Observer Design for Hybrid Dynamical Systems with Approximately Known Jump Times

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Abstract

This paper proposes a general framework for state estimation of plants modeled as hybrid dynamical systems with jumps occurring at (approximately) known times. A candidate observer is a hybrid dynamical system with jumps triggered when the plant jumps. With some information about the time elapsed between successive jumps, a Lyapunov-based analysis allows us to derive sufficient conditions for observer design. In particular, a high-gain flow-based observer, with innovation during flow only, can be designed for plants with an average dwell-time when the flow dynamics are strongly differentially observable. On the other hand, when the jumps are persistent, a jump-based observer, with innovation at jumps only, should be designed based on an equivalent discrete-time system corresponding to the hybrid system discretized at jump times. In the linear context, this reasoning leads us to a hybrid Kalman filter. These designs apply to a large class of hybrid systems, including cases where the time between successive jumps is unbounded or tends to zero – namely, Zeno behavior–, and cases where detectability only holds during flows, at jumps, or neither. We also study the robustness of this approach when the jumps of the observer are delayed with respect to those of the plant. Under some regularity and dwell-time conditions, we show that the estimation error is semiglobally practically asymptotically stable over time intervals after such delays. The results are illustrated in examples and applications, including mechanical systems with impacts, spiking neurons, and switched systems.

1 Introduction

1.1 Context

In many applications, estimating the state of a system is crucial, whether it be for control, supervision, or fault diagnosis purposes. Unfortunately, the problem of designing observers for hybrid systems of the form (Goebel et al. (2012))

$$\dot{x} \in f(x) \quad x \in C\,, \qquad\qquad x^+ \in g(x) \quad x \in D \qquad (1)$$

presenting both a continuous-time behavior in C and a discrete-time behavior in D, is still largely unsolved, even when the flow/jump maps f and g are linear. A major difficulty lies in the fact that the plant's jump times, that is, the times at which discrete events occur in the plant's solution, generally depend on its initial condition, which is unknown in the context of observer design. From there, one may distinguish two scenarios: when the jump times of the plant are detected by sensors (or known a priori), and when these jump times are truly unknown.

In the second scenario, the jumps of the observer cannot be triggered when the jumps of the plant occur. It follows that the domain of definition of the solutions to the plant and observer are different and a standard error system approach for observer design does not apply. This mismatch of time domains makes the formulation of observability/detectability and, in turn, observer design very challenging (Bernard and Sanfelice (2020a)). Very few observer results for plants of the form (1) exist apart from particular settings as in Forni et al. (2013), which requires the composition $g \circ g$ to be the identity map, and in Kim et al. (2019), thanks to a change of coordinates transforming the jump map g into the identity map, in this way, removing the jumps. Note that in the particular context of switched systems, this mismatch issue translates into the problem of estimating the switching signal. The observability properties of such a signal have been studied in Vidal et al. (2003); Küsters and Trenn (2017). Observer designs based on the so-called *mode location observers*, capable of detecting and identi-

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fying properties of the switching signal appeared in Balluchi et al. (2013); Lee et al. (2013); Battistelli (2013); Gómez-Gutiérrez et al. (2015); Ping et al. (2017); Zammali et al. (2019), to list a few, which include the broad literature of fault tolerant control.

On the other hand, in the first scenario where the jump times of the plant are known/detected, the observer jumps can be triggered at the same time as those of plant (up to small detection delays). The difficulties due to a possible mismatch of time domains thus disappear, and observability analysis also reduces to comparing solutions with same output on the same time domain.

A first class of systems falling into this first scenario is the so-called *impulsive systems*. It consists of continuous-time dynamical systems (possibly switching among different flow dynamics) with state jumps (or switch) that occur at pre-specified times, which are usually assumed to be separated by nonzero periods of flow – in particular, to avoid Zeno behavior. In that setting, observability/determinability have been extensively studied in Guan et al. (2002); Xie and Wang (2004); Medina and Lawrence (2008); Zhao and Sun (2009); Tanwani et al. (2015). As for observer design, results first appeared assuming observability of each flow dynamics Alessandri and Coletta (2001), and then more generally in Medina and Lawrence (2009) (resp. in Tanwani et al. (2015)), for impulsive systems (resp. switched impulsive systems) that are observable (resp. determinable) for any impulse time sequence containing more than a known finite number N of jumps.

Another important class of hybrid systems falling into the second scenario is when the system itself has continuous-time dynamics, but the measurements are sampled and available intermittently at specific time instances. For such a class of systems with sporadic events, observers have been designed under specific assumptions on the time elapsed between successive events or, in the case of periodic events, the sampling period. From Sur and Paden (1997); Deza et al. (1992), convergence of an impulsive observer with innovation terms triggered by the measurement events can be guaranteed when the sampling period is sufficiently small. Designs were then developed in Raff and Allgöwer (2007); Dinh et al. (2015) for any constant sampling period provided that appropriate matrix inequalities are satisfied, and further extended in Raff et al. (2008); Ahmed-Ali et al. (2014); Ferrante et al. (2016); Etienne et al. (2017); Sferlazza et al. (2019) to the case of sporadic measurements, i.e., when the time elapsed between sampling events varies in a known interval.

In this paper, we propose to address the problem of state estimation for general hybrid systems (1), in the context of the first scenario, namely when the jump times of the plant are (approximately) known, and in an attempt to unify most of the previously cited approaches. Preliminary results in this direction were given in Bernard and Sanfelice (2018); Ríos et al. (2020), in the particular case where f and g are linear, and when at least either the flow dynamics or the jump dynamics are detectable. We extend those results here to nonlinear dynamics, also when neither the continuous nor the discrete dynamics of the plant are detectable, but, the hybrid plant as a whole is.

1.2 Content and Contributions

First, under the assumption that the jumps of the plant are instantaneously detected, a candidate observer is defined as a hybrid system that jumps at the same time as the plant does, and is fed with the measured output in either the flow map, the jump map, or both (Section 2). Assuming the plant has an average dwell-time or a reverse average dwell-time, or simply that the time between its successive jumps belongs to a known (possibly unbounded) closed set, we derive Lyapunov-based sufficient conditions so as to ensure uniform pre-asymptotic stability of the zero estimation error set (Section 3). Then, we provide additional design conditions for special cases of the general observer problem and proposed hybrid observer. In Section 4, we consider the case when measurements are only used during flow, for which we propose a hybrid observer, which we call *flow-based hybrid observer*. Similarly, but for the situation when output measurements are used only at jumps, Section 5 introduces a *jump-based hybrid observer* and associated design conditions. Motivated by the fact that, in practice, the jumps of the plant cannot always be *instantaneously* detected, we study the robustness of the observer when the jumps of the observer are slightly delayed relative to those of the plant (Section 6). Finally, we demonstrate how those results can be used in several examples and applications, including mechanical systems with impacts, spiking neurons, and switched systems (Section 7). Our main contributions compared to the literature are as follows:

1. General hybrid systems (1) are considered in a unified framework, only assuming knowledge about the time between successive jumps, which allows any type of solutions, from Zeno and eventually discrete, to eventually continuous trajectories;
2. When the plant has an average dwell-time and its continuous dynamics are strongly differentially observable, we prove that a hybrid observer can be obtained by copying the discrete dynamics of the plant and designing a high-gain observer for its continuous dynamics, as long as the gain is taken sufficiently large compared to the average dwell-time and the Lipschitz constants of the flow and jump maps;
3. When the output measurements are only injected in the observer at jumps, we highlight that the inno-

vation term in the observer, which only plays a role at jumps, should be designed based on an equivalent discrete-time system that models the hybrid plant sampled at jumps. In the linear context, this reasoning leads us to a constructive hybrid Kalman filter;
4. A robustness analysis with respect to delays in the triggering of the jumps of the observer jumps is provided: under some regularity and dwell-time conditions, we show that the estimation error remains bounded and semi-global practical stability holds outside the delay intervals between the jumps of the plant and of the observer;
5. The generality of the framework enables us to recover and unify a significant part of the literature. In particular, the results apply well to switched systems with state-triggered switches: we show in the report version of this paper Bernard and Sanfelice (2020b) how a high-gain observer can be designed for switched systems with observable modes and average dwell-time, or how the output at the switching instants can be used when each mode is not observable on its own but the combination of them is.

1.3 Notation and Preliminaries

The set $\mathbb{R}$ (resp. $\mathbb{N}$) denotes the set of real numbers (resp. integers), $\mathbb{R}_{\geq 0} = [0,+\infty)$, $\mathbb{R}_{>0} = (0,+\infty)$, and $\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\}$. For $P \in \mathbb{R}^{n\times n}$, $\underline{\lambda}(P)$ (resp. $\overline{\lambda}(P)$) stands for its smallest (resp. largest) eigenvalue. The symbol $\star$ in a matrix denotes the symmetric blocks. A map $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ map if $\alpha(0) = 0$ and α is continuous and increasing, and a class-$\mathcal{K}^\infty$ map if it is also unbounded. A map $\beta : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ map if for all $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t)$ is class-$\mathcal{K}$ and for all $r \in \mathbb{R}_{\geq 0}$, $\beta(r, \cdot)$ is nonincreasing with $\lim_{t\to\infty} \beta(r, t) = 0$. For a set valued map $S : \mathbb{R}^{d_x} \rightrightarrows \mathbb{R}$ and a scalar c, writing $S(x) \leq c$ for some $x \in \mathbb{R}^{d_x}$ means that $s \leq c$ for all $s \in S(x)$. For a C^1 map $V : \mathbb{R}^{d_x} \to \mathbb{R}$, $L_S V(x)$ denotes the set of Lie derivatives along vector fields $s \in S(x)$, i.e. $\left\{\frac{dV}{dx}(x)s\, , s \in S(x)\right\}$. We consider hybrid dynamical systems of the form (1) where f (resp. g) is the flow (resp. jump) map, and C (resp. D) is the flow (resp. jump) set. Solutions to such systems are defined on *hybrid time domains*. A subset E of $\mathbb{R}_{\geq 0}\times\mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$, and it is a hybrid time domain if for any $(T, J) \in E$, $E\cap[0, T]\times\{0, 1, \ldots, J\}$ is a compact hybrid time domain. For a solution $(t, j) \mapsto x(t, j)$ (see (Goebel et al., 2012, Definition 2.6)), we denote $\operatorname{dom} x$ its domain, $\operatorname{dom}_t x$ (resp.$\operatorname{dom}_j x$) its projection on the time (resp. jump) component, and for $j \in \mathbb{N}$, $t_j(x)$ the only time defined by $(t_j, j) \in \operatorname{dom} x$ and $(t_j, j-1) \in \operatorname{dom} x$. Also, $N(t, s)$ denotes the number of jumps occurring between times t and s. We say that x is *complete* if $\operatorname{dom} x$ is unbounded and *Zeno* if it is complete and $\sup \operatorname{dom}_t x < +\infty$.

2 Synchronized Hybrid Observer

2.1 Mathematical Modeling

We consider a hybrid plant of the form

$$\mathcal{H}\left\{\begin{array}{ll} \dot{x} \in f(x)\ ,\ y_c = h_c(x)\ , & x \in C \\ x^+ \in g(x)\ ,\ y_d = h_d(x)\ , & x \in D \end{array}\right. \tag{2}$$

with state $x \in \mathbb{R}^{d_x}$, output $y = (y_c, y_d) \in \mathbb{R}^{d_{y_c}} \times \mathbb{R}^{d_{y_d}}$, with y_c available during flow and y_d during jumps. We are interested in estimating the state of (or part of the state of) $\mathcal{H}$ when its solutions are initialized in a given subset $\mathcal{X}_0 \subseteq C \cup D$. We denote by $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ the set of maximal solutions of $\mathcal{H}$ with initial condition in $\mathcal{X}_0$.

Definition 2.1 *For a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$ and a positive scalar $\tau^\star$, we will say that*
- *solutions have* flow length within $\mathcal{I}$ *if, for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$,*
  - $0 \leq t - t_j(x) \leq \sup \mathcal{I} \quad \forall(t, j) \in \operatorname{dom} x$
  - $t_{j+1}(x) - t_j(x) \in \mathcal{I}$ *holds*
    $\forall j \in \mathbb{N}_{>0}$ *if* $\sup \operatorname{dom}_j x = +\infty$,
    $\forall j \in \{1, \ldots, \sup \operatorname{dom}_j x - 1\}$ *otherwise.*

  For simplicity, we say that $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds;
- *solutions have an* average dwell-time (ADT) $\tau^\star$ *if there exists $N_0 \in \mathbb{N}_{>0}$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$,*

$$N(t, s) \leq \frac{(t-s)}{\tau^\star} + N_0 \qquad \forall t \geq s \geq 0\ .$$

  For simplicity, we say that $\mathcal{C}^{\mathrm{av}}_{\mathcal{X}_0}[\tau^\star]$ holds;
- *solutions have a* reverse average dwell-time (rADT) $\tau^\star$ *if there exists $N_0 \in \mathbb{N}_{>0}$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$,*

$$N(t, s) \geq \frac{(t-s)}{\tau^\star} - N_0 \qquad \forall t \geq s \geq 0\ .$$

  For simplicity, we say that $\mathcal{C}^{\mathrm{rav}}_{\mathcal{X}_0}[\tau^\star]$ holds.

In the definition of $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$, the set $\mathcal{I}$ describes the possible lengths of the flow intervals between successive jumps. Properties $\mathcal{C}^{\mathrm{av}}_{\mathcal{X}_0}[\tau^\star]$ and $\mathcal{C}^{\mathrm{rav}}_{\mathcal{X}_0}[\tau^\star]$ correspond to the standard notions of *average dwell-time* and *reverse average dwell-time* respectively (Goebel et al. (2012); Hespanha et al. (2008)). They enforce that the solutions jump, on average, at most (resp. at least) once per time interval of length $\tau^\star$. A particular case of $\mathcal{C}^{\mathrm{av}}_{\mathcal{X}_0}[\tau^\star]$ is when all the intervals of flow last at least $\tau^\star$, namely they have a *dwell-time*, which can also be modeled by $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ with $\mathcal{I} = [\tau^\star, +\infty)$.

We are now ready to state the observer problem of interest. Our goal is to design an observer assuming we know: 1) when the jumps of the plant occur, 2) the outputs y_c during flows and/or y_d at jumps, 3) some information about the flow time between successive jumps of the type $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$, $\mathcal{C}^{\mathrm{av}}_{\mathcal{X}_0}[\tau^\star]$, or $\mathcal{C}^{\mathrm{rav}}_{\mathcal{X}_0}[\tau^\star]$.

Example 2.2 *A Lagrangian mechanical system with impacts is typically modeled as* $\mathcal{H}$ *with state* $x = (\theta, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ *capturing its (generalized) position and velocity, flow of the form* $f(x) = (\omega, \alpha(x))$*, jump map* g *translating the velocity discontinuity at the impact,* D *characterizing the impact condition and* $C = \mathrm{cl}(\mathbb{R}^d \setminus D)$*. A complete model is given in Section 7.1. If there is loss of energy at impacts, we typically know that for a bounded set of initial conditions* $\mathcal{X}_0$*, the time between impacts is bounded, so that* $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ *holds with* $\mathcal{I}$ *of the form* $\mathcal{I} = [0, \tau_{\max}]$*, with* $\tau_{\max} > 0$*. This case does not exclude Zeno behavior close to the jump set* D*. On the other hand, we may know that solutions have a dwell-time, for instance if at least* $\tau_{\min} > 0$ *amount of time is needed to flow from* $g(D)$ *to* D*. Then,* $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ *holds with* $\mathcal{I}$ *of the form* $\mathcal{I} = [\tau_{\min}, +\infty)$ *or* $\mathcal{I} = [\tau_{\min}, \tau_{\max}]$*.*

Example 2.3 *The important class of* switched systems *also falls in the framework of this paper with*

$$x = \begin{pmatrix} x_p \\ q \end{pmatrix} \ , \ f(x) = \begin{pmatrix} f_q(x_p) \\ 0 \end{pmatrix} \ , \ g(x) = \begin{pmatrix} g_q(x_p) \\ Q \end{pmatrix}$$
$$C = \bigcup_{q \in Q} C_q \times \{q\} \quad , \quad D = \bigcup_{q \in Q} D_q \times \{q\} \tag{3}$$

where $Q = \{1, 2, \ldots, q_{max}\}$ *and the discrete signal* q *indicates the* mode *in which the system evolves. When* x_p *is in* D_q *and a jump occurs, the mode either stays the same or is "switched" to a new value in* Q*. The plant then evolves according to the flow map* f_q *and jump map* g_q*, until* q *is switched to another value. The switches are triggered by the state being in a certain region* D_q*: it is a* state-dependent *switching. The switches can also be triggered by an external signal called* switching signal*, in which case the switches are said* time-dependent*. This case could also be modeled by* (2) *by making some assumptions about the time between successive switches, which can take the form of* $\mathcal{C}^{\mathrm{av}}_{\mathcal{X}_0}[\tau^\star]$*,* $\mathcal{C}^{\mathrm{rav}}_{\mathcal{X}_0}[\tau^\star]$*, or* $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$*. See Liberzon (2003); Goebel et al. (2012) for more detail. In this paper, we assume the switching times are known or detected. The output map is defined depending on the available information: known/unknown mode* q*, measurements of* x_p*, etc. See (Bernard and Sanfelice, 2020b, Section 7.3) for a detailed analysis on the way the results of this paper apply to switched systems.*

Example 2.4 *The proposed framework applies also to continuous-time systems*

$$\dot{x}_p = f_p(x_p) \quad , \quad y = h_p(x_p)$$

whose output y *is only available at discrete times* t_j *(hence* $h_c(x) = \emptyset$ *in* (4)*), which do not necessarily occur periodically. Assuming we know bounds on the time elapsed between two successive sampling events, or more generally that it belongs to a closed bounded set* $\mathcal{I}$*, namely* $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ *holds, such a system can be modeled by* $\mathcal{H}$ *with state* $x = (x_p, \tau)$*,*

$$\begin{aligned} f(x) &= (f_p(x_p), 1) \quad , \quad g(x) = (x_p, 0) \\ C &= \mathbb{R}^{d_{x_p}} \times [0, \max \mathcal{I}] \ , \quad D = \mathbb{R}^{d_{x_p}} \times \mathcal{I} \\ h_c(x) &= \emptyset \quad , \quad h_d(x) = (h_p(x_p), \tau) \end{aligned} \tag{4}$$

where τ *models the (known) time elapsed since the previous jump. For instance,* $\mathcal{I}$ *is a singleton in the case of periodic sampling Raff and Allgöwer (2007); Dinh et al. (2015), and* $\mathcal{I}$ *is a compact interval of* $\mathbb{R}_{>0}$ *in the case of aperiodic sampling with known bounds as considered for linear systems in Ferrante et al. (2016); Sferlazza et al. (2019) and classes of nonlinear Lipschitz systems in Raff et al. (2008); Ahmed-Ali et al. (2014); Farza et al. (2014); Mazenc et al. (2015); Etienne et al. (2017). Similarly, we could say that* $\mathcal{C}^{\mathrm{rav}}_{\mathcal{X}_0}[\tau^\star]$ *holds if we know that measurements occur at most every* $\tau^\star$ *units of time and adapt the model* (4) *accordingly.*

2.2 Problem Statement and Proposed Hybrid Observer

Since the jump times of the plant are assumed to be known, it is natural to use an observer for (2) of the form

$$\hat{\mathcal{H}} \left\{ \begin{array}{rll} \dot{z} &\in F(z, y_c) & \text{when } \mathcal{H} \text{ flows} \\ z^+ &\in G(z, y_d) & \text{when } \mathcal{H} \text{ jumps} \end{array} \right. \tag{5}$$

that is synchronized with the plant, for some maps $F : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_c}} \to \mathbb{R}^{d_z}$ and $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \to \mathbb{R}^{d_z}$ to be designed such that z asymptotically enables to reconstruct the plant state x, or part of it, as formalized next. Since the plant and the observer jump simultaneously, the observer analysis and design can be carried out on the cascade system

$$\mathcal{H} - \hat{\mathcal{H}} \left\{ \begin{array}{l} \left. \begin{array}{rl} \dot{x} &\in f(x) \\ \dot{z} &\in F(z, h_c(x)) \end{array} \right\} (x, z) \in C \times \mathbb{R}^{d_z} \\ \left. \begin{array}{rl} x^+ &\in g(x) \\ z^+ &\in G(z, h_d(x)) \end{array} \right\} (x, z) \in D \times \mathbb{R}^{d_z} \end{array} \right. \tag{6}$$

whose flow and jump map we denote

$$\mathcal{F}(x, z) = (f(x), F(z, h_c(x)) \tag{7a}$$
$$\mathcal{G}(x, z) = (g(x), G(z, h_d(x))) \tag{7b}$$

The observer problem can then be reformulated as a stabilization problem of a set $\mathcal{A} \subseteq \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$, which depends on the observation goal. For instance, if we want to estimate the full state x, we can first try to take $d_z = d_x$ and stabilize the *zero estimation error set* given by

$$\mathcal{A} = \left\{ (x, z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \ : \ x = z \right\} , \tag{8a}$$

which is nothing but the diagonal. In that case, z directly provides an asymptotic estimate of x. But sometimes, as for continuous-time systems, we need to change coordinates, or add some degrees of freedom, thus leading to $d_z \geq d_x$ and

$$\mathcal{A} = \left\{(x,z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \ : \ z = T(x)\right\} , \tag{8b}$$

for some map $T : \mathbb{R}^{d_x} \to \mathbb{R}^{d_z}$. In that case, an estimate for x may be recovered from z by left-inversion if T is injective. We may also be interested in estimating only a part x_p of the state x, in the context of switched systems for instance, or use in z states that are not directly functions of x, such as varying gains. This can be translated into an appropriate choice of $\mathcal{A}$, i.e., more generally

$$\mathcal{A} = \left\{(x,z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \ : \ T(x,z) = 0\right\} , \tag{8c}$$

for some map $T : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \to \mathbb{R}^p$. The goal of this paper is finally to solve the following problem.

Problem (O) : Given a set of initial conditions $\mathcal{X}_0 \subseteq \mathbb{R}^{d_x}$, a closed subset $\mathcal{A}$ of $\mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$ as in (8), and assuming one of the conditions of Definition 2.1 holds, design maps $F : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_c}} \to \mathbb{R}^{d_z}$ and $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \to \mathbb{R}^{d_z}$ such that there exist a class-$\mathcal{KL}$ function β and a subset $\mathcal{Z}_0$ of $\mathbb{R}^{d_z}$ such that for every $\phi = (x,z) \in \mathcal{S}_{\mathcal{H}-\hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$,

$$\left|\phi(t,j)\right|_{\mathcal{A}} \leq \beta\left(\left|\phi(0,0)\right|_{\mathcal{A}}, t+j\right) \tag{9}$$

for all $(t,j) \in \operatorname{dom}\phi$, namely $\mathcal{A}$ is uniformly pre-asymptotically stable (UpAS) for $\mathcal{H} - \hat{\mathcal{H}}$ with basin of attraction including $\mathcal{X}_0 \times \mathcal{Z}_0$.

Note that the set $\mathcal{A}$ should also ensure that

$$x \text{ bounded and } \left|(x,z)\right|_{\mathcal{A}} \text{ bounded} \quad \Longrightarrow \quad z \text{ bounded}$$

to guarantee from (9) that z cannot escape in finite time before x does. In other words, the observer solution is indeed defined as long as the plant solution is. This is verified for $\mathcal{A}$ defined in (8a) or (8b) if T is continuous.

Remark 2.5 *The implementation of the observer $\hat{\mathcal{H}}$ requires a perfect jump synchronization with the plant $\mathcal{H}$. Unfortunately, the practical detection of the plant jumps often involves measurements and transmission of information which might entail some delays in the triggering of the observer jumps. The robustness of the UpAS property of $\mathcal{A}$ given by Problem* (O) *with respect to delays is thus analyzed in Section 6.*

Remark 2.6 *We have restricted Problem* (O) *to sets $\mathcal{A}$ of the forms* (8) *in order to remain in the observation context, even though some results presented in this paper, such as Theorem 3.1 below, guarantee* (9) *for any choice of $\mathcal{A}$. On the other hand, some of the results are restricted to specific forms of $\mathcal{A}$ defined in* (8a) *or* (8b).

3 A General Sufficient Condition

The following theorem gives a first Lyapunov-based sufficient condition to solve Problem (O).

Theorem 3.1 *Assume there exist scalars $a_c, a_d \in \mathbb{R}$, class-$\mathcal{K}^\infty$ maps $\underline{\alpha}, \overline{\alpha}$, and a C^1 map $V : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \to \mathbb{R}$ verifying*

$$\begin{cases} \underline{\alpha}\left(|(x,z)|_{\mathcal{A}}\right) \leq V(x,z) \ \forall (x,z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \\ V(x,z) \leq \overline{\alpha}\left(|(x,z)|_{\mathcal{A}}\right) \ \forall (x,z) \in \mathcal{X}_0 \times \mathcal{Z}_0 \end{cases} \tag{10a}$$

$$L_{\mathcal{F}} V(x,z) \leq a_c\, V(x,z) \quad \forall (x,z) \in C \times \mathbb{R}^{d_z} \tag{10b}$$

$$V(\mathcal{G}(x,z)) \leq e^{a_d}\, V(x,z) \quad \forall (x,z) \in D \times \mathbb{R}^{d_z} \tag{10c}$$

with $\mathcal{F}$ and $\mathcal{G}$ defined in (7). *Then, Problem* (O) *is solved if any of the following conditions holds, referred to as conditions* (C)*:*

(C1a) $a_c < 0$ *and* $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ *holds with* $\min \mathcal{I} > \frac{a_d}{|a_c|}$.

(C1b) $a_c < 0$ *and* $\mathcal{C}^{\mathrm{av}}_{\mathcal{X}_0}[\tau^\star]$ *holds with* $\tau^\star > \frac{a_d}{|a_c|}$.

(C2a) $a_d < 0$ *and* $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ *holds with* $a_c \sup \mathcal{I} < |a_d|$.

(C2b) $a_d < 0$ *and* $\mathcal{C}^{\mathrm{rav}}_{\mathcal{X}_0}[\tau^\star]$ *holds with* $a_c \tau^\star < |a_d|$.

PROOF. The proof consists in modeling the jumps by a hybrid timer τ with appropriate dynamics depending on the condition (C). Then, similarly to Liberzon et al. (2014), maps of the form $e^{a\tau} V(x,z)$ with a appropriately chosen provide strict Lyapunov functions satisfying the conditions in (Goebel et al., 2012, Theorem 3.18). See report version Bernard and Sanfelice (2020b).

Remark 3.2 *In (Goebel et al., 2012, Theorem 3.18), Condition* (10a) *is strengthened into*

$$\underline{\alpha}\left(|(x,z)|_{\mathcal{A}}\right) \leq V(x,z) \leq \overline{\alpha}\left(|(x,z)|_{\mathcal{A}}\right) \qquad \forall (x,z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \tag{11}$$

for easiness of presentation but the upper inequality is only needed on the initial conditions in the proof. It turns out to be useful to relax it to (10a) *in what follows.*

Conditions (C) imply that in the case of a reverse average dwell-time or if $0 \in \mathcal{I}$, the innovation term in the discrete dynamics of the observer makes the error contract at jumps ($a_d < 0$), due to possible Zeno solutions. Similarly in the case of average dwell-time or if $\sup \mathcal{I} = +\infty$, then the innovation term in the continuous dynamics makes the error contract during flow ($a_c < 0$).

Example 3.3 *The case of linear flow/jump/output maps, where $f(x) = A_c x$, $g(x) = A_d x$, $h_c(x) = H_c x$,*

$h_d(x) = H_d x$, has been studied in Bernard and Sanfelice (2018). We consider $\mathcal{A}$ defined in (8a) *and*

$$F(z, y_c) = A_c z + L_c(y_c - H_c x) \tag{12a}$$
$$G(z, y_d) = A_d z + L_d(y_d - H_d x) \tag{12b}$$

with $L_c \in \mathbb{R}^{d_x \times d_{y_c}}$ and $L_d \in \mathbb{R}^{d_x \times d_{y_d}}$. Then, the conditions in (10) *hold for a quadratic Lyapunov function $V(x, z) = (x - z)^\top P(x - z)$ if there exist scalars a_c and a_d, and a positive definite symmetric matrix $P \in \mathbb{R}^{d_x \times d_x}$ such that*

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \tag{13a}$$
$$(A_d - L_d H_d)^\top P(A_d - L_d H_d) \leq e^{a_d} P \tag{13b}$$

If a solution to (13) *exists and one of the conditions* (C) *holds, Problem* (O) *is solved. Note that if both* (A_c, H_c) *and* (A_d, H_d) *are detectable,* (13) *may be solvable with both $a_c \leq 0$ and $a_d \leq 0$, and* (C) *then holds directly if at least one of them is nonzero. By the Schur complement, this is equivalent to solving the LMIs*

$$A_c^\top P + P A_c - (\tilde{L}_c H_c + H_c^\top \tilde{L}_c^\top) < 0$$
$$\begin{pmatrix} P & (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \tag{14}$$

in $(P, \tilde{L}_c, \tilde{L}_d)$ and take $L_c = P^{-1}\tilde{L}_c$ and $L_d = P^{-1}\tilde{L}_d$. This has been done in (Bernard and Sanfelice, 2018, Example 3.3) for a bouncing ball with a restitution coefficient $\lambda < 1$, and position measured at all (hybrid) times.

Remark 3.4 *In the favorable case where both the flow and jump dynamics of $\mathcal{H}$ are detectable, it is not sufficient to choose independently a map F as a continuous-time observer of the flow and a map G as a discrete-time observer of the jumps. Indeed, jumps could destroy what has been achieved during flow, or vice versa. See Bernard and Sanfelice (2020b) for an example. To avoid this phenomenon,* (10b) *and* (10c) *should hold with the same Lyapunov function V.*

A drawback of Theorem 3.1 is that design conditions on the observer flow and jump maps are coupled through a_c and a_d in conditions (C) and it is not clear how they can be solved in the general nonlinear context. Even in the linear case as in Example 3.3, the conditions are nonlinear, unless both a_c and a_d can be taken negative and (14) can be solved. In Sections 4 and 5, we show how this loop can be broken by using innovation only in flow or only at jumps, through high-gain in flows and by considering an equivalent discrete-time system at jumps.

Another drawback is that Theorem 3.1 requires at least a_c or a_d to be negative. Therefore, either the continuous or the discrete dynamics of $\mathcal{H}$ has to admit an observer and thus be detectable. But it could happen that neither the continuous nor the discrete dynamics is observable, and yet the system as a whole is indeed observable. An application featuring an hybrid system with such a property is given in Section 7.2. Actually, Section 5 will show that we should rather study an equivalent discrete-time system, containing both the continuous and discrete dynamics and providing insight for observer design.

4 Flow-based Hybrid Observer

When the continuous dynamics of $\mathcal{H}$ are detectable and *persistent* in the sense of an average dwell-time, it is tempting to use a continuous-time observer

$$\dot{z} = F(z, h_c(x)) \quad , \quad \hat{x} = \Theta(z) \tag{15}$$

during flow, and simply copy the discrete dynamics of $\mathcal{H}$ in the jump map of the observer. Indeed, intuitively, if the estimation error decreases more during flow than it increases at jumps, namely, if the continuous-time observer (15) is sufficiently fast, the error is expected to converge to zero asymptotically. We thus need persistence of flow, namely conditions of the type $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ with $\min \mathcal{I} > 0$, or more generally $\mathcal{C}^{\text{av}}_{\mathcal{X}_0}[\tau^\star]$.

4.1 Sufficiently Large Average Dwell-time

The first thing to notice is that if the continuous-time observer (15) verifies (10a)-(10b) with $a_c < 0$ and if G is chosen such that (10c) holds for some $a_d \in \mathbb{R}$, then Problem (O) is solved if the average dwell time (ADT) is sufficiently large to satisfy (C1b). This result is standard in the literature of switched systems as reviewed in (Bernard and Sanfelice, 2020b, Section 7.3). Actually, if (10a) is strengthened into (11) and if there exists a class-$\mathcal{K}^\infty$ map κ such that

$$|\mathcal{G}(x, z)|_{\mathcal{A}} \leq \kappa\left(|(x, z)|_{\mathcal{A}}\right) \quad \forall (x, z) \in D \times \mathbb{R}^{d_z} \tag{16a}$$
$$\overline{\alpha} \circ \kappa \circ \underline{\alpha}^{-1} \leq c\,\text{Id} \tag{16b}$$

for some positive scalar c, then (10c) automatically holds with $a_d = \ln(c)$. For instance, in the case where $d_z = d_x$ and $\mathcal{A}$ is simply the diagonal set (8a), a map G satisfying (16a) is a simple copy of the plant jump map g, namely,

$$G(z, y_d) = g(z) \ , \tag{17}$$

if g is single-valued and κ-continuous, namely,

$$|g(x) - g(\hat{x})| \leq \kappa(|x - \hat{x}|) \qquad \forall (x, \hat{x}) \in D \times \mathbb{R}^{d_x} \ . \tag{18}$$

In particular, if g is Lipschitz with Lipschitz constant k_G and V is quadratic with $\underline{\alpha} = \underline{\lambda}(\cdot)^2$ and $\overline{\alpha} = \overline{\lambda}(\cdot)^2$, then Problem (O) is solved if the ADT is larger than $\frac{1}{|a_c|} \ln\left(\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)} k_G^2\right)$. Note that if g is only locally Lipschitz and any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ remains in a compact set $\mathcal{X}$, it is enough to guarantee (18) on $(D \cap \mathcal{X}) \times \mathbb{R}^{d_x}$ by taking

$$G(z, y_d) = \text{sat}(g(z)) \tag{19}$$

where sat is a saturation map active outside of $g(\mathcal{X})$.

However, apart from switched systems where the switching signal may be a controlled input, the ADT property cannot be imposed or controlled for a general hybrid system due to the jumps being state-dependent. Therefore, ADT (if it exists) is a property of the system and cannot be made "sufficiently fast." When the flow/jump/output maps are linear, this problem is overcome in Bernard and Sanfelice (2018); Ríos et al. (2020) by using (12) with $L_d = 0$ and L_c such that there exists $P = P^\top > 0$ solution to

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \tag{20a}$$
$$A_d^\top P A_d \leq e^{a_d} P \tag{20b}$$
$$a_c \tau^\star + a_d < 0 \tag{20c}$$

with $a_c < 0$. However existence of a solution to (20) is a priori not guaranteed, and more importantly, this method is not viable for general nonlinear systems.

4.2 Arbitrary Average Dwell-time

Another way to satisfy (C1a) or (C1b) is to choose a sufficiently fast continuous-time observer (15); i.e., satisfying (10a)-(10b) with $|a_c|$ sufficiently large. However, increasing a_c may require to change V, which in turns, modifies a_d. The following corollary shows that this compromise can be achieved for "high-gain observers."

Corollary 4.1 *Assume $\mathcal{X}_0$ is compact and $\mathcal{C}^{\text{av}}_{\mathcal{X}_0}[\tau^\star]$ holds for some $\tau^\star > 0$. Suppose also there exist $\lambda > 0$, $\ell_0 > 0$, rational functions $\underline{c}$ and $\overline{c}$, a continuous map $T : \mathbb{R}^{d_x} \to \mathbb{R}^{d_z}$, and for all $\ell > \ell_0$, a map $F_\ell : \mathbb{R}^{d_z} \times \mathbb{R}^{d_y} \to \mathbb{R}^{d_z}$ and a C^1 map $V_\ell : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \to \mathbb{R}$ such that*

$$\underline{c}(\ell)|z - T(x)|^2 \leq V_\ell(x, z) \leq \overline{c}(\ell)|z - T(x)|^2 \qquad \forall (x, z) \in (C \cup D \cup g(D)) \times \mathbb{R}^{d_z} \tag{21a}$$

$$L_{\mathcal{F}_\ell} V_\ell(x, z) \leq -\ell\lambda\, V_\ell(x, z) \quad \forall (x, z) \in C \times \mathbb{R}^{d_z} \tag{21b}$$

with $\mathcal{F}_\ell(x, z) = (f(x), F_\ell(z, h_c(x))$. Then, for any compact set $\mathcal{Z}_0 \subset \mathbb{R}^{d_z}$, there exists $\ell^* \geq \ell_0$ such that for all $\ell > \ell^*$, Problem (O) *is solved with $\mathcal{A}$ defined in* (8b), *$F := F_\ell$, and any map $G : \mathbb{R}^{d_z} \times \mathbb{R}^{d_{y_d}} \to \mathbb{R}^{d_z}$, Lipschitz with respect to z (uniformly in $y_d \in h_d(D)$), verifying*

$$G\left(T(x), h_d(x)\right) = T \circ g(x) \qquad \forall x \in D\ . \tag{21c}$$

PROOF. Consider a compact set $\mathcal{Z}_0 \subset \mathbb{R}^{d_z}$. First, by definition of $\mathcal{A}$ in (8b), $|(x, z)|_{\mathcal{A}} \leq |z - T(x)|$ for all $(x, z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$, and by continuity of T, because $\mathcal{X}_0 \times \mathcal{Z}_0$ is compact, there exists a class-$\mathcal{K}^\infty$ map α such that $|z - T(x)| \leq \alpha(|(x, z)|_{\mathcal{A}})$ on $\mathcal{X}_0 \times \mathcal{Z}_0$. Therefore, (21a) implies (10a) for all $\ell > \ell_0$. In addition, (21b) implies (10b) with $a_c = -\ell\lambda$. Then, from the definition of $\mathcal{G}$ in (7b) and from (21a), for all $(x, z) \in D \times \mathbb{R}^{d_z}$,

$$\begin{aligned}
V_\ell(\mathcal{G}(x, z)) &\leq \overline{c}(\ell)\, |G(z, h_d(x)) - T(g(x))|^2 \\
&\leq \overline{c}(\ell)\, |G(z, h_d(x)) - G(T(x), h_d(x))|^2 \\
&\leq \overline{c}(\ell) k_G^2\, |z - T(x)|^2 \\
&\leq \frac{\overline{c}(\ell)}{\underline{c}(\ell)} k_G^2 V_\ell(x, z)
\end{aligned}$$

where k_G is the Lipschitz constant of G with respect to z. Therefore, (10c) holds for all $\ell > \ell_0$ with $a_d = \ln\left(k_G^2 \frac{\overline{c}(\ell)}{\underline{c}(\ell)}\right)$. Exploiting exponential over polynomial growth, $-\ell\lambda\tau^\star + \ln\left(k_G^2 \frac{\overline{c}(\ell)}{\underline{c}(\ell)}\right) < 0$ for ℓ sufficiently large and (C1b) holds. Therefore, the result follows from Theorem 3.1.

In other words, if we know a *high-gain* continuous-time observer for the continuous dynamics of $\mathcal{H}$, verifying (21a)-(21b), then a possible hybrid observer is made of this continuous-time observer and a copy of the jump dynamics (written in the high-gain coordinates $z = T(x)$, i.e. verifying (21c)), with a gain ℓ sufficiently large compared to the average dwell-time and the Lipschitz constant of the jump dynamics. Compared to Bernard and Sanfelice (2018); Ríos et al. (2020), this result guarantees the existence of a solution to (20) in the linear context and provides a constructive way to compute it as detailed in Example 4.2. More importantly, the result applies to general nonlinear dynamics whose flow dynamics are *strongly differentially observable* as detailed in Example 4.3.

Example 4.2 *Assume $f(x) = A_c x$ and $h_c(x) = H_c x$ with the pair (A_c, H_c) observable. The eigenvalues of the observer can then be assigned arbitrarily fast. For that, we define $\mathcal{V} \in \mathbb{R}^{d_x \times d_x}$ a change of coordinates transforming (A_c, H_c) into a block-diagonal observable form, namely such that*

$$\mathcal{V} A_c \mathcal{V}^{-1} = \boldsymbol{A} + \boldsymbol{D}\boldsymbol{H} \quad , \quad H_c \mathcal{V}^{-1} = \boldsymbol{H}$$

with

$$\boldsymbol{A} := \text{blkdiag}(A_1, \ldots, A_{d_{y_c}})\ ,\ \boldsymbol{D} := \text{blkdiag}(D_1, \ldots, D_{d_{y_c}})$$

$$\boldsymbol{H} := \text{blkdiag}(H_1, \ldots, H_{d_{y_c}})\ ,$$

$D_i \in \mathbb{R}^{d_i\times 1}$, $A_i \in \mathbb{R}^{d_i\times d_i}$, $H_i \in \mathbb{R}^{1\times d_i}$ of shape

$$A_i = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \ , \ H_i = (0 \ \dots \ 0 \ 1) \ ,$$

and d_i integers such that $\sum d_i = d_x$. Consider vectors K_i such that $A_i - K_i H_i$ is Hurwitz, and for a positive scalar ℓ, define $\mathcal{L}_i(\ell) := \mathrm{diag}(\ell^{d_i-1}, \dots, \ell, 1)$. Then, let us take F defined by (12a) with

$$L_c = \mathcal{V}^{-1}(\boldsymbol{D} + \ell \boldsymbol{\mathcal{L}}(\ell)\boldsymbol{K}) \tag{22}$$

$\boldsymbol{K} := \mathrm{blkdiag}(K_1, \dots, K_{d_{y_c}})$, $\boldsymbol{\mathcal{L}} := \mathrm{blkdiag}(\mathcal{L}_1, \dots, \mathcal{L}_{d_{y_c}})$. *Consider a positive definite matrix $P \in \mathbb{R}^{d_x\times d_x}$ such that*

$$(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{H})^\top P + P(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{H}) \leq -\lambda P$$

for some $\lambda > 0$. Then, (21a)-(21b) hold with $T = \mathrm{Id}$,

$$V_\ell(x,z) = (x-z)^\top \mathcal{V}^\top \boldsymbol{\mathcal{L}}(\ell)^{-1} P \boldsymbol{\mathcal{L}}(\ell)^{-1} \mathcal{V}\,(x-z) \ ,$$

$$\underline{c}(\ell) = \frac{\underline{\lambda}(\mathcal{V}^\top P \mathcal{V})}{\ell^{2(d-1)}} \quad , \quad \overline{c}(\ell) = \overline{\lambda}(\mathcal{V}^\top P \mathcal{V})$$

where $d = \max d_i$. Therefore, whatever the average dwell-time is, Problem (O) is solved for ℓ sufficiently large by taking G as in (17) (resp. (19)) if g is Lipschitz (resp. locally Lipschitz and the solutions x are bounded).

Example 4.3 *Assume that f and g are single valued, with a single output ($d_{y_c} = 1$), and the flow pair (f, h_c) of $\mathcal{H}$ is strongly differentially observable of order d_z on $C \cup D$, namely the map $T : \mathbb{R}^{d_x} \to \mathbb{R}^{d_z}$ defined by*

$$T(x) = (h_c(x), L_f h_c(x), ..., L_f^{d_z-1} h_c(x)) \tag{23}$$

is C^1 and an injective immersion on $C \cup D$. Assume also there exists a Lipschitz map $\Phi : \mathbb{R}^{d_z} \to \mathbb{R}$ verifying

$$\Phi(T(x)) = L_f^{d_z} h(x) \quad \forall x \in C \cup D \ .$$

This is guaranteed, in particular, if any $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0)$ evolves in a compact set $\mathcal{X} \subseteq C \cup D$, since there exists a Lipschitz map $\Theta : \mathbb{R}^{d_z} \to \mathbb{R}^{d_x}$ such that

$$\Theta(T(x)) = x \qquad \forall x \in \mathcal{X} \ ,$$

and Φ can simply be chosen as $\Phi = \mathrm{sat} \circ L_f^{d_z} \circ \Theta$ where sat saturates outside of $L_f^{d_z}(\mathcal{X})$. Then, following Khalil and Praly (2013), a high-gain observer can be built for the flow dynamics, defined by

$$F_\ell(z, y_c) = A\,z + B\Phi(z) + \ell\mathcal{L}(\ell)K(y_c - z_1) \ ,$$

$$A = \begin{pmatrix} 0 & 1 & & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{d_z} \ , \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{d_z} \ ,$$

$\mathcal{L}(\ell) = \mathrm{diag}(1, \ell, \ell^2, \dots, \ell^{d_z-1})$, and K such that $A - KH$ is Hurwitz with $H = (1, 0, \dots, 0)$. Classical high gain computations show that conditions (21a) and (21b) then hold for the Lyapunov function

$$V_\ell(x,z) = (T(x) - z)^\top \mathcal{L}(\ell)^{-1} P \mathcal{L}(\ell)^{-1} (T(x) - z) \ ,$$

with P a positive definite matrix such that

$$(A - KH)^\top P + P(A - KH) \leq -\lambda_0 P$$

for some $\lambda_0 > 0$, $\underline{c}(\ell) = \frac{\underline{\lambda}(P)}{\ell^{2(d_z-1)}}$, $\overline{c}(\ell) = \overline{\lambda}(P)$, $\lambda > 0$ depending on λ_0 and on the Lipschitz constant of Φ, and ℓ larger than a threshold ℓ_0 also depending on that Lipschitz constant. Selecting G Lipschitz verifying (21c), finally provides an observer relative to $\mathcal{A}$ defined in (8b), if the gain ℓ is sufficiently large according to Corollary 4.1. In particular, if any $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0)$ evolves in a compact set $\mathcal{X}$, we can choose $G(z, y_d) = \mathrm{sat} \circ T \circ g \circ \Theta(z)$ where sat saturates outside of $T \circ g(D)$, and an estimate of x is obtained by $\hat{x} = \Theta(z)$. This design is illustrated in Sections 7.1 and 7.2. Note that the same tools can be used for other (multi-output) triangular normal forms, as long as the nonlinearities are Lipschitz .

5 Jump-based Hybrid Observer

We now consider the case where the output is used to create contraction of the Lyapunov function at jump times, namely we rather exploit y_d. Without natural contraction in the continuous dynamics of $\mathcal{H}$, we thus need the jumps to be persistent and sufficiently frequent to inject sufficient information in the observer, i.e., conditions of the type $\mathcal{C}^{\mathrm{rav}}_{\mathcal{X}_0}[\tau^\star]$ or $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ with $\mathcal{I}$ bounded.

5.1 Sufficiently Small Reverse Dwell-time

Similarly to the previous section, we can start by noting that when the discrete dynamics of $\mathcal{H}$ admit a discrete-time observer verifying (10a) and (10c) with $a_d < 0$, we may choose F such that (10b) holds for some $a_c \in \mathbb{R}$ and Problem (O) will then be solved if a_d is sufficiently negative with respect to a_c and to the maximal amount of flow; or equivalently, if the jumps are sufficiently frequent, i.e. either if $\max \mathcal{I}$ is sufficiently small to satisfy (C2a), or the rADT is sufficiently small to satisfy (C2b). When f is single valued and $\mathcal{A}$ defined as in (8a), one may choose F single-valued so that

$$|f(x) - F(z, h_c(x))| \leq c\,|x - z| \quad \forall (x,z) \in C \times \mathbb{R}^{d_z}$$

for some scalar c. For instance, if $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0)$ evolves in a compact set $\mathcal{X}$ and f is locally Lipschitz, one may simply take $F(z, y_c) = \text{sat}(f(z))$, where sat saturates outside of $f(\mathcal{X})$. In other words, F is simply a flow predictor.

However, again, this method has an interest only when the jumps are naturally sufficiently frequent (Zeno solutions) or can be made so (switching systems). Otherwise, we need to take explicitly into account the potential increase of V during flow, to ensure the conditions (10) and (C2b) hold simultaneously. When the flow/jump/output maps are linear, one may choose F and G as in (12) with $L_c = 0$ and L_d such that there exists P positive definite solution to

$$A_c^\top P + PA_c \leq a_c P \tag{24a}$$
$$(A_d - L_d H_d)^\top P(A_d - L_d H_d) \leq e^{a_d} P \tag{24b}$$
$$a_c \tau^\star + a_d < 0 \tag{24c}$$

for some $a_c \in \mathbb{R}$ and $a_d < 0$ as in Bernard and Sanfelice (2018); Ríos et al. (2020), where $\tau^\star$ denotes the rADT or the maximal length of flow.

As noticed in Etienne et al. (2017) in the context of sampled systems ($A_d = I$), this design is extendable to particular classes of nonlinear continuous dynamics for which f is included in the convex hull of a finite number of linear maps. The LMI (24a) must then hold for each of those maps. Furthermore, Etienne et al. (2017) shows that (24) might be relaxed by allowing P and L_d to depend on the length τ of the flow intervals in a way that ensures contraction during both flows and jumps. But this requires the feasibility of some LMIs that are not clearly related to observability.

In any case, the methods mentioned in this section require the detectability of the discrete dynamics of $\mathcal{H}$ and a sufficient contraction of the error at jumps. When either the discrete dynamics are not detectable, or the coupling between flows and jumps makes the matrix inequalities not feasible, we show in the next section that we should rather analyze an equivalent discrete-time system made of the plant sampled at the jump times, which naturally contains the information of both flows and jumps.

5.2 *Arbitrary Reverse Dwell-Time*

We now assume the jumps are persistent, i.e. $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with $\mathcal{I}$ compact, but without any constraint on the upper bound of $\mathcal{I}$. We also suppose that maximal solutions of $\dot{x} \in f(x)$ are defined on $\mathbb{R}_{\geq 0}$ and we denote Ψ_f the flow operator alongside f, i.e, $\Psi_f(x_0, \tau)$ denotes the set of points that can be reached at time τ by solutions to $\dot{x} \in f(x)$ initialized at x_0 at $\tau = 0$.

Now consider a solution $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0)$ and notice that $x_k := x(t_k, k)$ sampled after each jump and the output $y_k := h_d(x(t_k, k-1))$ obtained before each jump verify

$$x_{k+1} \in g(\Psi_f(x_k, \tau_k)) \quad , \quad y_k \in h_d\left(\Psi_f(x_k, \tau_k)\right) \tag{25}$$

where $\tau_k = t_{k+1} - t_k$ denotes the length of the kth flow interval, $k \in \mathbb{N}_{>0}$. It follows that with the discrete output y_d obtained right before each jump, we are actually observing the discrete-time system (25). It is therefore the observability/determinability of (25) that counts, and we must look for F and G making $\mathcal{A}$ of Problem (O) UpAS for the reduced system

$$\begin{aligned} x_{k+1} &\in g(\Psi_f(x_k, \tau_k)) \\ z_{k+1} &\in G(\Psi_F(z_k, \tau_k), h_d(\Psi_f(x_k, \tau_k))) \end{aligned}$$

or equivalently

$$(x_{k+1}, z_{k+1}) \in \mathcal{G} \circ \Psi_\mathcal{F}\left((x_k, z_k), \tau_k\right) \ , \tag{26}$$

with $\mathcal{F}$ and $\mathcal{G}$ defined in (7). Indeed, the following theorem shows that it is sufficient to prove UpAS of $\mathcal{A}$ for (26) with sequences $(\tau_k) \in \mathcal{I}^\mathbb{N}$ to solve Problem (O) (see (28) below). But because the first interval of flow $t_1 \in [0, \max \mathcal{I}]$ is not necessarily in $\mathcal{I}$, the system (26) with $(\tau_k) \in \mathcal{I}^\mathbb{N}$ only captures the behavior of solutions after hybrid time $(t_1, 1)$. Hence, we need to consider (26) initialized in a superset $\widetilde{\mathcal{X}}_0 \times \widetilde{\mathcal{Z}}_0$ of $\mathcal{X}_0 \times \mathcal{Z}_0$ such that

$$\mathcal{G} \circ \Psi_\mathcal{F}(\mathcal{X}_0 \times \mathcal{Z}_0, \tau_0) \subseteq \widetilde{\mathcal{X}}_0 \times \widetilde{\mathcal{Z}}_0 \quad \forall \tau_0 \in [0, \max \mathcal{I}] \ .$$

In addition, because (26) only describes solutions at discrete times (t_k, k), we need a regularity property of solutions during flow (see (29) below).

Theorem 5.1 *Assume that $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with $\mathcal{I}$ compact and maximal solutions to $(\dot{x}, \dot{z}) \in \mathcal{F}(x, z)$ are defined on $\mathbb{R}_{\geq 0}$. Suppose the following properties hold.*

1. *There exists a class-$\mathcal{K}$ function ρ_0 such that for all $(x_0, z_0, \tau_0) \in \mathcal{X}_0 \times \mathcal{Z}_0 \times [0, \max \mathcal{I}]$,*
$$|\mathcal{G} \circ \Psi_\mathcal{F}((x_0, z_0), \tau_0)|_\mathcal{A} \leq \rho_0\left(|(x_0, z_0)|_\mathcal{A}\right) \ . \tag{27}$$
2. *There exist a class-$\mathcal{KL}$ function β and a superset $\mathcal{X} \times \mathcal{Z}$ of $\widetilde{\mathcal{X}}_0 \times \widetilde{\mathcal{Z}}_0$ such that any solution (x, z) to (26) initialized to $(x_1, z_1) \in \widetilde{\mathcal{X}}_0 \times \widetilde{\mathcal{Z}}_0$ and with input $k \mapsto \tau_k \in \mathcal{I}$, remains in $\mathcal{X} \times \mathcal{Z}$ and verifies*
$$\left|(x_k, z_k)\right|_\mathcal{A} \leq \beta\left(\left|(x_1, z_1)\right|_\mathcal{A}, k\right) \qquad \forall k \in \mathbb{N}_{>0} \ . \tag{28}$$
3. *There exists a class-$\mathcal{K}$ function ρ such that for all $(x_0, z_0) \in \mathcal{X} \times \mathcal{Z}$ and for all $\tau \in [0, \max \mathcal{I}]$,*
$$|\Psi_\mathcal{F}((x_0, z_0), \tau)|_\mathcal{A} \leq \rho\left(|(x_0, z_0)|_\mathcal{A}\right) \ . \tag{29}$$

Then, Problem (O) *is solved.*

PROOF. Consider $\phi = (x,z) \in \mathcal{S}_{\mathcal{H}-\hat{\mathcal{H}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$. Denote $J := \sup \mathrm{dom}_j\, \phi$ and $\tau_{\max} := \max \mathcal{I}$. The discrete trajectory $\tilde{\phi} = (\tilde{x},\tilde{z}) : \mathrm{dom}_j\, \phi \to \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$ defined by $\tilde{\phi}_k = \phi(t_k,k)$ verifies (26) with input τ defined by $\tau_k = t_{k+1} - t_k$, for all $k \in \mathrm{dom}_j\, \phi \setminus \{J\}$. It follows from $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ that $\tau_0 \leq \tau_{\max}$, and $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}_{>0}$ if $J = +\infty$ and for all $k \in \{1,\ldots,J-1\}$ otherwise. Therefore, $\tilde{\phi}_1 \in \widetilde{\mathcal{X}}_0 \times \widetilde{\mathcal{Z}}_0$, $\tilde{\phi}_k \in \mathcal{X} \times \mathcal{Z}$ for all k, and according to (27),(28), for all $k \in \mathrm{dom}_j\, \phi \geq 1$,

$$\begin{aligned}|\phi(t_k,k)|_{\mathcal{A}} &\leq \beta(|\phi(t_1,1)|_{\mathcal{A}},k)\\ &\leq \beta\big(\rho_0(|\phi(0,0)|_{\mathcal{A}}),k\big)\ .\end{aligned}$$

This latter inequality still holds for $k=0$, by redefining β so that $\beta(\rho_0(s),0) \geq s$. Then, by $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$, for all $(t,j) \in \mathrm{dom}\,\phi$, $t - t_j \in [0,\tau_{\max}]$ and from (29),

$$|\phi(t,j)|_{\mathcal{A}} \leq \rho\left(|\phi(t_j,j)|_{\mathcal{A}}\right) \leq \rho\left(\beta\big(\rho_0(|\phi(0,0)|_{\mathcal{A}}),j\big)\right)\ .$$

Besides, for all $(t,j) \in \mathrm{dom}\,\phi$, $t - t_j \leq \tau_{\max}$ and $t_j - t_{j-1} \leq \tau_{\max}$ for $j \geq 1$, so that $t_j \leq \tau_{\max} j$ and $t \leq \tau_{\max}(j+1)$. Thus,

$$|\phi(t,j)|_{\mathcal{A}} \leq \rho\left(\beta\big(\rho_0(|\phi(0,0)|_{\mathcal{A}}),\max\{a(t+j)+b,0\}\big)\right)$$

with $a = \frac{1}{\tau_{\max}+1}$ and $b = -\frac{\tau_{\max}}{\tau_{\max}+1}$. Therefore, Problem (O) is solved. □

The condition (29) guarantees that the distance of (x,z) to $\mathcal{A}$ during flow is continuous on the compact interval $[0,\max\mathcal{I}]$ with respect to the initial distance to $\mathcal{A}$. If $\mathcal{A}$ is defined by (8a) and $f = F$ is locally Lipschitz, this regularity property is always satisfied when $\mathcal{X}$ and $\mathcal{Z}$ are compact.

It is important to note that Ψ_f and Ψ_F need not be computed for the implementation of the observer (5) : they are only used in the analysis in order to design the maps F and G to be used in (5). Although the reduced system (26) may not be handier to use for design than (6), it helps to understand the observability conditions that are at stake here. In addition, when f is linear, i.e. $f(x) = A_c x$, we can choose $F(z) = A_c z$, so that

$$\Psi_f(x_k,\tau_k) = \exp(A_c\tau_k)x_k\ ,\ \Psi_F(z_k,\tau_k) = \exp(A_c\tau_k)z_k$$

and (29) immediately holds for $\mathcal{A}$ defined in (8a).

Corollary 5.2 *Assume that $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with $\mathcal{I}$ compact and f, g, h_d are linear defined by $f(x) = A_c x$, $g(x) = A_d x$ and $h_d(x) = H_d x$. Assume there exist a positive definite matrix $P \in \mathbb{R}^{d_x \times d_x}$ and a gain vector $L_d \in \mathbb{R}^{d_x \times d_{y_d}}$ such that*

$$(\exp(A_c\tau))^\top (A_d - L_d H_d)^\top P (A_d - L_d H_d)\exp(A_c\tau) < P \quad \forall \tau \in \mathcal{I}\ . \tag{30}$$

Then, F and G defined in (12) with $L_c = 0$, solve Problem (O) with $\mathcal{A}$ defined in (8a).

PROOF. Follows from Theorem 5.1 using the Lyapunov function $V(x,z) = (x-z)^\top P(x-z)$ since $e = z - x$ in (26) verifies $e_{k+1} = (A_d - L_d H_d)\exp(A_c\tau_k)e_k$. □

The existence of the matrix P verifying (30) for a given τ is equivalent to $(A_d - L_d H_d)\exp(A_c\tau)$ being Schur for some gain L_d, which in turn is equivalent to the detectability of the discrete-time system

$$x_{k+1} = A_d \exp(A_c\tau)x_k\ ,\ y_k = H_d\exp(A_c\tau)x_k\ . \tag{31}$$

Thus, having (30) for any $\tau \in \mathcal{I}$ requires detectability of (31) for any $\tau \in \mathcal{I}$. It is not sufficient, however, because (30) must be verified with the same L_d and P for all $\tau \in \mathcal{I}$. So (30) requires in fact the detectability of the LTV or LPV discrete-time system

$$x_{k+1} = A_d\exp(A_c\tau_k)x_k\ ,\ y_k = H_d\exp(A_c\tau_k)x_k \tag{32}$$

with input τ_k in the compact set $\mathcal{I}$, which is exactly (25). Actually, (30) is stronger because it requires a quadratic Lyapunov function with a matrix P, that is independent from the sequence $k \mapsto \tau_k$. This property is sometimes called "quadratic detectability" (see Wu (1995); Halimi et al. (2013); Bernussou et al. (1992)).

Remark 5.3 *By the Schur complement, finding P and L_d satisfying (30) is equivalent to finding P and $\tilde{L}_d$ satisfying the LMIs*

$$\begin{pmatrix} P & \exp(A_c\tau)^\top (PA_d - \tilde{L}_d H_d)^\top \\ \star & P\end{pmatrix} > 0 \quad \forall \tau \in \mathcal{I} \tag{33}$$

with $\tilde{L}_d = PL_d$. In the case where $\mathcal{I}$ has infinitely many elements, it is shown in Ferrante et al. (2016) that it is always possible to compute numerically a polytopic decomposition of $\exp(A_c\tau)$, namely a finite number of matrices $\{M_1, M_2, \ldots, M_\nu\}$ such that $\exp(A_c\tau)$ is in the convex hull of those matrices whenever $\tau \in \mathcal{I}$. Since (33) is convex in $\exp(A_c\tau)$, it is then sufficient to solve the finite number of LMIs

$$\begin{pmatrix} P & M_i^\top (PA_d - \tilde{L}_d H_d)^\top \\ \star & P\end{pmatrix} > 0 \quad \forall i \in \{1,2,\ldots,\nu\} \tag{34}$$

with common P and $\tilde{L}_d$. An example is given in Section 7.1. In particular, if A_c is nilpotent of order N, we have $\exp(A_c\tau) = \sum_{k=0}^{N-1}\frac{\tau^k}{k!}A_c^k$ so that for all τ in a compact subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$, $\exp(A_c\tau)$ is in the convex hull of the $\nu = 2^{N-1}$ matrices $\left\{I + \sum_{k=1}^{N-1}\frac{\tau_k^k}{k!}A_c^k\right\}$ with $\tau_k \in \{\min\mathcal{I}, \max\mathcal{I}\}$ for all k. See Section 7.2.

What makes the approach of Remark 5.3 work is the fact that the flow operator of the error $e = \hat{x} - x$ is contained in the convex hull of a finite number of linear maps. In the context of sampled nonlinear systems, Andrieu and Nadri (2010); Dinh et al. (2015) noticed that by copying the continuous dynamics in the observer, namely taking $F = f$, the error components evolve during flow according to

$$\dot{e}_i = f_i(\hat{x}) - f_i(x) = \frac{\mathrm{d}f_i}{\mathrm{d}x}(v(t))e$$

for some v depending on x and $\hat{x}$, thanks to the mean value theorem. For certain classes of maps f Dinh et al. (2015), the error reachable set within a time $\tau \in \mathcal{I}$ may then be included in the convex hull of a finite number of linear maps $\{e \mapsto M_i e\}_{i=1,..\nu}$ if the Jacobian components of f are bounded. If g is linear, the discrete error system in Corollary 5.2 is then replaced by

$$e_{k+1} = \sum_{i=1}^{\nu} \beta_{i,k}(A_d - L_d H_d) M_i e_k$$

with $\sum_{i=1}^{\nu} \beta_{i,k} = 1$, and following the same steps as in Andrieu and Nadri (2010) with the Lyapunov function of Corollary 5.2, it is enough to ensure (30) with $\exp(A_c\tau)$ replaced by M_i, for each $i \in \{1, 2, \dots \nu\}$, namely solve the LMIs (34).

The advantage of using a constant gain L_d is that it is sufficient to compute only once the vertices M_i of the polytopic decomposition of the flow operator for $\tau \in \mathcal{I}$ and solve offline the finite number of LMIs (34). However, as mentioned above, those LMIs might not be solvable since they require a stronger property than detectability of (32). In that case, we may allow L_d to be time-varying, by adapting L_d to τ_k, as done in the particular case of sampled-data observers in Sferlazza et al. (2019). Indeed, observe that the observer jump map G in (26) is applied after flowing τ_k units of time with F. Therefore, at the moment where G is used, τ_k represents the time elapsed since the previous jump and is known to the observer. This can be modeled in our framework with a timer τ added to the observer state, where τ flows according to $\dot{\tau} = 1$ and jumps according to $\tau^+ = 0$. It follows that at each jump, the gain L_d in the jump map G defined in Corollary 5.2 can be adapted to the length of the previous interval of flow, in a way that makes

$$\hat{x}_{k+1} = A(\tau_k)\hat{x}_k + L_{d,k}(y_k - H(\tau_k)\hat{x}_k) \tag{35}$$

an observer for (32), where

$$A(\tau_k) = A_d \exp(A_c\tau_k) \quad , \quad H(\tau_k) = H_d \exp(A_c\tau_k) \; .$$

Since H is not constant, we cannot use the results obtained for LPV systems (Halimi et al. (2013)). However, an even simpler approach is to consider (32) as a LTV system and design $L_{d,k}$ as the gain of a discrete Kalman filter. More precisely, we use an observer with state $z = (\hat{x}, \tau, K, P)$, flow dynamics $F(z) = (A_c\hat{x}\,,\, 1\,,\, 0\,,\, 0)$ and jump dynamics

$$G(z, y_d) = \begin{pmatrix} A_d\hat{x} + A(\tau)K(y_d - H_d\hat{x}) \\ 0 \\ \chi(P,\tau) \\ \big(I - \chi(P,\tau)H(\tau)\big)p(P,\tau) \end{pmatrix} \tag{36}$$

where the maps p and χ are defined by

$$\begin{aligned} p(P,\tau) &= A(\tau)PA(\tau)^\top + Q \\ \chi(P,\tau) &= p(P,\tau)H(\tau)^\top \Big(H(\tau)p(P,\tau)H(\tau)^\top + R\Big)^{-1} \end{aligned}$$

It is important to note that the innovation of $\hat{x}_{k+1}$ in (35) must use y_k, instead of y_{k+1} as in a standard Kalman filter. That is why we use in (36) a Kalman filter with prediction after innovation, where the gain writes $L_d = A(\tau)K$ with K the Kalman gain computed at the previous jump.

In the same spirit, one may note that if (32) is known to be observable after N jumps, Medina and Lawrence (2009) proposed to compute $L_{d,k}$ based on the weighted observability Grammian over the past N jumps.

6 Robustness with Respect to Delays in Jumps

We now study how the observer convergence is impacted if the jumps of the observer are delayed with respect to those of the plant, thus leading to a mismatch between the observer jump times and those of the plant. For this, we start from the following assumption.

Assumption 6.1 *$\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with $\mathcal{I}$ compact, $\min \mathcal{I} > 0$ (dwell-time), and Problem* (O) *has been solved, namely the set $\mathcal{A}$ is UpAS for $\mathcal{H} - \hat{\mathcal{H}}$ with basin of attraction including $\mathcal{X}_0 \times \mathcal{Z}_0$.*

We choose to study the particular case where the value of the innovation term, implemented in the observer at the delayed jump is the one that would have been computed at the actual jump time of the plant if there had been no delay. This covers the situations where the measurement and computation of the innovation $G(z, y_d)$ are instantaneous, but the implementation of the jump in the observer is delayed; or the measurement takes a known amount of time $\delta \geq 0$ to arrive to the observer, and the update of z is chosen as $G(z(t-\delta, j), y_d)$, thanks to a buffer in z or by backward integration of z. Inspired from Altin and Sanfelice (2020), for any delay $\Delta \in [0, \min \mathcal{I})$, this situation can be modeled as

$$
\hat{\mathcal{H}}(\Delta)\begin{cases}
\left.\begin{array}{l}\dot{x} \in f(x)\\ \dot{z} \in F(z, h_c(x))\\ \dot{\mu} = 0\\ \dot{\tau}_\delta = -\min\{\tau_\delta + 1\,,\, 1\}\end{array}\right\} \mathtt{x} \in \hat{C}(\Delta)\\[2ex]
\left.\begin{array}{l}x^+ \in g(x)\\ z^+ = z\\ \mu^+ \in G(z, h_d(x))\\ \tau_\delta^+ \in [0, \Delta]\end{array}\right\} \mathtt{x} \in \hat{D}_{-1}(\Delta)\\[2ex]
\left.\begin{array}{ll}x^+ = x, & \mu^+ = 0\ ,\\ z^+ = \mu, & \tau_\delta^+ = -1,\end{array}\right\} \mathtt{x} \in \hat{D}_0(\Delta)
\end{cases} \tag{37}
$$

with state $\mathtt{x} = (x, z, \mu, \tau_\delta)$, flow set

$$
\hat{C}(\Delta) = \left(\hat{C} \times \{0\} \times \{-1\}\right) \cup \left(\hat{C} \times \mathbb{R}^{d_z} \times [0, \Delta]\right)\ ,
$$

jump set $\hat{D}_{-1}(\Delta) \cup \hat{D}_0(\Delta)$ with

$$
\hat{D}_{-1}(\Delta) = \hat{D}\times\{0\}\times\{-1\}\,,\ \hat{D}_0(\Delta) = (\hat{C}\cup\hat{D})\times\mathbb{R}^{d_z}\times\{0\}
$$

$$
\hat{C} := C \times \mathbb{R}^{d_z} \quad , \quad \hat{D} := D \times \mathbb{R}^{d_z}\ .
$$

System $\hat{\mathcal{H}}(\Delta)$ contains two new states μ and τ_δ evolving in $\mathbb{R}^{d_z}$ and $[0, \Delta] \cup \{-1\}$ respectively. The state τ_δ is a timer modeling the delay between the jumps of the plant and of the observer. The role of μ is to store the update to be implemented in the observer at the end of the delay interval, when it actually jumps. More precisely, when $\tau_\delta = -1$ and x does not jump, $\hat{\mathcal{H}}(\Delta)$ flows, with (x, z) flowing according to $\mathcal{F}$ as in $\mathcal{H}-\hat{\mathcal{H}}$, while μ and τ_δ remain equal to 0 and -1 respectively. When the plant state x jumps, then the update in $G(z, h_d(x))$ that should have been instantaneously implemented in the observer state z is stored in the memory state μ, and τ_δ is set to a number in $[0, \Delta]$ thus starting a delay period: $\hat{\mathcal{H}}(\Delta)$ then flows and the time τ_δ decreases, until it reaches 0. At this point, a delay interval of length smaller than or equal to Δ has elapsed, and the observer state z is updated with the content of μ, while μ is reset to 0 and τ_δ switched back to -1.

Note that the plant state x is not allowed to jump again before the delay expressed by τ_δ has expired. That is why this model only works in the case where $\Delta < \min \mathcal{I}$, i.e., the maximal delay is smaller than the smallest possible time between successive jumps of the plant.

In order to study the robustness of this property in presence of delay, we need to resort to compact attractors and some regularity properties of $\mathcal{H} - \hat{\mathcal{H}}$.

Assumption 6.2 *There exists a compact subset $\mathcal{X}$ of $C \cup D$, such that any solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ remains in $\mathcal{X}$. In addition, $\mathcal{A}_{\mathcal{X}} := \mathcal{A} \cap (\mathcal{X} \times \mathbb{R}^{d_z})$ is compact.*

Assumption 6.3 *The interconnection $\mathcal{H} - \hat{\mathcal{H}}$ defined in* (6) *satisfies the hybrid basic conditions defined in (Goebel et al., 2012, Assumption 6.5), namely C and D are closed, $\mathcal{F}_{|\hat{C}}$ and $\mathcal{G}_{|\hat{D}}$ are outer semicontinuous and locally bounded, and $\mathcal{F}_{|\hat{C}}$ takes convex values.*

It follows that the plant solutions from $\mathcal{X}_0$ are also solution to (2) with flow set $C \cap \mathcal{X}$ and jump set $D \cap \mathcal{X}$, which are compact. The assumption that $\mathcal{A}_{\mathcal{X}}$ is compact is satisfied for $\mathcal{A}$ defined in (8b) if T is continuous, namely in all the examples considered above.

Let us define the sets

$$
\begin{aligned}
\mathcal{A}' &:= \left(\mathcal{A}_{\mathcal{X}} \times \{0\} \times \{-1\}\right) \cup \left(\widehat{G} \times \{0\}\right)\\
\widehat{G} &:= \big\{(\eta_x, z, \eta_z)\ :\ \eta_x \in g(x)\ ,\ \eta_z \in G(z, h_d(x))\\
&\qquad\qquad x \in D\ ,\ (x, z) \in \mathcal{A}_{\mathcal{X}}\big\}\ .
\end{aligned}
$$

Theorem 6.4 *Suppose Assumptions 6.1, 6.2 and 6.3 hold. Then, $\mathcal{A}'$ is UpAS for $\hat{\mathcal{H}}(0)$ with basin of attraction containing $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$. In addition, there exist a class-$\mathcal{KL}$ map β, scalars $t^\star \geq 0$ and $j^\star \in \mathbb{N}$, and for any $\epsilon > 0$, there exists $\Delta^* > 0$, such that any solution $\phi = (x, z, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}(\Delta)$ with $\Delta < \Delta^*$ and initialized in $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$ verifies*

$$
|\phi(t, j)|_{\mathcal{A}'} \leq \beta(|\phi(0, 0)|_{\mathcal{A}'}, t + j) + \epsilon\ , \tag{38}
$$

and $\left|(x, z)(t, j)\right|_{\mathcal{A}} \leq 2\epsilon$ for all $(t, j) \geq (t^\star, j^\star)$ such that $\tau_\delta(t, j) = -1$.

PROOF. Take a solution $\phi_\delta = (x, z, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}(\Delta)$ for some $\Delta \in [0, \min \mathcal{I})$ with $(x, z)(0, 0) \in \mathcal{X}_0 \times \mathcal{Z}_0$. Observe that the component x is not impacted by the delay mechanism, therefore, from Assumption 6.2, $x(t, j) \in \mathcal{X}$ for all $(t, j) \in \operatorname{dom} x$. It follows that ϕ_δ is solution to a hybrid system $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ which has same dynamics as $\hat{\mathcal{H}}(\Delta)$ but with flow set $\hat{C}_{\mathcal{X}}(\Delta) := \hat{C}(\Delta) \cap (\mathcal{X} \times \mathbb{R}^{2d_z+1})$ and jump set $\hat{D}_{\mathcal{X}}(\Delta) := \hat{D}(\Delta) \cap (\mathcal{X} \times \mathbb{R}^{2d_z+1})$. In the framework of Altin and Sanfelice (2020), $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ is then the delayed version of the nominal observer $\mathcal{H} - \hat{\mathcal{H}}$ with flow set $\hat{C}_{\mathcal{X}} = (C \cap \mathcal{X}) \times \mathbb{R}^{d_z}$, and jump set $\hat{D}_{\mathcal{X}} = (D \cap \mathcal{X}) \times \mathbb{R}^{d_z}$. By Assumption 6.1 (and by containment (Goebel et al., 2012, Theorem 3.32)), the set $\mathcal{A}_{\mathcal{X}}$ (that is compact according to Assumption 6.2) is still UpAS for this modified system. With the hybrid basic conditions in Assumption 6.3, we conclude from (Altin and Sanfelice, 2020, Proposition 4.3, Remark 4.4) that the set $\mathcal{A}'$ is UpAS for $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ with basin of attraction containing $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$. $\widehat{G}$ is compact by outer-semicontinuity and local boundedness of g and G. $\mathcal{A}'$ is therefore compact. Still from the hybrid basic conditions, $\mathcal{A}'$ is thus semi-globally practically robustly $\mathcal{KL}$

asymptotically stable for $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ according to (Goebel et al., 2012, Lemma 7.20). This means that there exists a $\mathcal{KL}$ function β such that for any $\epsilon > 0$, there exists $\rho > 0$ such that any solution ϕ to a ρ-perturbation of $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ initialized in $\mathcal{X}_0 \times \mathcal{Z}_0 \times \{0\} \times \{-1\}$, verifies (38). Since $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ can be included in any outer-perturbation of $\hat{\mathcal{H}}_{\mathcal{X}}(0)$ by taking Δ sufficiently small, (38) holds along solutions of $\hat{\mathcal{H}}_{\mathcal{X}}(\Delta)$ for Δ sufficiently small. Now for ϵ sufficiently small and for sufficiently large (t, j) (depending only on β and the compact set of initial conditions), $|\phi(t,j)|_{\mathcal{A}'} = |\phi(t,j)|_{\mathcal{A}'_{-1}}$ when $\tau_\delta = -1$, and thus $|(x,z)(t,j)|_{\mathcal{A}} \leq |\phi(t,j)|_{\mathcal{A}'} \leq 2\epsilon$. □

In other words, we achieve semi-global practical stability of $\mathcal{A}$ except possibly on the delay intervals. More precisely, for any $\epsilon > 0$, there exists a maximal delay Δ^* between the jumps of the plant and of the observer, such that the distance of (x, z) to $\mathcal{A}$ is asymptotically smaller than 2ϵ, except possibly during the delay intervals in-between those jumps, of length smaller than Δ^*. This is illustrated in Section 7.2.

In fact, if $\mathcal{A}$ is the diagonal set (8a), the mismatch during the delay intervals cannot be prevented if the jump map is not the identity. Indeed, after one jump of either x or z, one is close to x^- while the other is in $g(x^-)$, no matter how short the delay is. This well-known phenomenon, called *peaking*, was reported in the context of observation Forni et al. (2013), but also more generally output-feedback and tracking Biemond et al. (2013). This suggests that the Euclidian distance to evaluate the observer error is not appropriate and more general distances could be designed Biemond et al. (2016). In particular, the expression of $\mathcal{A}'$ shows that semi-global practical stability is ensured for the peaking free set

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \big\{(x,z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} : x \in g(x^-)\,,\ (x^-, z) \in \mathcal{A}\big\}\,.$$

Note that in the limit case where $\min \mathcal{I} = 0$, namely Zeno solutions could exist, then an arbitrarily small delay in the observer jumps could lead to several jumps of delay, namely, one would need to consider

$$\begin{aligned}\tilde{\mathcal{A}} = \mathcal{A} \cup \big\{(x,z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} : x \in g^{k^*}(x^-)\,,\ (x^-, z) \in \mathcal{A} \\ g^k(x^-) \cap D \neq \emptyset \quad \forall k \in \{1, \cdots, k^*-1\}\big\}\,.\end{aligned}$$

With an average dwell-time, k^* would be limited by N_0.

7 Applications

The results in the previous sections are exercised in applications. Section 7.1 introduces a model that covers a class of mechanical systems with impacts, including juggling systems and walking robots; see Sanfelice et al. (2007) and Short and Sanfelice (2018), and the references therein. Section 7.2 presents a second application that pertains to a parameterized model capturing the dynamics exhibited by a wide range of cortical neurons. This model, introduced in Izhikevich (2003), has been widely used by the neuroscience community due to its capabilities of reproducing a variety of spiking and bursting behaviors by properly choosing its parameters. The reader is also referred to the report version (Bernard and Sanfelice, 2020b, Section 7.3) for a detailed analysis of the way those results apply to the design of observers for the general class of switched systems defined in Example 2.3.

7.1 Mechanical system with impacts

Consider a system evolving according to

$$\begin{cases} \dot\theta = \omega \\ \dot\omega = \alpha(\theta,\omega) \end{cases} (\theta,\omega) \in C\,, \begin{cases} \theta^+ = g_\theta(\theta,\omega) \\ \omega^+ = g_\omega(\theta,\omega) \end{cases} (\theta,\omega) \in D$$

with $x = (\theta, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ the (bounded) positions and velocities, α, g_θ and g_ω locally Lipschitz functions, the position $y = \theta$ measured and jumps occurring at the impacts of θ on a surface $\mathcal{W}$, typically modeled by a jump set of the form

$$D = \{(\theta, w) \in \mathbb{R}^d \times \mathbb{R}^d\ , \theta \in \mathcal{W}\,,\ \langle \omega, \nabla_{\mathcal{W}} \rangle \leq 0\}$$

where the second condition ensures the velocity is pointing inwards $\mathcal{W}$. The flow dynamics are clearly strongly differentially observable of order $d_z = 2$, since $(y, \dot y) = x$ defines an injective immersion (with T simply the identity map). Therefore, if the impacts are detected (for instance through force sensors) and are known to have an average dwell-time, then an observer is simply given by

$$\begin{cases} \dot{\hat\theta} = \hat\omega - \ell(\hat\theta - y) \\ \dot{\hat\omega} = \mathrm{sat}\Big(\alpha(\hat\theta,\hat\omega)\Big) - \ell^2(\hat\theta - y) \end{cases} \quad \begin{cases} \hat\theta^+ = \mathrm{sat}\Big(g_\theta(\hat\theta,\hat\omega)\Big) \\ \hat\omega^+ = \mathrm{sat}\Big(g_\omega(\hat\theta,\hat\omega)\Big) \end{cases}$$

for ℓ sufficiently large, sat saturation functions saturating outside the bounds within which x is known to evolve, and jumps triggered at the detected impacts.

On the other hand, if the mechanical system possibly exhibits Zeno behavior (i.e. with $\sup \mathrm{dom}_t\, x < +\infty$ and $\sup \mathrm{dom}_j\, x = +\infty$), for instance due to gravity, a jump-based observer should be used instead. For instance, consider a vertical bouncing ball with

$$\begin{aligned} f(\theta,\omega) = (\omega, -\varrho\omega - \mathfrak{g}) \quad , \quad g(\theta,\omega) = (-\theta, -\lambda\omega) \quad (39) \\ C = \mathbb{R}_{\geq 0} \times \mathbb{R}\,, \quad D = \{(\theta,\omega) \in \mathbb{R}^2\ :\ \theta = 0\,,\ \omega \leq 0\} \end{aligned}$$

with $\mathfrak{g}$ the gravity constant, ϱ the friction coefficient, and $\lambda < 1$ the impact restitution coefficient. Assume the measurement $y_d = \theta$ is only available at jumps, namely only impact sensors are used. We know that any

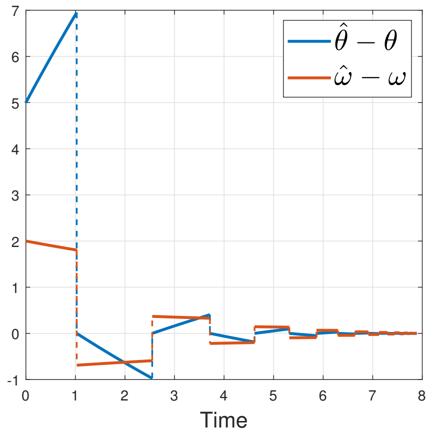

Fig. 1. Jump-based estimation of a Zeno bouncing ball (39).

maximal solution x is Zeno. More precisely, the time between two successive jumps $t_{j+1}(x) - t_j(x)$ tends to zero when j tends to $+\infty$, and its upper bound increases with $|x(0,0)|$. Hence, for any bounded set of initial conditions $\mathcal{X}_0$, $\mathcal{C}_{\mathcal{X}_0}[\mathcal{I}]$ holds with $\mathcal{I}$ of the form $\mathcal{I} = [0, \tau_{\max}]$, with $\tau_{\max} < +\infty$ depending on $\mathcal{X}_0$. Since the system is linear, we implement a linear observer (12), with $L_c = 0$ and L_d chosen such that (30) holds, where $A_c = \left(\begin{smallmatrix} 0 & 1 \\ 0 & -\varrho \end{smallmatrix}\right)$, $A_d = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -\lambda \end{smallmatrix}\right)$ and $H_d = (1, 0)$. As in Remark 5.3, we compute a polytopic decomposition of $\exp(A_c \tau)$ based on the residues of A_c. Because one eigenvalue of A_c equals zero and $\tau_{\min} = 0$, we obtain that $\exp(A_c \tau)$ is in the convex hull of only two matrices $M_1 = I$ and $M_2 = \left(\begin{smallmatrix} 1 & 3.9347 \\ 0 & 0.6065 \end{smallmatrix}\right)$ for $\tau_{\min} = 0$, $\tau_{\max} = 5$, $\lambda = 0.8$, and $\rho = 0.1$. Solving (34) with Yalmip then gives $L_d = (-1, -0.1085)^\top$. The result of a simulation [1] with initial condition $x_0 = (5, 0)$, $\hat{x}_0 = (10, 2)$ is shown on Figure 1. Note that one could also use the hybrid Kalman filter (36) with a varying gain L_d.

7.2 Spiking Neurons

The parameterized model of a spiking neuron in Izhikevich (2003) results in a hybrid system $\mathcal{H}$ as in (2) with state $(x_1, x_2) \in \mathbb{R}^2$ and data given by

$$\begin{aligned} f(x) &= \big(0.04x_1^2 + 5x_1 + 140 - x_2 + I_{\text{ext}}\ ,\ a(bx_1 - x_2)\big) \\ g(x) &= (c, x_2 + d) \qquad , \qquad h_c(x) = h_d(x) = x_1 \\ C &= \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 \leq v_m\} \\ D &= \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 = v_m\} \end{aligned} \tag{40}$$

where x_1 is the membrane potential, x_2 is the recovery variable, and I_{ext} represents the (constant) synaptic current or injected DC current. The value of the input I_{ext} and the model parameters a, b, c, and d, as well as the threshold voltage v_m characterize the neuron type and

[1] Available on `https://github.com/HybridSystemsLab/SyncHyObserverMechanicalSysImpacts`.

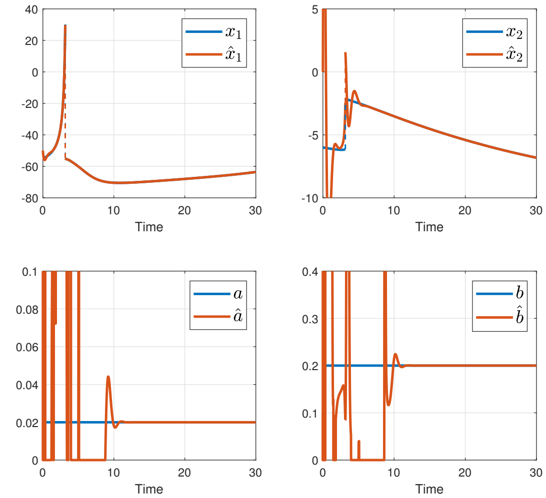

Fig. 2. Flow-based estimation of (x_1, x_2, a, b) in the neuron model (40) through high-gain observer.

its firing pattern Izhikevich (2003). The solutions are known to have an average dwell-time (actually a dwell-time), and the jump times can be detected from the discontinuities of the output $y = x_1$. We consider two scenarios [2] for estimation of state variables and parameters.

Let us first assume that c and d are known and we want to estimate the state x_2 as well as the parameters a and b. Adding the constant parameters a and b to the state, the nonlinear map T made of the successive derivatives of h along f defined by (23) for $d_z = 4$ is an injective immersion with respect to $x = (x_1, x_2, a, b)$ if the matrix $\left(\begin{smallmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{smallmatrix}\right)$ is invertible along the plant trajectories. Under this condition, we can thus use the high-gain design of Example 4.3. The result of a simulation with $I_{\text{ext}} = 10, a = 0.02, b = 0.2, c = -55, d = 4$, $x(0,0) = (-55, -6, a, b)$, $\hat{x}_0 = (-50, 0, 0.1, 0.1)$, $z(0,0) = T(\hat{x}_0)$, $\ell = 4$, $K = (3.0777, 4.2361, 3.0777, 1)$ and appropriate saturations is shown on Figure 2.

Let us now consider the case where a, b are known but d is not. Neither the continuous dynamics nor the discrete dynamics are observable for (x_1, x_2, d) with output x_1, so a flow-based observer cannot be used. However, x_2 is observable from the flow and d impacts x_2 at jumps, so the system as a whole could be observable thanks to the jumps. Actually, we observe that the model is linear in the unknowns x_2 and d, namely the dynamics of (x_1, x_2, d) are characterized by the matrices $A_c = \left(\begin{smallmatrix} 0 & -1 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $A_d = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $H_d = (1, 0, 0)$, modulo output injection. One can check that the equivalent discrete-time model given by (31), namely the pair $A(\tau) :=$

[2] Available on `https://github.com/HybridSystemsLab/SyncHyObserverSpikingNeurons`.

$A_d \exp(A_c\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-a\tau & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $H(\tau) := H_d \exp(A_c\tau) = \begin{pmatrix} 1 & -\tau & 0 \end{pmatrix}$ is observable for any nonzero τ. Let's say that from the measurements, the flow intervals in-between firing times are known to be within a compact set $\mathcal{I} = [\tau_{\min}, \tau_{\max}]$ with $\tau_{\min} > 0$. Since A_c is nilpotent of order 2, from Remark 5.3, it is enough to solve the two LMIs given by

$$\begin{pmatrix} P & (I+\tau A_c)^\top (PA_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \qquad (41)$$

for $\tau = \tau_{\min}$ and $\tau = \tau_{\max}$. If they are solvable (with common P and $\tilde{L}_d$), then with Corollary 5.2, we obtain an observer with state $z = (\hat{x}_1, \hat{x}_2, \hat{d})$ by taking

$$F(\hat{x}_1, \hat{x}_2, \hat{d}, y_c) = (f(y_c, \hat{x}_2), 0)$$
$$G(\hat{x}_1, \hat{x}_2, \hat{d}, y_d) = (c, \hat{x}_2 + \hat{d}, \hat{d}) + L_d(y_d - \hat{x}_1) \ .$$

For instance, for $\tau_{\min} = 30$ and $\tau_{\max} = 50$, solving the LMIs via Yalmip for P and $\tilde{L}_d$, we get $L_d = (0, 0.0028, -0.0063)^\top$. The results of a simulation are provided on the top and bottom left of Figure 3. Of course, because information from the output is injected at the jumps only, the estimation error takes a longer time to converge than the flow-based observer where instantaneous observability is guaranteed during flow. Note that theoretically, we could have used the estimate $\hat{x}_1$ instead of the measurement y_c in the observer flow F for the linear terms in x_1, and take it into account in A_c. However, this does not work well with the term $5x_1$ because it produces an error growing in $e^{5\tau}$ during flow, which reaches 10^{86} for $\tau = 40$. In other words, a jump-based observer will only work numerically if the eigenvalues of A_c are reasonable compared to the length of the flow intervals. Finally, we plot on the bottom right of Figure 3, the norm of the estimation error $\|(\hat{x}_1, \hat{x}_2, \hat{d}) - (x_1, x_2, d)\|$ in steady state, when the jumps of the observer are triggered with some delay after the plant state. We observe the error peaking to around 80 during the delay intervals whatever the delay's value ; and then the smaller the delay the smaller the error outside of the delay intervals, as predicted by Theorem 6.4.

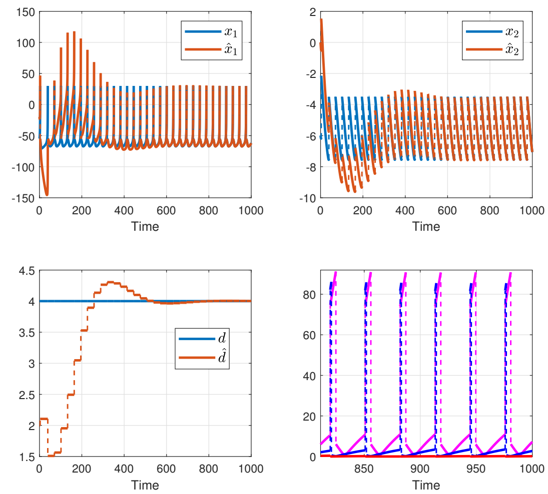

Fig. 3. Top and bottom left : jump-based estimation of (x_1, x_2, d) in the neuron model (40) without delay. Bottom right : norm of the residual estimation error without delay (red), with delay $\Delta = 1$ (blue), with delay $\Delta = 5$ (pink).

8 Conclusion

Under the assumption that the jumps of the plant can be detected, we have given Lyapunov-based sufficient conditions for asymptotic convergence of an observer for general hybrid systems. Design methods have been provided, in particular high-gain designs for nonlinear differentially observable continuous dynamics, and discrete-based designs when observability is ensured from the output at jump times. Jumps in the observer must be triggered at the same time as the plant jumps but we have shown their robustness with respect to detection delays, namely semi-global practical stability of the estimation error outside the delay intervals. Those results provide a new insight for the design of observers for switched systems.

However, unlike the flow-based designs which are inherently made for nonlinear dynamics, the nonlinear jump-based designs are limited by the computation of the flow reachable set, as well as the limits already existing for the design of nonlinear discrete-time observers. Future work consists in combining the flow-based and jump-based designs via high-gain ISS interconnections in order to enlarge the class of systems for which those designs are constructive. Moreover, further work is needed to evaluate how those observers can be used in the context of output-feedback, as was done on a particular example of biped robot in Grizzle et al. (2007), with precisely the high-gain observer of Example 4.3.

More importantly, observers able to synchronize automatically their jumps with those of the plant still need to be developed, at least locally, to avoid relying on the often noisy/delayed jump detection. Indeed, the robust practical stability result of Theorem 6.4 would then enable to combine such local auto-synchronizing observers with the global observers of this paper. This problem represents a significant challenge since the entire analysis needs to be rethought to handle non-simultaneous jumps and ensure contraction of the difference between jump times. Preliminary work in this direction is presented in Bernard and Sanfelice (2021).

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