Reference Governor for Hybrid Dynamical Systems

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***Abstract*— We formulate a reference governor algorithm for hybrid systems modeled as hybrid equations, in which the continuous dynamics are governed by a constrained differential equation and the discrete dynamics by a constrained difference equation. Basic definitions, models, and properties of the proposed hybrid reference governor approach are introduced and a time-based implementation is formulated. We apply the methodology to hybrid equations with linear right-hand side in both the differential and difference equations and with explicit logic variables. We illustrate the approach in examples.**

I. Introduction

Enforcing constraints by design is a practical challenge in many control applications. Due to this challenge, one of the most commonly used control approaches is model predictive control (MPC) [1], as it not only allows for constraint satisfaction but also leads to a feedback controller with optimality properties. Unfortunately, MPC tends to be computationally expensive, imposing the selection of low control rates and requiring expensive computing platforms. An alternative approach that safely handles constraints with lower computational requirements – possibly at the price of reduced closed-loop performance – is the reference governor (RG) approach [2].

RG is a state-feedback control law that manages the reference signal of a plant, usually pre-stabilized by a feedback controller. In simple words, the objective of the reference governor is to manipulate the reference with respect to a user-defined value so that the closed-loop system satisfies, at every point in time, its constraints. While MPC requires solving a multi-variable optimization problem, RG requires only the selection of a constant reference value, called *the command input*, which requires solving an optimization problem with few variables (often, a single one) or by simply evaluating inequalities at a set of values. Furthermore, being an add-on scheme, RG easily integrates with an existing controller without requiring a full redesign. On the other hand, RG usually does not allow for complex cost functions that are allowed by MPC, achieves a slower response being typically limited to constant references, and requires more stringent conditions on the plant, e.g., asymptotic stability of an equilibrium point. Nevertheless, RG is of interest especially for systems with legacy controllers, high sampling rates, and limited computational capabilities.

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Reference governors have been studied extensively for smooth systems, both linear and nonlinear, and applied successfully in several domains [2]. On the other hand, studies of RG for hybrid systems are limited. The key prior contributions focus on specific classes of systems and constraints exhibiting a nonsmooth or switching behavior, such as piecewise affine systems [3], switching systems with hysteresis [4], systems subject to disjunctive constraints for the approximation of nonlinear constraints [5], and hybrid designs combining governors for smooth systems to handle non-convex constraints [6]. Unfortunately, such classes of systems do not capture all features of hybrid systems, such as jumps at unknown times, especially when they are interleaved between intervals of continuous evolution.

Motivated by such gap and the emergence of hybrid behavior in control applications, in this paper we formulate a hybrid reference governor strategy for plants modeled as a *hybrid equations* [7], which are given by

$$\mathcal{H}_P \ : \ \begin{cases} (z,u) \in C_P & \dot{z} \ = \ F_P(z,u) \\ (z,u) \in D_P & z^+ \ = \ G_P(z,u) \\ & y \ = \ h(z) \end{cases} \tag{1}$$

The hybrid plant, denoted $\mathcal{H}_P$, is defined by the maps F_P and G_P, which govern the evolution of z along flows and at jumps, respectively, on the sets C_P and D_P, referred to as the flow set and the jump set, respectively. After presenting a motivating application for the need of RG for hybrid systems, we formulate a general reference governor strategy for this class of systems. Specifically, we introduce the maximal output admissible set (MOAS) for hybrid trajectories and show that it is forward invariant in an appropriate sense, see Section IV-A. Then, we formulate the numerical problem for computing the hybrid RG command at each event triggering its recomputation (see Section IV-B) and establish its feasibility (see Section IV-C). Based on these results, we introduce the notion of hybrid time window and formulate an algorithm implementing a time-driven version of hybrid RG; see Section IV-D. In Section V, we specialize the approach in two classes of hybrid systems: hybrid systems with linear maps and, more briefly, hybrid systems with modes transitions occurring in a sequence. Numerical examples illustrate the results.

II. Motivation

Consider the "impactor-runner" mechanism shown in Figure 1 used in several applications such as automotive valves [8] and circuit breakers [9], in which an actuated impactor mass (first mass, m_1) is controlled to collide with a runner mass (second mass, m_2) and to move it to a desired position.

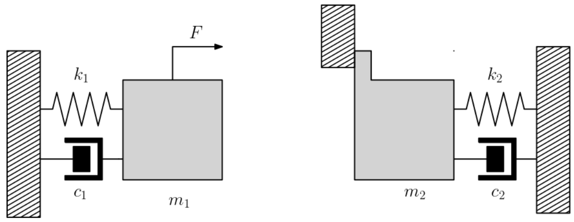


Fig. 1. Impactor-runner motion mechanism. The runner mass (left) is controlled and the impactor mass (right) is at rest at the stopper.

The change of position p_1 of the impactor mass moving alone, referred to as *mode 1*, is governed by

$$m_1\ddot{p}_1 + c_1\dot{p}_1 + k_1 p_1 = F \tag{2}$$

When joined to the runner mass, referred to as *mode 2*, p_1 is governed by

$$(m_1+m_2)\ddot{p}_1 + (c_1+c_2)\dot{p}_1 + k_1 p_1 + k_2(p_1 - \bar{p}) = F \tag{3}$$

The constants c_1, c_2 are the damping coefficients; k_1, k_2 are the spring stiffness on the first and second masses; and $\bar{p}$ is the pre-charge position of the spring acting on m_2.

The impacts between the impactor and runner masses are perfectly inelastic, that is, when the impactor collides with the runner, the two stick together. The stopper placed at $p_1 = p^*$ also undergoes inelastic impacts with m_2. This and the mechanical design ensure that the runner mass is only moving when in contact with the impactor mass, i.e., m_2 is captured at the stopper by m_1, and released only at the stopper, where it stays until the next impact with m_1. Thus, the position of the runner mass, denoted by p_2, when alone (mode 1) and when joined with the impactor mass (mode 2) is governed by $\dot{p}_2 = \ddot{p}_2 = 0$ and by $p_1 = p_2$, $\dot{p}_1 = \dot{p}_2$, $\ddot{p}_1 = \ddot{p}_2$, respectively. By momentum conservation, since $\dot{p}_2$ is initially zero (at rest), the inelastic impact between m_1 and m_2 results in the velocity update

$$\dot{p}_1^+ = \frac{m_1}{m_1+m_2}\dot{p}_1 \tag{4}$$

The masses are subject to the velocity constraints

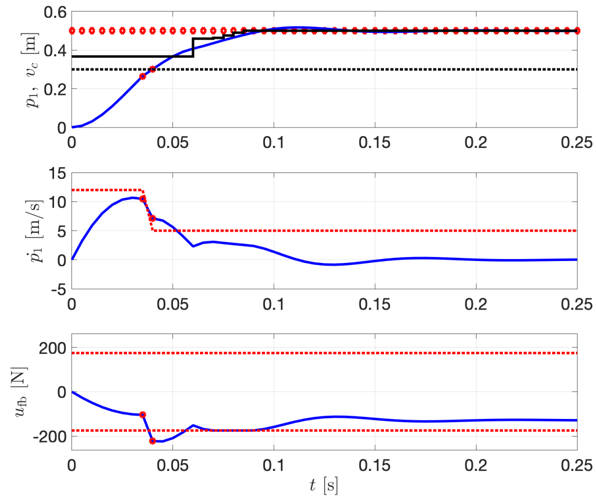
$$\dot{p}_1 \leq 12 \text{ m/s}, \quad \dot{p}_2 \leq 5 \text{ m/s} \tag{5}$$

where the constraint on $\dot{p}_2$ imposes a constraint on $\dot{p}_1$ when the masses are joined, since $\dot{p}_1 = \dot{p}_2$ in such condition.

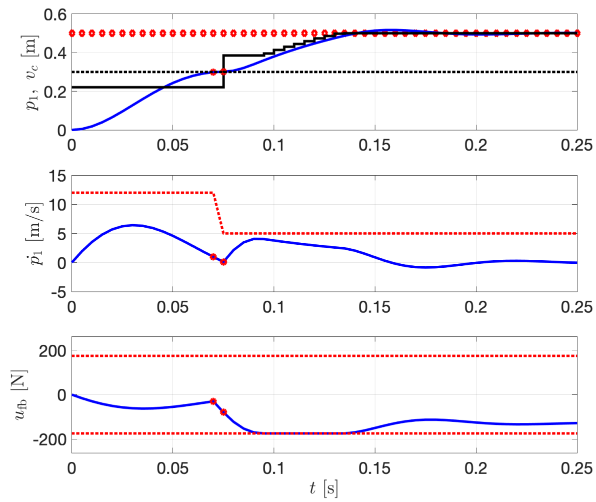
For each mode of operation, we design a feedback-feedforward controller $F = u_{\text{fb}} + u_{\text{ff}}$ that stabilizes the dynamics and provides a closed-loop system with unitary DC-gain from reference to position. The feedback component of the control signal is subject to the constraint

$$|u_{\text{fb}}| \leq 175\text{N} \tag{6}$$

Figure 2(a) shows closed-loop trajectories using two standard RGs designed separately aiming at reaching a reference of 0.5m: a RG for when the first mass moves alone (mode 1) according to (2) and another RG for when the two masses move together (mode 2) according to (3). As the plots show, after the impact at $\approx$ 0.03s, the constraints are violated, because the governor for mode 1 ignores the jump and the constraints of mode 2. Figure 2(b) shows the result of using two different (standard) RGs that achieve safe operation by first aiming to reach the jump set with low speed, and only after the jump aiming at the actual reference. In the latter case, the approaching phase to the jump set is rather slow, causing an impact at low speed, before start tracking the actual reference, which results in overall slow tracking. In both cases, the control loop operates at 200Hz, i.e., with a period of 5ms, which rules out computationally expensive control algorithms [10], as the embedded processors for these applications have limited computing power [11]. The constraint violations and limited performance obtained using standard RGs motivate the need for hybrid RG strategies.



(a)



(b)

Fig. 2. Impact-runner motion mechanism simulations: (a) two RGs designed separately cause constraints violations at impact; (b) two RGs designed separately satisfy the constraints by having m_1 cautiously approaching m_2, to the jump set, but tracking is slowed. Trajectories (solid blue), reference r (red diamonds), virtual reference from RG (solid black) constraints (dash red), m_2 rest position (black dot), boundaries of controller sampling interval when impacts occur (red circles).

III. Preliminaries

A. Reference Governor

Consider a discrete-time system

$$x^+ = f(x, v_c), \quad y = h(x, v_c) \tag{7}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ are the state and constrained output vectors, respectively, and $v_c \in \mathbb{R}$. For every constant v_c, (7) admits a unique globally asymptotically stable equilibrium, possibly due to (7) being already a closed-loop system. The reference governor (RG)

$$v_c = \kappa(x, r) \tag{8}$$

determines the virtual reference $v_c \in \mathbb{R}$ that is closest to r such that the output constraint $y \in Y$ is satisfied for all resulting output trajectories of (7), where Y is the output admissible set. Most RG designs augment (7) with a constant virtual reference $v_c^+ = v_c$ and construct an invariant set for the augmented system in the state-reference space, denoted $O \subset \mathbb{R}^n \times \mathbb{R}$, such that $h(O) \subset Y$. At any time instant $k \in \{0, 1, \ldots\}$, the RG chooses $v_c(k)$ such that $(x(k), v_c(k)) \in O$, by minimizing a distance between $v_c(k)$ and $r(k)$ – commonly, the Euclidean distance $|v_c(k) - r(k)|$. The forward invariance property of O ensures recursive feasibility and constraint satisfaction. Different approaches have been proposed for designing O, including reachability tools and Lyapunov functions [2].

B. Hybrid Systems

Following [7], a hybrid plant $\mathcal{H}_P$ is defined by (1), where $z \in \mathbb{R}^{n_P}$, $u \in \mathbb{R}^{m_P}$, and $y \in \mathbb{R}^{r_P}$ are the state, input, and output, respectively, and the hybrid system data are given by C_P, the flow set of the plant; F_P, the flow map of the plant; D_P, the jump set of the plant; G_P, the jump map of the plant; and h, the output map of the plant. Solutions to $\mathcal{H}_P$ are given by pairs $(t, j) \mapsto (\phi_z(t, j), \phi_u(t, j))$, where ϕ_z and ϕ_u are the state and input trajectories, respectively. Solutions are parameterized by ordinary time $t \in \mathbb{R}_{\geq 0} := [0, \infty)$ and jump time $j \in \mathbb{N} := \{0, 1, \ldots, \}$, and given in terms of hybrid arcs, which are functions defined on a hybrid time domain [12], [7]. A solution is nontrivial if it flows for some nonzero amount of time or jump at least once. It is maximal if it cannot be further extended. See [12], [7] for details.

When the input u to $\mathcal{H}_P$ is assigned by a (potentially hybrid) controller, the resulting closed-loop system is

$$\mathcal{H} \; : \; \begin{cases} (x, v_c) \in C & \dot{x} = F(x, v_c) \\ (x, v_c) \in D & x^+ = G(x, v_c) \\ & \chi = h_{cl}(x, v_c) \end{cases} \tag{9}$$

where $x \in \mathbb{R}^n$ is the state of the resulting closed-loop system, $\chi \in \mathbb{R}^s$ is its output, and v_c is its input. We denote the set of maximal solution pairs (ϕ_x, ϕ_{v_c}) to $\mathcal{H}$ by $\mathcal{S}_\mathcal{H}$. Given an initial condition x_0 for $\mathcal{H}$, we also define the set of maximal solutions pairs to $\mathcal{H}$ with initial state x_0 and denote it $\mathcal{S}_\mathcal{H}(x_0)$, and the set of inputs for which a solution to $\mathcal{H}$ exists as $\mathcal{V}_\mathcal{H}(x_0)$.

IV. Reference Governor for Hybrid Systems

Motivated by the example in Section II, we formulate a reference governor strategy for hybrid systems.

A. Admissible Sets for Hybrid Reference Governor

Given a hybrid plant and a hybrid controller defining the closed-loop system (9), the *output admissible set* is

$$Y \subset \mathbb{R}^s \tag{10}$$

which allows to formulating constraints on the state x and the command input v_c. The maximal output admissible set (MOAS) is the set of all initial states and hybrid command inputs (not necessarily constant) such that the resulting output response satisfies the constraints, at any future time.

Definition 4.1 (Maximal output admissible set): Given the hybrid closed-loop system $\mathcal{H}$ in (9) and the output admissible set Y in (10), the maximal output admissible set is

$$O_\infty := \{(x_0, \phi_{v_c}) \; : \; \operatorname{rge} h_{cl}(\phi_x, \phi_{v_c}) \subset Y \quad \forall \phi_x \; : \; (\phi_x, \phi_{v_c}) \in \mathcal{S}_\mathcal{H}(x_0)\} \tag{11}$$

where rge denotes range, namely, $\operatorname{rge} h_{cl}(\phi_x, \phi_{v_c}) := \{h_{cl}(\phi_x(t, j), \phi_{v_c}(t, j)) \; : \; (t, j) \in \operatorname{dom}(\phi_x, \phi_{v_c}) \; \}$.

The construction of O_∞ in Definition 4.1 collects all possible hybrid command input signals ϕ_{v_c} that, for the given state x_0, lead to outputs to $\mathcal{H}$ that remain in Y. It is a generalization of the conventional case, which typically restrict the command inputs to constant signals of time [2].

Lemma 4.2: (invariance of MOAS) For each $(x_0, \phi_{v_c}) \in O_\infty$, each ϕ_x such that $(\phi_x, \phi_{v_c}) \in \mathcal{S}_\mathcal{H}(x_0)$ satisfies $\operatorname{rge}(\phi_x, \phi_{v_c}) \subset O_\infty$.

B. The Hybrid Reference Governor Problem

With the constructions given in Section IV-A, we formulate the hybrid reference governor problem as follows.

Problem $(\star)$. Given a hybrid closed-loop system $\mathcal{H}$ (9), its current state x_0, a MOAS Y (11), a nontrivial hybrid reference signal $(t, j) \mapsto r(t, j)$, and the previously applied command $\phi_{\widetilde{v}_c} \in \mathcal{V}_\mathcal{H}(x_0)$, determine a hybrid command input

$$\phi_{v_c} = \phi_{\widetilde{v}_c} + \gamma(r - \phi_{\widetilde{v}_c}) \tag{12}$$

such that $(t, j) \mapsto \gamma(t, j)$ is a hybrid input that solves

$$\gamma = \sup \quad \bar{\gamma} \tag{13a}$$
$$\text{s.t. } (x_0, \phi_{\widetilde{v}_c} + \bar{\gamma}(r - \phi_{\widetilde{v}_c})) \in O_\infty, \tag{13b}$$
$$\operatorname{rge} \bar{\gamma} \subset [0, 1] \tag{13c}$$

C. Feasibility

The following feasibility result is a direct consequence of the properties of O_∞ and the formulation of Problem $(\star)$.

Lemma 4.3: (feasibility) Suppose $(t, j) \mapsto r(t, j)$ is constant. If x_0 is such that there exists a nontrivial hybrid input ϕ_{v_c}' such that $(x_0, \phi_{v_c}') \in O_\infty$ then Problem $(\star)$ has a nontrivial solution, namely, there exist $(t, j) \mapsto \phi_{\widetilde{v}_c}(t, j)$ and $(t, j) \mapsto \gamma(t, j)$ such that $\operatorname{dom} \phi_{\widetilde{v}_c} = \operatorname{dom} \gamma$ has more than one point, the conditions in (13) *are satisfied, and γ is the result of the maximization formulated in* (13a).

D. A (Hybrid) Time-Driven Implementation of Hybrid RG

Next, we propose a hybrid time-driven implementation of the reference governor for $\mathcal{H}$, where Problem $(\star)$ is solved at the initial hybrid time, $(t, j) = (0, 0)$, and then at (T_1, J_1), (T_2, J_2), etc., which are pairs in $\mathbb{R}_{\geq 0} \times \mathbb{N}$. One way to determine these time instances is by separating them by a finite amount of hybrid time, namely, a *hybrid time window*

$$\mathcal{T} = \{(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} : \max\{t/\delta, j\} = \tau\} \tag{14}$$

where $\tau \in \mathbb{N}$ defines the number of jumps allowed in the hybrid time window and $\delta > 0$ adjusts the length of the window relative to ordinary time t. With the finite-hybrid time horizon structure, a hybrid signal can be defined to have a hybrid time domain that fits within the hybrid time window.

Using the definitions above, we obtain a hybrid reference governor with time-driven recomputation events, based on the parameters δ and τ of the generic hybrid time window in (14) to calculate the instants (T_i, J_i) at which the hybrid command input is recomputed.

Algorithm 1 Hybrid Reference Governor

```
1: Set i = 0.
2: Set initial computation time (T_0, J_0) = (0, 0).
3: Set x_0 = φ_x(0, 0).
4: Set φ_{ṽ_c}^{-1} such that (x_0, φ_{ṽ_c}^{-1}) ∈ O_∞.
5: while true do
6:     Solve Problem (⋆) to obtain a command input φ_{ṽ_c}^i.
7:     while max{(t − T_i)/δ, j − J_i} ≤ τ do
8:         Apply φ_{ṽ_c}^i to H to track r under constraints.
9:     end while
10:    i = i + 1.
11:    (T_i, J_i) = (t, j).
12:    x_0 = x(T_i, J_i).
13: end while
```

In Algorithm 1, the resulting hybrid command input $\phi_{\widetilde{v}_c}$ is a sequence of hybrid command inputs $\{\phi_{\widetilde{v}_c}{}^i\}_{i=0}^{\infty}$. The entry $\phi_{\widetilde{v}_c}{}^i$ of this sequence corresponds to the value of the input from hybrid time (T_i, J_i) to (T_{i+1}, J_{i+1}), and is obtained by solving Problem $(\star)$ (Line 6) using the following data:

- The current state $x(T_i, J_i)$;
- The shifted version of the hybrid reference signal r, denoted $\widetilde{r}^i$, which is defined by shifting r by (T_i, J_i), namely, for each $(t, j) \in \operatorname{dom} r - \{(T_i, J_i)\} =: \operatorname{dom} \widetilde{r}^i$, $\widetilde{r}^i(t, j) := r(t - T_i, j - J_i)$;
- The previously applied command $\phi_{\widetilde{v}_c}{}^{i-1}$.

For $i = 0$, the optimization problem is solved at $(T_0, J_0) = (0, 0)$ using a predefined value of the hybrid command input, which is denoted $\phi_{\widetilde{v}_c}{}^{-1}$ (see Line 4). The recomputation times $\{(T_i, J_i)\}_{i=0}^{\infty}$ are regulated online, through the definition of the hybrid time window $\mathcal{T}$. Due to the condition in Line 7, $(T_{i+1} - T_i, J_{i+1} - J_i)$ is not necessarily constant. However, when the jump set D is empty, the algorithm simplifies to sampled-data reference governor for continuous-time systems. Similar observations can be made when C is empty, with $\tau = 1$ recovering the standard (discrete-time) implementation of reference governor. We emphasize that the hybrid command inputs $\phi_{\widetilde{v}_c}{}^i$ are defined over hybrid time domains and therefore are not discretized during flows.

V. Solving the Hybrid Reference Governor Problem For Special Classes of Hybrid Systems

A. Reference Governor for Hybrid Systems with Linear Maps and Constant Reference

In this section, we consider the case when $\mathcal{H}$ in (9) has linear flow, jump, and output map, i.e.,

$$\mathcal{H}_\ell \ : \ \begin{cases} x \in C & \dot{x} = A_c x + B_c v_c \\ x \in D & x^+ = A_d x \\ & \chi = Mx + Hv_c \end{cases} \tag{15}$$

where $x \in \mathbb{R}^n$, $\chi \in \mathbb{R}^s$, and $v_c \in \mathbb{R}^m$, and the output admissible set is $Y \subset \mathbb{R}^s$. For solving Problem $(\star)$ numerically, we first need to construct the set O_∞.

To construct O_∞, we augment the system with constant reference dynamics and apply tools for the characterization of forward invariance of sets for autonomous hybrid systems [13]. The state variable μ models the constant dynamics of the reference signal r, which take values from $S \subset \mathbb{R}^{m_r}$. The resulting augmented hybrid system with state $w = (x, \mu)$ is

$$\mathcal{H}_\ell^{\text{aug}} \ : \ \begin{cases} x \in C, \ \mu \in S & \begin{cases} \dot{x} = A_c x + B_c \mu \\ \dot{\mu} = 0 \end{cases} \\ x \in D, \ \mu \in S & \begin{cases} x^+ = A_d x \\ \mu^+ = \mu \end{cases} \\ & \chi = Mx + H\mu \end{cases} \tag{16}$$

The chosen dynamics for μ guarantee that, for a given initial condition $\mu(0, 0) = r$, the resulting trajectory for μ is constant and equal to r, along both flows and jumps. The set O_∞ in (11) associated to (16) is denoted O_∞^{aug} and is given by

$$\Big\{ w_0 : \forall w = (x, \mu) \ : \ w \in \mathcal{S}_{\mathcal{H}_\ell^{\text{aug}}}(w_0), \ \operatorname{rge} Mx + H\mu \subset Y \Big\}$$

which collects all initial conditions $w_0 = (x_0, \mu_0)$ for which the output of the trajectory of $\mathcal{H}_\ell$ in (15) belongs to Y. With the construction of the augmented system, we characterize O_∞^{aug} as

$$K := \Big\{ (x, \mu) \in C \times S : Mx + H\mu \in Y, (x, \mu) \in \hat{L} \Big\} \tag{17}$$

where the set $\hat{L}$, which is to be determined, restricts the values of x and μ. Specifically, the set $\hat{L}$ should be chosen so that the set K is forward invariant for $\mathcal{H}_\ell^{\text{aug}}$.

There are several tools in the literature of hybrid systems that certify forward invariance of a set and that can be applied to characterize O_∞^{aug}. One approach is to use tangent cone conditions certifying forward invariance and proceed as in [14]. An alternative approach is the (multiple) barrier functions approach proposed in [13] that characterizes K using a sublevel set of a barrier function. We illustrate the latter approach in an example.

Example 5.1: Consider the model of second-order system with continuous dynamics $\dot{z}_1 = z_2$, $\dot{z}_2 = u$ when $z_1 \geq 0$ and discrete dynamics $z_1^+ = z_1$, $z_2^+ = -\lambda z_2$ when $z_1 = 0$ and $z_2 \leq 0$, where z_1 represents position, z_2 represents velocity, u is the control input, $y = h(z) := z$ is the output, and λ is a parameter modeling the impact restitution coefficient. This model resembles the well-known bouncing ball with actuation during the continuous dynamics [7]. Denoting by $r = r^*$, $r^* \geq 0$, the desired constant reference to track, a suitable tracking controller is given by the static control law $z \mapsto \kappa(z) := -k_1 z_1 - k_2 z_2 + k_1 r^*$, with k_1 and k_2 such that the matrix $A_c = \left[\begin{smallmatrix} 1 & 0 \\ -k_1 & -k_2 \end{smallmatrix}\right]$ is Hurwitz, cf. (15), where, in particular, $x = z$. When $\lambda \in (0, 1]$, κ guarantees global asymptotic stability of r for the resulting closed loop.

Now, consider the maximal output admissible set $Y := \left\{ z \in \mathbb{R}^2 \ : \ |z_1| \leq z_{\max} \right\}$ where $z_{\max} \geq 0$. Noting that $C = \left\{ z \in \mathbb{R}^2 \ : \ z_1 \geq 0 \right\}$ and $S = \{ r \in \mathbb{R} \ : \ r \geq 0 \}$, the associated set K in (17) is given by

$$K = \{ (z, \mu) \ : \ z_1 \in [0, z_{\max}], \mu \geq 0 \} \cap \hat{L}$$

Using the certificate

$$B(z, \mu) := \begin{bmatrix} z_1 - \mu & z_2 \end{bmatrix} P \begin{bmatrix} z_1 - \mu \\ z_2 \end{bmatrix} \qquad \forall (z, \mu) \in \mathbb{R}^3$$

with $P = P^\top > 0$ such that $A_c^\top P + P A_c < 0$ it follows that for[1] $\mu \mapsto \overline{c}(\mu) := B(z_{\max}, -\frac{p_{12}}{p_{22}}(z_{\max} - \mu))$ and $\hat{S} := \{ \mu \in \mathbb{R} \ : \ \mu \in [0, z_{\max}] \}$, the set

$$\left\{ (z, \mu) \in C \times \hat{S} \ : \ B(z, \mu) \leq \overline{c}(\mu) \right\} \tag{18}$$

is forward invariant and a subset of $Y \times S$. To show this property, note that (16) is given by

$$\left\{ \begin{array}{ll} z_1 \geq 0, \ \mu \geq 0 & \left\{ \begin{array}{rcl} \dot{z} & = & \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} z \\ & & + \begin{bmatrix} 0 \\ k_1 \end{bmatrix} \mu \\ \dot{\mu} & = & 0 \end{array} \right. \\ z_1 = 0, z_2 \leq 0, \mu \geq 0 & \left\{ \begin{array}{rcl} z^+ & = & \begin{bmatrix} 1 & 0 \\ 0 & -\lambda \end{bmatrix} z \\ \mu^+ & = & \mu \end{array} \right. \\ & \quad \chi = z \end{array} \right. \tag{19}$$

Since $\dot{\mu} = 0$, the variation of $B - \overline{c}$ along the flows of (19) is given by

$$\left\langle \nabla \left\{ B(z, \mu) - \overline{c}(\mu) \right\}, \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} z + \begin{bmatrix} 0 \\ k_1 \end{bmatrix} \mu \\ 0 \end{bmatrix} \right\rangle \leq 0 \tag{20}$$

for each $z_1 \geq 0$ and each $\mu \in \mathbb{R}$. Moreover, the change of $B - \overline{c}$ at jumps of (19) satisfies

$$B(z^+, \mu^+) - \overline{c}(\mu^+) - (B(z, \mu) - \overline{c}(\mu)) \leq 0 \tag{21}$$

when $\lambda \in (0, 1]$. Then, by [13, Theorem 1], for each $c \geq 0$, $\{ (z, \mu) \in C \times S \ : \ B(z, \mu) - \overline{c}(\mu) \leq c \}$ is forward invariant for the augmented system (19). In particular, this set is forward invariant for the augmented system when $c = 0$. The last step is to show that $\overline{c}$ and $\hat{S}$ are such that[2]

$$\left\{ (z, \mu) \in C \times \hat{S} \ : \ B(z, \mu) \leq \overline{c}(\mu) \right\} \subset Y \times S \tag{22}$$

This property holds since, from the definition of $\overline{c}$, $\overline{c}(\mu)$ defines the largest level for the sublevel set of the function B that is contained in $\left\{ z \in \mathbb{R}^2 \ : \ z_1 \leq z_{\max} \right\}$.

We perform simulations for $\lambda = 0.9$, $z_{\max} = 10$, and $r^* = 9.999$, which is close to the boundary of the constraint defined by Y; hence, it puts our hybrid RG strategy to the test. With the controller parameters designed as $k_1 = k_2 = 1$, Figure 3 shows the trajectories for the states of the plant z_1 and z_2. The stars represent times at which the hybrid command input v_c are updated, as well as jumps of the hybrid plant. Following the implementation formulated in Section IV-D, v_c is updated every time that the timer reaches zero. To test robustness of the approach, these events are separated by δ seconds of flow, which changes after each event within the range of values $[0.8s, 1.5s]$. Figure 3 shows that v_c monotonically approaches r^* after each event.

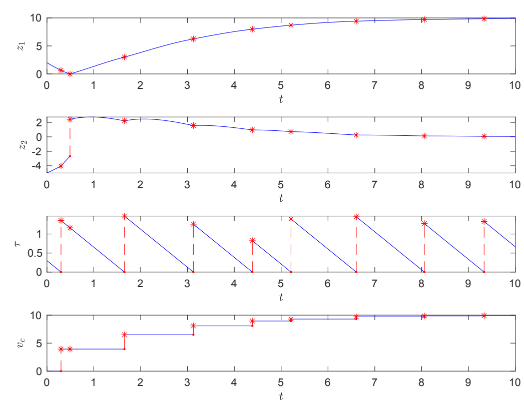


Fig. 3. Trajectories for (z_1, z_2) in Example 5.1 as a function of ordinary time (top two plots), timer triggering the events at which the hybrid command input is computed (third plot), and hybrid command input v_c.

B. Reference Governor for Hybrid Systems with Modes

Next, we consider an example for when $\mathcal{H}$ has linear flow and jump maps, linear output map, and an explicit logic variable q denoting the mode of the system. Specifically, we consider the following special case of (9):

$$\mathcal{H}_m \ : \ \left\{ \begin{array}{ll} z \in C_q & \left\{ \begin{array}{rcl} \dot{z} & = & A_q z + B_q v_c \\ \dot{q} & = & 0 \end{array} \right. \\ z \in D_q & \left\{ \begin{array}{rcl} z^+ & = & E_q z + N_q v_c \\ q^+ & = & J_q(z) q + K_q(z) \end{array} \right. \\ & \chi = M_q z + H_q v_c \end{array} \right. \tag{23}$$

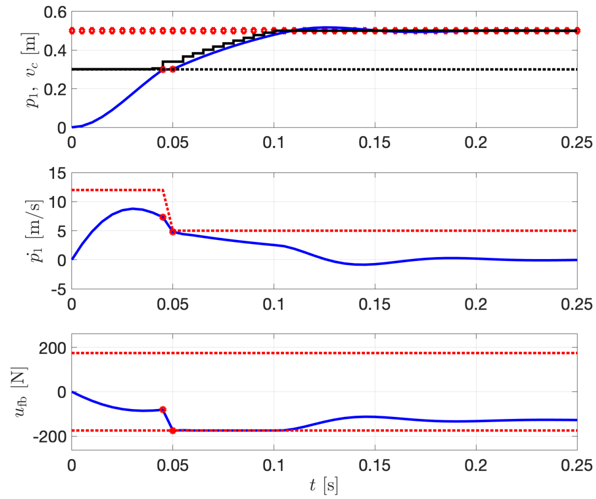
with $z \in \mathbb{R}^n$ being the continuous state, $q \in Q$ the discrete (logic) state, $v_c \in \mathbb{R}$ is the command, $\chi \in \mathbb{R}^s$ the constrained output, $\{(A_q, B_q, E_q, N_q, M_q)\}_{q \in Q}$ constant matrices, $\{(J_q, K_q)\}_{q \in Q}$ functions of the continuous state, $C_q :=$

[1] The function $\overline{c}$ returns the largest sublevel set of B that is contained in Y.

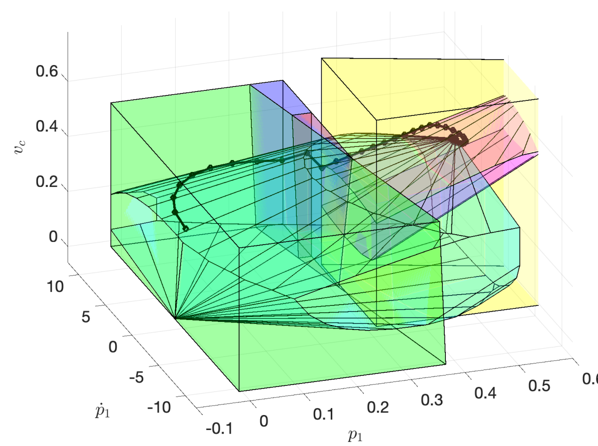
[2] The set $\hat{L}$ in (17) is omitted, but it is given by $\hat{L} := \left\{ (z, \mu) \in \mathbb{R}^3 \ : \ B(z, \mu) \leq \overline{c}(\mu) \right\}$.

$\{z \ : \ M_{C,q}z \le L_{C,q} \ \}$, and $D_q := \{z \ : \ M_{C,q}z \le L_{D,q} \ \}$, for given constants $\{(L_{C_q}, L_{D_q})\}_{q\in Q}$. We consider a given sequence $\mathcal{Q}_N = (q_1, q_2, \dots, q_N)$ to achieve a constant reference r, where the jump times are not fixed, and implement the hybrid RG in discrete time that enforces the inequalities pointwise-in-time with period T_s. We construct the sequence of MOASs $O^{q_i}_\infty$ for each for $i = 1, 2, \dots N$, where for each $O^{q_i}_\infty$ we further constrain the jump set to the constraint admissible jump set (CAJS), the set of states that "lands" into $O^{q_{i+1}}_\infty$. The additional constraint is enforced by the RG. This guarantees that during flow the constraints of q_i are satisfied, and after the jump, the constraint of q_{i+1} are also satisfied, and that all the problems are convex.

Example 5.2: Figure 4 shows the results of the hybrid RG for the impactor-runner motion mechanism in Section II. As guaranteed by the RG design, the constraints are satisfied even after the mode changes, and the virtual reference approaches the actual reference. The time for v_c to nearly reach the reference r is a little longer (about 15%) than the one obtained by the two separated RGs shown in Figure 2(a), because these ignore, and in fact violate the constraints at the jump. On the other hand, the hybrid RG command nearly reaches the reference sooner (about 30%) than the conservative design of Figure 2(b), which is overly cautious to avoid constraint violation after the jump, which cannot be handle by design using the traditional approach. As shown in Figure 4, the invariant set for the mode of the impactor mass moving alone intersects the jump set only in the CAJS. In fact, the trajectory reaches the jump set in the CAJS, which ensures landing in the invariant set for mode 2, in this way guaranteeing the satisfaction of the constraints after the jump.



(a)



(b)

Fig. 4. Application of hybrid RG to the impactor-runner mechanism. (a) closed-loop trajectories using hybrid RG: state trajectories (blue solid), reference r (red diamonds), virtual reference (black solid), constraints (red dash), m_2 rest position (black dots), boundaries of controller sampling period when impact occurs (red circles). (b) impactor moving alone: flow set (green), jump set (blue), CAJS (red), $O^{q_1}_\infty$ (cyan); impactor and runner moving together: flow set (yellow), $O^{q_2}_\infty$; closed-loop trajectory obtained with hybrid RG (black).

VI. Conclusion

In this paper, a hybrid reference governor strategy for a broad class of hybrid dynamical systems is presented. The optimization to solve involving hybrid signal as well as basic properties are introduced. Future work includes extending, beyond those in Section V, the class of systems for which the hybrid RG strategy can be solved.

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