, input $B : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^{n\times p}$, and data

$$\mathcal{H} = \begin{cases} \dot{\xi} = F(\xi, B(t)) & \xi \in C \\ \xi^+ = G(\xi) & \xi \in D \end{cases} \tag{13}$$

where

$$F(\xi, B(t)) = \begin{bmatrix} A_0 x + B(t)\theta \\ h(x, \hat{z}_1, \Gamma_1, B(t)) \\ (1-q)h(x, \hat{z}_2, \Gamma_2, B(t)) + qh(x, \hat{z}_2, \Gamma_1, B(t)) \\ \Psi(\Gamma_1, B(t))\Phi_1 \\ (1-q)\Psi(\Gamma_2, B(t))\Phi_2 + q\Psi(\Gamma_1, B(t))\Phi_2 \\ 0 \end{bmatrix}$$

$$G(\xi) = (x, R(\xi), R(\xi), I, I, 1)$$

and

$$C := \{\xi \in \mathcal{X} : |\det(\Phi_1 - \Phi_2)| \leq \varepsilon\}$$
$$D := \{\xi \in \mathcal{X} : |\det(\Phi_1 - \Phi_2)| \geq \varepsilon,\ q = 0\}.$$

The logic variable q is used to ensure that the algorithm jumps only one time, when $q = 0$, and to prevent the solution components $\hat{z}_2$ and Φ_2 from growing unbounded.

The following proposition provides sufficient conditions for the invertibility of the matrix $\Phi_1 - \Phi_2$.

Proposition 3.1: Given a hybrid system $\mathcal{H}$ with data as in (13) *where the matrices A_1 and $-A_2$ are Hurwitz, $\gamma_1, \gamma_2 > 0$, and $Q_1, Q_2 \in S^n_{++}$, suppose the regressor $t \mapsto B(t)$ is persistently exciting as in Definition 2.2. Then, there exists a time $T > 0$ such that for each maximal solution ξ to $\mathcal{H}$ from $\xi(0, 0) \in \{\xi \in \mathcal{X} : \Phi_1 = \Phi_2 = I,\ q = 0\}$, the gains K_1 and K_2 are well defined at hybrid time $\{T\} \times \{0\}$, and there exists $\varepsilon > 0$ such that $|\det(\Phi_1(T, 0) - \Phi_2(T, 0))| = \varepsilon$.*

Sketch of Proof: The proof is sketched in Appendix I. □

Remark 3.2: The persistence of excitation condition imposed in Proposition 3.1 is sufficient for the existence of a time T such that the matrix $\Phi_1(T, 0) - \Phi_2(T, 0)$ is invertible, but it is not necessary. Indeed, Section V-A shows an example where finite-time convergence of the parameter estimate is achieved using the proposed hybrid algorithm when the regressor is exciting on only a finite time interval, in the sense of Definition 2.1. A formal study of the invertibility of the matrix $\Phi_1 - \Phi_2$ under relaxed excitation conditions is left as future research.

Next, we study the stability properties induced by the proposed estimator. To this end, we define the following set:

$$\mathcal{A} := \{\xi \in \mathcal{X} : \hat{z}_1 = z\}. \tag{14}$$

Theorem 3.3: Given a hybrid system $\mathcal{H}$ with data as in (13) *where the matrices A_1 and $-A_2$ are Hurwitz, $\gamma_1, \gamma_2 > 0$, and $Q_1, Q_2 \in S^n_{++}$, suppose the regressor $t \mapsto B(t)$ is such that there exists a time $T > 0$ when $|\det(\Omega_1(T, 0) - \Omega_2(T, 0))| = \varepsilon$, where Ω_1 and Ω_2 are the state transition matrices for, respectively, the systems in* (10). *Then, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that each solution ξ to $\mathcal{H}$ from $\xi(0, 0) \in \mathcal{X}_0 := \{\xi \in \mathcal{X} : \hat{z}_1 = \hat{z}_2,\ \Phi_1 = \Phi_2 = I,\ q = 0\}$ satisfies*

$$|\xi(t, j)|_{\mathcal{A}} \leq \alpha_1^{-1} \circ \alpha_2(|\xi(0, 0)|_{\mathcal{A}}) \tag{15}$$

for all $(t, j) \in \operatorname{dom}\xi$. Moreover, $\xi(t, j) \in \mathcal{A}$ for all $(t, j) \in \operatorname{dom}\xi$ satisfying $t \geq T$ and $j \geq 1$.

Sketch of Proof: For each solution ξ to $\mathcal{H}$ from $\mathcal{X}_0$, we have by assumption that the regressor $t \mapsto B(t)$ is such that $|\det(\Omega_1(T,0) - \Omega_2(T,0))| = \varepsilon$, and, since Φ_1 and Φ_2 are initialized as the identity matrix, it follows that $\xi(T,0) \in D$. Then, at the jump, according to the jump map, we have that $\hat{z}_1(T,1) = z$ from (12). Thus, the set $\mathcal{A}$ is finite-time attractive from $\mathcal{X}_0$ for $\mathcal{H}$.

To show stability of $\mathcal{A}$, consider the Lyapunov function

$$V(\xi) = \tilde{x}_1^\top P_1 \tilde{x}_1 + \tfrac{1}{\gamma_1}|\tilde{\theta}_1|^2$$

where $P_1 \in S^n_{++}$ solves the equation $A_1^\top P_1 + P_1 A_1 = -Q_1$ with $Q_1 \in S^n_{++}$. It can be shown that $\langle \nabla V(\xi), F(\xi, B(t)) \rangle \le 0$ for all $\xi \in C$. Furthermore, for all $\xi \in D$, we have from (12) that $V(\xi(t,j+1)) - V(\xi(t,j)) = -V(\xi(t,j)) \le 0$. Hence, it follows that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (15) is satisfied. □

Theorem 3.3 states that if the unknown parameter θ is constant, then the proposed estimator converges to the exact value θ after one jump. Proposition 3.1 shows that the persistence of excitation condition in Definition 2.2 is sufficient to ensure the existence of a time T such that $|\det(\Omega_1(T,0) - \Omega_2(T,0))| = \varepsilon$.

B. Piecewise constant unknown parameter

When the unknown parameter θ is piecewise constant, it is also possible to estimate it in finite time. However, one jump is not sufficient for the estimate to converge. Hence, we consider the following adaptation law, denoted $\mathcal{H}$, with state $\xi := (x, \hat{z}_1, \hat{z}_2, \Phi_1, \Phi_2) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^{n+p} \times \mathbb{R}^{n+p} \times \mathbb{R}^{(n+p)\times(n+p)} \times \mathbb{R}^{(n+p)\times(n+p)}$, input $B : \mathbb{R}_{\ge 0} \mapsto \mathbb{R}^{n\times p}$, and data

$$\mathcal{H} = \begin{cases} \dot{\xi} = \begin{bmatrix} A_0 x + B(t)\theta \\ h(x, \hat{z}_1, \Gamma_1, B(t)) \\ h(x, \hat{z}_2, \Gamma_2, B(t)) \\ \Psi(\Gamma_1, B(t))\Phi_1 \\ \Psi(\Gamma_2, B(t))\Phi_2 \end{bmatrix} =: F(\xi, B(t)) & \xi \in C \\ \xi^+ = (x, R(\xi), R(\xi), I, I) =: G(\xi) & \xi \in D \end{cases} \quad (16)$$

where

$$C := \{\xi \in \mathcal{X} : |\det(\Phi_1 - \Phi_2)| \le \varepsilon\}$$
$$D := \{\xi \in \mathcal{X} : |\det(\Phi_1 - \Phi_2)| \ge \varepsilon\}.$$

The new structure of the set D makes it possible to have multiple jumps, which are necessary to estimate the piecewise constant parameter.

The following theorem states the stability properties induced by $\mathcal{H}$ in (16) for a piecewise constant unknown parameter.

Theorem 3.4: Given a hybrid system $\mathcal{H}$ with data as in (16) *where the matrices A_1 and $-A_2$ are Hurwitz, $\gamma_1, \gamma_2 > 0$, and $Q_1, Q_2 \in S^n_{++}$, suppose that the following conditions hold:*

1. *The regressor $t \mapsto B(t)$ is such that there exists a sequence of time instants $\{\tau_k\}_{k\in\mathbb{N}_{>0}}$ such that, for all $k \in \mathbb{N}_{>0}$, $0 \le \tau_k < \tau_{k+1}$ and $|\det(\Omega_1(\tau_{k+1}, \tau_k) - \Omega_2(\tau_{k+1}, \tau_k))| = \varepsilon$, where Ω_1 and Ω_2 are the state transition matrices for, respectively, the systems in* (10)*.*
2. *The unknown parameter θ is piecewise constant, where the time instants at which the value of θ changes are defined by a sequence of times $\{\delta_i\}_{i\in\mathbb{N}_{>0}}$ satisfying $0 \le \delta_i < \delta_{i+1}$ for all $i \in \mathbb{N}_{>0}$.*

Then, for each solution ξ to $\mathcal{H}$ from $\xi(0,0) \in \{\xi \in \mathcal{X} : \Phi_1 = \Phi_2 = I\}$ and each $i, k \in \mathbb{N}_{>0}$ such that

I. *$[\tau_k, \tau_{k+2}] \times \{k, k+1\} \subset \operatorname{dom}\xi$;*
II. *$\delta_i \in [\tau_k, \tau_{k+1}]$;*
III. *$(\tau_{k+1}, \tau_{k+2}) \cap \{\delta_i\}_{i\in\mathbb{N}_{>0}} = \varnothing$,*

the following property holds: $\xi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom}\xi$ satisfying $t \in [\tau_{k+2}, \delta_{i+1})$ and $j \ge k+2$.

Sketch of Proof: From items II and III of Theorem 3.4, we have that the value of the unknown parameter vector θ changes in the time interval $[\tau_k, \tau_{k+1}]$ and is constant in the interval (τ_{k+1}, τ_{k+2}). By assumption, we have that $\xi(\tau_{k+1}, k) \in D$ and at the jump, according to the jump map, $\xi(\tau_{k+1}, k+1) \in \{\xi \in \mathcal{X} : \hat{z}_1 = \hat{z}_2,\ \Phi_1 = \Phi_2 = I\}$. Finally, $\xi(\tau_{k+2}, k+1) \in D$ by assumption, and the result follows from Theorem 3.3. □

Theorem 3.4 shows that, each time the value of the parameter θ changes, the proposed estimator converges to the exact value θ after no more than two jumps. Let us also mention that as in Theorem 3.3, the bound in (15) can be established.

IV. Robustness to measurement noise

In this section, we analyze the robustness of the proposed hybrid parameter estimator with respect to bounded time-varying measurement noise. For the sake of readability, we focus on constant unknown parameters. However, the robustness results can be generalized using the same approach to deal with piecewise constant unknown parameters.

Starting from the noise-free hybrid estimator $\mathcal{H}$ in (13), we denote our proposed estimator under the effect of the state measurement noise $t \mapsto \nu(t) \in \mathbb{R}^n$ as a hybrid system, $\mathcal{H}_\nu$, with state $\xi_\nu := (x, \hat{z}_1, \hat{z}_2, \Phi_1, \Phi_2, q) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^{n+p} \times \mathbb{R}^{n+p} \times \mathbb{R}^{(n+p)\times(n+p)} \times \mathbb{R}^{(n+p)\times(n+p)} \times \{0,1\}$, input $B : \mathbb{R}_{\ge 0} \mapsto \mathbb{R}^{n\times p}$, and data

$$\mathcal{H}_\nu = \begin{cases} \dot{\xi}_\nu = F_\nu(\xi_\nu, B(t)) & \xi_\nu \in C_\nu \\ \xi_\nu^+ = G_\nu(\xi_\nu) & \xi_\nu \in D_\nu \end{cases} \quad (17)$$

where $C_\nu := C$ and $D_\nu := D$ given below (13), and

$$F_\nu(\xi_\nu, B(t)) := \begin{bmatrix} A_0 x + B(t)\theta \\ h(x+\nu(t), \hat{z}_1, \Gamma_1, B(t)) \\ (1-q)h(x+\nu(t), \hat{z}_2, \Gamma_2, B(t)) \\ \quad + q h(x+\nu(t), \hat{z}_2, \Gamma_1, B(t)) \\ \Psi(\Gamma_1, B(t))\Phi_1 \\ (1-q)\Psi(\Gamma_2, B(t))\Phi_2 + q\Psi(\Gamma_1, B(t))\Phi_2 \\ 0 \end{bmatrix}$$

$$G_\nu(\xi_\nu) := (x, R(\xi_\nu), R(\xi_\nu), I, I, 1).$$

Next, we show that if the regressor is persistently exciting, the noisy system $\mathcal{H}_\nu$ is input-to-state stable (ISS) for any essentially bounded measurement noise $t \mapsto \nu(t)$. The question of establishing ISS properties under relaxed excitation conditions is left as future research.

Proposition 4.1: Given a hybrid system $\mathcal{H}_\nu$ with data as in (17) *where the matrices A_1 and $-A_2$ are Hurwitz, $\gamma_1, \gamma_2 > 0$, and $Q_1, Q_2 \in S^n_{++}$, suppose the regressor $t \mapsto B(t)$ is persistently exciting as in Definition 2.2. Then, for any solution ξ_ν to $\mathcal{H}_\nu$ from $\xi_\nu(0,0) \in \mathcal{X}_0 := \{\xi_\nu \in \mathcal{X} : \hat{z}_1 = \hat{z}_2,\ \Phi_1 = \Phi_2 = I,\ q = 0\}$, there exist constants $w_M, T, \Phi_M > 0$, $\sigma_1, \sigma_2, \sigma_3 > 0$ with $\sigma_1 \leq \sigma_2$, and $\upsilon > 0$ such that, for all $(t,j) \in \operatorname{dom} \xi_\nu$,*

$$\begin{aligned} |\xi_\nu(t,j)|_{\mathcal{A}} \leq \rho(j)(\beta(|\xi_\nu(0,0)|_{\mathcal{A}}, t) + \alpha_1(|\nu|_\infty)) \\ + (1-\rho(j))(\alpha_2(|\nu|_\infty) + \alpha_1(|\nu|_\infty)) \end{aligned} \tag{18}$$

where $\rho(0) = 1$ and $\rho(j) = 0$ for $j \in \mathbb{N}_{>0}$,

$$\beta(s,t) = \sqrt{\tfrac{\sigma_2}{\sigma_1}} \exp(-\tfrac{\sigma_3}{2\sigma_2} t) s, \qquad \alpha_1(s) = \sqrt{\tfrac{\sigma_2}{\sigma_1}} \tfrac{2\sigma_2}{\sigma_3} w_M s,$$
$$\alpha_2(s) = \left((1-\kappa\upsilon)^{-1}\left(\sqrt{\tfrac{\sigma_2}{\sigma_1}}\tfrac{2\sigma_2}{\sigma_3} + \Phi_M\right) + \Phi_M\right)\sqrt{\tfrac{\sigma_2}{\sigma_1}} w_M s$$

with $\kappa = \beta(1, T)$.

V. Examples

In this section, we present simulation results that demonstrate the practicality of the proposed algorithms. Simulations are performed using the Hybrid Equations Toolbox [11].

A. Constant parameter

Consider a scalar system with dynamics as in (1), where $A_0 = -0.5$, $\theta = -2$, and

$$B(t) = \begin{cases} 0 & \text{if} \quad t < 1, \\ 1 & \text{if} \quad t \in [1,2], \\ 0 & \text{if} \quad t > 2, \end{cases}$$

with measurement noise $\nu(t) = 4\sin(30t)$ added to the state. Note that B does not satisfy the persistence of excitation condition imposed in Lemma 3.1. Instead, B is exciting over only a finite time interval, in the sense of Definition 2.1, with $\eta_1 = 1$ and $\eta_2 = 1$.

The proposed finite-time parameter estimation algorithm $\mathcal{H}$ in (13) is applied to estimate θ, with parameters $A_1 = -A_2 = -1$, $\gamma_1 = \gamma_2 = 1$, $Q_1 = Q_2 = 1$, and $\varepsilon = 1$, which verifies the conditions of Theorem 3.3. The system is simulated from initial conditions $x(0,0) = 1$, $\hat{z}_1(0,0) = \hat{z}_2(0,0) = (0,0)$, $\Phi_1(0,0) = \Phi_2(0,0) = I$, $q(0,0) = 0$, producing the results in Figure 1 (only the trajectory of $\tilde{z}_1$ is shown).[1] Due to the fact that B is zero over the interval $[0,1)$, the dynamics of $\tilde{\theta}_1$ are zero until $t = 1$, followed by finite-time convergence of the state and parameter estimates in accordance with Theorem 3.3.

[1] Code at https://github.com/HybridSystemsLab/MRAC_FT_Estimator

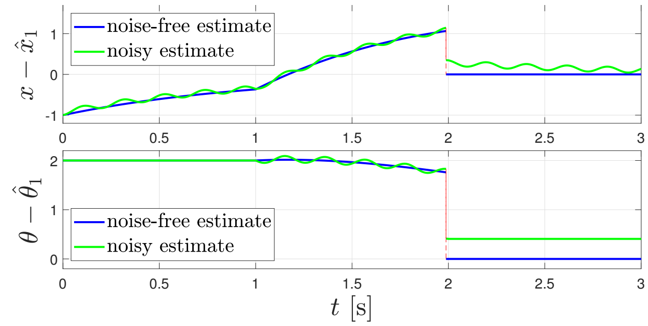

Fig. 1: The projection onto t of the state estimation error $x - \hat{x}_1$ and parameter estimation error $\theta - \hat{\theta}_1$.

B. Piecewise constant parameter

Consider the following frequency estimation problem: given a signal $t \mapsto \zeta(t) = \Upsilon \sin(\omega t)$ where $\Upsilon \in \mathbb{R}_{>0}$ is the magnitude and $\omega \in \mathbb{R}_{>0}$ is the frequency, estimate ω from measurements of ζ. To formulate this problem as the problem of estimating a parameter θ of a model as in (1), let $x = (x_1, x_2)$ be such that $x_1 = \zeta$ and $\dot{x}_1 = x_2$, and Υ is related to the initial condition. Note that a similar problem was studied in [12]. Then, we obtain the following parametric form:

$$\dot{x} = A_0 x + B(x)\theta \tag{19}$$

where $A_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $B(x) = \left[\begin{smallmatrix} 0 \\ -x_1 \end{smallmatrix}\right]$, and $\theta = \omega^2 - 1$. Then, with $\eta_1 = \frac{\pi}{\omega}$, it can be shown that for all $t \geq 0$, $\int_t^{t+\eta_1} B^\top(x(s))B(x(s))ds = \frac{\pi\Upsilon^2}{2\omega}$. Hence, B is persistently exciting in the sense of Definition 2.2 with $\eta_2 = \frac{\pi\Upsilon^2}{2\omega}$. The unknown parameter θ is piecewise constant, with a value of $\theta = 3$ if $t \in [0,2)$ and $\theta = 1$ if $t \geq 2$, and measurement noise $\nu(t) = \begin{bmatrix}\sin(50t) & \cos(50t)\end{bmatrix}^\top$ is added to the state.

The proposed estimation algorithm $\mathcal{H}$ in (16) is applied to estimate θ in (19), with parameters $A_1 = -A_2 = -I$, $\gamma_1 = \gamma_2 = 10$, $Q_1 = Q_2 = I$, and $\varepsilon = 0.5$, which verifies the conditions of Theorem 3.4. The system is simulated from initial conditions $x(0,0) = (1,0)$, $\hat{z}_1(0,0) = (0.5,0,0)$, $\hat{z}_2(0,0) = (0,0,-0.2)$, $\Phi_1(0,0) = \Phi_2(0,0) = I$, producing the results in Figure 2.[1] The estimation errors $\tilde{x}_1$ and $\tilde{\theta}_1$ converge to zero first at $t = 1.6$ and, after the value of θ changes at $t = 2$, converge again at $t = 3.23$ in accordance with Theorem 3.4. The figure also illustrates the robustness of the proposed estimator to time-varying measurement noise.

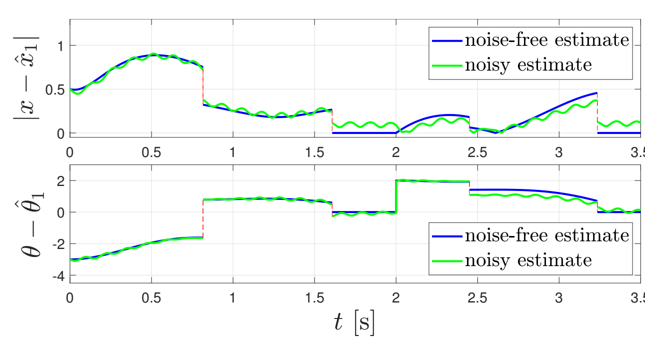

Fig. 2: The projection onto t of the state prediction error $|x - \hat{x}_1|$ and the parameter estimation error $\theta - \hat{\theta}_1$.

Appendix I
Sketch of proof for Proposition 3.1

To prove Proposition 3.1, we first provide the following auxiliary result:

Lemma A.1: Consider the system given by

$$\dot{\tilde{x}} = A\tilde{x} + B(t)\tilde{\theta}, \qquad \dot{\tilde{\theta}} = -\gamma B^\top(t)P\tilde{x}. \tag{20}$$

Suppose the regressor $t \mapsto B(t)$ is persistently exciting as in Definition 2.2. Denote $\tilde{z} = (\tilde{x}, \tilde{\theta})$ and let $\Omega(t, t_0)$ be the state transition matrix of the system (20) *describing the continuous evolution of the state between t_0 and t with $t \geq t_0 \geq 0$. Then, we have the following:*

1. *If the matrix A is Hurwitz and $\gamma > 0$, then there exist constants $\sigma_1, \sigma_2, \sigma_3 > 0$ with $\sigma_2 \geq \sigma_1$ such that, for each solution $t \mapsto \tilde{z}(t)$ to* (20)*,*
$$|\Omega(t,0)| \leq \sqrt{\frac{\sigma_2}{\sigma_1}} \exp\left(-\frac{\sigma_3}{2\sigma_2}t\right) \quad \forall\, t \geq 0. \tag{21}$$
2. *If the matrix $-A$ is Hurwitz and $\gamma > 0$, then there exists $\upsilon > 0$ such that, for each solution $t \mapsto \tilde{z}(t)$ to* (20)*,*
$$|\Omega^{-1}(t,0)| \leq \upsilon \quad \forall t \geq 0. \tag{22}$$

Sketch of Proof: To show item 1 of Lemma A.1, let the matrix A be Hurwitz and $\gamma > 0$. Then, inspired by [13], we consider the function

$$V(\tilde{z}, t) = cV_1(\tilde{z}) + W_1(\tilde{z}, t) + \frac{1}{4}W_2(\tilde{z}, t) \tag{23}$$

where $c > 0$ is a design parameter, $V_1(\tilde{z}) := \tilde{x}^\top P\tilde{x} + \frac{1}{\gamma}|\tilde{\theta}|^2$, $W_1(\tilde{z}, t) := -\tilde{x}^\top B(t)\tilde{\theta}$, and $W_2(\tilde{z}, t) := -\tilde{\theta}^\top M(t)\tilde{\theta}$, where $P \in S^n_{++}$ solves the equation $A^\top P + PA = -Q$ with $Q \in S^n_{++}$ and $M(t) := \int_t^\infty \mathrm{e}^{t-s}B^\top(s)B(s)ds$

By analyzing the decrease in V and using the fact that $t \mapsto B(t)$ is persistently exciting as in Definition 2.2, it can be shown that, for each solution $t \mapsto \tilde{z}(t)$ to (20), there exist constants $\sigma_1, \sigma_2, \sigma_3 > 0$ with $\sigma_2 \geq \sigma_1$ such that

$$|\tilde{z}(t)| \leq \sqrt{\frac{\sigma_2}{\sigma_1}} \exp\left(-\frac{\sigma_3}{2\sigma_1}t\right)|\tilde{z}(0)| \quad \forall\, t \geq 0.$$

Hence, (21) holds.

To show item 2 of Lemma A.1, let the matrix $-A$ be Hurwitz and $\gamma > 0$, and recall V_1 given below (23). Since $-A$ is Hurwitz, there exists $P \in S^n_{++}$ that solves the equation $-A^\top P - PA = -Q$ with $Q \in S^n_{++}$. Then, it can be shown that, for each solution $t \mapsto \tilde{z}(t)$ to (20), there exist constants $\upsilon_1, \upsilon_2 > 0$ such that

$$|\tilde{z}(t)| \geq \sqrt{\upsilon_1/\upsilon_2}|\tilde{z}(0)| \quad \forall\, t \geq 0.$$

Hence, (22) holds with $\upsilon := \sqrt{\upsilon_2/\upsilon_1}$. □

Next, we recall the following lemma from [14].

Lemma A.2: Given a matrix A, if $|A| < 1$ then $(I - A)$ is invertible and $|(I - A)^{-1}| \leq (1 - |A|)^{-1}$.

We now have all the ingredients to sketch a proof of Proposition 3.1.

Sketch of proof for Proposition 3.1: First, from Lemma A.1, the fact that B satisfies Definition 2.2 implies that (21) and (22) are satisfied for all $t \geq 0$. Since the state variables Φ_1 and Φ_2 are initialized as the identity matrix, at any time $t \in I^0$, where $I^j := \{t : (t,j) \in \operatorname{dom}\xi\}$, they are equivalent to the state-transition matrices Ω_1 and Ω_2 for, respectively, the systems in (10). Then, from the invertibility of the state transition matrix Ω_2, for all $t \in I^0$, we have

$$|\Phi_1(t,0)\Phi_2^{-1}(t,0)| \leq \upsilon\sqrt{\frac{\sigma_2}{\sigma_1}} \exp\left(-\frac{\sigma_3}{2\sigma_2}t\right). \tag{24}$$

Next, let $T > 0$ be such that $T > \frac{2\sigma_2}{\sigma_3}\ln\left(\upsilon\sqrt{\frac{\sigma_2}{\sigma_1}}\right)$. Therefore, there exists $s > 0$ such that $T = (1 + s)\frac{2\sigma_2}{\sigma_3}\ln\left(\sqrt{\frac{\sigma_2\upsilon_2}{\sigma_1\upsilon_1}}\right)$. Substituting T into t in (24) yields

$$|\Phi_1(T,0)\Phi_2^{-1}(T,0)| \leq \exp\left(-s\frac{2\sigma_2}{\sigma_3}\ln\left(\upsilon\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\right) < 1.$$

Finally, we rewrite K_1 using the Woodbury matrix identity as $K_1(\xi(T,0)) = \left(I - \Phi_1(T,0)\Phi_2^{-1}(T,0)\right)^{-1}$ and, in view of Lemma A.2, the matrix $(I - \Phi_1(T,0)\Phi_2^{-1}(T,0))$ is invertible. Hence, there exists $\varepsilon > 0$ such that $|\det(\Phi_1(T,0) - \Phi_2(T,0))| = \varepsilon$. □

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