Explaining the "mystery" of periodicity in inter-transmission times in two-dimensional event-triggered controlled systems

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***Abstract*— Motivated by scenarios where the communication or the computation resources are limited, event-triggered control consists of transmitting data between the plant and the controller according to the actual system needs, and not the elapsed time since the last transmission instant as in traditional sampled-data control, so that the desired control objective is achieved. A range of techniques are nowadays available to design event-triggered controllers. However, we generally have only very little information about the actual behaviour of the transmission instants and thus about the amount of transmissions being actually generated, though this is a key feature of the design. In this paper, we analyse the inter-event times, i.e., the times between two successive transmission instants, when the plant is modeled as a two-dimensional linear time-invariant system. The controller is a state-feedback law and the triggering rule is the relative threshold policy, which is allowed to be time-regularized. One of the main results in this paper is the explanation of the oscillatory behaviour of the inter-event times when the constant used to define the threshold is small relative to 1, a phenomenon commonly observed in simulations but never explained so far. More generally, the presented results help to understand the behaviour of the inter-event times, instead of solely relying on numerical simulations, and thereby can be exploited to rigorously evaluate the performance of the considered triggering condition in terms of (average) inter-transmission times.**

***Index Terms*— Event-triggered control, sampled-data, hybrid systems**

I. Introduction

Event-triggered control is a transmission paradigm, which consists in generating communications between the plant and the controller using a state-dependent criterion that is continuously monitored [22]. The basic idea is to adapt plant-controller communication based on the current system needs, and not the time elapsed since the last transmission as in traditional time-triggered control. Event-triggered control is relevant in scenarios where the control system is subject to communication or computation constraints, as in networked control systems or embedded systems see, e.g., [23], [25], [50].

While various event-triggered control techniques are available in the literature, very little is known about the actual behaviour of the inter-event times, i.e., the time between two successive transmission instants. This is problematic as the inter-event times directly relate to the amount of transmissions generated and is therefore of primary importance in view of the raison d'être of event-triggered control. In most cases, the analysis of the inter-event times only ensures the existence of a dwell-time also sometimes called "minimum inter-event time", that is a (uniform) strictly positive amount of time between any two successive transmissions. This property allows avoiding the Zeno phenomenon and is required by practical hardware limitations. Besides the existence of a dwell-time, we generally do not know how the inter-event times behave. Numerical simulations are thus often carried out to get an idea of it. Exceptions exist though. The work in [46] provides conditions under which the inter-event times approximately converge to a constant value when the triggering rule satisfies a homogeneity property and when zero-order hold devices are used to implement the controller. This reference also analyses stability properties of the inter-event times assuming it exhibits a periodic pattern. Similarly, conditions for the inter-event times function to exhibit continuity and periodicity properties have been very recently proposed in [38] for two-dimensional linear time-invariant systems. The works on/based on discrete-time systems in, e.g., [4], [10], [11], which rely on model predictive control techniques, provide analytical guarantees regarding the average inter-event times. When the plant dynamics evolve in continuous-time and smart actuators are available, properties on the inter-event times can be derived when using model-based holding functions [34], as advocated in [5], [27] for fixed threshold policies, even in the presence of stochastic disturbances. Interestingly, in the absence of disturbances, model-based implementations [34] can lead to a single transmission to stabilize the system in the ideal state-feedback control case. Also, some schemes ensure that inter-event times grow larger or converge to a constant as the solution converges to the origin [24], [32], and [37, Section V.B], or as time grows [35]. Another recent relevant line of

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research is based on symbolic abstractions see, e.g., [18], [19], [29]. The general idea is to partition either the state space or the inter-event times and then to construct an automaton that schedules transmissions with guaranteed properties on (the long term behaviour of) the inter-event times. Lastly, it has to be noted that several works on event-triggered control under bit-rate constraints and also on event-triggered stochastic estimation analyse the inter-event times, see, e.g., [28], [31], [33] and, e.g., [21], [30], [45], [48], respectively.

Besides the aforementioned works, our understanding of the inter-event times remains limited, while it is a key characteristic of the event-triggered controlled system. Phenomena such as when the inter-event time describes a periodic-like pattern, which is often seen in simulations (see, e.g., [6, Figure 3], [8, Figure 3], [40, Figure 4], [44, Figure 1], [49, Figure 4]), remain unexplained. Interestingly, inter-event times oscillations were observed in one of the earliest works in the field: more than twenty years ago in [7] it was stated that "*Several interesting phenomena have been observed during the simulations. One example is limit cycles in the actual sampling interval*", which is still not elucidated as far as we know. More generally, understanding the behaviour of the inter-event times is essential to appreciate the features of the considered triggering technique and to evaluate its performance in terms of transmissions.

There is a simple reason for our limited understanding of the inter-event times: the question is notoriously challenging technically. In this paper, we focus on plant dynamics given by two-dimensional continuous-time linear time-invariant systems and we will see that the problem becomes quickly technically involved. The controller is a static state-feedback law implemented using zero-order hold devices. The triggering rule is the one in [42], which is one of the pillars of the literature that has been used and extended in various contexts see, e.g., [2], [13], [15], [17], [40], [47]. This triggering law relies on the condition $|x - \hat{x}| \geq \sigma |x|$, where x is the current plant state, $\hat{x}$ is the plant state at the last transmission instant and $\sigma \in \mathbb{R}_{>0}$ is a tunable parameter. Our results also apply for a time-regularized version of [42], in the sense that a given minimum time is enforced between any two transmissions, see, e.g., [1], [9], [14], [16], [39], [41], [43]. This is relevant when we want to have a direct control on the minimum inter-event time as well as for robustness reasons, see [2], [8], [9], [12], [14]. The idea of including time-regularization is to check the condition above once $T \geq 0$ units of times have elapsed since the last transmission instant: if it is satisfied, a transmission between the plant and the controller is triggered. We only talk of time-regularization when $T > 0$, as, for $T = 0$, the "classical" relative triggering law of [42] is obtained.

Our results require σ to be small relative to 1, which is typically the case to ensure the stability of the origin of the closed-loop system, see, e.g., [1], [14], [41], [42]. We will see that accurate results are obtained on examples even when σ is taken close to its maximum admissible value ensuring stability. We first establish key properties of the inter-event times functions, which apply to system of *any* dimension, not only two-dimensional ones. In particular, we provide an expression of the inter-event time, which allows to derive new lower and upper bounds; this result has its own interest and could be exploited for scheduling purposes for instance. We then specialize to two-dimensional systems and distinguish different cases depending on the nature of the eigenvalues λ_1 and λ_2 of the state matrix of the continuous-time closed-loop system in the absence of sampling. In summary, when λ_1 and λ_2 are complex conjugates, we show that the inter-event times oscillate with a period close to $\frac{\pi}{\omega}$, where ω is the absolute value of the imaginary parts of λ_1 and λ_2. This provides for the first time, as far as we know, an explanation of the oscillatory nature of the inter-event times. In addition, we demonstrate that the values taken by the inter-event times over any time interval of length longer than $\frac{\pi}{\omega}$ are almost insensitive to the considered initial condition. This result has important implications: not only the periodicity of inter-event times is explained and analysed, but this means that a single simulation over a time interval of length $\frac{\pi}{\omega}$ is enough to rigorously know the behaviour of the inter-event times for all initial conditions and all times. Compared to [46, Section IV] where periodic patterns of the inter-transmission times are mentioned, (i) we do prove the existence of such patterns, instead of assuming it, (ii) we provide an easy-to-compute expression of the period and (iii) we analyse the impact of the initial conditions on the inter-event times, while [46] assumes exact periodicity, which cannot occur in general as we show, and studies the stability properties of the inter-event times. On the other hand, when λ_1 and λ_2 are real, the inter-event times either converge to a neighborhood of $\max\left\{\frac{\sigma}{|\lambda_1|}, T\right\}$ as time tends to infinity or lies in a neighborhood of $\max\left\{\frac{\sigma}{|\lambda_2|}, T\right\}$ for all positive times. The only case that we do not treat is when $\lambda_1 = \lambda_2$ and the corresponding geometric multiplicity is equal to one because significant technical difficulties arise in this case as we explain. We conjecture that the inter-event times converge to $\max\left\{\frac{\sigma}{|\lambda_1|}, T\right\}$ in this case, which is confirmed by simulations. These results are consistent with [46, Proposition 1] where non-time-regularized homogeneous triggering rules are discussed. We go further here as (i) we carefully analyse the impact of σ (and T) on the inter-event times, (ii) we prove that the inter-event times are close to given values for all positive times in some cases, instead of providing asymptotic properties only, (iii) we address time-regularization. Compared to [38], we provide constructive and easy-to-compute estimates on the behaviour of the inter-event times, we reveal the relationship between these properties and the eigenvalues of the closed-loop state matrix and we analyse the impact of the initial conditions on the inter-event times. The provided simulation results confirm and show the strength of the obtained theoretical guarantees.

Compared to preliminary version of this work [36], the main novelty is the time regularization of the triggering law of [42], which is important as the relative threshold strategy of [42] is known to be non-robust [8] as mentioned above. We also present several new results, including new lower and upper-bounds on the inter-event times (see Lemma 1), discussions about the application or the extension of the results to other classes of systems (see Section V), as well as new examples including a nonlinear one (Section VI).

The remainder of the paper is organized as follows. The problem is formally stated in Section II. Then, key properties of the inter-event time function are established in Section III. The main results are given in Section IV. Discussions on the extension of the results are proposed in Section V. The results are confronted to numerical simulations in Section VI. Section VII provides conclusions. Finally, lengthy proofs are presented in the appendix.

Notation. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_{\geq 0} := [0,\infty)$, $\mathbb{R}_{>0} := (0,\infty)$, $\mathbb{R}_{<0} := (-\infty,0)$, $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_{\geq 0} := \{0,1,2,\ldots\}$ and $\mathbb{Z}_{>0} := \{1,2,\ldots\}$. Given a set $\mathcal{E} \subseteq \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$, we use $\mathcal{E}^*$ to denote $\mathcal{E}\setminus\{0\}$. We denote the set of unit norm vectors of $\mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$, as $\mathbb{S}_n$, i.e., $\mathbb{S}_n := \{x \in \mathbb{R}^n : |x| = 1\}$ where $|\cdot|$ stands for the Euclidean norm. The notation (x,y) stands for $[x^\top, y^\top]^\top$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ right continuous and $t \geq 0$, we write $f(t^-)$ to denote $\lim_{t' \to t,\, t'<t} f(t')$. We use I to denote the identity matrix of appropriate dimension according to the context. For a matrix $A \in \mathbb{R}^{n \times n}$ with $n \in \mathbb{Z}_{>0}$, we respectively denote its maximum and minimum singular values as $\sigma_{\max}(A) := \sqrt{\lambda_{\max}(A^\top A)}$ and $\sigma_{\min}(A) := \sqrt{\lambda_{\min}(A^\top A)}$, where $\lambda_{\max}(A^\top A)$ and $\lambda_{\min}(A^\top A)$ are the maximal and the minimal eigenvalues of $A^\top A$, respectively. The argument[1] of $x = (x_1, x_2) \in \mathbb{R}^{2*}$ is defined as

$$
\arg : \mathbb{R}^{2*} \to [-\pi, \pi]
$$
$$
x \mapsto \arg(x) := \begin{cases} \arctan(\frac{x_2}{x_1}), & \text{when } x_1 > 0 \\ \arctan(\frac{x_2}{x_1}) + \pi, & \text{when } x_1 < 0 \text{ and } x_2 \geq 0 \\ \arctan(\frac{x_2}{x_1}) - \pi, & \text{when } x_1 < 0 \text{ and } x_2 < 0 \\ \frac{\pi}{2}, & \text{when } x_1 = 0 \text{ and } x_2 > 0 \\ -\frac{\pi}{2}, & \text{when } x_1 = 0 \text{ and } x_2 < 0 \end{cases}
$$

By argument, we mean here the angle of the two-dimensional vector x, which, without loss of generality, is treated as a complex number.

II. Problem statement

Consider the plant model

$$
\dot{x} = Ax + Bu, \tag{1}
$$

where $x \in \mathbb{R}^n$ is the state with $n \in \mathbb{Z}_{>0}$, $u \in \mathbb{R}^m$ is the control input with $m \in \mathbb{Z}_{>0}$, and (A, B) is stabilizable. We restrict n to be equal to 2 later, in Section IV. The control input u is given by the feedback law

$$
u = Kx, \tag{2}
$$

where the matrix $K \in \mathbb{R}^{m \times n}$ is such that $A + BK$ is Hurwitz; such a matrix does exist since (A, B) is stabilizable.

We study the scenario where controller (2) is implemented on a digital platform and communicates with system (1) at time instants t_i, $i \in \mathcal{I}$ with $\mathcal{I} := \{1, 2, \ldots, N\} \subseteq \mathbb{Z}_{\geq 0}$ with $N \in \mathbb{Z}_{>0} \cup \{\infty\}$; this will be clarified in Section III. Between two successive transmission instants, the control input is held constant using a zero-order hold device, and it is updated at every t_i, $i \in \mathcal{I}$, which leads to

$$
u = K\hat{x} \tag{3}
$$

with $\hat{x}$ being given by the solution to

$$
\begin{aligned} \dot{\hat{x}}(t) &= 0 && \text{for all } t \in (t_i, t_{i+1}) \\ \hat{x}(t_i^+) &= x(t_i). \end{aligned} \tag{4}
$$

We also introduce the clock variable $\tau \in \mathbb{R}_{\geq 0}$ to measure the time elapsed since the last transmission instant. This variable is needed when the triggering law is time-regularized. Its dynamics are given by

$$
\begin{aligned} \dot{\tau}(t) &= 1 && \text{for all } t \in (t_i, t_{i+1}) \\ \tau(t_i^+) &= 0. \end{aligned} \tag{5}
$$

The overall system is

$$
\left.\begin{aligned} \dot{x}(t) &= Ax(t) + BK\hat{x}(t) \\ \dot{\hat{x}}(t) &= 0 \\ \dot{\tau}(t) &= 1 \end{aligned}\right\} \text{ for all } t \in (t_i, t_{i+1})
$$
$$
\left.\begin{aligned} x(t_i^+) &= x(t_i) \\ \hat{x}(t_i^+) &= x(t_i) \\ \tau(t_i^+) &= 0 \end{aligned}\right\} \text{ for all } i \in \mathcal{I}. \tag{6}
$$

To obtain a solution to (6) in the Carathéodory sense, for each $i \in \mathcal{I}$, the latter flows on $[t_i, t_{i+1}]$ and experiences a jump at t_{i+1}, and so on. Also, by a solution, we mean a maximal solution, i.e., one that cannot be extended.

The sequence of transmission instants t_i, $i \in \mathcal{I}$, is defined implicitly by a state-dependent triggering rule. In particular, we consider the law in [42], possibly time-regularized, to define these instants as proposed in, e.g., [1], [14], [16]. Hence, a transmission occurs whenever

$$
|\hat{x}(t) - x(t)| \geq \sigma |x(t)| \quad \text{and} \quad \tau(t) \geq T, \tag{7}
$$

where $\sigma > 0$ and $T \geq 0$ are design parameters. We only talk of time-regularization when $T > 0$ as mentioned in Section I, and we note that, when $T = 0$, the second condition in (7) is always verified. The first inequality in (7) guarantees that the error $|\hat{x} - x|$ induced by sampling is smaller than $\sigma|x|$ as in [42], after T units of times have elapsed since the last transmission; otherwise a transmission is triggered. On the other hand, the inequality $\tau(t) \geq T$ in (7) enforces a minimum time between successive transmissions of at least T units of time, which we design, whenever $T > 0$. Constants σ and T are selected to ensure that the origin of system (6)-(7) is uniformly globally exponentially stable, as formalized next.

Standing Assumption 1 (SA1): There exist $d_1 \geq 1$, $d_2, \sigma^*, T^* > 0$ such that for all $(\sigma, T) \in (0, \sigma^*) \times [0, T^*)$, for all solutions $(x, \hat{x}, \tau)$ to (6)-(7) and $t \geq 0$, $|(x(t), \hat{x}(t))| \leq d_1 e^{-d_2 t} |(x(0), \hat{x}(0))|$. □

Various techniques are available in the literature to compute the bounds σ^* and T^* to ensure SA1, see,[2] e.g., [1], [14], [16], [39], [41], [42], [42], [43].

[1] Often, the argument is defined as $\arg(x) = \arctan(\frac{x_2}{x_1})$, but this is only true when $x_1 > 0$.

[2] Although the work in [42] does not consider time-regularized triggering laws, SA1 does hold by taking $T^* = 0$ where σ^* is given in [42, Corollary IV.1].

We assume that $t_0 = 0$, which means that the initial time $t = 0$ is a sampling time. We therefore concentrate on solutions to (6)-(7) initialized at time 0 with initial state of the form $(x_0, x_0, 0)$ where $x_0 \in \mathbb{R}^n$, since after a sampling instant $\hat{x}$ is equal to x and τ to 0. The first inter-transmission time is the time, greater than or equal to T, such that $|\hat{x} - x|$ is larger than or equal to $\sigma|x|$. Since $t_0 = 0$, and $x(0) = \hat{x}(0) = x_0$, this time only depends on x_0, and is parameterized by σ and T, we therefore denote it $\tau_{\sigma,T}(x_0)$. The first inter-transmission time is defined as, given x_0,

$$\tau_{\sigma,T}(x_0) := \inf\{t \geq T : |x_0 - \phi(t, x_0)| \geq \sigma|\phi(t, x_0)|\}, \qquad (8)$$

where $\phi(t, x_0)$ denotes the solution[3] to $\dot{x} = Ax + BKx_0$ at time $t \geq 0$, initialized at time zero at state x_0. By induction, we denote the i^{th} inter-transmission time, with $i \in \mathcal{I}$, as $\tau_{\sigma,T}(x(t_i))$ which only depends on $x(t_i)$, as $x(t_i^+) = \hat{x}(t_i^+) = x(t_i)$ and $\tau(t_i^+) = 0$. The mathematical definition of $\tau_{\sigma,T}(x(t_i))$ is given by (8) by simply replacing x_0 by[4] $x(t_i)$. Noting that $\hat{x}(t) = x(t_i)$ for $t \in (t_i, t_{i+1}]$ in view of (6), we can write $\tau_{\sigma,T}(x(t_i)) = \tau_{\sigma,T}(\hat{x}(t))$.

Problem Statement: The objective is to analyse the properties of $\tau_{\sigma,T}(\hat{x}(\cdot))$ along solutions to the hybrid system (6)-(7) initialized at $(x_0, x_0, 0)$ for some $x_0 \in \mathbb{R}^n$ when $n \geq 2$ and for σ small relative to 1. $\blacksquare$

The only guarantee on the inter-event times we find in the literature for the triggering condition (7) is the existence of a minimum inter-event time. More precisely, when $T = 0$, we know from [42] that there exists $\varepsilon > 0$ such that $\tau_{\sigma,T}(x_0) \geq \varepsilon$ for any $x_0 \in \mathbb{R}^{n,*}$, and, when $T > 0$, $\tau_{\sigma,T}(x_0) \geq T$ for any $x_0 \in \mathbb{R}^n$, which directly follows from (7). We aim at going further in the analysis of the function $\tau_{\sigma,T}$: we want to provide analytical characterizations of the behaviour of $t \mapsto \tau_{\sigma,T}(\hat{x}(t))$ along the solutions to (6)-(7). In that way, we would be able to rigorously quantify the amount of transmissions generated by the triggering rule.

For this purpose, we view system (6)-(7) as a family of systems parameterized by σ and T, and the presented results apply for small σ in (7), which we justify as follows. First, σ typically needs to be small for the closed-loop system in (6)-(7) to exhibit stability properties, see SA1. Second, our line of analysis exploits properties of the limit case when $\sigma \to 0$. This allows us to derive simple and accurate properties on the inter-event times, which are corroborated by numerical simulations in Section VI even when σ is taken close to σ^* defined in SA1.

The next section establishes preliminary instrumental properties of the map $\tau_{\sigma,T}$.

III. Properties of the map $\tau_{\sigma,T}$

We first need to make sure that $\tau_{\sigma,T}$ cannot be equal to ∞. In other words, we want to guarantee that $\tau_{\sigma,T}(\mathbb{R}^n) \subseteq [0, \infty)$. This is ensured by the next proposition.

[3] We abandon in the following the notation ϕ to denote a solution, and use instead directly x (or $\hat{x}$).

[4] We can still consider the time from $t = T$ in (8) in this case, and not from $t = t_i + T$, as system (6)-(7) is time-invariant and satisfies the semi-group property.

Proposition 1: For any $x_0 \in \mathbb{R}^n$, $\sigma \in (0, \sigma^*)$ and $T \in [0, T^*)$ where σ^* and T^* come from SA1, $\tau_{\sigma,T}(x_0) \in [0, \infty)$. $\blacksquare$

Proof: Let $\sigma \in (0, \sigma^*)$ and $T \in [0, T^*)$. We first note that $\tau_{\sigma,T}(0) = T$ in view of (6), (7) and (8). To prove that $\tau_{\sigma,T}$ takes finite values on $\mathbb{R}^{n,*}$, we proceed by contradiction and we suppose that there exists $x_0 \in \mathbb{R}^{n,*}$ such that $\tau_{\sigma,T}(x_0) = \infty$. This means that the solution $(x, \hat{x}, \tau)$ to system (6)-(7) initialized at $(x_0, x_0, 0)$ never jumps. By SA1, $x(t)$ is defined for all positive times and converges to zero as t tends to infinity. On the other hand, $|\hat{x}(t) - x(t)| = |x_0 - x(t)| \geq \sigma|x(t)|$ for any $t \geq T$ since no jump occurs. By taking the limit as $t \to \infty$ on both sides of the latter inequality, we obtain $|x_0| \leq 0$, which is impossible since $x_0 \neq 0$. This proves the desired result.

Proposition 1 implies that $\mathcal{I} = \mathbb{Z}_{\geq 0}$ as introduced in Section II, for any $x_0 \in \mathbb{R}^n$ and any pair (σ, T), which satisfies SA1.

Second, we state a homogeneity property of $\tau_{\sigma,T}$, which is established in [3, Theorem 4.11 and Remark 4.12] for the case where $T = 0$. The proof directly follows when $T > 0$, and is therefore omitted.

Proposition 2: For any $x_0 \in \mathbb{R}^{n,*}$, $\lambda \in \mathbb{R}_{\neq 0}$, $\sigma \in (0, \sigma^*)$ and $T \in [0, T^*)$, $\tau_{\sigma,T}(x_0) = \tau_{\sigma,T}(\lambda x_0)$. $\blacksquare$

Proposition 2 states that $\tau_{\sigma,T}$ is constant along lines passing through the origin, excluding the origin.

Third, we derive an approximate expression of $\tau_{\sigma,T}$ on $\mathbb{R}^{n,*}$ for small σ. We distinguish two cases for this purpose whether, given $m > 0$, the pairs (σ, T) belong to the set

$$\mathcal{S}_m(\sigma^*, T^*) := \left\{(\sigma, T) : \sigma \in (0, \sigma^*), T \in [0, \min\{m\sigma, T^*\})\right\} \qquad (9)$$

or not. While the set $\mathcal{S}_m$ imposes no extra condition on σ compared to SA1, it requires that, when σ is small, so is T (which implies that T depends on σ). Note that, when no time-regularization mechanism is implemented, $T = 0$ and any pair $(\sigma, 0)$ belongs to $\mathcal{S}_m(\sigma^*, T^*)$. The next proposition provides approximate expressions of $\tau_{\sigma,T}$ on $\mathbb{R}^{n,*}$ for small σ in the general case first, and then provides additional expressions when the pairs (σ, T) belong to $\mathcal{S}_m(\sigma^*, T^*)$ for a given $m > 0$.

Proposition 3: There exist $r^1 : \mathbb{R}^n \times [0, 1) \to \mathbb{R}$, $c_r^1 > 0$ and $\sigma_1^1 \in (0, \min\{1, \sigma^*\})$ such that for any $\sigma \in (0, \sigma_1^1)$, $T \in [0, T^*)$ and any $x_0 \in \mathbb{R}^{n,*}$, $\tau_{\sigma,T}(x_0) = T + r^1(x_0, \sigma)$ and $|r^1(x_0, \sigma)| \leq c_r^1 \sigma$. Moreover, for any fixed $m > 0$, there exist $r : \mathbb{R}^n \times [0, 1) \to \mathbb{R}$, $c_r > 0$ and $\sigma_1 \in (0, \min\{1, \sigma^*\})$ such that for any $(\sigma, T) \in \mathcal{S}_m(\sigma_1, T^*)$, $\tau_{\sigma,T}(x_0) = \max\left\{\frac{|x_0|}{|A_c x_0|}\sigma + r(x_0, \sigma), T\right\}$ and $|r(x_0, \sigma)| \leq c_r \sigma^2$, where $A_c := A + BK$. $\blacksquare$

Proposition 3 states that $\tau_{\sigma,T}(x_0)$ can be written as T plus a term of the order of σ when (σ, T) is selected as in SA1 and σ is small compared to 1. This result implies that when T is "big" compared to σ, we essentially have periodic sampling as $\tau_{\sigma,T}(x_0)$ is then well approximated by T for all x_0 in this case, since $r^1(x_0, \sigma)$, which is of the order of σ, is negligible compared to T. Because of that, we concentrate on the case where the pairs (σ, T) belong to $\mathcal{S}_m(\sigma_1, T^*)$ for a given fixed $m > 0$ in the remainder of the paper. In this case, Proposition 3 states that $\tau_{\sigma,T}(x_0)$ is well approximated by $\max\left(\frac{|x_0|}{|A_c x_0|}, T\right)$ for small $\varepsilon > 0$, for any $x_0 \in \mathbb{R}^{n_x}$, which is of the order of ε, while the error term $r(x_0, \varepsilon)$ is of the order of ε^2. The fact that the constant c_r, which appears in the upper-bound of the norm $r(x_0, \varepsilon)$, is independent of x_0 and (ε, T) (but does depend on m), is crucial in the following. We will see via examples in Section VI that the forthcoming analytical guarantees on the inter-event times may provide accurate estimations even when ε and T are taken close to their respective maximal admissible values ε and T according to SA1.

Interestingly, Proposition 3 can be used to derive a new lower bound as well as an upper bound on $\tau_{\varepsilon,T}(x_0)$, which have their own interest.

Lemma 1: Given $m > 0$, for any $(\varepsilon, T) \in S_m(\varepsilon_1, T)$ with T from SA1 and ε_1 from Proposition 3, and any $x_0 \in \mathbb{R}^{n_x}$, $\underline{\tau}_{\varepsilon,T} \leq \tau_{\varepsilon,T}(x_0) \leq \overline{\tau}_{\varepsilon,T}$ where $\underline{\tau}_{\varepsilon,T} := \max\left(\frac{\varepsilon}{\lambda_{\max}(A_c)} - c_r\varepsilon^2, T\right)$, $\overline{\tau}_{\varepsilon,T} := \max\left(\frac{\varepsilon}{\lambda_{\min}(A_c)} + c_r\varepsilon^2, T\right)$ and c_r as in Proposition 3. ■

Proof: Let $m > 0$, $x_0 \in \mathbb{R}^{n_x}$ and $(\varepsilon, T) \in S_m(\varepsilon_1, T)$. In view of Proposition 3, $\tau_{\varepsilon,T}(x_0) = \max\left\{\varepsilon\frac{|x_0|}{|A_c x_0|} + r(x_0, \varepsilon), T\right\}$. On the other hand, $\frac{|x_0|}{|A_c x_0|} \leq \max_{x_0^1 \neq 0} \frac{|x_0^1|}{|A_c x_0^1|} = \max_{x_0^1 \text{ s.t. } |x_0^1| = 1} \frac{|x_0^1|}{|A_c x_0^1|}$ in view of Proposition 2, and thus $\frac{|x_0|}{|A_c x_0|} \leq \frac{1}{\lambda_{\min}(A_c)}$. Consequently, since $r(x_0, \varepsilon) \leq c_r\varepsilon^2$, $\tau_{\varepsilon,T}(x_0) \leq \overline{\tau}_{\varepsilon,T}$ with $\overline{\tau}_{\varepsilon,T}$ defined in Lemma 1. We follow similar lines to derive the lower bound inequality on $\tau_{\varepsilon,T}(x_0)$ in Lemma 1.

Lemma 1 provides a global lower-bound on the inter-transmission times when ε is small. Compared to the exact expression of the (global) minimum inter-event time we find in [15, Theorem IV.1], which addresses non-time regularized triggering conditions, i.e. $T = 0$, the bound in Lemma 1 is more conservative a priori but easier to compute. Indeed, we can simply take it as $\max\left(\frac{\varepsilon}{\lambda_{\max}(A_c)}, T\right)$ as the term $c_r\varepsilon^2$ is negligible compared to it for small ε. Lemma 1 also gives a global upper-bound on the inter-event times, for the first time as far as we know, which is similarly well-approximated by $\max\left(\frac{\varepsilon}{\lambda_{\min}(A_c)}, T\right)$. Both bounds of Lemma 1 may be very accurate and even exact, as illustrated in Section VI-A.

Remark 1: To know lower and upper-bounds on the inter-event times may be precious in practice, as it provides guarantees on the window of time at which the transmissions occur, which can be used for scheduling purposes when the plant and the controller communicate over a shared digital network for instance. ■

On the other hand, Proposition 3 and Lemma 1 apply for $x_0 \in \mathbb{R}^{n_x}$. The case where $x_0 = 0$ was ignored as some of the above expressions above are not well-defined in this case. Now, when $x_0 = 0$ and $T = 0$, $\tau_{\varepsilon,0}(0) = 0$, which means that an infinite number of jumps occurs in finite time at the origin[5]. This potential issue is clarified when writing the overall system using the hybrid formalism [20], see [15] and [37, Section IV.B] for more details. On other hand, when $T > 0$, $\tau_{\varepsilon,0}(0) = T$ and this implies that $\tau_{\varepsilon,0}(\hat{x}(t)) = T$ for all $t \geq 0$ in view of (6)-(7). In other words, a solution initialized at state $(0, 0, 0)$ at time 0 experiences jumps every T units of time: we have periodic sampling. These singularities invite us to discard the case where x_0 is equal to 0 in the sequel. This is fine according to the next proposition, which ensures the $(x, \hat{x})$-component of any solution to system (6)-(7) initialized at $(x_0, x_0, 0)$ with $x_0 \neq 0$ will never reach $(0, 0)$. We can therefore indeed exclusively consider x and $\hat{x}$ on $\mathbb{R}^{n_x}$ in the rest of this study.

Proposition 4: Given $m > 0$, for any $(\varepsilon, T) \in S_m(\varepsilon_1, T)$, any solution $(x, \hat{x}, \tau)$ to system (6)-(7) initialized at $(x_0, x_0, 0)$ with $x_0 \in \mathbb{R}^{n_x}$ verifies $x(t) \neq 0$ and $\hat{x}(t) \neq 0$ for all $t \geq 0$. ■

Proof: The proof relies on the next claims, whose proofs are given in the appendix.

Claim 1: Given $m > 0$, there exists $\mu > 0$ such that for any $(\varepsilon, T) \in S_m(\varepsilon_1, T)$, any solution $(x, \hat{x}, \tau)$ to system (6)-(7) initialized at $(x_0, x_0, 0)$ with $x_0 \in \mathbb{R}^{n_x}$ verifies $|\hat{x}(t) - x(t)| \leq \mu|x(t)|$ for all $t \geq 0$. ■

Claim 2: Given $m > 0$, for any $x_0 \in \mathbb{R}^{n_x}$, the solution x to $\dot{x} = Ax + BKx_0$ initialized at x_0 satisfies $x(t) \neq 0$ for all $t \in [0, \tau_{\varepsilon,T}(x_0)]$. ■

The desired result follows by applying Claim 2 on each inter-transmission interval, since x is not affected by jumps in view of (6)-(7) and $\hat{x}(t) = x(t_i)$ for any $t \in [t_i, t_{i+1})$ and any $i \in \mathbb{I}$.

Remark 2: We recall that Proposition 4 applies in the absence of exogenous perturbations; otherwise it may not be true, see, e.g., [8], [12]. ■

We end this section with a continuity-like property with respect to time of $\tau_{\varepsilon,T}$ along the x-component of solutions to (6)-(7).

Lemma 2: Given $m > 0$, there exist $c_{cont,1}, c_{cont,2} \geq 0$ such that for any $(\varepsilon, T) \in S_m(\varepsilon_1, T)$ with ε_1 from Proposition 3, any $x_0 \in \mathbb{R}^{n_x}$, the x-component of the solution to (6)-(7) initialized at $(x_0, x_0, 0)$ verifies for any $t, t^1 \geq 0$, $|\tau_{\varepsilon,T}(x(t)) - \tau_{\varepsilon,T}(x(t^1))| \leq \varepsilon c_{cont,1}|t - t^1| + \varepsilon^2 c_{cont,2}$. ■

Lemma 2 implies that for close times, $\tau_{\varepsilon,T}(x)$ takes close values. This result plays a key role in some of the forthcoming proofs.

Remark 3: In Proposition 3 (and Lemmas 1 and 2) as well as in the forthcoming statements, the results rely on the existence of some upper-bound on ε (ε_1 in Proposition 3). Estimates of these bounds can be derived from the proofs. However, these estimates are typically subject to some conservatism and may not be easy to compute, which is the reason why these are not provided explicitly. ■

IV. Main results

From now on $n_x = 2$. We distinguish different cases according to the type of eigenvalues of $A_c = A + BK$, which are denoted by λ_1 and λ_2, under SA1. Note that the real parts of λ_1 and λ_2 are strictly negative, otherwise SA1 would not hold.

[5] We consider Carathéodory solutions in this work as mentioned in Section I, which leads to a slight inconsistency because the solution initialized at the origin is trivial, as it cannot flow. We nevertheless show in the following that we can exclude the origin in the forthcoming analysis.

A. When λ_1 and λ_2 are complex conjugates and non-real

We write $\lambda_1 = \sigma + i\omega$ and $\lambda_2 = \sigma - i\omega$ where $\omega \neq 0$ and $\sigma > 0$.

The next theorem explains the oscillatory behaviour of the inter-event times often observed in simulations, see Section I for references.

Theorem 1: Given $m > 0$, when λ_1 and λ_2 are non-real, complex conjugates, there exist $\varepsilon_{complex} \in (0,1]$, $\bar{c}_r > 0$, $\bar{c}_{complex} \geq 0$ such that for any initial condition $(x_0, x_0, 0)$ with $x_0 \in \mathbb{R}^2 \setminus \{0\}$, and any $(\varepsilon, T) \in S_m(\varepsilon_{complex}, T)$, the corresponding solution $(x, \hat{x}, \tau)$ to (6)-(7) verifies the next property. For any $t \geq 0$, there exist $\hat{\theta}(t) \in \left[-\frac{\pi}{\omega}\bar{c}_{complex}\varepsilon, \frac{\pi}{\omega}\bar{c}_{complex}\varepsilon\right]$ and $r_{complex}(t, x_0, \varepsilon)$ such that

$$\tau_{\varepsilon,T}(\hat{x}(t)) = \tau_{\varepsilon,T}\left(\hat{x}\left(t - \tfrac{\pi}{\omega} - \hat{\theta}(t)\right)\right) + r_{complex}(t, x_0, \varepsilon), \tag{10}$$

and $|r_{complex}(t, x_0, \varepsilon)| \leq \bar{c}_r \varepsilon^2$. ∎

Theorem 1 implies that the inter-event time function $t \mapsto \tau_{\varepsilon,T}(\hat{x}(t))$ describes an "almost" periodic pattern of period $\frac{\pi}{\omega}$ for any initial condition $(x_0, x_0, 0)$ with $x_0 \in \mathbb{R}^2 \setminus \{0\}$, for small enough $\varepsilon > 0$ and $T > 0$. Note that $\bar{c}_{complex}\varepsilon$, which is the order of ε, is negligible with respect to $\frac{\pi}{\omega}$, as ε is taken small. Also, $r_{complex}(t, x_0, \varepsilon)$ is of the order of ε^2 and is therefore negligible with respect to $\tau_{\varepsilon,T}(\hat{x}(t - \hat{\theta}(t)))$ which is of the order of ε according to Proposition 3. Theorem 1 thus explains why periodic patterns can arise when plotting the time evolution of the inter-event times: because the eigenvalues of A_c are complex, non-real, conjugates.

The next natural question is whether the values taken by the inter-event times depend on the value x_0. The next theorem ensures that this is not the case, more precisely that x_0 has a negligible impact of the inter-event times.

Theorem 2: Given $m > 0$, when λ_1 and λ_2 are non-real, complex conjugates, for any $x_0, x_0^1 \in \mathbb{R}^2 \setminus \{0\}$, there exist $c_{r,1}, c_{r,2} > 0$ such that for any $(\varepsilon, T) \in S_m(\varepsilon_{complex}, T)$ with $\varepsilon_{complex}$ from Theorem 1, the solutions $(x, \hat{x}, \tau)$ and $(x^1, \hat{x}^1, \tau^1)$ to (6)-(7) initialized at $(x_0, x_0, 0)$ and $(x_0^1, x_0^1, 0)$, respectively, are such that for any $t \in [0, \frac{\pi}{\omega} + \bar{c}_{complex}\varepsilon]$, there exists $\hat{r}_{complex}(t, x_0, x_0^1, \varepsilon)$ such that

$$\tau_{\varepsilon,T}(\hat{x}(t)) = \tau_{\varepsilon,T}(\hat{x}^1(t - c_{r,1})) + \hat{r}_{complex}(t, x_0, x_0^1, \varepsilon), \tag{11}$$

and $|\hat{r}_{complex}(t, x_0, x_0^1, \varepsilon)| \leq c_{r,2}\varepsilon^2$. ∎

Only the time interval $[0, \frac{\pi}{\omega} + \bar{c}_{complex}\varepsilon]$ is considered in Theorem 2 as this suffices to study the values taken by the inter-event times over any time interval of length $\frac{\pi}{\omega} + \bar{c}_{complex}\varepsilon$ in view of Theorem 1. Hence, Theorem 2 implies that changing the initial condition x_0 essentially leads to a phase shift $c_{r,1}$ of the inter-event times. As a result, different initial conditions essentially gives the same inter-event times over any interval of length of the order of $\frac{\pi}{\omega}$, for small enough $\varepsilon > 0$ and $T > 0$ as $c_{r,1}$ is of the order of ε, and thus for all positive times in view of Theorem 1. As a consequence, the amount of transmissions is almost the same for any $x_0 \in \mathbb{R}^2 \setminus \{0\}$.

We derive from the above results that a single simulation for a single value of $x_0 \in \mathbb{R}^2 \setminus \{0\}$ over $\frac{\pi}{\omega}$ units of time can be run to accurately determine the inter-event times for all initial conditions and all future times, and thus to estimate the average inter-transmission time. This average inter-transmission time is defined as the limit of time t over the number of triggering instants, which have occurred on the interval $[0, t]$, as t goes to infinity, like in [19], [27]. This corresponds, for a given solution $(x, \hat{x}, \tau)$ to (6)-(7) initialized at $(x_0, x_0, 0)$ with $x_0 \in \mathbb{R}^2 \setminus \{0\}$, to

$$\tau^{avg}_{\varepsilon,T}(x_0) := \lim_{t \to \infty} \frac{t}{N(t, x_0)}, \tag{12}$$

where the number of triggering instants in the time window $[0, t]$ for $t \geq 0$ is given by $N(t, x_0) := \max\{i \in \mathbb{Z}_{\geq 0} : t \geq t_i\}$ with $t_0 = 0$ the initial time, and $t_k - t_{k-1} = \tau_{\varepsilon,T}(x(t_{k-1}))$ for any $k \in \mathbb{Z}_{>0}$, the k^{th} inter-event time.

We thus have a rigorous, numerical way to estimate the amount of transmissions generated by the event-triggered controller in this case as $\tau^{avg}_{\varepsilon,T}(x_0)$ is well approximated by $\frac{\pi}{\omega}\frac{1}{N(\frac{\pi}{\omega}, x_0)}$ in view of Theorem 1, and this value is essentially the same for all initial conditions according to Theorem 2, which can thus be evaluated by performing a single simulation as illustrated in Section VI-B.

B. When λ_1 and λ_2 are real and distinct

We assume without loss of generality that $\lambda_1 > \lambda_2$. Proposition 2 reveals an important feature of the inter-event time function: it only depends on which line passing through the origin the state $\hat{x}$ lies and not on its actual value. To analyse $\tau_{\varepsilon,T}(\hat{x}(t))$ along the solutions to (6)-(7), we can therefore study the argument of $\hat{x}(t)$ and then exploit the results of Section III. The next proposition characterizes the (asymptotic) behaviour of the argument of $\hat{x}$ along the solutions to (6)-(7).

Proposition 5: Given $m > 0$, when $\lambda_1 > \lambda_2$, there exist $c_{distinct} > 0$ and $\varepsilon_{distinct} \in (0,1]$ such that for any initial condition $(x_0, x_0, 0)$ with $x_0 \in \mathbb{R}^2 \setminus \{0\}$, and any $(\varepsilon, T) \in S_m(\varepsilon_{distinct}, T)$, the corresponding solution $(x, \hat{x}, \tau)$ to (6)-(7) verifies one of the following properties.

(i) There exists v_1, a non-zero eigenvector of A_c associated with λ_1, such that $\limsup_{t \to \infty} |\arg(\hat{x}(t)) - \arg(v_1)| \leq c_{distinct}\varepsilon$.

(ii) There exists v_2, a non-zero eigenvector of A_c associated with λ_2, such that $|\arg(\hat{x}(t)) - \arg(v_2)| \leq c_{distinct}\varepsilon$ for all $t \geq 0$. ∎

Proposition 5 approximately recovers the properties of the argument of the solutions for the continuous-time closed-loop system in the absence of sampling $\dot{x}_c = A_c x_c$ and $x_c(0) \neq 0$, see [26, Chapter 2.1]. Indeed, when λ_1 and λ_2 are real and distinct, the argument of x_c converges to $\arg(v_1)$ for v_1 some non-zero eigenvector of A_c associated with λ_1 when x_0 is not in the eigenspace associated to λ_2. Otherwise, it is constant and equal to $\arg(v_2)$ at all times, with v_2 some non-zero eigenvector of A_c associated with λ_2. Similar results are recovered in Proposition 5 up to a perturbation of the order of ε due to sampling.

Properties of $\tau_{\varepsilon,T}(\hat{x})$ along solutions to (6)-(7) are established in the next theorem.

Theorem 3: Given $m > 0$, when $\lambda_1 > \lambda_2$, there exist $c_1, c_2 > 0$ and $\varepsilon_{distinct} \in (0,1]$ such that for any initial condition $(x_0, \hat{x}_0, 0)$ with $x_0 \in \mathbb{R}^{2}\setminus\{0\}$, and any $(\varepsilon, T) \in \mathcal{S}_m(\varepsilon_{\text{distinct}}, T)$, the corresponding solution $(x, \hat{x}, \tau)$ to (6)-(7) verifies one of the following properties.

(i) $\left|\limsup_{t\to\infty} \tau_{\varepsilon,T}(\hat{x}(t)) - \max\left\{\frac{\varepsilon}{|\lambda_1|}, T\right\}\right| \le c_1 \varepsilon^2$.

(ii) $\left|\tau_{\varepsilon,T}(\hat{x}(t)) - \max\left\{\frac{\varepsilon}{|\lambda_2|}, T\right\}\right| \le c_2 \varepsilon^2$ for all $t \ge 0$. ∎

Theorem 3 means that, when the eigenvalues of A_c are real and distinct, the inter-event time of system (6)-(7) either tends to $\max\left\{\frac{\varepsilon}{|\lambda_1|}, T\right\}$ or it takes values close to $\max\left\{\frac{\varepsilon}{|\lambda_2|}, T\right\}$ for all positive times, up to a perturbation of the order of ε^2 in both cases, which is negligible for small $\varepsilon > 0$ (and $T > 0$) as, again, the inter-event time is of the order of ε according to Proposition 3. As a result, $\tau^{avg}_{\varepsilon,T}$ in (12) is well approximated either by $\max\left\{\frac{\varepsilon}{|\lambda_1|}, T\right\}$ or $\max\left\{\frac{\varepsilon}{|\lambda_2|}, T\right\}$.

C. When λ_1 and λ_2 are real, equal and of geometric multiplicity two

The next theorem follows from Proposition 3 and the properties of λ_1 and λ_2. Note that in this case $A_c = \lambda_1 I$.

Theorem 4: Given $m > 0$, when $\lambda_1 = \lambda_2$ and their geometric multiplicity is two, there exist $c_r > 0$ and $\varepsilon_1 \in (0, 1]$ such that for any initial condition $(x_0, \hat{x}_0, 0)$ with $x_0 \in \mathbb{R}^{2}\setminus\{0\}$, and any $(\varepsilon, T) \in \mathcal{S}_m(\varepsilon_1, T)$, the corresponding solution $(x, \hat{x}, \tau)$ to (6)-(7) verifies $\tau_{\varepsilon,T}(\hat{x}(t)) = \max\left\{\frac{\varepsilon}{|\lambda_1|} + r(\hat{x}(t), \varepsilon), T\right\}$ with $|r(\hat{x}(t), \varepsilon)| \le c_r \varepsilon^2$. ∎

Proof: Let $m > 0$, $x_0 \in \mathbb{R}^{2}\setminus\{0\}$, $(\varepsilon, T) \in \mathcal{S}_m(\varepsilon_1, T)$, $(x, \hat{x}, \tau)$ be the solution to (6)-(7) initialized at $(x_0, \hat{x}_0, 0)$, and $t \ge 0$. In view of Proposition 3, $\tau_{\varepsilon,T}(\hat{x}(t)) = \max\left\{\varepsilon\frac{|\hat{x}(t)|}{|A_c \hat{x}(t)|} + r(\hat{x}(t), \varepsilon), T\right\}$. Since $\lambda_1 = \lambda_2$ and their geometric multiplicity is two, the associated eigenspace is $\mathbb{R}^2$, consequently $A_c \hat{x}(t) = \lambda_1 \hat{x}(t)$. Hence, $\tau_{\varepsilon,T}(\hat{x}(t)) = \max\left\{\frac{\varepsilon}{|\lambda_1|} + r(\hat{x}(t), \varepsilon), T\right\}$, which corresponds to the desired result as r satisfies the properties stated in Theorem 4 in view of Proposition 3.

Theorem 4 ensures that, for any initial condition $(x_0, \hat{x}_0, 0)$ with $x_0 \in \mathbb{R}^{2}\setminus\{0\}$, the inter-event times are close to $\max\left\{\frac{\varepsilon}{|\lambda_1|}, T\right\}$ for all positive times when $\lambda_1 = \lambda_2$ and their geometric multiplicity is two for small $\varepsilon > 0$ and $T > 0$. Hence, the considered event-triggering rule essentially leads to periodic sampling, when ε is small, and $\tau^{avg}_{\varepsilon,T}$ in (12) is well approximated by $\max\left\{\frac{\varepsilon}{|\lambda_1|}, T\right\}$ for all $x_0 \in \mathbb{R}^{2}\setminus\{0\}$. The proof of Theorem 4 does not exploit the fact that the state x is of dimension two: the results apply to systems of any dimension. Hence, when x is of dimension $n \in \mathbb{Z}_{>0}$ and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A_c are equal and of geometric multiplicity n, $A + BK = \lambda_1 I$ and the same conclusions as in Theorem 4 apply. Also, function r and constants c_r, ε_1 are the same as in Proposition 3, which explains why the same notation is used.

When the geometric multiplicity of $\lambda_1 = \lambda_2$ is one, the arguments used in the proof of Theorem 4 no longer apply and significant technical difficulties arise, as explained in more detail next.

V. Discussions

A. When λ_1 and λ_2 are real, equal and of geometric multiplicity one

The results of Section IV eludes the case where $\lambda_1 = \lambda_2$ and their geometric multiplicity is one. The reason is that the argument of $\hat{x}$ along the solutions to (6)-(7) only exhibits an attractivity property in this case. As a result, the proof techniques used for the other cases, which rely on robustness arguments, do not apply. To see this, consider a non-zero solution z to $\dot{z} = Jz$ with $J := \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ like in the proof of Proposition 5, see Section H. The argument of z either converges to 0 or to π, see [26, Chapter 2.1]. This property is not an asymptotic stability property, as in the case where $\lambda_1 \neq \lambda_2$, see the proof of Proposition 5, but only a global attractivity property. If $z_2(0) > 0$ is very small and $z_1(0) > 0$ for instance, then the argument of the corresponding solution will monotonically converge to zero. However, if we change $z_2(0)$ so that it is very small but negative, the argument will converge to π. As a result, a small perturbation may destroy this convergence property, which explains the difficulty encountered in this case.

We conjecture that the inter-event times approximately converge to $\max\left\{\frac{\varepsilon}{|\lambda_1|}, T\right\}$ in this case, for any $x_0 \in \mathbb{R}^{2}\setminus\{0\}$, and $(\varepsilon, T) \in \mathcal{S}_m(\varepsilon_1, T)$, consistently with Theorem 3, and as also seen in simulations in Section VI-A.

B. Nonlinear systems

The results of Section IV apply mutatis mutandis to nonlinear event-triggered control systems, whose linearization around the origin is given by the considered linear model and triggering rules. More precisely, the analytical guarantees of Section IV apply asymptotically in time for such nonlinear systems assuming its origin is globally asymptotically stable and its linearization around the origin verifies SA1 and the considered pairs (ε, T) belong to $\mathcal{S}_m(\bar\varepsilon, T)$ for some given $m > 0$. In particular, the properties of the average inter-transmission times (12) presented in Section IV do apply in this case, as this quantity is related to the asymptotic behaviour of the inter-event times. An illustration is provided in Section VI-B.

C. Other system dimension

When the system is scalar, it is commonly known that the triggering rule in (7) leads to periodic sampling due to homogeneity (see Proposition 2). As we could not find this result formally stated in the literature, we formalize it in the next proposition.

Proposition 6: When $n = 1$, for any $\varepsilon \in (0, \infty)$, $T \in [0, \infty)$, and $x_0 \in \mathbb{R}$, $\tau_{\varepsilon,T}(x_0) = \max\left\{T, \frac{1}{A}\ln\left(\frac{\frac{\varepsilon A}{1+\varepsilon} + BK}{A + BK}\right)\right\}$ when $A \neq 0$, and $\tau_{\varepsilon,T}(x_0) = \max\left\{T, \frac{\varepsilon}{|BK|}(1+\varepsilon)\right\}$ when $A = 0$. ∎

Note that (ε, T) does not need to belong to $\mathcal{S}_m(\varepsilon_1, T)$ for some given $m > 0$ in Proposition 6. When the system dimension is larger than 2, the situation becomes much more complicated and the proofs in Section IV need to undergo major changes, unless A_c has a single eigenvalue of geometric multiplicity equal to the state dimension, in which case Theorem 4 applies as already mentioned. Still, we expect the key properties of the inter-event times established in Section III, which apply to systems of any dimension, to play an important role in future extensions of the present results.

VI. Numerical examples

A. Linear example in [42, Section V]

To illustrate the obtained theoretical results, we consider the same linear system as in [42, Section V], namely

$$\dot{x} \quad \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} x \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{13}$$

The matrix K is designed such that the corresponding matrix $A \quad BK$ is Hurwitz, and three cases are considered depending on the eigenvalues $_1;\ _2$ of $A \quad BK$ being (i) non-real, complex conjugates, (ii) real and distinct, (iii) real and equal. To design the triggering rule, we apply Proposition 1 in [1]. As a result, SA1 is satisfied[6] with

$$? \frac{1}{b} \overline{\min\ _{min}\ A_2^J A_2 \quad p\ "_1 \quad "_2 q} \quad \text{and}^{7}\ T \quad \overline{c}$$

$\frac{1}{Lr}$ arctan$p r q$, where $A_2 : \quad A_c$, $L : | B_2 |$, $r : \frac{}{L^2} \quad 1$ and $"_1; "_2; \quad ¡ \quad 0$ are obtained by solving [1, (16)] with $A_1 : \quad A_c$, $B_1 \quad BK$, $B_2 \quad BK$ and $\overline{C}_p \quad I$.

For each of these cases, we have studied numerically the impact of , T and of the initial conditions on the inter-event times[8]. We first present a comparison of the estimated lower and upper-bounds on the inter-event times established in Lemma 1 with the actual minimum and maximum values of the inter-event times obtained in simulations, which we denote $_{min}$ and $_{max}$, respectively. The estimated bounds are taken as $\hat{\ }_{;T} : \max \frac{}{\&_{max}\ pA_c q}; T$ and $\hat{\ }_{;T} : \max \frac{}{\&_{min}\ pA_c q}; T$, respectively, as explained after Lemma 1. The values of $_{min}$ and $_{max}$ were computed in simulations by taking 10 initial conditions on the unit circle and extracting the minimum and the maximum values of the inter-event times over the 10 runs. The results are summarized in Table I. We observe that both the estimated lower and the upper bounds are tight, actually exact for the former, even when is close to the maximum allowed value , which is specified in the following for each case. We now study the results of Section IV on simulations for each case.

Case (i): $K \quad r \quad 3 \quad 7s$, $_1 \quad 2 \quad j$ and $_2 \quad 2 \quad j$. Then $\quad 0{:}0844$ and $T \quad 0{:}1153$. We have selected different values of , namely $\quad P \quad t0{:}01; 0{:}04; 0{:}084u$, $T \quad \frac{}{2}$ so that $p; T q \ P \ S_m p \quad ; T \ q$ with $m \quad 1\{2$, with initial condition $px_0; x_0; 0q$ and $x_0 \quad p\ 1; 1q$. The obtained inter-event times are depicted in Figure 1. We observe a periodic-like behaviour in each case and that the "pseudo" period is getting closer to $\quad$ as decreases, in agreement with Theorem 1.

We have then selected $\quad 0{:}03$ and studied the inter-event times for different initial conditions $px_0; x_0 q$ with $x_0 \ P \ tp1; 1q, p1; \quad 2q, p1; \quad 1qu$, see Figure 2. The inter-event times describe similar though slightly different patterns of very similar periods, in agreement with Theorem 2.

Case (ii): $K \quad r \ 0 \quad 6s$, $_1 \quad 1$ and $_2 \quad 2$. Then $0{:}0761$ and $T \quad 0{:}1486$. Figure 3 shows the inter-event times for $\quad P \quad t0{:}01; 0{:}03; 0{:}076u$ and $T \quad \frac{}{2}$, and the initial condition $px_0; x_0; 0q$ with $x_0 \quad p\ 1; 1q$. According to Theorem 3, the inter-event times converge to a value close to $\max t\ ; T\ u$ as the time tends to infinity or is close to $\max \ \frac{}{2}; T \quad \frac{}{2}$ for all positive times. We see that the inter-event times indeed converge to a constant close to in all the cases considered in Figure 3, and that the mismatch between the limit value and is getting smaller as we decrease, which is in agreement with the conclusions of Theorem 3.

We might wonder whether there are solutions for which the inter-event times are close to $\frac{}{2}$ for all positive times, which is allowed by Item (ii) of Theorem 3. We have not been able to find such solutions for this example, even when taking x_0 in the eigenspace associated to $_2$.

Case (iii): $K \quad r \quad 2 \quad 7s$, $_1 \quad _2 \quad 2$. Then $\quad 0{:}0818$ and $T \quad 0{:}1228$. Note that this case is not covered by our analysis as the geometric multiplicity of the double eigenvalue is one, see Section V-A. We have considered the initial

Fig. 1. Inter-event times for different values of for the example of Section VI-A when $p\ _1;\ _2 q \quad p\ 2 \quad j; \quad 2 \quad j q$: 0:0845 (blue), 0:04 (green), 0:01 (yellow). The dotted lines represent the value of T for each selection of . The mismatch is the error percentage between and the observed period.

Fig. 2. Inter-event times for the example of Section VI-A for different values of x_0 when $p\ _1;\ _2 q \quad p\ 2 \quad j; \quad 2 \quad j q$: $p1; 1q$ (yellow), $p1; \quad 2q$ (green), $p1; \quad 1q$ (blue).

[6]Strictly speaking, Proposition 1 in [1] ensures that $tpx; \hat{x}; \quad q : x \quad \hat{x} \quad 0u$ is uniformly globally asymptotically stable, but this property is actually exponential due to the linearity of the flow dynamics.

[7]In this example, $\quad ¡ \quad L$ in all cases with the notation of [1], which explains the expression of T, see [1, (11)].

[8]In all the cases $T \ ¡ \ 0$, simulation results for $T \quad 0$ are presented in [36, Section V].

	p 1; 2q p 2 j; 2 jq			p 1; 2q p 1; 2q			p 1; 2q p 2; 2q		
	0:01	0:04	0:084	0:01	0:03	0:076	0:01	0:04	0:081
^ ;T	T	T	T	T	T	T	T	T	T
min	T	T	T	T	T	T	T	T	T
^ ;T	0:0129	0:0515	0:1081	0:0185	0:0555	0:1407	0:0143	0:0570	0:1155
max	0:0128	0:0504	0:1033	0:0184	0:0545	0:1340	0:0142	0:0556	0:1095

TABLE I
GUARANTEED AND ESTIMATED MINIMUM AND MAXIMUM VALUES OF THE INTER-EVENT TIMES FOR THE EXAMPLE OF SECTION VI-A.

Fig. 3. Inter-event times (solid lines) and value of $\frac{}{|\ 1|}$ (dashed line) for the example of Section VI-A when p 1; 2q p 1; 2q for different values of : 0:076 (blue), 0:03 (green), 0:01 (yellow). The dotted lines represent the value of T for each selection of . The mismatch is the error percentage between the limit value of the inter-event times and max $\frac{}{|\ 1|}$; T

Fig. 4. Inter-event times for different values of when 1 2 2: 0:085 (blue), 0:04 (green), 0:01 (yellow). The dotted lines represent the value of T for each selection of . The mismatch corresponds to the error percentage between max $\frac{}{|\ 1|}$; T and the limit value of the inter-event times.

condition px_0; x_0; 0q and x_0 p 1; 1q and different values of , namely P t0:01; 0:04; 0:081u, T $\frac{}{3}$, see Figure 4. We observe that the inter-event times converge in all cases to a constant, which is in a neighborhood of as conjectured in Section V-A, and that the mismatch reduces with like in case (ii).

We have also varied the initial conditions for 0:01. In particular, we have taken x_0 p 1; 2q, which is in the eigenspace associated with 1, and x_0 p 1; 1:9q and x_0 p1; 2:1q, which are, loosely speaking, on both sides of the eigenspace of 1. Again, in all cases the inter-event times converge to a constant close to, see Figure 5.

B. Nonlinear single-link robot arm in [1, Example 3]

We revisit [1, Example 3], which is nonlinear, in the light of Sections IV and V-B. We thus consider a single-link robot arm modeled as 9 Ax pxq Bu where x p x_1; x_2q P

Fig. 5. Inter-event times for the example of Section VI-A for different values of x_0 when 1 2 2: p1; 2q (blue), p1; 2:1q (green), p1; 1:9q (yellow). The dashed line corresponds to the value max $\frac{}{|\ 1|}$; T , and the dotted line to T .

R^2, x_1 is the angle, x_2 is the rotational velocity, u P R is the input torque, A $\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$, B $\begin{matrix} 0 \\ 1 \end{matrix}$ and pxq p0; sinp$x_1$qq. The designed state-feedback controller is given by u sinpx_1q Kx where K p 2 2q. We synthesize the triggering rule as in [1, Section VI], which can be written in the form of (7) in view [1, Example 3]. As a consequence SA1 is satisfied in view of [1, Corollary 1] with 0:1929 and T 0:0898; note that the stability property is exponential for the considered system.

The state matrix of the linearized continuous-time closed-loop model around the origin is given by A BK, whose eigenvalues are 1 1 j and 2 1 j. We have selected 0:19 and T 0:089. We have performed simulations for three initial conditions of the form px_0; x_0; 0q with x_0 P tp10; 0q, p10; 10q, p0; 10qu. The obtained inter-transmission times are depicted in Figure 6. We observe that these all exhibit a periodic-like behaviour and that the values taken over a "period" are very similar for the different initial conditions in agreement with Section IV-A. In particular, we obtain for the estimated values of $^{avg}_{,T}$ in (12) 0:1208, 0:1206, 0:1199 for x_0 p 10; 0q, p10; 10q and p0; 10q, respectively. These values are similar, in agreement with the statements in Section V-B. The observed period in simulation is around 2:9 in all cases, while the theory predicts we thus have a mismatch of only 7:7%. Note that these results have been obtained for and T close to their maximum value and T , respectively, even though the theory has been developed for small and T compared to 1.

VII. CONCLUSIONS

We have analysed the inter-event times for two-dimensional linear event-triggered control based on the relative threshold

Fig. 6. Inter-event times for the example of Section VI-B for different values of x_0: $(10,10)$ (blue), $(10,0)$ (green), $(0,10)$ (yellow). The dotted line corresponds to T .

technique of [42] with and without time regularization for small parameter . We have shown that these times (approximately): (i) describe a periodic pattern, which is essentially independent of the considered initial condition, when these eigenvalues are non-real, complex conjugates, and an estimation of the period is provided; (ii) converge to or lie for all positive times in a neighborhood of given constants when the eigenvalues of the state matrix of the closed-loop system in absence of sampling are real and distinct, or real, equal and of geometric multiplicity two.

It would be interesting, in future work, to adapt and extend the presented methodology to address other classes of control systems and triggering rules, and to go beyond the two-dimensional case.

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Appendix

A. Technical results

We first state the next claim, which essentially says that, given $m > 0$, $\tau_{\varepsilon,T}$ is of the order of when $(\varepsilon,T) \in S_m(\varepsilon_1,T)$, and which plays an instrumental role in the proof of Proposition 3.

Claim 3: Given $m > 0$, there exists $c_2 \in \mathbb{R}_{\geq 0}$ such that for any $x_0 \in S_n$, and any $(\varepsilon,T) \in S_m(\varepsilon_1,T)$, $\tau_{\varepsilon,T}(x_0) \geq T$ implies that $\tau_{\varepsilon,T}(x_0) \leq c_2$. □

Proof: Let $m > 0$, $x_0 \in S_n$ and $(\varepsilon,T) \in S_m(\varepsilon_1,T)$. We first consider the case where $\tau_{\varepsilon,T}(x_0) > T$. As a consequence, $|x(T) - x_0| \leq |x(T)|$, otherwise we would have $\tau_{\varepsilon,T}(x_0) \leq T$ in view of (7), which is excluded here. On the other hand, by the triangle inequality $|x(t)| \leq |x_0| + |x(t) - x_0|$. Thus, as $|x_0| = 1$, $|x(t) - x_0| \leq |x(t)| \leq |x_0| + |x(t) - x_0|$, from which we deduce $|x(t) - x_0| \leq \frac{}{1-}$ for $t \in [T, \tau_{\varepsilon,T}(x_0)]$ since $\in (0,1)$. Therefore, $\tau_{\varepsilon,T}(x_0)$ is less than T plus the time it takes for $|x(t) - x_0|$ to grow from $|x(T) - x_0|$ to $\frac{}{1-}$, which we denote $r_{\varepsilon,T}(x_0)$. We now study $r_{\varepsilon,T}(x_0)$. Let $e := x - x_0$. In view of (6), $\dot{e} = Ax + BKx_0 = A(x - x_0) + Ax_0 + BKx_0 = Ae + A_c x_0$ on $[T, \tau_{\varepsilon,T}(x_0))$. By integration, for $t \in [T, \tau_{\varepsilon,T}(x_0))$, $e(t) = e(T) + \int_T^t Ae(s)ds + \int_T^t A_c x_0 ds = e(T) + \int_T^t Ae(s)ds + (t - T)A_c x_0$. We deduce from the last equality and the fact that $|a - b| \geq |a| - |b|$ for any $a, b \in \mathbb{R}^n$ that

$$\begin{aligned} |e(t)| &\geq (t - T)|A_c x_0| - \left|e(T) + \int_T^t Ae(s)ds\right| \\ &\geq (t - T)|A_c x_0| - |e(T)| - \int_T^t |A||e(s)|ds \end{aligned} \tag{14}$$

Noting that $|e(t)| \leq \frac{}{1-}$ for $t \in [T, T + r_{\varepsilon,T}(x_0)]$, we derive that

$$\begin{aligned} |e(t)| &\geq (t - T)|A_c x_0| - \frac{}{1-} - \int_T^t |A|\frac{}{1-}ds \\ &= (t - T)\left(|A_c x_0| - |A|\frac{}{1-}\right) - \frac{}{1-}. \end{aligned} \tag{15}$$

Let $\underline{c} := \min\{|A_c x_0^1| : |x_0^1| = 1\}$. Since A_c is invertible (being Hurwitz), $\underline{c} > 0$. We derive from (15) $|e(t)| \geq (t - T)\left(\underline{c} - |A|\frac{}{1-}\right) - \frac{}{1-}$. For ε_1 sufficiently small, $\underline{c} - |A|\frac{}{1-} \geq \frac{1}{2}\underline{c}$ as $\in (0, \varepsilon_1)$. Thus,

$$|e(t)| \geq (t - T)\tfrac{1}{2}\underline{c} - \frac{}{1-}. \tag{16}$$

The lower-bound in (16) is equal to $\frac{}{1-}$ when $t = T + 4\underline{c}^{-1}\frac{}{1-}$ and this quantity upper-bounds $\tau_{\varepsilon,T}(x_0)$ in view of (16). Hence, $r_{\varepsilon,T}(x_0) \leq 4\underline{c}^{-1}\frac{}{1-}$. We deduce that $\tau_{\varepsilon,T}(x_0) \leq T + 4\underline{c}^{-1}\frac{}{1-}$, hence, since $T \leq m$ as $(\varepsilon,T) \in S_m(\varepsilon_1,T)$, there exists $c_2 \in \mathbb{R}_{\geq 0}$ such that $\tau_{\varepsilon,T}(x_0) \leq c_2$ for ε_1 sufficiently small, as $\in (0, \varepsilon_1)$.

When $\tau_{\varepsilon,T}(x_0) = T$, $\tau_{\varepsilon,T}(x_0) \leq m$ as $(\varepsilon,T) \in S_m(\varepsilon_1,T)$, and the desired result holds with $c_2 = m$. ■

The next lemma will also be used in the sequel.

Lemma 3: For any $a, b, c \in \mathbb{R}_{\geq 0}$, $|\max\{a,c\} - \max\{b,c\}| \leq |a - b|$. □

Proof: Let $a, b, c \in \mathbb{R}_{\geq 0}$. We distinguish several cases. If $\max\{a,c\} = c$ and $\max\{b,c\} = c$, then $\max\{a,c\} - \max\{b,c\} = 0$. If $\max\{a,c\} = a$ and $\max\{b,c\} = b$, then $|\max\{a,c\} - \max\{b,c\}| = |a - b|$. If $\max\{a,c\} = a$ and $\max\{b,c\} = c$, then $|\max\{a,c\} - \max\{b,c\}| = a - c \leq a - b$. If $\max\{a,c\} = c$ and $\max\{b,c\} = b$, then $|\max\{a,c\} - \max\{b,c\}| = b - c \leq b - a$. ■

B. Proof of Proposition 3

Let $m > 0$, $x_0 \in \mathbb{R}^{n_x}$ and $m > 0$. In view of Proposition 2, it suffices to prove the desired result for $|x_0| = 1$. Hence, consider $x_0 \in \mathbb{R}^n$ with $|x_0| = 1$, i.e., $x_0 \in S_n$, $\in (0, \varepsilon_1)$ with $\varepsilon_1 \in (0, 1)$ specified in the following and $T \in (0, T)$.

We start by proving the result for $(\varepsilon,T) \in S_m(\varepsilon_1,T)$. The Taylor expansion of the solution $t \mapsto x(t)$ to $\dot{x} = Ax + BKx_0$ initialized at x_0 at $t = 0$ and evaluated at $\tau_{\varepsilon,T}(x_0)$ is $x(\tau_{\varepsilon,T}(x_0)) = x_0 + \tau_{\varepsilon,T}(x_0)(Ax_0 + BKx_0) + \tau_{\varepsilon,T}(x_0) r_x(x_0,)$, where $r_x : \mathbb{R}^n \times (0,1) \to \mathbb{R}$ is such that $|r_x(x_0,)| \leq c_x \tau_{\varepsilon,T}(x_0)$ with $c_x > 0$ independent of x_0 and .[9] Since $A_c = A + BK$,

$$x(\tau_{\varepsilon,T}(x_0)) = x_0 + \tau_{\varepsilon,T}(x_0)A_c x_0 + \tau_{\varepsilon,T}(x_0) r_x(x_0,). \tag{17}$$

[9] The existence of such a function r_x follows from the expression of the remainder of the Taylor expansion of $x(\tau_{\varepsilon,T}(x_0))$ which can be upper-bounded by a uniform constant c_x times $\tau_{\varepsilon,T}(x_0)$ as $|x_0| = 1$ and $|x(t)| \leq d_1$ for any $t \geq 0$ in view of SA1 and the fact that $|x_0| = 1$.

Consider the case where $\tau_{\varepsilon,T}(x_0) < T$. Hence, by definition of $\tau_{\varepsilon,T}(x_0)$, $|x(\tau_{\varepsilon,T}(x_0)) - x_0| = \varepsilon|x(\tau_{\varepsilon,T}(x_0))|$. Consequently, in view of (17),

$$\big|\tau_{\varepsilon,T}(x_0)\left(A_c x_0 + r_x(x_0,\varepsilon)\right)\big| = \varepsilon\,\big|x_0 + \tau_{\varepsilon,T}(x_0)A_c x_0 + \tau_{\varepsilon,T}(x_0)\, r_x(x_0,\varepsilon)\big|. \tag{18}$$

We have $|A_c x_0 + r_x(x_0,\varepsilon)| \neq 0$. Indeed, otherwise we would have from (18) that $0 = \varepsilon|x_0|$, and thus $x_0 = 0$, which is excluded here as $x_0 \in \mathcal{S}_n$. Hence, in view of (18), we can write $\tau_{\varepsilon,T}(x_0) = \varepsilon\left|\frac{x_0}{|A_c x_0 + r_x(x_0,\varepsilon)|} + \tau_{\varepsilon,T}(x_0)\frac{A_c x_0 + r_x(x_0,\varepsilon)}{|A_c x_0 + r_x(x_0,\varepsilon)|}\right|$. This implies that

$$\begin{cases} \tau_{\varepsilon,T}(x_0) \le \varepsilon\dfrac{|x_0|}{|A_c x_0 + r_x(x_0,\varepsilon)|} + \varepsilon\,\tau_{\varepsilon,T}(x_0) \\ \tau_{\varepsilon,T}(x_0) \ge \varepsilon\dfrac{|x_0|}{|A_c x_0 + r_x(x_0,\varepsilon)|} - \varepsilon\,\tau_{\varepsilon,T}(x_0). \end{cases} \tag{19}$$

Since $\varepsilon \in (0,1)$ as $\varepsilon \in (0,\varepsilon_1)$ and $\varepsilon_1 \le 1$, these inequalities are equivalent to[10]

$$\begin{cases} \tau_{\varepsilon,T}(x_0) \le \dfrac{\varepsilon}{1-\varepsilon}\dfrac{|x_0|}{|A_c x_0 + r_x(x_0,\varepsilon)|} \\ \tau_{\varepsilon,T}(x_0) \ge \dfrac{\varepsilon}{1+\varepsilon}\dfrac{|x_0|}{|A_c x_0 + r_x(x_0,\varepsilon)|}. \end{cases} \tag{20}$$

To obtain the desired result, we are going to exploit the fact that $\tau_{\varepsilon,T}(x_0)$ is of the order of ε. This is not obvious from (20) because of the term $r_x(x_0,\varepsilon)$, which depends on $\tau_{\varepsilon,T}(x_0)$, in the denominator of the right hand-sides.

Returning to (20), we temporarily concentrate on the first inequality, which gives

$$|A_c x_0 + r_x(x_0,\varepsilon)|\,\tau_{\varepsilon,T}(x_0) \le \frac{\varepsilon}{1-\varepsilon}|x_0|. \tag{21}$$

As $(|A_c x_0| - |r_x(x_0,\varepsilon)|)\tau_{\varepsilon,T}(x_0) \le |A_c x_0 + r_x(x_0,\varepsilon)|\,\tau_{\varepsilon,T}(x_0)$,

$$(|A_c x_0| - |r_x(x_0,\varepsilon)|)\,\tau_{\varepsilon,T}(x_0) \le \frac{\varepsilon}{1-\varepsilon}|x_0|. \tag{22}$$

Since $|r_x(x_0,\varepsilon)| \le c_x\tau(x_0)$ and $\tau_{\varepsilon,T}(x_0) \le c_2\varepsilon$ according to Claim 3, (22) implies that

$$\begin{aligned} |A_c x_0|\,\tau_{\varepsilon,T}(x_0) - c_x c_2^2\varepsilon^2 &\le \frac{\varepsilon}{1-\varepsilon}|x_0| \\ \Leftrightarrow\ |A_c x_0|\,\tau_{\varepsilon,T}(x_0) &\le \frac{\varepsilon}{1-\varepsilon}|x_0| + c_x c_2^2\varepsilon^2. \end{aligned} \tag{23}$$

In view of the Taylor expansion of $\varepsilon \mapsto \frac{\varepsilon}{1-\varepsilon}$ around the origin and since $|A_c x_0| \neq 0$ as $x_0 \neq 0$ and A_c is invertible, we deduce from the above inequality that, for ε_1 sufficiently small as $\varepsilon \in (0,\varepsilon_1)$,

$$\tau_{\varepsilon,T}(x_0) \le \varepsilon\frac{|x_0|}{|A_c x_0|} + \bar c\,\varepsilon^2, \tag{24}$$

with $\bar c \ge 0$ independent of ε and x_0.

By following similar lines, we derive from the second inequality in (20) that

$$\tau_{\varepsilon,T}(x_0) \ge \varepsilon\frac{|x_0|}{|A_c x_0|} - \underline{c}\,\varepsilon^2, \tag{25}$$

with $\underline{c} \ge 0$ independent of ε and x_0. Consequently, in view of (24) and (25), $\tau_{\varepsilon,T}(x_0) = \varepsilon\frac{|x_0|}{|A_c x_0|} + r(x_0,\varepsilon)$ with $|r(x_0,\varepsilon)| \le c_r\varepsilon^2$ and $c_r > 0$ independent of x_0 and ε. Since $\tau_{\varepsilon,T}(x_0) < T$, $\tau_{\varepsilon,T}(x_0) = \max\left\{\varepsilon\frac{|x_0|}{|A_c x_0|} + r(x_0,\varepsilon), T\right\}$.

So far, we have been addressing the case where $\tau_{\varepsilon,T}(x_0) < T$. Note that this case covers the scenario where $T = \infty$, as $\tau_{\varepsilon,T}(x_0) < \infty = T$ according to [42]. We now focus on the case where $\tau_{\varepsilon,T}(x_0) = T$ and $|x(T) - x_0| \ge \varepsilon|x(T)|$. Let $t^* \le T$ be the first time instant in $[0,T)$ such that $|x(t^*) - x_0| = \varepsilon|x(t^*)|$. We derive from the above developments that $t^* = \varepsilon\frac{|x_0|}{|A_c x_0|} + r(x_0,\varepsilon)$ for all $\varepsilon \in (0,\varepsilon_1)$ and small enough ε_1. Since $t^* \le T$, $\varepsilon\frac{|x_0|}{|A_c x_0|} + r(x_0,\varepsilon) \le T$ and $\tau_{\varepsilon,T}(x_0) = \max\left\{\varepsilon\frac{|x_0|}{|A_c x_0|} + r(x_0,\varepsilon), T\right\}$, which completes the proof of Proposition 3.

For general pairs $(\varepsilon,T) \in (0,\varepsilon_1^1) \times [0,T)$ (not necessarily in $\mathcal{S}_m(\varepsilon_1,T)$), either $\tau_{\varepsilon,T}(x_0) = T$, or, in view of the the proof of Claim 3, $\tau_{\varepsilon,T}(x_0) \in [T, \underline{c}^1\frac{\varepsilon}{1-\varepsilon})$. Therefore, $\tau_{\varepsilon,T}(x_0) = T + r^1(x_0,\varepsilon)$ for some $r^1 : \mathbb{R}^n \times (0,1) \to \mathbb{R}$ satisfying $|r^1(x_0,\varepsilon)| \le c_r^1\varepsilon$ with $c_r^1 > 0$ independent of x_0 and ε, for small enough ε_1^1.

C. Proof of Claim 1

Let $m > 0$, $x_0 \in \mathbb{R}^{n_\xi}$, $(\varepsilon,T) \in \mathcal{S}_m(\varepsilon_1,T)$ and $t \ge 0$. We either have $|\hat x(t) - x(t)| \le \varepsilon|x(t)|$ or $\tau(t) \le T$ in view of (7). When $|\hat x(t) - x(t)| \le \varepsilon|x(t)|$, the desired result holds. On the other hand, by following similar arguments as in the proof of [42, Theorem III.1] (p. 1682), we derive that[11] $|\hat x(t) - x(t)| \le \frac{Lt}{1-Lt}|x(t)|$ where $L := \max\{|A_c|, |BK|\}$. The map $s \mapsto \frac{Ls}{1-Ls}$ is increasing and well-defined on $[0,T) \subset [0,\frac{1}{L})$ for ε_1 sufficiently small, as $\varepsilon \in (0,\varepsilon_1)$, $L > 0$ as A_c is Hurwitz and is independent of ε, and T of the order of ε as $(\varepsilon,T) \in \mathcal{S}_m(\varepsilon_1,T)$ so that $T < \frac{1}{L}$. Hence, as $\varepsilon \in (0,\varepsilon_1)$, there exists $\kappa^1 > 0$ independent of ε such that $|\hat x(t) - x(t)| \le \frac{LT}{1-LT}|x(t)| \le \frac{Lm\varepsilon}{1-Lm\varepsilon}|x(t)| \le \kappa^1\varepsilon|x(t)|$. The desired result holds by taking $\kappa := \max\{1, \kappa^1\}$.

D. Proof of Claim 2

Let $m > 0$. We proceed by contradiction and suppose that there exist $x_0 \in \mathbb{R}^{n_\xi}$ and $t \in [0, \tau_{\varepsilon,T}(x_0))$ such that the solution x to $\dot x = Ax + BK\hat x$ initialized at x_0 satisfies $x(t) = 0$. In view of Claim 1, $|x_0 - x(t)| \le \varepsilon|x(t)| = 0$ since $t \in [0, \tau_{\varepsilon,T}(x_0))$. Hence, $x(t) = x_0$, but $x(t) = 0$ while $x_0 \neq 0$. We have obtained a contradiction, which proves the claim.

[10] We could replace $|x_0|$ by 1 in (20), but we do not do so to obtain a, what we believe, simpler and clearer expression in Proposition 3.

[11] There is a typo in the expression of $\phi(\sigma;0)$ in [42, p.1682], it should be $\phi(\sigma;0) = \frac{L\sigma}{1-L\sigma}$ and not $\frac{L}{1-L}$.

E. Proof of Lemma 2

Let m ¡ 0, p; T q P S_m p $_1$; T q with $_1$ from Proposition 3, x_0 P $R^{n;}$, and px; x̂; q be the solution to (6)-(7) initialized at px$_0$; x_0; 0q. Let t; t^1 ¥ 0, according to Proposition 3, $_{;T}$ pxptqq

$_{;T}$ pxpt^{1}qq max $\frac{|xptq|}{|A_c xptq|}$ r pxptqq q; T

max $\frac{|xpt^1q|}{|A_c xpt^1q|}$ r pxpt^{1}q; q; T . In view of the properties of r stated in Proposition 3 and using Lemma 3 given in Appendix A with a $\frac{|xptq|}{|A_c xptq|}$ r pxptq; q, b $\frac{|xpt^1q|}{|A_c xpt^1q|}$ r pxpt^{1}q and c T, we derive

$$| _{;T} pxptqq \quad _{;T} pxpt^1qq| \leq \frac{|xptq|}{|A_c xptq|} \quad \frac{|xpt^1q|}{|A_c xpt^1q|} \quad 2c_r \quad ^2; \qquad (26)$$

where c_r ¡ 0. The function x ÞÑ $\frac{|xptq|}{|A_c xptq|}$ is continuously differentiable on $R_{¥0}$ as x never cancels according to Proposition 4 and A_c is invertible, being Hurwitz. Hence,

$$\frac{d}{dt}\frac{|xptq|}{|A_c xptq|} \quad \frac{\frac{9xptq^J xptq}{|xptq|}|A_c xptq| \quad | xptq|\frac{9xptq^J A_c^J A_c xptq}{|A_c xptq|}}{|A_c xptq|^2} \qquad (27)$$

Since A_c is invertible, there exist \$ $_1$; \$ $_2$ ¡ 0 independent of t; x_0; such that \$ $_1$|xptq| ¤ |$A_c$xptq| ¤ \$ $_2$|xptq|. Therefore,

$$\frac{d}{dt}\frac{|xptq|}{|A_c xptq|} \leq \frac{\frac{|9xptq||xptq|}{|xptq|}\$_2|xptq| \quad | xptq|\frac{|9xptq||A_c^J A_c||xptq|}{\$_1|xptq|}}{\$_1^2|xptq|^2}$$
$$\frac{\$_2|9xptq|}{\$_1^2|xptq|} \quad \frac{|A_c^J A_c||9xptq|}{\$_1^3|xptq|}. \qquad (28)$$

We have 9xptq $A_c$xptq BK pxpt$_i$q xptqq and |xpt$_i$q xptq| ¤ |xptq| in view of Claim 1 where t$_i$ is such that t P rt$_i$; t$_{i\ 1}$q, hence |9xptq| ¤ p|A_c| |BK |q|xptq| ¤ p|A_c| |BK |q|xptq| as P p0; 1q. Consequently,

$$\frac{d}{dt}\frac{|xptq|}{|A_c xptq|} \leq \frac{\$_2}{\$_1^2} \quad \frac{|A_c^J A_c|}{\$_1^3} \quad p|A_c| \quad |BK |q$$
$$: \ c_{cont,1}: \qquad (29)$$

Notice that $c_{cont,1}$ is independent of t, x_0 and . This implies, by application of the mean value theorem, that, in view of (26), $_{;T}$ pxptqq $_{;T}$ pxpt^{1}qq ¤ c $_{cont,1}$|t t^1| $c_{cont,2}$ ^{2} with $c_{cont,2}$: 2c_r.

F. Proof of Theorem 1

We first derive properties of $_{;T}$ pxq, which differs from the inter-event time function $_{;T}$ px̂q, along solutions to (6)-(7). We then exploit these properties to derive the desired result on $_{;T}$ px̂q in Theorem 1.

Proposition 7: Given m ¡ 0, when $_1$ and $_2$ are non-real, complex conjugates, there exist $_{complex}$ ¡ 0 and $_{complex}$ P p0; 1s such that for any initial condition px$_0$; x_0; 0q with x_0 P $R^{2;}$, and any p; T q P S_m p $_{complex}$; T q, the corresponding solution px; x̂; q to (6)-(7) verifies the next property. For any t ¥ 0, there exists ptq P $c_{complex}$; $c_{complex}$ such that $_{;T}$ pxptqq $_{;T}$ pxpt ptqq l

The proof of Proposition 7 is given in Appendix K.

Let m ¡ 0, x_0 P $R^{2;}$, p; T q P S_m p $_{complex}$; T q and px; x̂; q be the solution to (6)-(7) initialized at px$_0$; x_0; 0q, and t ¥ 0. There exists i P $Z_{¥0}$ such that t P rt$_i$; t$_{i\ 1}$q. Hence, x̂ptq xpt$_i$q in view of (6)-(7) and

$$_{;T} px̂ptqq \quad _{;T} pxpt_i qq \qquad (30)$$

According to Proposition 7, there exists pt$_i$q P $c_{complex}$; $c_{complex}$ such that pxpt$_i$qq pxpt$_i$q pt$_i$qq. Therefore,

$$_{;T} px̂ptqq \quad _{;T} pxpt_i \quad pt_i qqq \qquad (31)$$

Let ^ptq : t$_i$ t pt$_i$q so that t ^ptq t$_i$ pt$_i$q and thus

$$_{;T} px̂ptqq \quad _{;T} pxpt \quad ^ptqqq \qquad (32)$$

By adding and subtracting $_{;T}$ px̂pt ^ptqqq we obtain

$$_{;T} px̂ptqq \quad _{;T} px̂pt \quad ^ptqqq \quad _{;T} pxpt \quad ^ptqqq$$
$$_{;T} px̂pt \quad ^ptqqq \qquad (33)$$

We note that ^ptq ¤ pt$_i$q as t$_i$ ¤ t. Hence, ^ptq ¤ $c_{complex}$ as pt$_i$q ¤ $c_{complex}$. On the other hand, as t ¤ t$_{i\ 1}$ and pt$_i$q ¥ $c_{complex}$, ^ptq ¥ t$_i$ t$_{i\ 1}$ $c_{complex}$ $_{;T}$ pxpt$_i$qq $c_{complex}$. According to Proposition 3, $_{;T}$ pxpt$_i$qq max $\frac{|xpt_i q|}{|A_c xpt_i q|}$ r px$_0$; q; T ¤ max t ; m u c_r ^{2} where : max$_{zPS_2}$ $\frac{|z|}{|A_c z|}$ ¡ 0, as |r px$_0$; q| ¤ c_r ^{2}. Since 1, max t ; m u c_r ^{2} ¤ $c_{complex}$ with $c_{complex}$ max t ; m u c_r ¡ 0. As a result, ^ptq ¥ $c_{complex}$ $c_{complex}$. Denoting $\hat{c}_{complex}$: $c_{complex}$ $c_{complex}$, we have proved that

$$^ptq \quad P \quad \hat{c}_{complex}; \quad \hat{c}_{complex} \quad : \qquad (34)$$

Returning to (33), we now concentrate on the term $_{;T}$ pxpt ptqq $_{;T}$ px̂pt ptqq. Denoting i^1 the element of $Z_{¥0}$ such that t ^ptq P rt$_{i^1}$; t$_{i^1\ 1}$q, we have $_{;T}$ pxpt ptqq $_{;T}$ px̂pt ptqq $_{;T}$ pxpt ptqq $_{;T}$ pxpt$_{i^1}$qq. By application of Lemma 2, we derive that | $_{;T}$ pxpt ptqq $_{;T}$ pxpt$_{i^1}$qq| ¤ c $_{cont,1}$|t ^ptq t$_i^1$| ^{2}$c_{cont,2}$. By definition of ^ptq, t ^ptq t$_i^1$ t$_i$ pt$_i$q t$_{i^1}$ and, since t$_i$ pt$_i$q ¤ t$_{i^1\ 1}$, |t ^ptq t$_i^1$| ¤ t$_{i^1\ 1}$ t$_{i^1}$ $_{;T}$ pxpt$_i^1$qq. By following similar lines as above, we derive that pxpt$_{i^1}$qq ¤ ¯ with ¯ ¡ 0 independent of t; ; x $_0$. As a result,

$$| _{;T} pxpt \quad ptqqq \quad _{;T} px̂pt \quad ptqq| \leq {}^2 c_{cont,1}¯ \quad {}^2 c_{cont,2}$$
$$\hat{c}_r \quad ^2; \qquad (35)$$

with $\hat{c}_r$: $c_{cont,1}$¯ $c_{cont,2}$. Therefore $_{;T}$ pxpt ptqq $_{;T}$ px̂pt ptqq r$_{complex}$pt; x$_0$; q with |r$_{complex}$pt; x$_0$; q| ¤ $\hat{c}_r$ ^{2}. As a consequence, in view of (33), $_{;T}$ px̂ptqq $_{;T}$ px̂pt ^ptqqq r$_{complex}$pt; x$_0$; q. The last equation together with (34) ensures that the desired result holds.

G. Proof of Theorem 2

Let m ¡ 0, $x_0; x_0^1$ P $R^{2;}$ and p; T q P S_m p $_{complex}$; T q. We denote by px; x̂; q and px^1; x̂^{1}; ^{1}q the solutions to (6)-(7) initialized at px$_0$; x$_0$; 0q and px$_0^1$; x$_0^1$; 0q, respectively. We define for any t ¥ 0, zptq M ^{1}xptq and z^{1}ptq M ^{1}x^{1}ptq as in the proof of Proposition 7 given in Appendix K. Let c$_{r;1}$: ^{1}pargpz^{1}p0qq argpzp0qqand t P r0; – ĉ$_{complex}$ s.

We first show that there exists e$_{r;2}$pt; x$_0$; x$_0^1$; q, e$_{r;2}$ptq for short, such that

$$;_T \ pxptqq \quad ;_T \ px^1pt \quad c_{r;1} \quad e_{r;2}ptqqq \qquad (36)$$
$$|e_{r;2}ptq| \ ¤ \ ^2p– \quad ĉ_{complex} \ qc_{complex};$$

where c$_{complex}$ comes from the proof of Proposition 7. In view of (the proof of) Proposition 7, the range of $_T$ pxp qqis equal to the range of $_T$ px^{1}p qqand there exists e$_{r;2}$ptq such that the first inequality in (36) holds. We now need to prove that |e$_{r;2}$ptq| ¤ ^{2}p– ĉ$_{complex}$ qc$_{complex}$. We exploit for this purpose the fact that (36) is equivalent to

$$argpzptqq \quad argpz^1pt \quad c_{r;1} \quad e_{r;2}ptqqq \qquad (37)$$

by Proposition 2. In view of (46) in Appendix K, argpzptqq argpzp0qq t ptqc$_{complex}$ where | ptq| ¤ – ĉ$_{complex}$. Consequently, argpzptqq argpzp0qq pt c$_{r;1}$ e$_{r;2}$ptqq ptqc$_{complex}$ pc$_{r;1}$ e$_{r;2}$ptqq By definition of c$_{r;1}$ and the fact that argpz^{1}pt c$_{r;1}$ e$_{r;2}$ptqqq argpz^{1}p0qq pt c$_{r;1}$ e$_{r;2}$ptqq ^{1}pt c$_{r;1}$ e$_{r;2}$ptqqc$_{complex}$ with | ^{1}pt c$_{r;1}$ e$_{r;2}$ptqq| ¤ – ĉ$_{complex}$,

$$argpzptqq \quad argpz^1pt \quad c_{r;1} \quad e_{r;2}ptqq \quad e_{r;2}ptq$$
$$p \quad ptq \quad ^1pt \quad c_{r;1} \quad e_{r;2}ptqqqc_{complex}: \qquad (38)$$

Since argpzptqq argpz^{1}pt c$_{r;1}$ e$_{r;2}$ptqq and by the properties of ptq and ^{1}pt c$_{r;1}$ c$_{r;2}$ptqq we derive that |e$_{r;2}$ptq| ¤ ^{2}p– ĉ$_{complex}$ qc$_{complex}$. We have thus proved (36).

Second, $_{;T}$ px̂ptqq $_{;T}$ pxpt$_i$ qq with i P I such that t P rt$_i$; t$_i$ $_1$q. We derive from (36) that $_{;T}$ px̂ptqq $_{;T}$ px^{1}pt$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ qqq Let j P I be such that t$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ q P rt$_j^1$; t$_j^1$ $_1$q, $_{;T}$ px̂ptqq $_{;T}$ px^{1}pt$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ qqq $_{;T}$ px^{1}pt$_j^1$qq $_{;T}$ px^{1}pt$_j^1$qq $_{;T}$ px̂^{1}pt$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ qqq $_{;T}$ px^{1}pt$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ qqq $_{;T}$ px^{1}pt$_j^1$qq We derive, by proceeding like in the end of the proof of Theorem 1, that | $_{;T}$ px^{1}pt$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ qqq $_{;T}$ px^{1}pt$_j^1$qq| ¤ ĉ$_r$ ^{2}. Consequently, | $_{;T}$ px̂ptqq $_{;T}$ px̂^{1}pt$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ qqq| ¤ ĉ$_r$ ^{2}.

Finally, by Lemma 2, we derive that | $_{;T}$ px̂ptqq $_{;T}$ px̂^{1}pt$_i$ c$_{r;1}$qq| ¤ ĉ$_r$ ^{2} | $_{;T}$ px̂^{1}pt$_i$ c$_{r;1}$ e$_{r;2}$pt$_i$ qqq $_{;T}$ px̂^{1}pt$_i$ c$_{r;1}$qq| ¤ ĉ$_r$ ^{2} c$_{cont,1}$|e$_{r;2}$pt$_i$ q| ^{2}c$_{cont,2}$ ¤ ĉ$_r$ ^{2} ^{2}c$_{cont,1}$ ^{2}p– ĉ$_{complex}$ qc$_{complex}$ ^{2}c$_{cont,2}$ as |e$_{r;2}$ptq| ¤ ^{2}p– ĉ$_{complex}$ qc$_{complex}$, from which we obtain the desired result.

H. Proof of Proposition 5

Let m ¡ 0, x$_0$ P $R^{2;}$, p; T q P S_m p $_{distinct}$; T q with $_{distinct}$ P p0; $_1$s specified in the following. We write matrix A$_c$ in a Jordan form. Let M r w$_1$; w$_2$s, where w$_1$; w$_2$ are non-zero eigenvectors of A$_c$ associated with $_1$ and $_2$, respectively, z p z$_1$; z$_2$q: M ^{1}x and ẑ : M ^{1}x̂. Hence, on ows ż Jz " where J : $\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}$ and " p"$_1$; "$_2$q: M ^{1}BKM pẑ zq. Equivalently, ż$_1$ $_1$z$_1$ "$_1$ and ż$_2$ $_2$z$_2$ "$_2$. Note that |"| ¤ ¯|z| for some ¯ ¡ 0 independent of , whenever |x̂ x| ¤ |x|, which holds along solutions to (6)-(7) in view of Claim 1.

We are going to study the variation of $_1$pzq: $\frac{z_2^2}{|z|^2}$ for any z p z$_1$; z$_2$q P$R^{2;}$, which is equal to sinpargpzqq^2, along the solutions to ż Jz "; recall that z 0 so that $_1$pzq is well-defined. The obtained properties will allow us to derive that Items (i)-(ii) of Proposition 5 hold. Let z P$R^{2;}$ and " PR^2 be such that |"| ¤ ¯|z|. It holds that xr $_1$pzq; Jz "y $\frac{1}{|z|^4}$ 2z$_2$p $_2$z$_2$ "$_2$q|z|^2 2z$_2^2$p $_1$z$_1^2$ z$_1$"$_1$ $_2$z$_2^2$ z$_2$"$_2$q . We obtain by adding and subtracting $_2$z$_1^2$ in the second term inside the brackets above xr $_1$pzq; Jz "y $\frac{1}{|z|^4}$ 2z$_2$p $_2$z$_2$ "$_2$q|z|^2 2z$_2^2$p $_1$z$_1^2$ $_2$z$_1^2$ $_2$z$_1^2$ z$_1$"$_1$ $_2$z$_2^2$ z$_2$"$_2$q $\frac{1}{|z|^4}$ 2p $_1$ $_2$qz$_1^2$z$_2^2$ 2"$_1$z$_1$z$_2^2$ "$_2$p2z$_2$|z|^2 2z$_2^3$q . Since |"| ¤ ¯ |z|, there exists $_1$ ¡ 0 independent of such that $\frac{1}{|z|^4}$ 2"$_1$z$_1$z$_2^2$ "$_2$p2z$_2$|z|^2 2z$_2^3$q ¤ $_1$. Consequently,

$$xr \ _1pzq; Jz \ "y \ ¤ \ \frac{2p \ _1 \ _2qz_1^2z_2^2}{|z|^4} \ _1: \qquad (39)$$

We have $\frac{z_1^2z_2^2}{|z|^4}$ cospargpzqq^2 sinpargpzqq^2 p1 sinpargpzqq^{2}qsinpargpzqq^2 p1 $_1$pzqq $_1$pzq. Therefore,

$$xr \ _1pzq; Jz \ "y \ ¤ \ 2p \ _1 \ _2qp1 \ _1pzqq \ _1pzq \ _1: \qquad (40)$$

Let p q : 1 2 $\frac{1}{1 \quad 2}$. We note that p q 1 as $_1$ ¡ $_2$. Also p q ¡ $\frac{1}{2}$ for $_{distinct}$ small enough, as P p0; $_{distinct}$q . Hence p q P $\frac{1}{2}$; 1 . Furthermore, $\frac{1}{p \ _1 \ _2qp1 \ p qq}$ $\frac{1}{2}$ p q. Consequently, for any $_1$pzq P $\frac{1}{p \ _1 \ _2qp1 \ p qq}$; p q ,

$$xr \ _1pzq; Jz \ "y \ ¤ \ p \ _1 \ _2qp1 \ p qq \ _1pzq. \qquad (41)$$

System (6)-(7) in the coordinates pz; ẑ; q becomes pż; ẑ̇; 9q p Jz "; 0; 1q for all t P pt$_i$; t$_i$ $_1$q and pzpt$_i^+$ q; ẑpt$_i^+$ q; pt$_i^+$ qq pzpt$_i$ q; zpt$_i$ q; 0q. Consider a solution pz; ẑ; q initialized at pz$_0$; z$_0$; 0q P$R^{2;}$, where z$_0$ M ^{1}x$_0$.

If $_1$pzqp0q ¤ p q, we derive from (41) using standard Lyapunov techniques and the fact that $_1$pzq is not affected by jumps that lim sup$_{t\to\infty}$ $_1$pzptqq ¤ $\frac{1}{p \ _1 \ _2qp1 \ p qq}$. This means that either lim sup$_{t\to\infty}$ |argpzptqq| ¤ c$_{z;arg}$ or lim sup$_{t\to\infty}$ |argpzptqq | ¤ c$_{z;arg}$ for some constant c$_{z;arg}$ ¡ 0 independent of and z$_0$, since $_1$pzq sinpargpzqq^2. This implies that, in the original coordinates px; x̂q, there exists a non-zero eigenvector v$_1$ (w$_1$) associated with $_1$

such that $\limsup_{t\to\infty} |\arg(x(t)) - \arg(v_1)| \leq c_{\arg}$ for some $c_{\arg} > 0$ independent of and x_0. On the other hand, if $_1(z(0)) \in (\ ,\ )\setminus\{\}$ there are two options: (a) there exists $t > 0$ such that $_1(z(t)) \leq (\ ,\ )$; (b) $_1(z(t)) \in (\ ,\ )\setminus\{\}$ for all $t \geq 0$. In case (a), we deduce from the reasoning above that there exists a non-zero eigenvector v_1 associated with $_1$ such that $\limsup_{t\to\infty} |\arg(x(t)) - \arg(v_1)| \leq c_{\arg}$. In case (b), $_1(z(t)) \in (\ ,\ )\setminus\{\} \quad 1 \quad 2 \frac{1}{1\ 2};1$ for all $t \geq 0$, which means that $|\arg(z(t)) - \frac{}{2}| \leq c^1_{\text{distinct}}$ for all $t \geq 0$ with $c^1_{\text{distinct}} > 0$ independent of and x_0. Returning to the original coordinates, this means that there exists a non-zero eigenvector v_2 (w_2) associated with $_2$ such that $|\arg(x(t)) - \arg(v_2)| \leq c_{\text{distinct}}$ for all $t \geq 0$. Since $\hat{x}(t) = x(t_i)$ for any $t \in [t_i, t_{i+1})$ and the sequence t_i, $i \in \mathbb{Z}_{\geq 0}$, is unbounded according to Propositions 1 and 3, we deduce from the properties established in this paragraph that the desired result holds.

I. Proof of Theorem 3

Let $m > 0$, $x_0 \in \mathbb{R}^{2}$ and $(,T) \in S_m(\ _{\text{distinct}}, T)$. Let $t \geq 0$ and consider $(x, \hat{x},)$ the solution to system (6)-(7) initialized at $(x_0, x_0, 0)$. In view of Proposition 3, $_{,T}(\hat{x}(t)) = \max\left(\frac{|\hat{x}(t)|}{|A_c\hat{x}(t)|} \ r(\hat{x}(t),), T\right)$, $\max\left(A_c\frac{\hat{x}(t)}{|\hat{x}(t)|} \right)^{-1} r(\hat{x}(t),), T$, recall that we have $\hat{x}(t) \neq 0$ according to Proposition 4. In polar coordinates, the above equation becomes $_{,T}(\hat{x}(t)) = \max\left(A_c \cos(\arg(\hat{x}(t))), \sin(\arg(\hat{x}(t))) \right)^{-1} r(\hat{x}(t),), T$. Suppose Item (i) of Proposition 5 holds, and let v_1 be the corresponding unit eigenvector of A_c associated with $_1$. Consider the case where $_{,T}(\hat{x}(t)) = A_c \cos(\arg(\hat{x}(t))), \sin(\arg(\hat{x}(t)))^{-1} r(\hat{x}(t),)$. It holds that

$$\frac{|v_1|}{|A_c v_1|} = A_c \cos(\arg(v_1)), \sin(\arg(v_1))^{-1}; \tag{42}$$

thus $_{,T}(\hat{x}(t)) - \frac{|v_1|}{|A_c v_1|} = r(\hat{x}(t),) A_c \cos(\arg(\hat{x}(t))), \sin(\arg(\hat{x}(t)))^{-1} - A_c \cos(\arg(v_1)), \sin(\arg(v_1))^{-1}$. Noting that $z \mapsto |A_c z|^{-1}$ is Lipschitz with some constant $\ell \geq 0$ on the compact set S_2, and since $|r(z,)| \leq c_r\ ^2$ for any $z \in \mathbb{R}^2$ according to Proposition 3, we deduce that $_{,T}(\hat{x}(t)) - \frac{|v_1|}{|A_c v_1|} \leq \ell \ \cos(\arg(\hat{x}(t))), \sin(\arg(\hat{x}(t))) - \cos(\arg(v_1)), \sin(\arg(v_1)) + c_r\ ^2$. Exploiting the global Lipschitz properties of the cosine and sine functions, we have

$$_{,T}(\hat{x}(t)) - \frac{|v_1|}{|A_c v_1|} \leq 2\ell\ |\arg(\hat{x}(t)) - \arg(v_1)| + c_r\ ^2. \tag{43}$$

By applying Lemma 3 given in the appendix with $a = A_c \cos(\arg(\hat{x}(t))), \sin(\arg(\hat{x}(t)))^{-1} r(\hat{x}(t),)$, $b = \frac{|v_1|}{|A_c v_1|}$ and $c = T$, we derive that, when $_{,T}(\hat{x}(t)) = \max\left(A_c \cos(\arg(\hat{x}(t))), \sin(\arg(\hat{x}(t)))^{-1} r(\hat{x}(t),), T\right)$, $_{,T}(\hat{x}(t)) - \max\left(\frac{|v_1|}{|A_c v_1|}; T\right) \leq 2\ell\ |\arg(\hat{x}(t)) - \arg(v_1)| + c_r\ ^2$. As a result, we obtain by invoking Item (i) of Proposition 5 and the fact that $\frac{1}{|\ _1|} = \frac{|v_1|}{|A_c v_1|}$ as v_1 is an eigenvector for $_1$,

$$\limsup_{t\to\infty} \left| _{,T}(\hat{x}(t)) - \max\left(\frac{1}{|\ _1|}; T\right)\right| \leq 2\ell c_{\text{distinct}}\ ^2 + c_r\ ^2; \tag{44}$$

we conclude that Item (i) of Theorem 3 holds with $c_1 = 2\ell c_{\text{distinct}} + c_r$ in this case.

Similar arguments apply when Item (ii) of Proposition 5 is verified, which leads to the satisfaction of Item (ii) of Theorem 3.

J. Proof of Proposition 6

Let $x_0 \in \mathbb{R}$, $\in (0,)$ and $T \in (0, T)$. We first assume that $A \neq 0$. Consider the case where $_{,T}(x_0) > T$. Hence, $|x(_{,T}(x_0)) - x_0| = |x(_{,T}(x_0))|$ in view of (7). Since $x(_{,T}(x_0)) = e^{A\ _{,T}(x_0)} x_0 + \int_0^{_{,T}(x_0)} e^{A(_{,T}(x_0) - s)} BKx_0\,ds = e^{A\ _{,T}(x_0)} x_0 + A^{-1}(1 - e^{A\ _{,T}(x_0)}) BKx_0$ and $x_0 \in \mathbb{R}$, we have at $t = _{,T}(x_0)$, $e^{A\ _{,T}(x_0)} + A^{-1}(1 - e^{A\ _{,T}(x_0)}) BK - 1 = e^{A\ _{,T}(x_0)} + A^{-1}(1 - e^{A\ _{,T}(x_0)}) BK$. By squaring the last inequality and introducing $:= e^{A\ _{,T}(x_0)} + A^{-1}(1 - e^{A\ _{,T}(x_0)}) BK$, we obtain a second order polynomial in , namely $(1 - ^2)\ ^2 - 2 + 1 = 0$. This equation has two strictly positive roots, denoted $:= \frac{1}{1 +}$ and $:= \frac{1}{1 -}$. Since $|x(T) - x_0| \leq |x(_{,T}(T))|$, necessarily . By solving , i.e., $e^{A\ _{,T}(x_0)} + A^{-1}(1 - e^{A\ _{,T}(x_0)}) BK = \frac{1}{1 +}$, we derive that $_{,T}(x_0) = \frac{1}{A} \ln \frac{\frac{A}{1 +} - BK}{A - BK}$, which is strictly greater than T this is the case here.

When $_{,T}(x_0) = T$, this means that $|x(T) - x_0| \geq |x(T)|$, which implies that $\geq$, which is equivalent to $_{,T}(x_0) \geq \frac{1}{A} \ln \frac{\frac{A}{1 +} - BK}{1 - BK}$. Hence $_{,T}(x_0) = \max\left(T; \frac{1}{A} \ln \frac{\frac{A}{1 +} - BK}{A - BK} \right)$.

We follow similar lines as above when $A = 0$ to obtain the expression of the inter-event time in Proposition 6.

K. Proof of Proposition 7

Let $m > 0$, $x_0 \in \mathbb{R}^{2}$ and $(,T) \in S_m(\ _{\text{complex}}, T)$ with $_{\text{complex}} \in (0, \ _1)$ specified in the following. We write matrix A_c in the real Jordan form. Let $M = [w_1, w_2]$ where w_1 and w_2 are non-zero eigenvectors of A_c associated with the pair of complex conjugates eigenvalues $\pm i$, respectively, $z = (z_1, z_2) := M^{-1}x$ and $\hat{z} := M^{-1}\hat{x}$. Hence, system (6)-(7) becomes

$$\left.\begin{array}{l} \dot{z} = Jz \\ \dot{\hat{z}} = 0 \\ \dot{} = 1 \end{array}\right\} \text{ for all } t \in [t_i, t_{i+1}), \quad \left\{\begin{array}{l} z(t_i^+) = z(t_i) \\ \hat{z}(t_i^+) = z(t_i) \\ (t_i^+) = 0 \end{array}\right. \tag{45}$$

where J : and $\varepsilon = (\varepsilon_1, \varepsilon_2) = M^{-1}BKM(\hat{z} - z)$ as in the proof of Proposition 5.

The inter-event time function at time t becomes in these coordinates $\tau_{\sigma,T}(\hat{z}(t))$ with $\tau_{\sigma,T}(z_0) := \inf\{\tau \geq T : |Mz_0 - M\phi(\tau, z_0)| \geq \sigma|M\phi(\tau, z_0)|\}$, where $\phi(\tau, z_0)$ is the solution to $\dot{z} = Jz - M^{-1}BKM(z_0 - z)$ at time $\tau \geq 0$, initialized at z_0. Hence, for the solutions $(x, \hat{x}, \tau)$ and $(z, \hat{z}, \tau)$ to (6)-(7) and (45) initialized at $(x_0, x_0, 0)$ and $(z_0, z_0, 0)$, respectively, $\tau_{\sigma,T}(x(t)) = \tau_{\sigma,T}(z(t))$ for all $t \geq 0$. Moreover, there exists $c_3 > 0$, independent of σ, such that $|\varepsilon(t)| \leq \sigma c_3|z(t)|$ in view of Claim 1 and the definition of z.

We investigate the argument of the z-component of the solution to (45) initialized at $(z_0, z_0, 0)$, where $z_0 = M^{-1}x_0$. In view of its definition in Section I, the argument function is differentiable everywhere except on $\mathbb{R}_{\geq 0}\times\{0\}$, which is of Lebesgue measure zero. On the other hand, the set $\{t \geq 0 : z(t) \in \mathbb{R}_{\geq 0}\times\{0\}\}$ is also of Lebesgue measure zero. Indeed, suppose there exists $t \geq 0$ such that $z_1(t) \geq 0$ and $z_2(t) = 0$. Then $\dot{z}_2(t) = -z_2(t) + z_1(t) + \varepsilon_2(t) = z_1(t) + \varepsilon_2(t)$. Suppose $\dot{z}_2(t) = 0$ to obtain a contradiction. This means that $z_1(t) = -\varepsilon_2(t)$, which implies that $|z_1(t)| = |\varepsilon_2(t)|$, but $|\varepsilon_2(t)| \leq \sigma c_3|z(t)| = \sigma c_3|z_1(t)|$. Hence, we derive $|z_1(t)| \leq \sigma c_3|z_1(t)|$, which is impossible as $z_1(t) \neq 0$, in view of Proposition 4, when taking σ_{complex} and thus σ small enough. We conclude that the set $\{t \geq 0 : z(t) \in \mathbb{R}_{\geq 0}\times\{0\}\}$ is of Lebesgue measure zero. Consequently, for almost all $t \geq 0$, $\frac{d}{dt}\arg(z(t)) = \frac{1}{|z(t)|^2}\left(z_1(t)(z_2(t) - z_1(t) + \varepsilon_2(t)) - z_2(t)(-z_1(t) - z_2(t) + \varepsilon_1(t))\right) = \frac{1}{|z(t)|^2}\left(|z(t)|^2 + z_1(t)\varepsilon_2(t) - z_2(t)\varepsilon_1(t)\right)$, and

$$\frac{d}{dt}\arg(z(t)) = 1 + r_{\text{complex}}(z(t), \varepsilon(t)) \tag{46}$$

where $r_{\text{complex}}(z, \varepsilon) := \frac{1}{|z|^2}(z_1\varepsilon_2 - z_2\varepsilon_1)$. Since $|\varepsilon| \leq \sigma c_3|z|$, by Cauchy-Schwarz inequality, there exists $\bar{c}_{\text{complex}} > 0$ such that $|r_{\text{complex}}(z(t), \varepsilon(t))| \leq \sigma\bar{c}_{\text{complex}}$ for any $t \geq 0$.

Equation (46) and the properties of r_{complex} imply that z describes spirals “converging” to the origin in the phase portrait and that it spends at most $\frac{\pi}{1 - \sigma\bar{c}_{\text{complex}}}$ and at least $\frac{\pi}{1 + \sigma\bar{c}_{\text{complex}}}$ units of time to successively intersect twice any given line passing through the origin. The inter-event time function $\tau_{\sigma,T}$ satisfies the same homogeneity[12] as $\tau_{\sigma,T}$ stated in Proposition 2. Consequently, for any $t \geq 0$, there exists $\theta(t) \in \left[\frac{\pi}{1 + \sigma\bar{c}_{\text{complex}}}, \frac{\pi}{1 - \sigma\bar{c}_{\text{complex}}}\right]$ such that $\tau_{\sigma,T}(z(t)) = \tau_{\sigma,T}(z(t + \theta(t)))$. In view of the Taylor series of $\sigma \mapsto \frac{\pi}{1 + \sigma\bar{c}_{\text{complex}}}$ and $\sigma \mapsto \frac{\pi}{1 - \sigma\bar{c}_{\text{complex}}}$, as $\sigma \in (0, \sigma_{\text{complex}})$ and σ_{complex} is taken small, there exists $c_{\text{complex}} > 0$ independent of (σ, x_0) such that $\left[\frac{\pi}{1 + \sigma\bar{c}_{\text{complex}}}, \frac{\pi}{1 - \sigma\bar{c}_{\text{complex}}}\right] \subseteq [\pi - \sigma c_{\text{complex}}, \pi + \sigma c_{\text{complex}}]$. Therefore, since $\tau_{\sigma,T}(x(t)) = \tau_{\sigma,T}(z(t))$ for any $t \geq 0$, where z and x are components of the solutions to (45) and (6)-(7) initialized at $(M^{-1}x_0, M^{-1}x_0, 0)$ and $(x_0, x_0, 0)$, respectively, the desired result follows.

[12] It suffices to use the definition of $\tau_{\sigma,T}$ and to compute explicitly ϕ to see that $\tau_{\sigma,T}(\lambda z_0^1) = \tau_{\sigma,T}(z_0^1)$ for any $\lambda \in \mathbb{R}$ and any $z_0^1 \in \mathbb{R}^{2,\star}$.

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