A Class of Hybrid Geometric Controllers for Robust Global Asymptotic Stabilization on $\mathbb{S}^1$

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***Abstract*— This paper proposes a hybrid geometric control scheme for the classical problem of globally stabilizing a point-mass system on a unit circle, as it is impossible to design a smooth globally asymptotically stable controller for this problem. Unlike most existing solutions that rely on coordinates and rely on a particular controller construction, our proposed solution is coordinate free (or geometric) and belongs to a class of controllers that we also characterize. Specifically, we propose a geometric hybrid controller that uses a local geometric controller (from the said class) and an open-loop geometric controller. The system achieves global asymptotic stability when each controller from the local geometric class is combined with the geometric open-loop controller using a hybrid systems framework. Moreover, the hybrid geometric controller guarantees robust asymptotic stability. Simulations validate the stability properties of the proposed hybrid geometric controller.**

I. Introduction

In the context of classical control and dynamical systems [1], trajectories of the plant evolve on a n-dimensional Euclidean space, i.e., $\mathbb{R}^n$. However, there exist several dynamical, mechanical, and robotic systems whose states evolve on a more general structure, such as a smooth manifold or Lie groups [2]–[4]. The underlying manifold structure (Lie group structure) poses extra challenges in analysis and controller design for such systems. One natural way to study such a system is using a local coordinate chart approach [5], and then express the state of the system, locally, in $\mathbb{R}^n$. However, coordinate-based local approaches lead to local results and often suffer singularities [6], [7].

Although geometric controllers can be designed to avoid singularities associated with a local chart, designing a smooth global controller on a compact manifold or a compact Lie group is nontrivial [5], [8]. It is impossible to design a smooth (even continuous) global controller using geometric tools, even for one of the simplest Lie groups, the unit circle, because of topological obstructions [9]. Though a smooth almost global controller[1] design is possible, such a controller may not be robust to perturbations, even if they are arbitrarily small. Fortunately, it is possible to achieve global robust asymptotic stability using hybrid controllers [10].

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[1]Informally, almost global controllers exhibit global properties everywhere on the state space except on sets whose Lebesgue measure is zero.

Recently, hybrid controllers that are geometric, or coordinate free, have been developed for the stabilization of systems on specific manifolds [11], [12]. Moreover, in [13], [14], the authors design a geometric hybrid controller for systems on $\mathsf{SO}(3)$ and $\mathsf{SE}(3)$. The method in [13] relies on potential synergistic functions, which are used to design global hybrid attitude stabilization controllers. Their method requires the explicit computation of the so-called synergistic gap. In [12], [15], the control method is based on a central family of potential functions, which requires an explicit construction of these functions for controller design.

In this paper, we consider the classical control problem of globally and robustly stabilizing a point-mass to a point on the unit circle $\mathbb{S}^1$. In [10], [16], [17], the authors solve this problem by designing a hybrid controller. However, as mentioned above, the solution is based on first embedding the unit circle in the two-dimensional Euclidean space and then using the Euclidean coordinates to design a controller. Although the results are global, their method is not geometric or coordinate-free and, in a sense, restrictive as it is hard to generalize these results for Lie groups of higher dimensions. We want to underscore that this classical problem is simple yet rich enough because the unit circle is a Lie group, which is a building block of many more complex robotic, mechanical, and physical systems. Unlike the state of the art [10], [16], [17], we set up the problem in a powerful hybrid geometric framework, which would help solve more complicated control problems on Lie groups. We exploit the fact that the unit circle is isomorphic to the Special orthogonal group $\mathsf{SO}(2)$ and apply hybrid system tools to achieve global asymptotic stability with robustness.

Another significant difference between our work and [10], [16], [17] is that we introduce the concept of a novel family of Lie algebra valued functions on $\mathsf{SO}(2)$ and a novel geometric controller class. Specifically, we propose a geometric controller class in which every controller guarantees convergence to the desired point from a neighborhood of it, and a geometric global open-loop controller that forces the system to enter in that neighborhood. By employing hybrid systems tools, we show that the proposed controller class induces robust and global asymptotic stability of the closed-loop system.

The main contributions of this paper are as follows: i) a novel kinematic family of Lie algebra valued function $\mathcal{F}_k$ on $\mathbb{S}^1$ (Definition 5.4); ii) a geometric kinematic controller class $\mathcal{C}_k$ that provides asymptotic stability (Lemma 5.7); iii) a class of hybrid controllers that guarantee robust global asymptotic stability to the desired point (Theorem 5.12).

A. Notation and Math Preliminaries

The n-dimensional Euclidean space is represented by $\mathbb{R}^n$. For a point $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $|x|$, and the distance of a point x from a subset $S \subset \mathbb{R}^n$ is represented by $|x|_S := \inf_{y \in S} |x - y|$. The closed unit ball of appropriate dimension in Euclidean norm is denoted by $\mathbb{B}$. For a matrix $A \in \mathbb{R}^{m \times n}$, its Frobenius norm is given by $\|A\|_F$. We denote the inner product of two vectors $x, y \in \mathbb{R}^n$ as $\langle x, y \rangle$. A k-dimensional vector x is represented as $(x_1, x_2, \ldots, x_k) := \begin{bmatrix} x_1, & x_2, & \cdots & x_k \end{bmatrix}^\top$, where $^\top$ denotes transposition. The domain of a map f is represented by $\operatorname{dom} f$. The value of the gradient of the map $f : \mathbb{R}^m \to \mathbb{R}^n$ with respect to its argument evaluated at x is given by $\nabla f(x)$. The trace and determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is represented by $\operatorname{trace}(A)$ and $\det(A)$, respectively. The set of 2×2 rotation matrices is defined as $\mathsf{SO}(2) = \left\{ R \in \mathbb{R}^{2\times 2} : R^\top = R^{-1}, \det(R) = +1 \right\}$ and it has a Lie group structure. The associated Lie algebra of $\mathsf{SO}(2)$ is the set of 2×2 skew-symmetric matrices $\mathfrak{so}(2) = \left\{ A \in \mathbb{R}^{2\times 2} : A = -A^\top \right\}$, which is isomorphic to $\mathbb{R}$. The isomorphism is denoted by[2] $\widehat{\cdot} : \mathbb{R} \to \mathfrak{so}(2)$ and its inverse is denoted $(\cdot)^\vee : \mathfrak{so}(2) \to \mathbb{R}$. The unit sphere, or unit circle, is defined as $\mathbb{S}^1 := \left\{ x \in \mathbb{R}^2 : |x| = 1 \right\}$. It is well known that $\mathbb{S}^1$ is a Lie group and is isomorphic to $\mathsf{SO}(2)$. Finally, the matrix exponential is an analytic diffeomorphism [2] between $U_{\mathfrak{so}(2)} := \{ \widehat{\omega} \in \mathfrak{so}(2) : \omega \in \mathbb{R}, |\omega| < \pi \}$ and $U_{\mathsf{SO}(2)} := \{ R \in \mathsf{SO}(2) : \operatorname{trace}(R) \neq -2 \}$. The inverse map from $U_{\mathsf{SO}(2)}$ to $U_{\mathfrak{so}(2)}$ is the principal matrix logarithm and is denoted by $\log(\cdot)$. Let $\mathcal{M}$ be a smooth manifold and p be a point on $\mathcal{M}$. The tangent space of $\mathcal{M}$ at the point p is denoted by $\mathsf{T}_p\mathcal{M}$, and the tangent bundle of $\mathcal{M}$ is denoted by $\mathsf{T}\mathcal{M}$.

II. Motivation

Informally, first, we consider the classical point stabilization problem on the unit circle in terms of Euclidean coordinates. To be precise, consider a point mass with coordinates $x = (x_1, x_2) \in \mathbb{R}^2$ restricted to evolve on the unit circle $\mathbb{S}^1$ with controllable angular velocity $\omega \in \mathbb{R}$. The kinematics equations of the point-mass system are given by

$$\dot{x}_1 = -\omega x_2, \quad \dot{x}_2 = \omega x_1, \tag{1}$$

where $x \in \mathbb{S}^1$. The goal is to design a controller $\kappa : \mathbb{R}^2 \to \mathbb{R}$ that assigns ω and, robustly and globally, asymptotically stabilizes the system to a point $(x_1^\star, x_2^\star)$. Without loss of generality, we assume that the point to be stabilized is $(x_1^\star, x_2^\star) = e_1 := (1, 0)$. Although the problem seems simple, it is far from being trivial due to topological obstructions [9]. Since $\mathbb{S}^1$ is a compact manifold, it is not possible to design a continuous time-invariant state-feedback controller that globally asymptotically stabilizes any equilibrium [9]. While discontinuous control laws can attain global asymptotic stability, they lack robustness, even to small noise.

The representation of the point-mass system evolving on the unit circle given in (1) has two issues. First, the underlying manifold is of dimension one; therefore, we should express the position of the point mass on the circle by one coordinate, rather than two coordinates and a constraint. Second, the representation is not geometric or coordinate free; to be precise, Euclidean coordinates are used to express the system. To express the position of the point mass without coordinates, i.e., in a geometric setting, we rewrite the kinematic model (1) as

$$\dot{x} = \begin{bmatrix} -\omega x_2 \\ \omega x_1 \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x. \tag{2}$$

From (2), we make the following observations. Since $x \in \mathbb{S}^1$ and the unit circle is isomorphic [18] to $\mathsf{SO}(2)$, i.e., $\mathbb{S}^1 \simeq \mathsf{SO}(2)$, x belongs to $\mathsf{SO}(2)$. As we know that $\mathsf{SO}(2)$ is the set of rotation matrices, we relabel $x \in \mathsf{SO}(2)$ with $R \in \mathsf{SO}(2)$ for clarity. Moreover, we have a skew-symmetric matrix appearing in (2), so it must belongs to $\mathfrak{so}(2)$. Let the skew-symmetric matrix in (2) be denoted by $\Omega_r \in \mathfrak{so}(2)$. We can rewrite (2) in the geometric form given by

$$\dot{R} = \Omega_r R. \tag{3}$$

Formally this is an equation representing a right-invariant vector field. Using an adjoint map $\mathrm{Ad} : \mathfrak{so}(2) \to \mathfrak{so}(2)$, $\Omega_l \mapsto R \Omega_l R^\top$, we define $\Omega_r = R \Omega_l R^\top$, which transforms the right-invariant[3] system (3) into a left-invariant system $\dot{R} = R\Omega_l$. For the rest of this article, we consider the left-invariant vector field on $\mathsf{SO}(2)$, and for the notational simplification we drop the subscript from Ω_l, i.e.,

$$\dot{R} = R\Omega. \tag{4}$$

III. Hybrid Systems on Manifolds

Informally, a hybrid control system consists of a hybrid plant and a hybrid controller whose variables may evolve continuously, called *flow*, or change instantaneously, called *jump*. We refer the reader to [10], [19] for more details. First, we provide the notion of hybrid time.

Definition 3.1 (hybrid time and hybrid time domain): Hybrid time is defined by pairs (t, j), where $t \in \mathbb{R}_{\geq 0}$ captures the duration of flows and $j \in \mathbb{N}$ indicates the number of jumps. A set E is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ is a compact hybrid time domain; i.e., it can be written as $\cup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 \leq t_0 \leq t_1 \leq \ldots \leq t_J$.

Definition 3.2 (hybrid plant): A hybrid equation model of a plant with hybrid dynamics is given by

$$\mathcal{H}_P : \begin{cases} (z, u) \in C_P & \dot{z} = F_P(z, u) \\ (z, u) \in D_P & z^+ = G_P(z, u) \\ & y = h(z), \end{cases} \tag{5}$$

where the state z takes values on a smooth manifold, i.e., $z \in \mathcal{M}_P$, the inputs to the plant takes values on a subset of the Euclidean space, i.e., $u \in \mathcal{U}_P \subset \mathbb{R}^{m_P}$. Moreover, the set $C_P \subset \mathcal{M}_P \times \mathcal{U}_P$ is called the *flow set*, the set $D_P \subset \mathcal{M}_P \times \mathcal{U}_P$ is called the *jump set*, the single-valued mapping

[2] Given $\omega \in \mathbb{R}$, we express $\widehat{\omega} \in \mathfrak{so}(2)$ or equivalently $(\omega)^\wedge \in \mathfrak{so}(2)$.

[3] In terms of properties, left- and right-invariant systems are similar.

F_P: $\mathcal{M}_P \times \mathcal{U}_P \to \mathsf{T}\mathcal{M}_P$ is called the *flow map*, and the single-valued mapping $G_P : \mathcal{M}_P \times \mathcal{U}_P \to \mathsf{T}\mathcal{M}_P$ is called the *jump map*. The data of the hybrid plant is defined by the tuple (C_P, F_P, D_P, G_P, h).

Unlike [10], the states of the plant evolve on the smooth manifold $\mathcal{M}_P$. It should be noted that (4) is a special case of the hybrid plant $\mathcal{H}_P$ because the system only flows, i.e., $D_P = \emptyset$. Let the state and input of the system be $z := R \in \mathsf{SO}(2)$ and $u := \Omega \in \mathfrak{so}(2)$, respectively. Moreover, the data (C_P, F_P, D_P, G_P) is given as $C_P := \mathsf{SO}(2) \times \mathfrak{so}(2)$, $F_P(R, \Omega) := R\Omega$, $D_P = \emptyset$, and G_P can be any arbitrary mapping. This definition above captures a continuous-time plant evolving on a manifold; for details see [10]. Similarly, a hybrid controller model can be defined as follows.

Definition 3.3 (hybrid controller): A hybrid equation model of a controller with hybrid dynamics is given by

$$\mathcal{H}_K : \begin{cases} (v, \eta) \in C_K & \dot{\eta} \ = F_K(v, \eta) \\ (v, \eta) \in D_K & \eta^+ = G_K(v, \eta) \\ & \zeta \ = \kappa(v, \eta), \end{cases} \tag{6}$$

where η is the state, v is the input, and ζ is the output of the controller. Moreover, C_K is the flow set, D_K is the jump set, F_K is the flow map, and G_K is the jump map. The data of the hybrid controller is defined by the tuple $(C_K, F_K, D_K, G_K, \kappa)$.

The control of the plant $\mathcal{H}_P$ via the controller $\mathcal{H}_K$ defines an interconnection through the following simple rule: $u = \zeta$ and $v = y$. Similar to the hybrid plant and the controller, a hybrid closed-loop system can be defined as follows.

Definition 3.4 (hybrid closed-loop system): A hybrid equation model of the closed-loop system is given by

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \ = F(x) \\ x \in D & x^+ = G(x) \end{cases} \tag{7}$$

where x is the state evolving on the manifold $\mathcal{M}$, C is the flow set, D is the jump set, $F : \mathcal{M} \to \mathsf{T}\mathcal{M}$ is the flow map, and $G : \mathcal{M} \to \mathsf{T}\mathcal{M}$ is the jump map. The data of the hybrid closed-loop system is defined by the tuple (C, F, D, G).

Solutions to hybrid systems are given by hybrid arcs which are trajectories defined on hybrid time domains.

Definition 3.5 (hybrid arc): A hybrid arc x is a function whose values belong to $\mathcal{M}$, is defined on a hybrid time domain $\operatorname{dom} x$, and is such that $t \mapsto x(t, j)$ is locally absolutely continuous for every j such that $(t, j) \in \operatorname{dom} x$.

Hybrid time domains impose a specific structure on the domains of solutions to hybrid systems. In simple words, solutions to $\mathcal{H}$ are defined on intervals of flow $[t_j, t_{j+1}]$ indexed by the jump counter j when $t_{j+1} > t_j$. Hybrid arcs specify the functions that define solutions to hybrid systems when the following conditions are satisfied.

Definition 3.6 (solution): A hybrid arc ϕ is a solution to the hybrid system $\mathcal{H}$ if $\phi(0, 0) \in \overline{C} \cup D$ and for all $j \in \mathbb{N} := \{0, 1, 2, \dots\}$ and almost all t such that $(t, j) \in \operatorname{dom} \phi$, $\phi(t, j) \in C$, $\dot{\phi}(t, j) = F(\phi(t, j))$; for all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$, $\phi(t, j) \in D$, $\phi(t, j+1) = G(\phi(t, j))$.

A solution ϕ to $\mathcal{H}$ is said to be nontrivial if $\operatorname{dom} \phi$ contains at least two points. A solution ϕ to $\mathcal{H}$ is said to be complete if $\operatorname{dom} \phi$ is unbounded. A solution ϕ to $\mathcal{H}$ is said to be Zeno if it is complete and the projection of $\operatorname{dom} \phi$ onto $\mathcal{M}$ is bounded. A solution ϕ to $\mathcal{H}$ is said to be maximal if there does not exist another solution φ to $\mathcal{H}$ such that $\operatorname{dom} \varphi$ is a proper subset of $\operatorname{dom} \phi$, and $\varphi(t, j) = \phi(t, j)$ for all $(t, j) \in \operatorname{dom} \phi$.

Definition 3.7: (hybrid basic conditions) A hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies the hybrid basic conditions if i) C and D are closed subsets of $\mathcal{M}$; ii) $F : C \to \mathsf{T}\mathcal{M}$ is continuous; iii) $G : D \to \mathsf{T}\mathcal{M}$ is continuous.

Definition 3.8 (stability notions): Given a hybrid closed-loop system $\mathcal{H}$, a nonempty set $\mathcal{A} \subset \mathcal{M}$ is said to be 1) stable for $\mathcal{H}$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t, j)|_{\mathcal{A}} \leq \epsilon$ for all $(t, j) \in \operatorname{dom} x$; 2) attractive for $\mathcal{H}$ if there exists $\mu > 0$ such that every maximal solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \mu$ is complete and satisfies $\lim_{(t,j) \in \operatorname{dom} x, t+j \to \infty} |x(t, j)|_{\mathcal{A}} = 0$; 3) asymptotically stable for $\mathcal{H}$ if it is stable and attractive.

Definition 3.9: (robust stability) Given a hybrid closed-loop system $\mathcal{H}$, a nonempty closed set $\mathcal{A} \subset \mathcal{M}$ and an open set $\mathcal{U} \subset \mathcal{M}$ such that $\mathcal{A} \subset \mathcal{U}$, the set $\mathcal{A}$ is said to be robustly stable for $\mathcal{H}$ on $\mathcal{U}$ if for every proper indicator function ϖ of $\mathcal{A}$ on $\mathcal{U}$, every function $\beta \in \mathcal{KL}$ such that

$$\varpi(x(t, j)) \leq \beta(\varpi(x(0, 0)), t + j) \quad \forall (t, j) \in \operatorname{dom} x$$

for the solutions to $\mathcal{H}$ from $\mathcal{U}$, and every continuous function $\rho^* : \mathcal{M} \to \mathbb{R}_{\geq 0}$ that is positive on $\mathcal{U} \backslash \mathcal{A}$, the following holds: for each compact set $K \subset \mathcal{U}$ and each $\epsilon > 0$, there exists $\delta^* > 0$ such that for each solution x_ρ the perturbed system $\mathcal{H}_\rho$ with $\rho = \delta^* \rho^*$, starting from $x_\rho(0, 0) \in K$ satisfies

$$\varpi(x_\rho(t, j)) \leq \beta(\varpi(x_\rho(0, 0)), t + j) + \epsilon \quad \forall (t, j) \in \operatorname{dom} x_\rho.$$

IV. Problem Formulation

For the point-mass system evolving on the circle, global asymptotic stability of a point in $\mathsf{SO}(2)$ is not possible with a continuous state-feedback law [9]. Global asymptotic stabilization of this point is possible by discontinuous feedback, although the resulting closed-loop system may not be robust to arbitrarily small measurement noise [16]. A hybrid controller can be designed to achieve robust, global asymptotic stabilization of the point, even in the presence of noise. Without loss of generality, let $I \in \mathsf{SO}(2)$ be the point we want to stabilize, where I is the two-by-two identity matrix.

Problem 1: Given a point-mass evolving on the unit circle as in (4), design a controller with state η, input v, and output ζ of the form (6) where C_K, D_K, F_K and D_K are the flow set, jump set, flow map, and jump set, respectively [10], such that each solution component $(t, j) \mapsto R(t, j)$ of the closed-loop system globally asymptotically converges to the desired point $I \in \mathsf{SO}(2)$, i.e., for all $R(0, 0) \in \mathsf{SO}(2)$

$$\lim_{t+j \to \infty} R(t, j) = I,$$

with robustness.

In the next section, we characterize the class of controllers that solves this problem.

V. Controller Design

As discussed earlier, it is impossible to design a global smooth continuous feedback controller that globally asymptotically stabilizes, without loss of generality, the point $I \in \mathsf{SO}(2)$. Specifically, the point $-I \in \mathsf{SO}(2)$ renders a topological singularity. To overcome this issue, we divide the unit circle in two regions, C_0 and C_1, as shown in Figure 1, such that $-I \in C_1 \subset \mathsf{SO}(2)$ and $I \in C_0 \subset \mathsf{SO}(2)$. As shown in Figure 1, the desired point $I \in \mathsf{SO}(2)$ is indicated by a red star, and the singular point $-I \in \mathsf{SO}(2)$ is indicated by a solid black dot. The region to the left of the red dot-dashed line is indicated by C_1 and the region to the right of the green dashed line is indicated by C_0. The region in between the dot-dashed red line and the dashed green line is the hysteresis region as indicated by the yellow shaded region in Figure 1. We precisely quantify the sets C_0, C_1, and the hysteresis region later in this section. The idea of

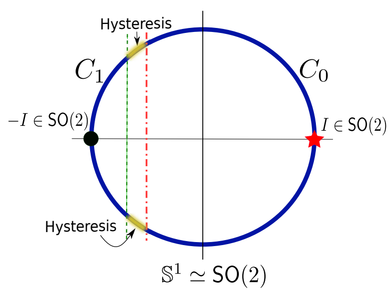

Fig. 1. Point stabilization problem on the unit circle.

the global geometric controller design is the following: if the system is initialized in the region C_0, employ a geometric feedback controller that asymptotically stabilizes the system to the point $I \in \mathsf{SO}(2)$ and if the system is initialized in the region C_1, use a geometric open-loop controller that forces the system to enter in the C_0 region, and switch the control authority to the first controller.

First we define a map that measures distance between two elements in $\mathsf{SO}(2)$. Let

$$\begin{aligned} d_{\mathsf{SO}(2)} : \mathsf{SO}(2) \times \mathsf{SO}(2) &\to \mathbb{R} \\ (R_1, R_2) &\mapsto \|I - R_1^\top R_2\|_F. \end{aligned} \tag{8}$$

Remark 5.1: It is easy to verify that $d_{\mathsf{SO}(2)}$ is a metric. The image of $\mathsf{SO}(2) \times \mathsf{SO}(2)$ under the map $d_{\mathsf{SO}(2)}$ is the closed interval $[0, 2\sqrt{2}]$. Moreover, $d_{\mathsf{SO}(2)}(R_1, R_2) = 2\sqrt{2}$ when R_1 and R_2 are "furthest apart" or "antipodal." One such example of antipodal points are when $R_1 = -I$ and $R_2 = I$.

Next, we present two elementary, yet useful definitions.

Definition 5.2 (closed-ball): The unit closed ball in $\mathsf{SO}(2)$ is defined as $\mathbb{B}_{\mathsf{SO}(2)} := \left\{R \in \mathsf{SO}(2) : d_{\mathsf{SO}(2)}(R, I) \leq 1\right\}$.

Definition 5.3: Given a point $R^\star \in \mathsf{SO}(2)$, for some $\epsilon > 0$, an open ϵ-neighborhood of $R^\star$ is defined as $\mathcal{N}_\epsilon(R^\star) := \left\{R \in \mathsf{SO}(2) : d_{\mathsf{SO}(2)}(R, R^\star) < \epsilon\right\}$.

It should be noted that the ϵ-neighbourhood of every point in $\mathsf{SO}(2)$ is a set of nonzero Lebesgue measure. To design a class of geometric controllers on $\mathsf{SO}(2)$, we first propose a novel family of Lie algebra valued functions on $\mathsf{SO}(2)$.

Definition 5.4: A function $f : \mathbb{D} \subset \mathsf{SO}(2) \to \mathfrak{so}(2)$ is said to belong to the kinematic family of Lie algebra valued functions $\mathcal{F}_k$ if it satisfies the following properties:

1) f is at least C^1;
2) $f^{-1}(0) = \{R \in \mathbb{D} : R = I\}$;
3) $\mathrm{d}_R f$, the derivative of f with respect to R, is nonsingular at least in a neighbourhood of I;
4) $\mathbb{D}$ contains an open neighborhood of I and is connected.

The kinematic family of Lie algebra valued functions $\mathcal{F}_k$ leads to the following definition.

Definition 5.5: Given a function f belonging to the family $\mathcal{F}_k$ and the domain of f containing an open neighborhood of I, the set C_0 is defined as

$$C_0 := \{R \in \mathbb{D} : \det(\mathrm{d}_R f(R)) \neq 0\} \setminus \mathcal{N}_\epsilon(-I). \tag{9}$$

Remark 5.6: An example of a function belonging to this family is $\log : \mathbb{D} \subset \mathsf{SO}(2) \to \mathfrak{so}(2)$. The $\log$ map is defined everywhere on $\mathsf{SO}(2)$ except at $-I \in \mathsf{SO}(2)$ and $\mathrm{d}_R f$ is nonsingular everywhere except at $-I \in \mathsf{SO}(2)$. It is easy to check that this function satisfies all three conditions of Definition 5.4 and the set in (9) reduces to $C_0 = \mathsf{SO}(2) \setminus \mathcal{N}_\epsilon(-I)$. Another example of a function belonging to this family $\mathcal{F}_k$ is given by $f(R) = R^\top - R$.

Lemma 5.7: Given the point mass system in (4) and a kinematic family of Lie algebra valued functions $\mathcal{F}_k$, each function $f \in \mathcal{F}_k$ induces a controller given by

$$\Omega = \kappa(R) = R^{-1} \left(\mathrm{d}_R f\right)^{-1} \left(-f(R)\right)^\vee, \tag{10}$$

such that, for the resulting closed-loop system

$$\dot{R} = R f(R),$$

the singleton set $\{R \in \mathbb{D} : R = I\}$ is asymptotically stable with the basin of attraction equal to

$$\mathcal{B}_f := \{R \in \mathbb{D} : \det(\mathrm{d}_R f(R)) \neq 0\}. \tag{11}$$

Remark 5.8: Every function f contained in the kinematic family $\mathcal{F}_k$ gives rise to a local geometric controller, which by Lemma 5.7 renders I locally asymptotically stable. The collection of all such controllers constitute a class, denoted by $\mathcal{C}_k$ that we call the kinematic controller class. We claim that there exists functions in the family $\mathcal{F}_k$ that lead to controllers whose domain of attraction is as large as possible, i.e., $\mathcal{B}_f = \mathsf{SO}(2) \setminus \{-I\}$.

Example 5.9: In this example, we select the following function from the $\mathcal{F}_k$ family:

$$f : \mathsf{SO}(2) \setminus \{-I\} \to \mathfrak{so}(2), \quad R \mapsto \log(R)$$

In the light of Lemma 5.7, it is straightforward to verify that $\mathrm{d}_R f = I$. Therefore, f is invertible everywhere and produces the following controller:

$$\Omega = \kappa_0(R) = -\log(R). \tag{12}$$

The controller κ_0 is defined everywhere on $\mathsf{SO}(2)$ except of a "small" set of measure zero. Precisely, that set of zero measure is given by $Z := \{R \in \mathsf{SO}(2) : R = -I\}$. In other words, the controller κ_0 has the largest possible basin of attraction, i.e., $\mathcal{B}_f = \mathsf{SO}(2) \setminus Z$. Therefore, this controller is almost globally asymptotically stabilizing. Finally, this ϵ-neighborhood leads to the characterization of the set $C_0 = \mathsf{SO}(2) \setminus \mathcal{N}_\epsilon(-I)$.

For other candidate functions of the family $\mathcal{F}_k$ the domain of attraction can be smaller than the one considered in Example 5.9. However, it must be noted that Definition 5.4 and Lemma 5.7 guarantee that the region of attraction will be a non empty open neighbourhood of $I \in \mathsf{SO}(2)$.

Next, we define the region C_1.

Definition 5.10: Given a function f belonging to the family $\mathcal{F}_k$, the set C_1 is chosen as

$$C_1 \subset \mathsf{SO}(2) \setminus (\{R \in \mathbb{D} : \det(\mathrm{d}_R f(R)) \neq 0\} \setminus \mathcal{N}_\epsilon(-I)), \tag{13}$$

such that C_1 is connected and $C_1 \cup C_0 = \mathsf{SO}(2)$.

An explicit construction of C_1 is provided later in this section. We define a global open-loop controller such that, when the state is in C_1, the open-loop controller forces the system to enter the region C_0 in finite time.

Lemma 5.11: Let C_0 and C_1 be the sets defined as in (9) and (13), respectively. For each, $R(0) \in C_1$, the open-loop controller

$$\kappa_1(R) = \widehat{1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{14}$$

is such that the solution $t \mapsto R(t)$ to system (4) under the effect of κ_1 reaches the set C_0 in finite time, i.e., for each $R(0) \in \mathsf{SO}(2)$, there exists $T > 0$ such that the solution $t \mapsto R(t)$ satisfies $R(T) \in C_0$.

In summary, roughly speaking, any controller κ_0 from the controller class $\mathcal{C}_k$ asymptotically stabilizes the equilibrium point if the initial state $R(0)$ is in the basin of attraction $\mathcal{B}_f$. If $R(0)$ is outside the basin of attraction, the controller κ_1 can be used to push R into $\mathcal{B}_f$. At first glance, it looks that a discontinuous (non-hybrid) switching scheme would be sufficient to achieve global stabilization. Nevertheless, such a solution would be sensitive to even arbitrarily small noise and hence nonrobust. In other words, in the presence of noise, solutions of the system may exhibit chattering at the switching surface when a discontinuous controller is used.

To avoid this issue, we introduce hysteresis in the switching mechanism and model it as hybrid control $\mathcal{H}_K$ as in (6). To create hysteresis and achieve robustness, we construct our hybrid geometric controller as follows. Let $V(R) = \frac{1}{2}\left((f(R))^\vee\right)^2$ be a real-valued map. By Definition 5.4 and Lemma 5.7, $\mathcal{B}_f$ is nonempty and has nonzero measure. Moreover, the point $I \in \mathsf{SO}(2)$ is an interior point of $\mathcal{B}_f$. This implies that for $0 < c_1 < c_{1,0} < c_0$, there exists an open set $\mathcal{U}_0$ such that

$$\mathcal{U}_0 := \{R \in \mathsf{SO}(2) : V(R) < c_0\}, \ \ \mathcal{U}_0 \subset \mathcal{B}_f.$$

The set $\mathcal{T}_{1,0}$ is defined by a $c_{1,0}$-sublevel set of V, such that $\mathcal{T}_{1,0}$ is contained in the interior of $\mathcal{U}_0$, i.e.,

$$\mathcal{T}_{1,0} := \{R \in \mathsf{SO}(2) : V(R) \leq c_{1,0}\} \subset \mathcal{U}_0.$$

Let $C_0 := \overline{\mathcal{U}_0}$ and $C_1 := \overline{\mathsf{SO}(2) \setminus \mathcal{T}_{1,0}}$, which lead to the hysteresis region $C_0 \setminus \mathcal{T}_{1,0}$. With the above mentioned sets, the hybrid controller $\mathcal{H}_K$ has state $\eta = q \in Q := \{0, 1\}$, input $v = z := R \in \mathsf{SO}(2)$, output $\zeta := \Omega \in \mathfrak{so}(2)$, and data $(C_K, F_K, D_K, G_K, \kappa)$ as follows:

$$C_K = \bigcup_{q \in Q} (C_{K,q} \times \{q\}), \quad \begin{cases} C_{K,0} := C_0 \\ C_{K,1} := C_1 \end{cases} \tag{15}$$

$$F_K(z, q) = 0 \quad \forall (z, q) \in C_K \tag{16}$$

$$D_K = \bigcup_{q \in Q} (D_{K,q} \times \{q\}), \quad \begin{cases} D_{K,0} := \overline{\mathsf{SO}(2) \setminus \mathcal{U}_0} \\ D_{K,1} := \mathcal{T}_{1,0} \end{cases} \tag{17}$$

$$G_K(z, q) = 1 - q \quad \forall (z, q) \in D_K \tag{18}$$

$$\kappa(z, q) = \kappa_1(z) + (1 - q)\kappa_0(z), \tag{19}$$

where the controller κ_0 belong the controller class $\mathcal{C}_k$, induced by the kinematic family of functions $\mathcal{F}_k$, and the open-loop controller κ_1 give in (14). The above mentioned construction of the sets $\mathcal{U}_0$ and $\mathcal{T}_{0,1}$ creates a hysteresis, with boundary of $\mathcal{U}_0$ and $\mathcal{T}_{0,1}$ being the outer and inner portion of the hysteresis region, respectively.

Controlling the continuous-time plant (4), defined on a Lie group, by the hybrid controller results in a hybrid closed-loop system with states $x = (z, q)$ and dynamics

$$\dot{z} = F_P(z, \kappa(z, q)) := R\kappa(z, q), \quad \dot{q} = 0 \tag{20}$$

during flows, and at jumps, the state is updated according to

$$z^+ = z, \quad q^+ = 1 - q. \tag{21}$$

Finally, the hybrid closed-loop system $\mathcal{H} = (C, F, D, G)$ with the state $x = (z, q) \in \mathsf{SO}(2) \times Q =: X$ has data given as

$$\begin{aligned} C &:= \{(z, q) \in X : (z, \kappa_q(z)) \in C_P, z \in C_{K,q}\} \\ F(x) &:= \begin{bmatrix} F_P(z, \kappa_q(z)) \\ 0 \end{bmatrix} \quad \forall x \in C \\ D &:= \{(z, q) \in X : (z, \kappa_q(z)) \in C_P, z \in D_{K,q}\} \\ G(x) &:= \begin{bmatrix} z \\ 1 - q \end{bmatrix} \quad \forall x \in D, \end{aligned} \tag{22}$$

where $C_P := \mathsf{SO}(2) \times \mathfrak{so}(2)$.

Theorem 5.12: Given $I \in \mathsf{SO}(2)$ and the continuous-time plant (1) defined on a Lie group $\mathsf{SO}(2)$, the following hold:

1) The closed-loop system $\mathcal{H} = (C, F, D, G)$ with data in (22) satisfies the hybrid basic conditions;
2) Every maximal solution to $\mathcal{H}$ from $C \cup D$ is complete and exhibits no more than two jumps;
3) The set $\mathcal{A} = \{I\} \times \{0\}$ is robustly globally asymptotically stable for $\mathcal{H}$.

VI. Simulation Results

In this section, we provide simulation results of the hybrid controller $\mathcal{H}_K$. Informally, we unite geometric controller κ_0 and the open-loop controller κ_1, given in (12) and (14), respectively, through the hybrid framework. The system is initialized at the most challenging position, i.e., $R(0) = -I$. No controller from the $\mathcal{C}_k$ controller class can make the system states converge asymptotic to the desired point. In other words, the system is initialized on the set $C_{K,1} \times \{1\}$; therefore, we invoke the controller κ_1. The system trajectories flow for about 1 sec, as seen in Figure 3, and then enter in the set $C_{K,0} \times \{0\}$. After that, the control authority is given to the controller κ_0, which makes the system asymptotically converge to the desired point $I \in \mathsf{SO}(2)$, as seen in Figure 2. All the errors converge to zero, as shown in Figure 3. Since the system is simulated under persistent random white noise, the effect of noise can be seen in the steady-state in Figure 3, and the system demonstrates robustness. Finally, as shown in the bottom plot of Figure 3, around 1 sec, the control authority switches from controller κ_1 to κ_0.

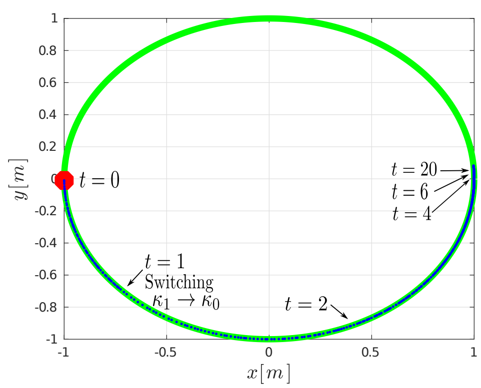

Fig. 2. Hybrid control scheme achieves global asymptotic stability on $\mathbb{S}^1$.

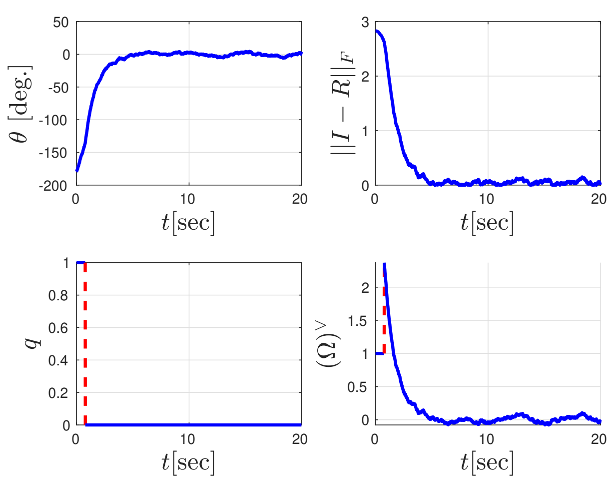

Fig. 3. The top left and top right plots show the errors converging to zero. The bottom left plot shows the logic variable switching from 1 to 0 around $t = 1$ secs and the bottom right plot shows the control input.

VII. Conclusion

We developed a hybrid geometric controller to globally robustly and asymptotically stabilize a desired point on the unit circle. First, we introduce a notion of a family of Lie algebra valued function on $\mathsf{SO}(2)$. This family of functions induces a geometric controller class, such that each controller is in this class is locally asymptotically stable. Using the tools of hybrid systems, we combine an open-loop controller with this geometric controller class and define a geometric hybrid controller for each function belonging to the geometric controller class. We proved that the resulting closed-loop hybrid system stabilizes the desired set and is robust. For future work, we will extend these results for systems defined on more general Lie groups.

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