On the Feasibility and Continuity of Feedback Controllers Defined by Multiple Control Barrier Functions for Constrained Differential Inclusions

Axton Isaly, Masoumeh Ghanbarpour, Ricardo G. Sanfelice, Warren E. Dixon

***Abstract*—Control barrier functions are a popular method for encoding safety specifications for dynamical systems. In this paper, a notion of control barrier function is defined that permits vector-valued barrier functions and flow constraints involving both the state and the control input. Control barrier functions induce constraints on the control input that, when satisfied, guarantee forward invariance of a safe set of states. The constraints are enforced using a pointwise-optimal feedback controller, and sufficient conditions for the continuity of the controller are given. The existence of a control barrier function is defined to be equivalent to the feasibility of the optimal feedback controller. Polynomial optimization problems based on sums of squares are formulated that can be used to certify that a given function is a control barrier function.**

I. Introduction

The use of control barrier functions (CBF) to synthesize feedback controllers that render sets of states forward invariant, analogous to attaining asymptotic stability via control Lyapunov functions (CLF), has recently gained significant interest because of the tight relationship between forward invariance and safety [1], [2]. Forward invariance is a property indicating that trajectories of a dynamical system starting within a given set stay in the set for all time. In many applications, such as [2] and [3, Sec. V], multiple CBFs are used to describe the control objective, whereas the majority of theoretical results are developed for scalar barrier functions. While it is possible to combine multiple barrier functions using max and min operations, as in works like [2] and [4], the resulting functions are generally nonsmooth, in which case the resulting controllers are discontinuous. A framework for studying forward invariance with multiple barrier functions was developed in [5] in the context of uncontrolled systems. For controlled systems, the conditions therein can be interpreted as constraints on the control input that can be enforced using optimization-based controllers; see [6, Ch. 11]. Enforcing multiple input constraints defined by multiple continuously differentiable CBF candidates is a promising way to obtain control laws that are continuous functions of the state.

Axton Isaly and Warren E. Dixon are with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville FL 32611-6250, USA. Email: {axtonisaly1013, wdixon}@ufl.edu

Masoumeh Ghanbarpour and Ricardo G. Sanfelice are with the Department of Electrical and Computer Engineering, University of California, Santa Cruz, USA. Email: {maghanba, ricardo}@ucsc.edu

This research is supported in part by Office of Naval Research Grant N00014-21-1-2481; AFOSR award numbers FA9550-18-1-0109, FA9550-19-1-0169, FA9550-19-1-0053, and FA9550-20-1-0238; NSF Grants no. ECS-1710621, CNS-1544396, and CNS-2039054; and by CITRIS and the Banatao Institute at the University of California. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the sponsoring agency.

Traditionally, a CBF is defined to guarantee that a safety-ensuring controller exists [1], yet tools for verifying that a given function has this property are not fully developed. While analytical conditions exist to determine whether a scalar-valued function is a CBF (cf. [7], [8, Prop. 1]), the problem is significantly more challenging when multiple constraints must be satisfied simultaneously. In general, there exists a set of constraints in the decision variable (control input) that vary with an external parameter (the state of the dynamic system), and it must be verified that feasible solutions to the constraints exist for all states in a given set. The authors of [9] leverage a tool for checking that multiple constraints have at least one feasible solution at a particular point in the state space, but it is not clear how to verify this property on a given (uncountable) subset of the state space. To address the feasibility problem, we use sum of squares programming, which requires the more restrictive assumptions that the constraints defining the feasible set are polynomials and affine in the control input [10], [11]. Our technique verifies feasibility on level-sets of a given function (typically the CBF candidate), which is useful for safe synthesis and computationally simpler than techniques that search simultaneously for a controller and CLF/CBF as in [12] and [13]. In [13], an iterative procedure is developed to search for a scalar CBF describing a safe set that avoids given unsafe regions. The safe set was rendered forward invariant by a feasible controller. This technique is valuable when a CBF candidate is unavailable, whereas our approach is targeted towards verification that a given candidate is a CBF.

The goal of this paper is to synthesize forward invariance-ensuring controllers for continuous-time differential inclusions with flow constraints on the state and the control input. A special case of the flow constraints are state-dependent input constraints. In Section III, we define a notion of vector-valued CBF that is equivalent to the feasibility of the CBF-induced pointwise optimal control law. We show that forward (pre-)invariance of the safe set defined by a CBF is guaranteed using any continuous selection of the feasible set mapping. In Section IV, we provide sufficient conditions under which the CBF-induced pointwise optimal control law is continuous. These conditions generalize available results by allowing broader classes of cost functions and not necessarily requiring the set-valued mapping describing the feasible control inputs to be locally bounded. In Section V, we develop sum of squares optimization tools that can be used to verify that a CBF candidate is a CBF.

II. Preliminaries

For vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $(x,y) \triangleq \left[x^\top, y^\top\right]^\top$. The shorthand $[d] \triangleq \{1, 2, \ldots, d\}$ is used. Given a function $B : \mathbb{R}^n \to \mathbb{R}^d$, the components are indexed as $B(x) \triangleq (B_1(x), B_2(x), \ldots, B_d(x))$ and the inequality $B(x) \leq 0$ means that $B_i(x) \leq 0$ for all $i \in [d]$. For a set $A \subset \mathbb{R}^n$, the notation ∂A denotes its boundary, $\overline{A}$ its closure, $\mathrm{Int}(A)$ its interior, and $U(A)$ denotes an open neighborhood around A.

Given a set $X \subset \mathbb{R}^n$, a set-valued mapping $M : X \rightrightarrows \mathbb{R}^m$ associates every point $x \in X$ with a set $M(x) \subset \mathbb{R}^m$. The mapping M is called locally bounded if, for every $x \in X$, there exists a neighborhood $U_X(x) \triangleq U(x) \cap X$ such that $M(U_X(x))$ is bounded, M is outer semicontinuous if $\mathrm{Graph}_X(M) \triangleq \{(x,u) \in X \times \mathbb{R}^m : u \in M(x)\}$ is relatively closed in $X \times \mathbb{R}^m$, and M is lower semicontinuous if, for any open set $\mathcal{G} \subset \mathbb{R}^m$, the inverse image $M^{-1}(\mathcal{G}) \triangleq \{x \in X : M(x) \cap \mathcal{G} \neq \emptyset\}$ is open.

III. Control Barrier Functions

Consider an open-loop constrained differential inclusion (F, C_u) with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ modeled by

$$\dot{x} \in F(x,u) \qquad (x,u) \in C_u \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the set-valued flow map and $C_u \subset \mathbb{R}^n \times \mathbb{R}^m$ is the flow set. To facilitate the subsequent development, let $\Pi(C_u) \triangleq \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \ s.t.\ (x,u) \in C_u\}$ denote the set of all states for which flowing is possible, and let

$$\Psi(x) \triangleq \{u \in \mathbb{R}^m : (x,u) \in C_u\} \tag{2}$$

denote the set of admissible control inputs at each state.

CBFs are defined to guarantee the existence of control inputs that ensure forward invariance (i.e., safety) of a given set of states $\mathcal{S} \subset \Pi(C_u)$. Compared to works such as [1], we use a notion of CBF that accommodates safe sets defined by multiple scalar functions. For notational convenience, we use vector-valued functions $B : \mathbb{R}^n \to \mathbb{R}^d$ to represent multiple CBFs. Defining a CBF in this case requires special care because there are multiple constraints on the control input that must be satisfied simultaneously. Our development is based on the work for closed-loop hybrid systems in [5] and for hybrid systems with inputs in [6], which we specialize to the continuous-time dynamics in (1).

Definition 1. A vector-valued function $B : \mathbb{R}^n \to \mathbb{R}^d$ is called a CBF candidate defining the safe set $\mathcal{S} \subset \Pi(C_u)$ if

$$\mathcal{S} = \{x \in \Pi(C_u) : B(x) \leq 0\}.$$

Also define $\mathcal{S}_i \triangleq \{x \in \mathbb{R}^n : B_i(x) \leq 0\}$ for every $i \in [d]$.

We restrict our attention to continuously differentiable CBF candidates because of advantages they offer towards synthesizing continuous controllers. Given a continuously differentiable CBF candidate, define a function $\Gamma : C_u \to \mathbb{R}^d$ such that the i-th component is

$$\Gamma_i(x,u) \triangleq \sup_{f \in F(x,u)} \langle \nabla B_i(x), f \rangle \quad \forall (x,u) \in C_u. \tag{3}$$

The value of $\Gamma_i(x,u)$ represents the worst-case growth of $B_i(x)$ for any possible direction of flow in the set-valued map $F(x,u)$. When $F(x,u)$ is nonempty and bounded, the supremum in (3) is finite. Thus, the following mild assumption will be needed to ensure that Γ is well-defined.

Assumption 1. The set $F(x,u)$ is nonempty and bounded for every $(x,u) \in C_u$.

We also introduce the primary design parameter in the form of a function γ, which is used to define a set of control inputs that constrain the worst-case growth function Γ according to conditions derived from [5] that guarantee forward invariance of the safe set $\mathcal{S}$. We impose the following assumption.

Assumption 2. The function $\gamma : \Pi(C_u) \to \mathbb{R}^d$ is such that, for each $i \in [d]$, $\gamma_i(x) \geq 0$ for all $x \in (U(M_i) \setminus \mathcal{S}_i) \cap \Pi(C_u)$, where $M_i \triangleq \{x \in \partial\mathcal{S} : B_i(x) = 0\}$.

Definition 2. Let (F, C_u) satisfy Assumption 1. A continuously differentiable CBF candidate $B : \mathbb{R}^n \to \mathbb{R}^d$ defining the set $\mathcal{S} \subset \Pi(C_u)$ is a CBF for (F, C_u) and $\mathcal{S}$ on a set $\mathcal{O} \subset \Pi(C_u)$ with respect to a function $\gamma : \Pi(C_u) \to \mathbb{R}^d$ if there exists a neighborhood of the boundary of $\mathcal{S}$ such that $U(\partial\mathcal{S}) \cap \Pi(C_u) \subset \mathcal{O}$, γ satisfies Assumption 2, and the set

$$K_c(x) \triangleq \{u \in \Psi(x) : \Gamma(x,u) \leq -\gamma(x)\} \tag{4}$$

is nonempty for every $x \in \mathcal{O}$.

Remark 1. Assumption 2 imposes conditions on the function γ that must hold on a region outside the set $\mathcal{S}$. In contrast to conditions based on Nagumo's theorem such as those in [1], the conditions here are valid even if the gradients of the component CBF candidates are degenerate (i.e., $\nabla B_i(x) = 0$ for some $x \in M_i$). See [5, Thm. 2] for alternative conditions applicable to multiple barrier functions that do not require checking points outside of $\mathcal{S}$. Adapting the conditions in [5] to the setting of controlled systems is the subject of future work.

A. Forward pre-Invariance Using Selections of K_c

We next relate the notion of CBF in Definition 2 to forward pre-invariance of the safe set $\mathcal{S} = \{x \in \Pi(C_u) : B(x) \leq 0\}$. The result in this section is comparable to [14, Thm. 4]. While our result applies to a more general class of dynamics, the result in [14] does not require the control law to be continuous. We require continuity to ensure outer semicontinuity of the closed-loop dynamics as required by the results in [5].

Consider a closed-loop system (F_{cl}, C) defined by (F, C_u) in (1) and a control law $\kappa : \Pi(C_u) \to \mathbb{R}^m$ as

$$\dot{x} \in F(x, \kappa(x)) \triangleq F_{cl}(x) \quad x \in C \tag{5}$$

where $C \triangleq \{x \in \mathbb{R}^n : (x, \kappa(x)) \in C_u\}$. A solution to (F_{cl}, C) starting from $x_0 \in C$ is a locally absolutely continuous function $\phi : \mathrm{dom}\,\phi \to \mathbb{R}^n$ such that $\phi(t) \in C$ for all $t \in \mathrm{Int}(\mathrm{dom}\,\phi)$ and $\dot{\phi}(t) \in F_{cl}(\phi(t))$ for almost all $t \in \operatorname{dom}\phi$, where $\operatorname{dom}\phi \subset [0,\infty)$ is an interval containing zero. A solution is said to be complete if $\operatorname{dom}\phi$ is unbounded, and it is maximal if there is no solution ϕ' such that $\phi(t) = \phi'(t)$ for all $t \in \operatorname{dom}\phi$ with $\operatorname{dom}\phi$ a proper subset of $\operatorname{dom}\phi'$. The following notions of forward invariance are adapted from [5] for the case of constrained differential inclusions.

Definition 3. A set $\mathcal{S} \subset C$ is forward pre-invariant for (F_{cl}, C) if, for each $x_0 \in \mathcal{S}$ and each maximal solution ϕ starting from x_0, $\phi(t) \in \mathcal{S}$ for all $t \in \operatorname{dom}\phi$. The set $\mathcal{S}$ is forward invariant if it is forward pre-invariant and, for each $x_0 \in \mathcal{S}$, each maximal solution ϕ starting from x_0 is complete.

The following assumption and lemma relate regularity conditions imposed on the open-loop system (F, C_u) in (1) to common regularity conditions for the closed-loop dynamics that will be used in the next two theorems.

Assumption 3. A) The flow map $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semicontinuous, and has nonempty and convex images on C_u.
B) The flow set C_u is a closed subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Lemma 1. *Suppose $\kappa : \Pi(C_u) \to \mathbb{R}^m$ is continuous. If Assumption 3A) holds, then $F_{cl} : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded, outer semicontinuous, and has nonempty and convex images on C. If Assumption 3B) holds, then C is a closed subset of $\mathbb{R}^n$.*

The following result provides conditions under which continuous controllers selected from the mapping K_c in (4) render the set $\mathcal{S}$ forward pre-invariant for the closed-loop dynamics in (5). In Section IV we provide a constructive strategy for designing continuous safety-ensuring controllers using optimization.

Theorem 1. ***(Forward pre-Invariance)*** *Let Assumption 3A) hold for the open-loop dynamics (F, C_u). Suppose $B : \mathbb{R}^n \to \mathbb{R}^d$ is a CBF for (F, C_u) and $\mathcal{S} \subset \Pi(C_u)$ on $\mathcal{O} \subset \Pi(C_u)$ with respect to $\gamma : \Pi(C_u) \to \mathbb{R}^d$. Let the control law $\kappa : \mathcal{O} \to \mathbb{R}^m$ be continuous with $\kappa(x) \in K_c(x)$ for all $x \in \mathcal{O}$. If $\mathcal{S} = \{x \in \Pi(C_u) : B(x) \le 0\}$ is closed[1] in $\mathbb{R}^n$, then $\mathcal{S}$ is forward pre-invariant for the closed-loop dynamics defined in (5) by (F, C_u) and κ.*

When the performance function γ satisfies stronger conditions than those imposed in Assumption 2, selections of K_c, designed to enforce all of the barrier function-induced constraints, not only render $\mathcal{S}$ forward pre-invariant, but also some larger sets defined by a subset of the barrier functions. This situation is different from redefining K_c by removing some of the constraints.

Corollary 1. *Under the assumptions of Theorem 1, assume additionally that $\mathcal{O} = \Pi(C_u)$ and $\gamma_i(x) \ge 0$ for all $x \in \mathcal{O}\backslash\mathcal{S}_i$, for each $i \in [d]$. For any index set $\mathcal{I} \subset \{1, 2, \ldots, d\}$, if the set $\mathcal{S}_\mathcal{I} \triangleq \{x \in \Pi(C_u) : B_i(x) \le 0, \forall i \in \mathcal{I}\}$ is closed in $\mathbb{R}^n$, then $\mathcal{S}_\mathcal{I}$ is forward pre-invariant for the closed-loop dynamics defined in (5) by (F, C_u) and κ.*

[1]Since B is assumed to be continuous, a sufficient condition for $\mathcal{S}$ to be closed is that $\Pi(C_u)$ is closed.

The corollary follows by applying Theorem 1 to the CBF candidate $B_\mathcal{I} : \mathbb{R}^n \to \mathbb{R}^{|\mathcal{I}|}$ defined by only the components of B in $\mathcal{I}$. The result in Corollary 1 can also be specialized to a particular index set $\mathcal{S}_\mathcal{I}$ under weaker assumptions.

B. Forward Invariance Using Selections of K_c

The forward pre-invariance property does not guarantee that maximal solutions to the closed-loop dynamics are complete. In particular, solutions may escape in finite time inside of $\mathcal{S}$ or may be unable to continue flowing in $\Pi(C_u)$. To select control inputs that prevent solutions from terminating on the boundary of $\Pi(C_u)$, we define a map

$$\Theta(x) \triangleq \begin{cases} \{u \in \Psi(x) : F(x,u) \cap T_{\Pi(C_u)}(x) \neq \emptyset\} \\ \qquad\qquad\qquad\qquad \text{if } x \in \partial\Pi(C_u) \cap \mathcal{S}, \\ \Psi(x) \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where $T_{\Pi(C_u)}(x)$ denotes the tangent cone to $\Pi(C_u)$ at x [15, Def. 5.12]. Relative to the assumptions of Theorem 1, we will assume additionally that the flow set C_u is closed and that the closed-loop controller is continuous on a set that contains the entire safe set. Doing so allows us to satisfy the hybrid basic conditions in Assumption 6.5 of [15] and establish completeness of maximal solutions.

Theorem 2. ***(Forward Invariance)*** *Let Assumption 3A) and 3B) hold for the open-loop dynamics (F, C_u). Suppose $B : \mathbb{R}^n \to \mathbb{R}^d$ is a CBF for (F, C_u) and $\mathcal{S} \subset \Pi(C_u)$ on $\mathcal{O}$ with respect to γ when $K_c(x)$ in (4) is replaced with $K_c(x) \cap \Theta(x)$. Let $\mathcal{D} \triangleq \mathcal{O} \cup \mathcal{S}$, and suppose $\kappa : \mathcal{D} \to \mathbb{R}^m$ is continuous with $\kappa(x) \in K_c(x) \cap \Theta(x)$ for all $x \in \mathcal{O}$ and $\kappa(x) \in \Psi(x)$ for all $x \in \mathcal{D}\backslash\mathcal{O}$. If $\mathcal{S}$ is closed in $\mathbb{R}^n$ and one of the following conditions hold:*
2.1) $\mathcal{S}$ is compact,
2.2) $F_{cl}(x) \triangleq F(x, \kappa(x))$ is bounded on $\mathcal{S}$, or
2.3) F_{cl} has linear growth on $\mathcal{S}$, namely, there exists $c > 0$ such that, for all $x \in \mathcal{S}$, $sup_{v \in F_{cl}(x)} |v| \le c(|x| + 1)$,
then $\mathcal{S}$ is forward invariant for the closed-loop dynamics defined in (5) by (F, C_u) and κ.

Remark 2. At times, it might be difficult to compute the tangent cone $T_{\Pi(C_u)}$, making it challenging to make a selection from the mapping $K_c \cap \Theta$. When $\partial\Pi(C_u) \cap \mathcal{S} = \emptyset$, Theorem 2 is simplified since $K_c(x) \cap \Theta(x) = K_c(x)$ at any point where $K_c(x)$ is defined. There are a number of situations where the safe set $\mathcal{S}$ can be either changed or redefined, by adding components to a CBF candidate B, to ensure that $\partial\Pi(C_u) \cap \mathcal{S} = \emptyset$. Under appropriate assumptions, the problem could be handled more generally by using a CBF candidate to define the set $\Pi(C_u)$ (cf. Remark 1).

IV. Design of Optimal Safety-Ensuring Feedback

Theorems 1 and 2 show that continuous selections of the mapping K_c in (4) render the safe set forward (pre-)invariant.

In this section, we develop a constructive method for making such selections using optimization, and provide a result on when the optimal selection is a continuous function of the state. To obtain an implementable form for the controller, we impose the following condition on the set-valued map Ψ of admissible controls.

Assumption 4. There exists $\psi : \Pi(C_u) \times \mathbb{R}^m \to \mathbb{R}^k$ such that $\Psi(x) = \{u \in \mathbb{R}^m : \psi(x,u) \leq 0\}$ for all $x \in \Pi(C_u)$.

Assumption 4 is common when input constraints are present [9], [14]. If B is a CBF for (F, C_u) and $\mathcal{S}$ on $\mathcal{O}$ with respect to γ, define the controller $\kappa^* : \mathcal{O} \to \mathbb{R}^m$ as[2]

$$\kappa^*(x) \triangleq \underset{u\in\mathbb{R}^m}{\arg\min}\ Q(x,u) \qquad s.t.\ \Gamma(x,u) \leq -\gamma(x),\ \psi(x,u) \leq 0, \tag{6}$$

where $Q : \mathcal{O} \times \mathbb{R}^m \to \mathbb{R}$ is a cost function and Γ is defined in (3). Because K_c in (4) is the feasible set for (6), κ^* is a selection of K_c– we write (6) equivalently as $\kappa^*(x) = \arg\min_{u\in K_c(x)} Q(x,u)$. When K_c is nonempty on $\mathcal{O}$ as required in the definition of CBF in Definition 2, the optimization in (6) is feasible.

Remark 3. The optimization in (6) is generally a nonlinear program. It is a quadratic program if the cost function Q is quadratic and the constraints are affine in the control input. For the case of a quadratic program, $\kappa^*(x)$ can often be computed at the current state in real time, as characterized in works like [3].

Although $\kappa^*(x)$ is feasible at $x \in \mathcal{O}$ if $K_c(x) \neq \emptyset$, it is not necessarily continuous. The following lemma, applicable to an optimal selection from a generic set-valued map, is used in proving the next theorem. Lemma 2 is a specialization of the more general results in [16]. Our result generalizes the min-norm control result of [17, Prop. 2.19] by allowing a broader class of cost function and not necessarily requiring the feasible set mapping to be locally bounded. We relax the boundedness requirement on the feasible set by imposing the following property on the cost function.

Definition 4. [18, Def. 1.16] Given a set $X \subset \mathbb{R}^n$, a function $Q : X \times \mathbb{R}^m \to [-\infty, \infty]$ with values $Q(x,u)$ is level-bounded in u, locally uniformly in x, if for each $\bar{x} \in X$ and $\lambda \in \mathbb{R}$ there is a neighborhood $U_X(\bar{x}) \subset X$ such that the set $\{(x,u) \in U_X(\bar{x}) \times \mathbb{R}^m : Q(x,u) \leq \lambda\}$ is bounded.

Lemma 2. *Let $\mathcal{O} \subset \mathbb{R}^n$. Suppose $K : \mathcal{O} \rightrightarrows \mathbb{R}^m$ is lower and outer semicontinuous with nonempty, convex values, and the function $Q : \mathcal{O} \times \mathbb{R}^m \to \mathbb{R}$ is continuous with $u \mapsto Q(x,u)$ strictly convex for every $x \in \mathcal{O}$. Suppose that either 1) K is locally bounded, or 2) $(x,u) \mapsto Q(x,u)$ is level-bounded in u, locally uniformly in x. Then, $\kappa^* : \mathcal{O} \to \mathbb{R}^m$ defined as $\kappa^*(x) \triangleq \arg\min_{u\in K(x)} Q(x,u)$ is single valued and continuous.*

[2]For κ^* to be well-defined, the function Γ should be extended to points $(x,u) \in \Pi(C_u) \times \mathbb{R}^m$ where $u \notin \Psi(x)$. This extension can be done arbitrarily since such points are infeasible.

Next, we provide one of our main results establishing the continuity of the controller in (6). We impose the following assumptions on the constraints, which lead to the continuity properties of the feasible set required by Lemma 2.

Assumption 5. For each $i \in [d]$ and $j \in [k]$,
A) For each $x \in \mathcal{O}$, the functions $u \mapsto \Gamma_i(x,u)$ and $u \mapsto \psi_j(x,u)$ are convex on $\Psi(x)$;
B) The functions $(x,u) \mapsto \Gamma_i(x,u) + \gamma_i(x)$ and $(x,u) \mapsto \psi_j(x,u)$ are continuous on $C_u \cap (\mathcal{O} \times \mathbb{R}^m)$ and $\mathcal{O} \times \mathbb{R}^m$, respectively.

Theorem 3. ***(Continuity of κ^*)*** *Let $C_u \subset \mathbb{R}^n \times \mathbb{R}^m$, $\mathcal{O} \subset \Pi(C_u)$, $\Gamma : C_u \to \mathbb{R}^d$, and $\gamma : \Pi(C_u) \to \mathbb{R}^d$ be given. Suppose Assumptions 4 and 5 hold, the cost function $Q : \mathcal{O} \times \mathbb{R}^m \to \mathbb{R}$ is continuous and, for each $x \in \mathcal{O}$, $u \mapsto Q(x,u)$ is strictly convex, and the mapping*

$$K_c^\circ(x) \triangleq \left\{ \begin{array}{c} u \in \mathbb{R}^m : \Gamma(x,u) < -\gamma(x) \\ \psi(x,u) < 0 \end{array} \right\} \tag{7}$$

is nonempty for every $x \in \mathcal{O}$. Additionally, suppose that one of the following conditions holds:
3.1) $\Psi : \Pi(C_u) \rightrightarrows \mathbb{R}^m$ in (2) is locally bounded,
3.2) $(x,u) \mapsto Q(x,u)$ is level-bounded in u, locally uniformly in x.
Then, $\kappa^* : \mathcal{O} \to \mathbb{R}^m$ defined in (6) is continuous.

Remark 4. By invoking Proposition 2.9 of [19], the functions Γ_i in (3) are continuous when the flow map $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semicontinuous, and lower semicontinuous. When the needed regularity is not present in the dynamics, one can replace Γ with a continuous upper bound $\bar{\Gamma} : C_u \to \mathbb{R}^d$ such that $\bar{\Gamma}_i(x,u) \geq \Gamma_i(x,u)$ for every $(x,u) \in C_u \cap (\mathcal{O} \times \mathbb{R}^m)$ and $i \in [d]$. It follows that $K_c^S(x) \triangleq \{u \in \Psi(x) : \bar{\Gamma}(x,u) \leq -\gamma(x)\} \subset K_c(x)$ for all $x \in \Pi(C_u) \cap \mathcal{O}$, so that redefining κ^* to be a selection of the subset mapping K_c^S still leads to a selection of K_c. Similar replacements can be made for the functions γ and ψ.

Theorem 3 shows that κ^* is continuous even if the admissible control mapping Ψ is unbounded, provided the cost function Q has the level-bounded property in Definition 4. The next result shows that a commonly-used class of cost functions has this property. We plan to characterize more general classes of functions with the level-bounded property in future work.

Proposition 1. *If $\mathcal{O} \subset \mathbb{R}^n$ and $\kappa_{nom} : \mathcal{O} \to \mathbb{R}^m$ is continuous, then the cost function $Q(x,u) \triangleq |u - \kappa_{nom}(x)|$ is level-bounded in u, locally uniformly in x.*

V. Feasibility Verification with Sum of Squares

A challenging aspect of verifying that a given CBF candidate is a CBF is determining whether the set $K_c(x)$ is nonempty for every $x \in \mathcal{O}$. Since K_c is the feasible set for the control law κ^* in (6), checking if a function is a CBF is the same as checking if the optimization defining κ^* is feasible. Moreover, certifying that K_c° in (7) has nonempty values guarantees continuity of κ^* under the assumptions of Theorem 3. In this section, we develop polynomial optimization tools for certifying that K_c and K_c° have nonempty values under more restrictive assumptions on the constraints defining the mappings. Namely, we assume that the constraints are polynomials and affine in the control input. However, the tools can be used in the case of non-polynomial constraints to obtain conservative estimates of the feasible region by replacing the constraints with polynomial upper bounds. This procedure is similar to Remark 4 except the polynomial upper bounds are used only for verification and we do not need to redefine κ^*.

Let $\mathcal{P}[x]$ be the set of all polynomials in the variables $x \in \mathbb{R}^n$. The set of sum of squares (SoS) polynomials is $\Sigma[x] \triangleq \{p \in \mathcal{P}[x] : p = \sum_{i=1}^N f_i^2,\ f_1, \ldots, f_N \in \mathcal{P}[x]\}$, where $p \in \Sigma[x]$ implies that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. We will also use $\mathcal{P}^{m_1 \times m_2}[x]$ to denote the set of matrix-valued functions $p : \mathbb{R}^n \to \mathbb{R}^{m_1 \times m_2}$ with polynomial entries.

SoS programming involves a series of relaxations of originally NP-hard polynomial optimization problems that lead to tractable semidefinite programs [10]. The class of problems that can be solved involve optimizing the coefficients of polynomials $p_i \in \mathcal{P}[x]$ subject to constraints of the form $a_0 + \sum_{i=1}^N p_i a_i \in \Sigma[x]$, where $a_0, a_1, \ldots, a_N \in \mathcal{P}[x]$ are given, constant coefficient polynomials (see [11], SoS Program 2). The aforementioned constraint is linear in the coefficients of the polynomials p_i.

To describe how SoS optimization can be used to certify whether a given function is a CBF, first consider a global feasibility problem. Let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping defined by a system of inequalities as

$$K(x) \triangleq \{u \in \mathbb{R}^m : A(x)u + b(x) \leq 0\}, \tag{8}$$

where $A : \mathbb{R}^n \to \mathbb{R}^{n_c \times m}$ and $b : \mathbb{R}^n \to \mathbb{R}^{n_c}$ are polynomial, i.e., $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$. The assumption that the constraints are affine is needed to obtain a proper SoS program, as discussed above. Given constraints of the form in (8), the following SoS program will certify that $K(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$.

Problem 1. (Global Feasibility) Given $x \in \mathbb{R}^n$ and polynomials $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$, find a constant $\epsilon \geq 0$ and a polynomial $u \in \mathcal{P}^m[x]$ such that, for all $i \in [n_c]$,

$$-A_{i*}(x)u(x) - b_i(x) - \epsilon \in \Sigma[x],$$

where $A_{i*}(x)$ denotes that i-th row of $A(x)$. The parameter ϵ could either be a fixed value or a decision variable. If $\epsilon > 0$, then $K^\circ(x) \triangleq \{u \in \mathbb{R}^m : A(x)u + b(x) < 0\}$ is nonempty.

Although the polynomial controller u found in Problem 1 is a selection of K (i.e., $u(x) \in K(x)$), it is not an optimal selection like κ^* in Section IV. Thus, we use u only for feasibility verification purposes, while κ^* is used to define a closed-loop system for control purposes. To apply the techniques in this section to K_c in Section III, we will need to assume the existence of a polynomial and affine upper bound of the functions defining K_c.

Assumption 6. Given $\Gamma : C_u \to \mathbb{R}^d$, $\gamma : \Pi(C_u) \to \mathbb{R}^d$, and $\psi : \Pi(C_u) \times \mathbb{R}^m \to \mathbb{R}^k$, let $n_c \triangleq d + k$ and assume there exists $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$ such that $A(x)u + b(x) \geq (\Gamma(x,u) + \gamma(x), \psi(x,u))$ for all $(x,u) \in C_u$.

For many practical control problems, especially those involving constraints on the magnitude of the control input, one will likely not find (or need) a CBF that exists on the entire state space. More often, feasibility verification can be restricted to a particular operating region. Thus, a method is needed to verify that $K_c(x)$ in (4) is nonempty on a subset of $\mathbb{R}^n$. Because the system is expected to operate nearby the safe set, a natural way to define the operating region is with sublevel sets of a CBF candidate B defining $\mathcal{S} \subset \Pi(C_u)$. By certifying that $K_c(x)$ is nonempty on a set $\mathcal{L}_B(\beta) \triangleq \{x \in \mathbb{R}^n : B(x) \leq \beta\}$, with $\beta > 0$, we certify that B is a CBF on $\mathcal{L}_B(\beta)$, and that the controller κ^* in (6) exists on the entire safe set $\mathcal{S} \subset \mathcal{L}_B(\beta)$. Since working with B in this context requires assuming that B is polynomial, we subsequently consider a generic polynomial $\tilde{B} \in \mathcal{P}^{n_b}[x]$.

While being a SoS polynomial is a global property, there exist hierarchies of relaxations that have close relationships to the set of polynomials that are nonnegative only on a particular subset of $\mathbb{R}^n$ [10]. The relaxation that will be most useful for the feasibility verification problem is the following, based on Putinar's Positivstellensatz [20].

Lemma 3. *Let $\tilde{B} \in \mathcal{P}^{n_b}[x]$ and define $\mathcal{L}_{\tilde{B}}(\beta) \triangleq \{x \in \mathbb{R}^n : \tilde{B}(x) \leq \beta\}$ for some $\beta \in \mathbb{R}$. A function $p \in \mathcal{P}[x]$ is nonnegative on $\mathcal{L}_{\tilde{B}}(\beta)$ if there exists $s_0, s_1, \ldots, s_{n_b} \in \Sigma[x]$ such that, for all $x \in \mathbb{R}^n$,*

$$p(x) \geq s_0(x) + \sum_{j=1}^{n_b} s_j(x)\left(\beta - \tilde{B}_j(x)\right). \tag{9}$$

Proof: The result follows from the facts that $s_j(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\beta - \tilde{B}_j(x) \geq 0$ for all $x \in \mathcal{L}_{\tilde{B}}(\beta)$. ■

Putinar's Positivstellensatz shows that every polynomial that is strictly positive on $\mathcal{L}_{\tilde{B}}(\beta)$ can be decomposed in the form on the right-hand side of (9) under the assumption that the functions defining $\mathcal{L}_{\tilde{B}}(\beta)$ have an Archimedean property [10, Thm 3.20]. While results guaranteeing the existence of SoS decompositions when the Archimedean property is not present have been applied to controls problems in, e.g., [12], these methods scale poorly with the number of components in $\tilde{B}$. The multiplicative monoid in [12] is known to lead to multiplicative combinations of decision variables that require developing complex iterative procedures, thereby adding conservativeness to the problem.

Recalling the definition of the mapping K in (8), the following program certifies that the set $K(x)$ is nonempty for all $x \in \mathcal{L}_{\tilde{B}}(\beta) = \{x \in \mathbb{R}^n : \tilde{B}(x) \leq \beta\}$.

Problem 2. (Feasibility on Level Sets) Given $x \in \mathbb{R}^n$, $A \in \mathcal{P}^{n_c \times m}[x]$, $b \in \mathcal{P}^{n_c}[x]$, $\tilde{B} \in \mathcal{P}^{n_b}[x]$, and $\beta \in \mathbb{R}$, find polynomials $u \in \mathcal{P}^m[x]$, $s_0, s_1, \ldots, s_{n_b} \in \Sigma[x]$, and a constant $\epsilon \geq 0$ such that, for all $i \in [n_c]$,

$$-A_{i*}(x)\, u(x) - b_i(x) - \epsilon$$
$$-s_0(x) - \sum_{j=1}^{n_b} s_j(x) \left(\beta - \tilde{B}_j(x)\right) \in \Sigma[x]. \quad (10)$$

The main result of this section follows. It states that a CBF candidate B can be certified as a CBF on a set $\mathcal{L}_{\tilde{B}}(\beta) \supset U(\partial\mathcal{S}) \cap \Pi(C_u)$ by finding a feasible solution to Problem 2. Unfortunately, the inability to find a feasible solution does not mean that no such feasible solution exists. One major cause for conservativeness is that the degree of the involved polynomials must be restricted in practice, and feasible solutions may exist for higher degree polynomials.

Theorem 4. ***(Verification of CBF)*** *Consider the dynamical system (F, C_u) in (1) and a set $\mathcal{S} \subset \Pi(C_u)$. Suppose Assumption 4 holds for a function ψ, $B : \mathbb{R}^n \to \mathbb{R}^d$ is a CBF candidate defining $\mathcal{S}$, and $\gamma : \Pi(C_u) \to \mathbb{R}^d$ satisfies Assumption 2. Given Γ defined in (3), let Assumption 6 hold for some $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$. If Problem 2 has a solution for some $\tilde{B}$ and β for which there exists a neighborhood of the boundary of $\mathcal{S}$ such that $U(\partial\mathcal{S}) \cap \Pi(C_u) \subset \mathcal{L}_{\tilde{B}}(\beta)$, then $K_c(x)$ in (4) is nonempty for all $x \in \mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ and B is a CBF for (F, C_u) and $\mathcal{S}$ on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ with respect to γ. Moreover, if the solution to Problem 2 is such that $\epsilon > 0$, then $K_c^\circ(x)$ in (7) is nonempty for all $x \in \mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$.*

Proof: Using Definition 2 and the assumptions of the theorem, we need only show that K_c in (4) is nonempty on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ to prove that B is a CBF. Problem 2 and Lemma 3 tell us that there exists $u \in \mathcal{P}^m[x]$ such that $A(x)\, u(x) + b(x) \leq -\epsilon$ for all $x \in \mathcal{L}_{\tilde{B}}(\beta)$. From Assumption 6, $(\Gamma(x, u(x)) + \gamma(x), \psi(x, u(x))) \leq -\epsilon$ for all $x \in \mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$. It follows by definition that K_c is nonempty on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ and, if $\epsilon > 0$, so is K_c°. ■

VI. Conclusion

This paper defined a notion of vector-valued CBF that is amenable to problems where the mapping of safety-ensuring control inputs is defined by multiple constraints. Selections of the safety-ensuring map render the safe set of states forward (pre-)invariant under mild conditions. Tools for certifying the continuity and feasibility of optimal selections from the map were developed.

Future work will investigate situations where convergence to the safe set is also desired. To characterize broader classes of cost function with the level bounded property, we expect to generalize Proposition 1 significantly. Finally, we will investigate adaptations of the developed framework to accommodate tangent cone conditions.

References

[1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Trans Autom Control*, vol. 62, no. 8, pp. 3861–3876, 2016.

[2] P. Glotfelter, J. Cortés, and M. Egerstedt, "Nonsmooth barrier functions with applications to multi-robot systems," *IEEE Control Syst. Lett.*, vol. 1, no. 2, pp. 310–315, 2017.

[3] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *Proc. Eur. Control Conf.* IEEE, 2019, pp. 3420–3431.

[4] P. Glotfelter, J. Cortes, and M. Egerstedt, "A nonsmooth approach to controller synthesis for boolean specifications," *IEEE Trans. Autom. Control*, 2020.

[5] M. Maghenem and R. G. Sanfelice, "Sufficient conditions for forward invariance and contractivity in hybrid inclusions using barrier functions," *Automatica*, p. 109328, 2021.

[6] R. G. Sanfelice, *Hybrid Feedback Control*. Princeton University Press, 2020.

[7] E. D. Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization," *Systems & Control Letters*, vol. 13, no. 2, pp. 117–123, 1989.

[8] A. Isaly, O. Patil, R. Sanfelice, and W. E. Dixon, "Adaptive safety with multiple barrier functions using integral concurrent learning," in *Proc. Am. Control Conf.*, 2021, pp. 3719–3724.

[9] M. J. Powell and A. D. Ames, "Towards real-time parameter optimization for feasible nonlinear control with applications to robot locomotion," in *Proc. Am. Control Conf.* IEEE, 2016, pp. 3922–3927.

[10] M. Laurent, "Sums of squares, moment matrices and optimization over polynomials," in *Emerging applications of algebraic geometry*. Springer, 2009, pp. 157–270.

[11] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing sostools: A general purpose sum of squares programming solver," in *Proc. IEEE Conf. Decis. Control*, vol. 1. IEEE, 2002, pp. 741–746.

[12] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard, "Some controls applications of sum of squares programming," in *Proc. IEEE Conf. Decis. Control*, vol. 5. IEEE, 2003, pp. 4676–4681.

[13] L. Wang, D. Han, and M. Egerstedt, "Permissive barrier certificates for safe stabilization using sum-of-squares," in *Proc. of American Control Conf.* IEEE, 2018, pp. 585–590.

[14] J. Usevitch, K. Garg, and D. Panagou, "Strong invariance using control barrier functions: A Clarke tangent cone approach," in *Proc. IEEE Conf. Decis. Control*. IEEE, 2020, pp. 2044–2049.

[15] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems*. Princeton University Press, 2012.

[16] E. A. Feinberg, P. O. Kasyanov, and M. Voorneveld, "Berge's maximum theorem for noncompact image sets," *Journal of Mathematical Analysis and Applications*, vol. 413, no. 2, pp. 1040–1046, 2014.

[17] R. A. Freeman and P. V. Kokotovic, "Inverse optimality in robust stabilization," *SIAM journal on control and optimization*, vol. 34, no. 4, pp. 1365–1391, 1996.

[18] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*. Springer Science & Business Media, 2009, vol. 317.

[19] R. A. Freeman and P. V. Kokotovic, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Boston, MA: Birkhäuser, 1996.

[20] M. Putinar, "Positive polynomials on compact semi-algebraic sets," *Indiana University Mathematics Journal*, vol. 42, no. 3, pp. 969–984, 1993.

[21] W. Xiao, C. A. Belta, and C. G. Cassandras, "Feasibility-guided learning for constrained optimal control problems," in *Proc. IEEE Conf. Decis. Control*. IEEE, 2020, pp. 1896–1901.

[22] E. A. Basso, H. M. Schmidt-Didlaukies, and K. Y. Pettersen, "Hysteretic control Lyapunov functions with application to global asymptotic tracking for underwater vehicles," in *Proc. IEEE Conf. Decis. Control*. IEEE, 2020, pp. 5267–5274.