Sufficient Conditions for Optimality and Asymptotic Stability in Two-Player Zero-Sum Hybrid Games

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ABSTRACT

In this paper, we formulate a two-player zero-sum game under dynamic constraints given in terms of hybrid dynamical systems. We present sufficient conditions with Hamilton-Jacobi-Isaacs-like equations to guarantee attaining a solution to the game. It is shown that when the players select the optimal strategy, the value function can be evaluated without the need of computing solutions. Under additional conditions, we show that the optimal feedback laws render a set of interest asymptotically stable. Using this framework, we address an optimal control problem under the presence of an adversarial action in which the decision-making agents have dynamics that might exhibit both continuous and discrete behavior. Applications of this problem, as presented here, include disturbance rejection and security scenarios, for which the effect of the worst-case adversarial action is minimized.

CCS CONCEPTS

• Computer systems organization → Robotic control; **• Theory of computation → Solution concepts in game theory**; **Mathematical optimization**; **• Information systems → Process control systems**; **• Computing methodologies → Multi-agent systems**.

KEYWORDS

Game Theory, Optimal Control, Hybrid Systems, Robust Control, Security

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1 INTRODUCTION

Games involving multiple players with potentially different interests emerge in multi-agent systems, both in benign (or cooperative) and contested (or noncooperative) settings. A list of examples includes and is not limited to route selection in a road network [25], heavy duty vehicle platooning [12], control of smart grids [30], trading modeling in the stock market [9], and control of large populations of systems [16]. Generally speaking, a game is an optimization problem with multiple players, constraints that enforce the "rules" of the game, and payoff functions to be optimized through the selection of decision variables. Constraints on the actions played by the players formulated as dynamic relationships (i.e., involving time) lead to dynamic games. Differential games pertain to the case when these constraints are given in terms of differential equations; see, e.g., [4] and the references therein. Of particular interest is the contested setting, which occurs when the players have independent objectives, such as when one agent aims at minimizing a cost function and another agent aims at maximizing it under dynamic constraints. If the players select their actions seeking their own benefit, a dynamic noncooperative game emerges. This type of dynamic games have been thoroughly studied in the literature, when the dynamic constraints are given in terms of difference equations or differential equations – in general, referred to as differential games – including, to just list a few, [3, 11, 19, 20, 24, 33].

In recent years, significant progress has been made in the understanding of dynamic games with agents sharing information over networks; see, e.g., [22]. Interestingly, the combination of physics, computing, and networks leads to dynamic constraints that exhibit both continuous and discrete behavior. In particular, intermittent information availability, resets of variables, such as expiring timers, and other nonsmooth and instantaneous changes lead to dynamic constraints that can be conveniently captured using hybrid system models. The design of algorithms that guarantee optimality under such hybrid dynamic constraints requires new tools, since using tools from the differential games literature would most likely lead to suboptimal solutions. Unfortunately, tools for the design of algorithms for games with such hybrid dynamic constraints, which we refer to as hybrid games, are not as developed as those for differential games, as cited above. In [18, 32], a control design approach using game theory that is applicable to a class of hybrid automata models is presented. Specifically, the models considered therein are based on finite-state automata, the specifications are defined in terms of temporal logic formulae, and the payoff is solely given by a terminal cost. Decidability for hybrid automata given a winning condition are studied in [34]. Applications of the approach in [32] include reachability-based controller design [7, 10]. The work in [26, 27] pertains to a class of dynamic games in which the evolution of the variables associated to each of the players is modeled using

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differential equations, while the interactions between the players is modeled as switches that occur at isolated time instances, similar to switched systems. Conditional viability for impulsive systems with two competing input actions was considered in [2] and treated as an evolutionary game. Other efforts pertaining to differential games with impulsive (or discontinuous) elements include establishing continuity of bounds on value functions and (viscosity) solutions [6], formulating necessary and sufficient conditions for optimal strategies for the special case of bimodel linear-quadratic differential games [17], and a class of stochastic two-player differential games in the context of sail boat competitions [5].

Motivated by the lack of tools for the design of algorithms for hybrid games with dynamic constraints that are richer than those allowed by finite-state automata and switched systems, we formulate a framework for the study of two-player zero-sum games with generic hybrid dynamic constraints. Specifically, we formulate an infinite horizon optimization problem in which the cost functional includes a stage cost that penalizes the continuous evolution (or flow) and the discrete evolution (or jumps) of the variables, as well as their final value, via a terminal cost. The dynamic constraint is hybrid and given in terms of hybrid equations [14, 15, 29], which allows the modeling of continuous-time dynamics with logical modes, switching systems, hybrid automata, impulsive differential equations, and dynamics described by algebraic differential inclusions. More precisely, we model the hybrid dynamic constraints as a hybrid system denoted $\mathcal{H}$ given in terms of the hybrid equation

$$\mathcal{H}\begin{cases}\dot{x} &= F(x,u_{C1},u_{C2}) \quad (x,u_{C1},u_{C2})\in C\\ x^{+} &= G(x,u_{D1},u_{D2}) \quad (x,u_{D1},u_{D2})\in D\end{cases} \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $(u_{C1},u_{D1})\in\mathbb{R}^{m_{C1}}\times\mathbb{R}^{m_{D1}}$ is the input chosen by player P_1, and $(u_{C2},u_{D2})\in\mathbb{R}^{m_{C2}}\times\mathbb{R}^{m_{D2}}$ is the input chosen by player P_2. The *flow map* $F:\mathbb{R}^n\times\mathbb{R}^{m_C}\to\mathbb{R}^n$ captures the continuous evolution of the system on the *flow set* C. The *jump map* $G:\mathbb{R}^n\times\mathbb{R}^{m_D}\to\mathbb{R}^n$ describes the discrete evolution of the system on the *jump set* D. In this framework, the data of the hybrid system $\mathcal{H}$ is given by (C,F,D,G). For such broad class of systems, when solutions are unique, we consider a cost functional $\mathcal{J}:\mathbb{R}^n\times\mathbb{R}^{m_C}\times\mathbb{R}^{m_D}$ associated to the solution to $\mathcal{H}$ from ξ and study the problem

$$\min_{(u_{C1},u_{D1})}\ \max_{(u_{C2},u_{D2})}\ \mathcal{J}(\xi,u_{C1},u_{C2},u_{D1},u_{D2}) \tag{2}$$

as a zero-sum two-player hybrid game. This type of hybrid game emerges in several settings, as we illustrate next.

Application 1. (Robust Control) Given the system $\mathcal{H}$ as in (1) with state x, the disturbance rejection problem consists of finding the control input (u_{C1},u_{D1}) that upper bounds the cost of solutions to $\mathcal{H}$ in (1) in the presence of a disturbance (u_{C2},u_{D2}). This problem reduces to finding conditions such that the action of P_1 upper bounds the values of a cost functional $\mathcal{J}$, under the presence of any disturbance chosen by P_2. This bound also applies for the worst-case disturbance that seeks to maximize $\mathcal{J}$.

Application 2. (Security) Given the system $\mathcal{H}$ as in (1) with state x and

$$\begin{aligned}F(x,u_{C1},u_{C2}) &= f_d(x,u_{C1})+f_a(u_{C2})\\ G(x,u_{D1},u_{D2}) &= g_d(x,u_{D1})+g_a(u_{D2})\end{aligned}$$

the security problem consists of ensuring the control input (u_{C1},u_{D1}) renders $\mathcal{H}$ to minimize a cost functional $\mathcal{J}$ under the action of an attacker (u_{C2},u_{D2}), that knows f_d and g_d and is designed to harm the system as much as possible. This problem reduces to finding the conditions such that (u_{C1},u_{D1}) minimizes $\mathcal{J}$ under the attack (u_{C2},u_{D2}), which aims to maximize it.

The main contributions of this paper are summarized as follows.

- We present a framework for the study of two-player zero-sum games with generic hybrid dynamic constraints.
- We present sufficient conditions based on Hamilton-Jacobi-Isaacs-like equations to attain a saddle-point equilibrium and evaluate the game value function without computation of solutions.
- Connections between optimality and asymptotic stability of a closed set are revealed and framed in the game theoretical approach employed.
- We address an optimal control problem in robust and security scenarios as a two-player zero-sum dynamic game problem for the case in which the players might exhibit continuous and discrete behavior as in [14].

To the best of our knowledge, there are no results in the literature that can be used to solve two-player zero-sum games with hybrid dynamics modeled as in (1), following the framework introduced in [15]. Recent advances on optimality of such hybrid system models include the results in [8] providing cost evaluation techniques for adversarial scenarios [21]. Sufficient conditions to guarantee the existence of optimal solutions are provided in [13]. The results therein relate the cost functional to a Lyapunov-like function to guarantee optimality of the closed-loop system. An extension of these ideas to a receding-horizon algorithm is presented in [1].

The remainder of this paper is organized as follows. Section 2 presents preliminary definitions that will be used along the development of this article. In Section 3, we present a formulation of two-player zero-sum hybrid games and provide the main results of the paper in Theorem 3.8 and Corollary 3.15, which focus on infinite horizon games. A numerical example and application covering a type of hybrid systems are presented, displaying the versatility of the approach. Applications to a robust control problem and to a security problem are presented in Section 4. Section 5 provides conclusions, closing remarks and future work.

Notation. Given two vectors x,y, we use the equivalent notation $(x,y)=[x^\top y^\top]^\top$. The symbol $\mathbb{N}$ denotes the set of natural numbers including zero. The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative reals. Given a vector x and a nonempty set $\mathcal{A}$, the distance from x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}}=\inf_{y\in\mathcal{A}}|x-y|$. In addition, we denote with $\mathbb{S}^n_+$ the set of real positive definite matrices of dimension n, and with $\mathbb{S}^n_{0+}$ the set of real positive semidefinite matrices of dimension n. Given a nonempty set C, denote by int$C$ its interior and by $\overline{C}$ its closure. Given a symmetric matrix $A\in\mathbb{R}^{n\times n}$, the scalars $\underline{\lambda}(A)$ and $\overline{\lambda}(A)$ denote the minimum and largest eigenvalue of A, respectively.

2 PRELIMINARIES

2.1 Hybrid Systems with Inputs

Since solutions to the dynamical system $\mathcal{H}$ can exhibit both continuous and discrete behavior, we use ordinary time t to determine the amount of flow, and a counter $j \in \mathbb{N}$ that counts the number of jumps. Thus, the concept of a hybrid time domain, in which solutions are fully described, is proposed.

Definition 2.1. (Hybrid time domain) *A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if, for each $(T, J) \in E$, the set $E \cap ([0,T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain, i.e., it can be written in the form*

$$\bigcup_{j=0}^{J}([t_j, t_{j+1}] \times \{j\})$$

for some finite nondecreasing sequence of times $\{t_j\}_{j=0}^{J+1}$ with $t_{J+1} = T$. Each element $(t, j) \in E$ denotes the elapsed hybrid time, which indicates that t seconds of flow time and j jumps have occurred.

A hybrid signal is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I_\phi^j = \{t : (t, j) \in \operatorname{dom} \phi\}$.

Definition 2.2. (Hybrid arc) *A hybrid signal $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is called a hybrid arc if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I_ϕ^j. A hybrid arc ϕ is compact if $\operatorname{dom} \phi$ is compact.*

In this article, the same symbols are used to denote input actions and its values. The context clarifies the meaning of u, as follows: "the function u," "the signal u," or "the hybrid signal u" that appears in "the solution pair (ϕ, u)" refer to the input action, whereas "u" refers to the input value as a point in $\mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ in any other case. The reader can replace "the function u" by "u_ϕ," that is the input action yielding the system to a response described by the hybrid arc ϕ.

Definition 2.3. (Hybrid Input) *A hybrid signal u is a hybrid input if for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I_u^j.*

Let $\mathcal{X}$ be the set of hybrid arcs $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$, and $\mathcal{U} = \mathcal{U}_C \times \mathcal{U}_D$ the set of hybrid inputs $u = (u_C, u_D) : \operatorname{dom} u \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$. A solution to the hybrid system with input $\mathcal{H}$ is defined as follows.

Definition 2.4. (Solution to the hybrid system $\mathcal{H}$) *A hybrid signal (ϕ, u) defines a solution pair to the hybrid system (1) if $\phi \in \mathcal{X}$, $u = (u_C, u_D) \in \mathcal{U}$, $\operatorname{dom}\phi = \operatorname{dom} u$, and*

- *$(\phi(0, 0), u_C(0, 0)) \in \overline{C}$ or $(\phi(0, 0), u_D(0, 0)) \in D$,*
- *For each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior $\operatorname{int} I_\phi^j$, we have, for all $t \in \operatorname{int} I_\phi^j$,*
$$(\phi(t, j), u_C(t, j)) \in C$$
and, for almost all $t \in I_\phi^j$,
$$\frac{d}{dt}\phi(t, j) = F(\phi(t, j), u_C(t, j))$$
- *For all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$,*
$$\begin{aligned} (\phi(t, j), u_D(t, j)) &\in D \\ \phi(t, j + 1) &= G(\phi(t, j), u_D(t, j)) \end{aligned}$$

A solution pair (ϕ, u) is a compact solution pair if ϕ is a compact hybrid arc.

We say that a solution pair (ϕ, u) to $\mathcal{H}$ is maximal if it cannot be extended and we say it is complete when $\operatorname{dom} \phi$ is unbounded. We denote by $\hat{\mathcal{S}}_{\mathcal{H}}(M)$ the set of solution pairs (ϕ, u) to $\mathcal{H}$ as in (1) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_{\mathcal{H}}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}}(M)$ denotes all maximal solution pairs and $\mathcal{S}_{\mathcal{H}}^\infty(M) \subset \hat{\mathcal{S}}_{\mathcal{H}}(M)$ the set of complete solutions. Given $\xi \in \mathbb{R}^n$, we denote by $\mathcal{U}_{\mathcal{H}}^\infty(\xi)$ the set of input actions u such that maximal solutions to $\mathcal{H}$ from ξ for u are complete. For a given $u \in \mathcal{U}$, we denote the set of maximal state trajectories, or responses, to $\mathcal{H}$ from ξ for u by $\mathcal{R}(\xi, u) = \{\phi : (\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)\}$. We say u renders a maximal response ϕ to $\mathcal{H}$ from ξ if $\phi \in \mathcal{R}(\xi, u)$.

We define the projections of C and D onto $\mathbb{R}^n$, respectively as

$$\Pi(C) = \{\xi \in \mathbb{R}^n : \exists u_C \in \mathbb{R}^{m_C} \text{ s.t. } (\xi, u_C) \in C\}$$
$$\Pi(D) = \{\xi \in \mathbb{R}^n : \exists u_D \in \mathbb{R}^{m_D} \text{ s.t. } (\xi, u_D) \in D\}$$

We also define the set-valued maps

$$\Pi_u(x, C) = \{u_C \in \mathbb{R}^{m_C} : (x, u_C) \in C\}$$
$$\Pi_u(x, D) = \{u_D \in \mathbb{R}^{m_D} : (x, u_D) \in D\}$$

denoting the input values available for a given state. Likewise, we denote by $\sup_j \operatorname{dom} \phi := \{\sup j \in \mathbb{N}_{\geq 0} : \exists t \in \mathbb{R}_{\geq 0} \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$.

The following conditions guarantee uniqueness of solutions to $\mathcal{H}$ as in (1) [14, Proposition 2.11].

Proposition 2.5. (Uniqueness of Solutions) *Consider the hybrid system $\mathcal{H}$ as in (1). For every $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and each $u \in \mathcal{U}$ there exists a unique maximal response ϕ with $\phi(0, 0) = \xi$ provided that the following holds:*

($\star$) *for every $\xi \in \Pi(\overline{C}) \setminus \Pi(D)$, if two absolutely continuous functions $z_1, z_2 : [0, T] \to \mathbb{R}^n$ and a measurable function $u : [0, T] \to \mathbb{R}^{m_C}$ are such that $\dot{z}_i(t) = F(z_i(t), u(t))$ for almost all $t \in [0, T]$, $(z_i(t), u(t)) \in C$ for all $t \in (0, T]$, and $z_i(0) = \xi$, for each $i \in \{1, 2\}$, then $z_1(t) = z_2(t)$ for every $t \in [0, T]$.*

2.2 Closed-loop Hybrid Systems

Given a hybrid system $\mathcal{H}$ and a function $\kappa := (\kappa_C, \kappa_D)$ with $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, the autonomous hybrid system resulting from assigning $u = \kappa(x)$, namely, the hybrid closed-loop system, is given by

$$\mathcal{H}_\kappa \begin{cases} \dot{x} = F(x, \kappa_C(x)) & x \in C_\kappa \\ x^+ = G(x, \kappa_D(x)) & x \in D_\kappa \end{cases} \tag{3}$$

where $C_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_C(x)) \in C\}$ and $D_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_D(x)) \in D\}$.

A solution to the hybrid closed-loop system $\mathcal{H}_\kappa$ is defined as follows.

Definition 2.6. (Solution to the hybrid system $\mathcal{H}_\kappa$) *A hybrid arc ϕ defines a solution to the hybrid system $\mathcal{H}_\kappa$ in (3) if*

- $\phi(0,0) \in \overline{C_\kappa} \cup D_\kappa$,
- *For each $j \in \mathbb{N}$ such that I^j_ϕ has a nonempty interior $\text{int} I^j_\phi$, we have, for all $t \in \text{int} I^j_\phi$,*
$$\phi(t,j) \in C_\kappa$$
and, for almost all $t \in I^j_\phi$,
$$\frac{d}{dt}\phi(t,j) = F(\phi(t,j), \kappa_C(\phi(t,j)))$$
- *For all $(t,j) \in \text{dom}\,\phi$ such that $(t,j+1) \in \text{dom}\,\phi$,*
$$\begin{aligned} \phi(t,j) &\in D_\kappa \\ \phi(t,j+1) &= G(\phi(t,j), \kappa_D(\phi(t,j))) \end{aligned}$$

A solution ϕ is a compact solution if ϕ is a compact hybrid arc.

We denote by $\hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ the set of solutions ϕ to $\mathcal{H}_\kappa$ as in (3) such that $\phi(0,0) \in M$. The set $\mathcal{S}_{\mathcal{H}_\kappa}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ denotes all maximal solutions and $\hat{\mathcal{S}}^\infty_{\mathcal{H}_\kappa}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ the set of complete solutions.

3 TWO-PLAYER ZERO-SUM HYBRID GAMES

3.1 Formulation

Following the formulation in [4], for each $i \in \{1,2\}$, consider the i-th player P_i with dynamics described by $\mathcal{H}_i$ as in (1) with data (C_i, F_i, D_i, G_i), state $x_i \in \mathbb{R}^{n_i}$, and input $u_i = (u_{Ci}, u_{Di}) \in \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$, where $C_i \subset \mathbb{R}^{n_i} \times \mathbb{R}^{m_{Ci}}$, $F_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_{Ci}} \to \mathbb{R}^{n_i}$, $D_i \subset \mathbb{R}^{n_i} \times \mathbb{R}^{m_{Di}}$ and $G_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_{Di}} \to \mathbb{R}^{n_i}$. The maps F_i and G_i are single-valued maps. We denote by $\mathcal{U}_i = \mathcal{U}_{Ci} \times \mathcal{U}_{Di}$ the set of hybrid inputs for $\mathcal{H}_i$.

Definition 3.1. (Elements of a two-player zero-sum hybrid game) *A two-player zero-sum hybrid game is composed by*

1) *The state $x = (x_1, x_2) \in \mathbb{R}^n$, where, $n_1 + n_2 = n$ and, for each $i \in \{1,2\}$, $x_i \in \mathbb{R}^{n_i}$ is the state of player P_i.*
2) *The set of joint input actions $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ with elements $u = (u_1, u_2)$, where, for each $i \in \{1,2\}$, u_i is a hybrid input. For each $i \in \{1,2\}$, P_i selects u_i independently from u_{3-i}, thus allowing the joint input action u to have components u_i that are independently chosen by each player.*
3) *The dynamics of the game, described as in (1) and denoted by $\mathcal{H}$, with data*
$$\begin{aligned} C &:= C_1 \times C_2 \\ F(x,u_C) &:= (F_1(x,u_C), F_2(x,u_C)) \\ D &:= \{(x,u_D) \in \mathbb{R}^n \times \mathbb{R}^{m_D} : (x_i, u_{Di}) \in D_i, i \in \{1,2\}\} \\ G(x,u_D) &:= \{\hat{G}_i(x,u_D) : (x_i, u_{Di}) \in D_i, i \in \{1,2\}\} \end{aligned}$$
where $u_C = (u_{C1}, u_{C2})$, $m_{C1} + m_{C2} = m_C$, $u_D = (u_{D1}, u_{D2})$, $m_{D1} + m_{D2} = m_D$, $\hat{G}_1(x,u_D) = (G_1(x,u_D), I_{n_2})$, and $\hat{G}_2(x,u_D) = (I_{n_1}, G_2(x,u_D))$.
4) *For each $i \in \{1,2\}$, a strategy space K_i of P_i defined as a collection of mappings $\kappa_i : \mathbb{R}^n \to \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$. Each K_i is such that the strategy space of the game, namely, K, that is the collection of mappings with elements $\kappa = (\kappa_1, \kappa_2)$, where $\kappa_i \in K_i$ for each $i \in \{1,2\}$, is such that every maximal solution (ϕ, u) to $\mathcal{H}$ with input assigned as $\text{dom}\,\phi \ni (t,j) \mapsto u_i(t,j) = \kappa_i(\phi(t,j))$ for each $i \in \{1,2\}$ is complete. Each $\kappa_i \in K_i$ is said to be a permissible pure[1] strategy for P_i.*
5) *A scalar-valued functional $(\xi, u) \mapsto \mathcal{J}_i(\xi, u)$ defined for each $i \in \{1,2\}$, and called the cost associated to P_i. For each $u \in \mathcal{U}$, we refer to a single cost functional $\mathcal{J} = \mathcal{J}_1 = -\mathcal{J}_2$ as the cost associated to the unique solution to $\mathcal{H}$ from ξ for u, and its structure is specified for each type of game.*

We say that a game formulation is in normal (or matrix) form when it describes only the correspondences between different strategies and costs. On the other hand, we refer to the mathematical description of a game to be in the Kuhn's extensive form if the evolution of the game defined by the dynamical equations, the decision making process defined by the strategies, the sharing of information between the players defined by the communication network and their outcomes defined by the cost associated to each player, are described in the formulation. For the formulation in Definition 3.1 to be in Kuhn's extensive form, additional assumptions are required such that each strategy has a unique cost correspondence. For a given initial condition, a given strategy potentially leads to nonunique solutions to $\mathcal{H}$, each of which may have a different cost.

Given the formulation of the elements of a zero-sum hybrid game in Definition 3.1, its solution is defined as follows.

Definition 3.2. (Saddle-point equilibrium) *Consider a two-player zero-sum game, with dynamics $\mathcal{H}$ as in (1) with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$, for a given cost functional $\mathcal{J} : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$. We say a strategy $\kappa = (\kappa_1, \kappa_2) \in K$ is a saddle-point equilibrium if for each $\xi \in \Pi(\overline{C} \cup D)$, every $u^* = (u_1^*, u_2^*)$ rendering a maximal response ϕ^* to $\mathcal{H}$ from ξ, with components defined as $\text{dom}\,\phi^* \ni (t,j) \mapsto u_i^*(t,j) = \kappa_i(\phi^*(t,j))$, for each $i \in \{1,2\}$, satisfies*

$$\mathcal{J}(\xi, (u_1^*, u_2)) \le \mathcal{J}(\xi, u^*) \le \mathcal{J}(\xi, (u_1, u_2^*)) \tag{4}$$

for all u_1 such that there exists ϕ such that $(\phi, (u_1, u_2^*)) \in \mathcal{S}_\mathcal{H}(\xi)$, and for all u_2 such that there exists ϕ such that $(\phi, (u_1^*, u_2)) \in \mathcal{S}_\mathcal{H}(\xi)$.

Definition 3.2 is a generalization of the classical pure strategy Nash equilibrium [4, (6.3)] to the case where the players exhibit hybrid dynamics and opposite optimization goals. In words, we refer to the strategy $\kappa^* = (\kappa_1^*, \kappa_2^*)$ as a saddle-point when a player P_i cannot improve the cost $\mathcal{J}_i$ by playing any strategy different from κ_i^* when the player P_{3-i} is playing the strategy of the saddle-point, κ_{3-i}^*. Notice that the saddle-point, as a solution to the zero-sum *two*-player game, is a strategy in K, though the concept of a solution to a hybrid system $\mathcal{H}$, as in Definition 2.4, is a hybrid arc.

[1] This is in contrast to when K_i is defined as a probability distribution, in which case $\kappa_i \in K_i$ is referred to as a mixed strategy.

Remark 3.3. (Equivalent costs) *Given $\xi \in \Pi(\overline{C} \cup D)$ and a strategy $\kappa^* = (\kappa_1^* \kappa_2^*) \in K$, denote by $\mathcal{U}^*(\xi, \kappa^*)$ the set of joint actions $u = (u_1, u_2)$ rendering a maximal response ϕ to $\mathcal{H}$ from ξ with components defined as $\operatorname{dom}\phi \ni (t,j) \mapsto u_i(t,j) = \kappa_i^*(\phi(t,j))$ for each $i \in \{1,2\}$. By expressing the cost associated to every solution to $\mathcal{H}$ from ξ under the strategy κ^* as $\hat{\mathcal{J}}(\xi, \kappa^*) := \sup_{u \in \mathcal{U}^*(\xi,\kappa^*)} \mathcal{J}(\xi, u)$, an equivalent condition to (4) for when $\hat{\mathcal{J}}(\xi, \kappa^*) = \mathcal{J}(\xi, u^*)$ for every $u^* \in \mathcal{U}^*(\xi, \kappa^*)$ is*

$$\hat{\mathcal{J}}(\xi, (\kappa_1^*, \kappa_2)) \leq \hat{\mathcal{J}}(\xi, \kappa^*) \leq \hat{\mathcal{J}}_1(\xi, (\kappa_1, \kappa_2^*))$$

for all $\kappa_i \in K_i$, $i \in \{1,2\}$.

Remark 3.4. (Relation to the literature) *Given a discrete-time two-player zero-sum game with final time $(0, J)$, f_k and X defining the jump map and jump set, respectively, as in [4], setting the data of $\mathcal{H}$ as $C = \emptyset$, $G = f_k$ for $k \in \mathbb{N}_{\leq J}$, and $D = X$ reduces Definition 3.1 to [4, Def. 5.1] for the case in which the output of each player is equal to its state and there is a feedback information structure as in [4, Def. 5.2]. Thus, items $(vi) - (ix)$ in [4, Def. 5.1] are omitted in the formulation herein and items $(i) - (v)$ and $(x) - (xi)$ are covered by Definition 3.1, the definition of the hybrid time domain with final time $(0, J)$, and the set $\mathcal{S}_{\mathcal{H}}$.*
Given a continuous-time two-player zero-sum game with final time $(T, 0)$, f and $\mathcal{S}^0$ defining the flow map and flow set, respectively, as in [4], setting the data of $\mathcal{H}$ as $D = \emptyset$, $F = f$, and $C = \mathcal{S}^0$ reduces Definition 3.1 to [4, Def. 5.5] for the case in which the output of each player is equal to its state and there is a feedback information structure as in [4, Def. 5.6]. Thus, items $(vi) - (vii)$ in [4, Def. 5.5] are omitted in the formulation herein and items $(i) - (v)$ and $(viii)$, (ix) are covered by Definition 3.1, the definition of the hybrid time domain with final time $(0, T)$, and the set $\mathcal{S}_{\mathcal{H}}$.
By considering a discrete-time system with the single-valued function G or by considering a continuous-time system with F Lipschitz continuous in $\overline{C}$, and by removing the initial condition as an argument of the cost functionals and specifying it in the state equation, Remark 3.3 presents equivalent conditions to those in [4, (6.3)]. Thus, Definition 3.2 covers the definitions of a pure strategy Nash equilibrium in [4, Sec. 6.2, 6.5] for the zero-sum case.

Next, we formulate an infinite-horizon optimization problem to solve the two-player zero-sum hybrid game and provide the sufficient conditions to characterize the solution. Consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (1) for given (C, F, D, G). The cost evaluation tools employed in approaches based on dynamic programming require uniqueness of solutions to $\mathcal{H}$ for a given input action u from an initial condition ξ. This justifies the following assumption.

Assumption 3.5. *The flow map F is Lipschitz continuous on $\Pi(\overline{C})$. The jump map G is single valued, i.e., $D_1 = D_2$.*

Under Assumption 3.5, the conditions in Proposition 2.5 are satisfied, so for a given $u \in \mathcal{U}$, the solution to $\mathcal{H}$ from ξ is unique.

Given $\xi \in C \cup D$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$ such that maximal solutions to $\mathcal{H}$ from ξ for u are complete, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ, under Assumption 3.5, as

$$\begin{aligned}\mathcal{J}(\xi, u) := &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt \\ + &\sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + \limsup_{\substack{t+j\to\infty \\ (t,j)\in\operatorname{dom}\phi}} q(\phi(t,j))\end{aligned} \tag{5}$$

where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ; see Definition 2.2. Under the said assumptions, the solution to the two-player zero-sum game consists of solving the following problem.

Problem ($\diamond$): Given $\xi \in \mathbb{R}^n$, under Assumption 3.5, solve

$$\underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}{\underset{u_1}{\text{minimize}}\ \underset{u_2}{\text{maximize}}} \qquad \mathcal{J}(\xi, u) \tag{6}$$

where $\mathcal{U}_{\mathcal{H}}^{\infty}$ is the set of joint input actions yielding maximal complete solutions to $\mathcal{H}$, as defined in Section 2.2.

Remark 3.6. (Saddle-point equilibrium and min-max control) *A solution to Problem ($\diamond$), when it exists, can be expressed in terms of the pure strategy saddle-point equilibrium $\kappa = (\kappa_1, \kappa_2)$ for the two-player zero-sum infinite-horizon game. Each $u^* = (u_1^*, u_2^*)$ rendering a response ϕ^* such that $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$, defined as $\operatorname{dom}\phi^* \ni (t,j) \mapsto u_i^*(t,j) = \kappa_i(\phi^*(t,j))$ for each $i \in \{1,2\}$, satisfies*

$$u^* = \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}{\arg\min_{u_1}\max_{u_2}} \qquad \mathcal{J}(\xi, u)$$

and it is referred to as a min-max control at ξ.

Definition 3.7. (Value function) *Given $\xi \in \Pi(\overline{C} \cup D)$, under Assumption 3.5, the value function at ξ is given by*

$$\mathcal{J}^*(\xi) := \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}{\min_{u_1}\max_{u_2}} \mathcal{J}(\xi, u) = \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}{\max_{u_2}\min_{u_1}} \mathcal{J}(\xi, u) \tag{7}$$

3.2 Design of Saddle-Point Equilibrium for Two-player Zero-sum Infinite-horizon Hybrid Games

The following result provides sufficient conditions to characterize the value function, and the feedback law that attains it. It addresses the solution to Problem ($\diamond$) for each $\xi \in \Pi(\overline{C} \cup D)$ showing that the optimizer is the saddle-point equilibrium.

Theorem 3.8. (Hamilton-Jacobi-Isaacs (HJI) for Problem ($\diamond$)) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (1) described by (C, F, D, G), satisfying Assumption 3.5, stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is* *continuously differentiable on a neighborhood of* $\Pi(C)$ *that satisfies the Hamilton-Jacobi-Isaacs hybrid equations given as*

$$\begin{aligned} 0 &= \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u(x,C)}} \max_{u_{C2}} \{L_C(x,u_C) + \langle \nabla V(x), F(x,u_C)\rangle\} \\ &= \max_{\substack{u_{C2} \\ u_C=(u_{C1},u_{C2})\in\Pi_u(x,C)}} \min_{u_{C1}} \{L_C(x,u_C) + \langle \nabla V(x), F(x,u_C)\rangle\} \qquad (8) \\ & \qquad \forall x \in \Pi(C), \end{aligned}$$

$$\begin{aligned} V(x) &= \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u(x,D)}} \max_{u_{D2}} \{L_D(x,u_D) + V(G(x,u_D))\} \\ &= \max_{\substack{u_{D2} \\ u_D=(u_{D1},u_{D2})\in\Pi_u(x,D)}} \min_{u_{D1}} \{L_D(x,u_D) + V(G(x,u_D))\} \qquad (9) \\ & \qquad \forall x \in \Pi(D), \end{aligned}$$

and for each $\xi \in \Pi(\overline{C} \cup D)$, *each* $(\phi, u) \in \mathcal{S}^\infty_{\mathcal{H}}(\xi)$ *satisfies*

$$\limsup_{\substack{t+j\to\infty \\ (t,j)\in\mathrm{dom}\phi}} V(\phi(t,j)) = \limsup_{\substack{t+j\to\infty \\ (t,j)\in\mathrm{dom}\phi}} q(\phi(t,j)), \qquad (10)$$

then

$$\mathcal{J}^*(\xi) = V(\xi) \qquad \forall \xi \in \Pi(\overline{C} \cup D), \qquad (11)$$

and any stationary feedback law $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ *with values*

$$\kappa_C(x) \in \arg \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u(x,C)}} \max_{u_{C2}} \{L_C(x,u_C) + \langle \nabla V(x), F(x,u_C)\rangle\} \qquad (12) \\ \forall x \in \Pi(C)$$

and

$$\kappa_D(x) \in \arg \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u(x,D)}} \max_{u_{D2}} \{L_D(x,u_D) + V(G(x,u_D))\} \qquad (13) \\ \forall x \in \Pi(D)$$

is a pure strategy saddle-point equilibrium for the two-player zero-sum hybrid game with infinite horizon and $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$.

Proof Sketch. To show the claim we apply cost evaluation tools built upon dynamic programming approaches and proceed as follows:

1) Pick an initial condition ξ and evaluate the cost associated to any solution yielded by $\kappa = (\kappa_C, \kappa_D)$, with values as in (12) and (13), from ξ. Show that this cost coincides with the value of the function V at ξ.
2) Lower bound the cost associated to any solution from ξ when P_2 plays $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$ by the value of the function V evaluated at ξ.
3) Upper bound the cost associated to any solution from ξ when P_1 plays $\kappa_1 := (\kappa_{C1}, \kappa_{D1})$ by the value of the function V evaluated at ξ.
4) By showing that the cost of any solution from ξ when P_1 plays κ_1 is not less than the cost of any solution yielded by κ from ξ, and by showing that the cost of any solution from ξ when P_2 plays κ_2 is not larger than the cost of any solution yielded by κ from ξ, we show optimality of κ in Problem $(\diamond)$ in the min-max sense.

□

Notice that when the players select the optimal strategy, the value function equals the function V evaluated at the initial condition. This makes evident the independence of the result from computing solutions/trajectories.

Remark 3.9. (Connections between Theorem 3.8 and Problem $(\diamond)$) *Given* $\xi \in \Pi(\overline{C} \cup D)$, *if there exist a function* V *satisfying the conditions in Theorem 3.8, then a solution to Problem* $(\diamond)$ *exists, namely there exists an input action* $u^* = (u^*_C, u^*_D) = ((u^*_{C1}, u^*_{C2}), (u^*_{D1}, u^*_{D2})) \in \mathcal{U}^\infty_{\mathcal{H}}(\xi)$ *such that* $\mathcal{J}(\xi, u^*) < \infty$, *that attains the min-max in (6), and as a consequence satisfies (4) in Definition 3.2. In addition, the strategy* $\kappa \in K$ *with values as in (12) and (13) is such that every complete solution to the closed-loop system* $\mathcal{H}_\kappa$ *from* ξ *has a cost that is equal to the min-max in (6).*

3.3 Linear Quadratic Hybrid Games

Next, we consider a special case of our result that emerges in hybrid systems with linear flow and jump maps and periodic jumps. We introduce a state variable τ that plays the role of a timer. Once τ reaches a fixed threshold $\bar{T}$, it triggers a jump in the state and resets τ to 0.

Given a time $\bar{T} \in \mathbb{R}$, consider a two-player zero-sum game with state $x = (x_p, \tau) = (x_{p1}, x_{p_2}, \tau) \in \mathbb{R}^n \times [0, \bar{T}]$, input $u = (u_C, u_D) = ((u_{C1}, u_{C2}), (u_{D1}, u_{D2})) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (1), described by

$$\begin{aligned} C &= \mathbb{R}^n \times [0, \bar{T}] \times \mathbb{R}^{m_C} \\ F(x, u_C) &= (A_C x_p + B_C u_C, 1) \\ &=: \left(\begin{bmatrix} A_{C1} & 0 \\ 0 & A_{C2} \end{bmatrix} \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} + \begin{bmatrix} B_{C1} & B_{C2} \end{bmatrix} \begin{bmatrix} u_{C1} \\ u_{C2} \end{bmatrix}, 1 \right) \\ D &= \mathbb{R}^n \times \{\bar{T}\} \times \mathbb{R}^{m_D} \\ G(x, u_D) &= (A_D x_p + B_D u_D, 0) \\ &=: \left(\begin{bmatrix} A_{D1} & 0 \\ 0 & A_{D2} \end{bmatrix} \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} + \begin{bmatrix} B_{D1} & B_{D2} \end{bmatrix} \begin{bmatrix} u_{D1} \\ u_{D2} \end{bmatrix}, 0 \right) \end{aligned}$$

where $C \cup D$ is nonempty. The input $u_1 = (u_{C1}, u_{D1})$ is assigned by P_1 and the input $u_2 = (u_{C2}, u_{D2})$ is assigned by P_2. The problem of finding conditions for u_1 to minimize a cost functional $\mathcal{J}$ in the presence of the action u_2 that seeks to maximize it, is formulated as a two-player zero-sum game. Thus, by solving Problem $(\diamond)$ for every $\xi \in \Pi(C \cup D)$, the control objective is achieved.

With the aim of pursuing minimum energy and distance to the origin, consider the cost functions $L_C(x, u_C) := x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2}$, $L_D(x, u_D) := x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, and terminal cost $q(x) := x_p^\top P(\tau) x_p$ where $Q_C, Q_D \in \mathbb{S}^n_+$, $R_{C1} \in \mathbb{S}^{m_{C_1}}_+$, $-R_{C2} \in \mathbb{S}^{m_{C_2}}_+$, $R_{D1} \in \mathbb{S}^{m_{D_1}}_+$, $-R_{D2} \in \mathbb{S}^{m_{D_2}}_+$ and $P(\tau) \in \mathbb{S}^n_+ \forall \tau \in [0, \bar{T}]$. These functions define $\mathcal{J}$ as in (5). Inspired by [8] and [28], the following result presents a tool for the solution of the optimal control problem for hybrid systems with linear maps and periodic jumps under an adversarial action.

Corollary 3.10. (Hybrid Riccati equation for periodic jumps) *Given* $\bar{T} \in \mathbb{R}$, $A_C, A_D \in \mathbb{R}^{n\times n}$, $B_C := [B_{C1}\ B_{C2}] \in \mathbb{R}^{n\times m_C}$, $B_D := [B_{D1}\ B_{D2}] \in \mathbb{R}^{n\times m_D}$, $Q_C, Q_D \in \mathbb{S}^n_+$, $R_{C1} \in \mathbb{S}^{m_{C_1}}_+$, $-R_{C2} \in \mathbb{S}^{m_{C_2}}_+$,

$R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$, *suppose there exists a matrix* $P : [0, \bar{T}] \to \mathbb{S}_+^n$ *continuously differentiable such that*

$$-\frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} = -P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau) + Q_C + P(\tau)A_C + A_C^\top P(\tau) \quad \forall \tau \in (0, \bar{T}), \tag{14}$$

$$\begin{aligned} -R_{D2} - B_{D2}^\top P(0) B_{D2} &\in \mathbb{S}_{0+}^{m_D}, \\ R_{D1} + B_{D1}^\top P(0) B_{D1} &\in \mathbb{S}_{0+}^{m_D}, \end{aligned} \tag{15}$$

the matrix $R_v = \begin{bmatrix} R_{D1}+B_{D1}^\top P(0)B_{D1} & B_{D1}^\top P(0)B_{D2} \\ B_{D2}^\top P(0)B_{D1} & R_{D2}+B_{D2}^\top P(0)B_{D2} \end{bmatrix}$ *is invertible, and*

$$P(\bar{T}) = Q_D + A_D^\top P(0) A_D - \begin{bmatrix} A_D^\top P(0) B_{D1} & A_D^\top P(0) B_{D2} \end{bmatrix} R_v^{-1} \begin{bmatrix} B_{D1}^\top P(0) A_D \\ B_{D2}^\top P(0) A_D \end{bmatrix} \tag{16}$$

Then, the feedback law $\kappa := (\kappa_C, \kappa_D)$*, with values*

$$\kappa_C(x) = (-R_{C1}^{-1}B_{C1}^\top P(\tau)x_p, -R_{C2}^{-1}B_{C2}^\top P(\tau)x_p) \quad \forall x \in \Pi(C), \tag{17}$$

$$\kappa_D(x) = -R_v^{-1}\begin{bmatrix} B_{D1}^\top P(0) A_D \\ B_{D2}^\top P(0) A_D \end{bmatrix} x_p \qquad \forall x \in \Pi(D) \tag{18}$$

is the pure strategy saddle-point equilibrium for the two-player zero-sum hybrid game with periodic jumps. In addition, for each $x = (x_p, \tau) \in \Pi(\overline{C} \cup D)$*, the value function is equal to* $V(x) := x_p^\top P(\tau) x_p$.

By following the same modeling approach and imposing conditions of the hybrid time domains, games for switching systems can be covered by Corollary 3.10. By selecting appropriate stage costs, optimality is encoded in the satisfaction of the infinitesimal conditions instead of in the knowledge of specific solutions/trajectories. Note that for switching systems, the function V might be independent of the timer state if the stage costs are independent of it as well.

As illustrated next, there are useful families of hybrid systems for which a pure strategy saddle-point equilibrium exists. The following example characterizes both the pure strategy saddle-point equilibrium and the value function in a two-player zero-sum game with a one-dimensional state, that is associated to player P_1. Thus, $n_1 = 1, n_2 = 0$, and the role of player P_2 reduces to select the action u_2.

Example 3.11. (Hybrid game with nonunique solutions) Consider a system with state $x \in \mathbb{R}$, input $u_C := (u_{C1}, u_{C2}) \in \mathbb{R}^2$, and dynamics $\mathcal{H}$ as in (1) described by

$$\begin{aligned} \dot{x} &= F(x, u_C) := ax + Bu_C & x \in [0, \delta] \\ x^+ &= G(x) := \sigma & x = \mu \end{aligned} \tag{19}$$

where $a < 0, B = [b_1\ b_2]$ and let $\mu > \delta > \sigma > 0$. Consider the cost functions $L_C(x, u_C) := x^2 Q_C + u_C^\top R_C u_C$, $L_D(x) := P(x^2 - \sigma^2)$, and terminal cost $q(x) := Px^2$, defining $\mathcal{J}$ as in (5), with $R_C := \begin{bmatrix} R_{C1} & 0 \\ 0 & R_{C2} \end{bmatrix}$, Q_C, R_{C1}, $-R_{C2}$, $P > 0$ and $Q_C + 2Pa - P^2(b_1^2 R_{C1}^{-1} + b_2^2 R_{C2}^{-1}) = 0$. Here, u_{C1} is designed by player P_1 which aims to minimize $\mathcal{J}$ while by means of u_{C2}, player P_2 seeks to maximize it. This is formulated as a two-player zero-sum hybrid game. The function $V(x) := Px^2$ is such that

$$\begin{aligned} \min_{u_{C1}} \max_{u_{C2}}_{u_C = (u_{C1}, u_{C2}) \in \mathbb{R}^2} & \{L_C(x, u_C) + \langle \nabla V(x), F(x, u_C) \rangle\} \\ &= \min_{u_{C1} \in \mathbb{R}} \max_{u_{C2} \in \mathbb{R}} \big\{(Q_C + 2Pa)x^2 + R_{C1}u_{C1}^2 + R_{C2}u_{C2}^2 + 2xP(b_1 u_{C1} + b_2 u_{C2})\big\} \\ &= 0 \end{aligned} \tag{20}$$

holds for all $x \in [0, \delta]$. In fact, (20) is attained by $\kappa_C(x) = (-R_{C1}^{-1}b_1 Px, -R_{C2}^{-1}b_2 Px)$. In particular, given that $Q_C + 2Pa = P^2(b_1^2 R_{C1}^{-1} + b_2^2 R_{C2}^{-1})$, we have

$$\begin{aligned} L_C(x, \kappa_C(x)) &+ \langle \nabla V(x), F(x, \kappa_C(x)) \rangle \\ &= \big[P^2(b_1^2 R_{C1}^{-1} + b_2^2 R_{C2}^{-1}) + R_{C1}^{-1}b_1^2 P^2 + R_{C2}^{-1}b_2 P^2 - 2P(b_1 R_{C1}^{-1} b_1 P + b_2 R_{C2}^{-1} b_2 P)\big] x^2 \\ &= 0 \end{aligned}$$

Then, $V(x) = Px^2$ is a solution to (8). In addition, the function V is such that

$$\begin{aligned} \min_{u_{D1}} \max_{u_{D2}}_{(u_{D1}, u_{D2}) \in \mathbb{R}^2} \{L_D(x) + V(G(x))\} &= P(x^2 - \sigma^2) + P\sigma^2 \\ &= Px^2 \end{aligned} \tag{21}$$

at $x = \mu$, which makes $V(x) = Px^2$ a solution to (9). Thus, given that V is continuously differentiable on $\mathbb{R}$, and that (8) and (9) hold thanks to (20) and (21), from Theorem 3.8 we have that the value function is $\mathcal{J}^*(\xi) := P\xi^2$ for any $\xi \in [0, \delta] \cup \{\mu\}$.

To investigate the case of nonunique solutions, now assume that $\delta \geq \mu > \sigma > 0$ and notice that solutions can potentially flow or jump at $x = \mu$. The set of all maximal responses from $x = \delta$ is denoted $\mathcal{R}_\kappa(\delta) = \{\phi_\kappa, \phi_h\}$, where the continuous response ϕ_κ is such that $\operatorname{dom}\phi_\kappa = \mathbb{R}_{\geq 0} \times \{0\}$, and is given by $\phi_\kappa(t, 0) = \delta \exp((a - R_{C1}^{-1}b_1 P - R_{C2}^{-1}b_2 P)t)$ for all $t \in [0, \infty)$. In simple words, ϕ_κ flows from $x = \delta$ towards $x = 0$. The maximal response ϕ_h is such that $\operatorname{dom}\phi_h = ([0, t^h] \times \{0\}) \cup ([t^h, \infty) \times \{1\})$, and is given by $\phi_h(t, 0) = \delta \exp((a - R_{C1}^{-1}b_1 P - R_{C2}^{-1}b_2 P)t)$, $\phi_h(t, 1) = \sigma \exp((a - R_{C1}^{-1}b_1 P - R_{C2}^{-1}b_2 P)(t - t^h))$. In simple words, the response ϕ_h flows from $x = \delta$ towards $x = \mu$, then it jumps to $x = \sigma$, and flows towards $x = 0$. Figure 1 illustrates this behavior. By denoting the corresponding input signals as $u_\kappa = \kappa(\phi_\kappa)$ and $u_h = \kappa(\phi_h)$, we show in Figure 1(c) that the costs of the solutions (ϕ_κ, u_κ) and (ϕ_h, u_h) are equal to $P\delta^2$. □

3.4 Asymptotic Stability for Hybrid Games

Next, we introduce definitions of some classes of functions to present a result that connects optimality and asymptotic stability for two-player zero-sum hybrid games.

Definition 3.12. (Class-$\mathcal{K}_\infty$ functions) *A function* $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is a class-*$\mathcal{K}_\infty$ *function, also written as* $\alpha \in \mathcal{K}_\infty$*, if* α *is zero at zero, continuous, strictly increasing, and unbounded.*

Definition 3.13. (Positive definite functions) *We say that a function* $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is positive definite, also written as* $\rho \in \mathcal{PD}$*, if* $\rho(s) > 0$ *for all* $s > 0$ *and* $\rho(0) = 0$*. We say that a function* $\rho : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ *is positive definite with respect to a set* $\mathcal{A} \subset \mathbb{R}^n$*, in composition with* $\kappa : \mathbb{R}^n \to \mathbb{R}^m$*, also written as* $\rho \in \mathcal{PD}_\kappa(\mathcal{A})$*, if* $\rho(x, \kappa(x)) > 0$ *for all* $x \in \mathbb{R}^n \setminus \mathcal{A}$ *and* $\rho(\mathcal{A}, \kappa(\mathcal{A})) = \{0\}$.

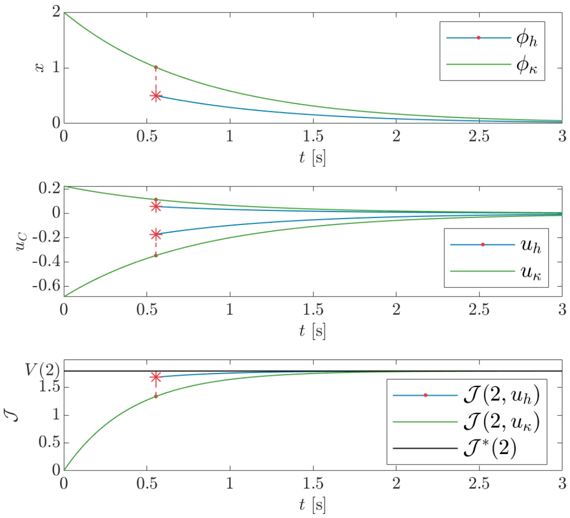

Figure 1: Nonnunique solutions attaining minmax optimal cost for $a = -1, b_1 = b_2 = 1, \delta = \xi = 2, \mu = 1, \sigma = 0.5, Q_C = 1, R_{C1} = 1.304, R_{C2} = -4$, and $P = 0.4481$. Continuous solution (green). Hybrid solution (blue and red).

Lemma 3.14. (Equivalent conditions) *Given $\mathcal{H}_\kappa$ as in (3) described by (C, F, D, G) and $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\Pi(C)$ such that $C_\kappa = \Pi(C), D_\kappa = \Pi(D)^2$, then (8), (9), (12), and (13) are satisfied if and only if*

$$L_C(x, \kappa_C(x)) + \langle \nabla V(x), F(x, \kappa_C(x)) \rangle = 0 \qquad \forall x \in C_\kappa, \tag{22}$$

$$\begin{aligned} L_C(x, (u_{C1}, \kappa_{C2}(x))) + \langle \nabla V(x), F(x, (u_{C1}, \kappa_{C2}(x))) \rangle \geq 0 \\ \forall (x, u_{C1}) : \ (x, (u_{C1}, \kappa_{C2}(x))) \in C, \end{aligned} \tag{23}$$

$$\begin{aligned} L_C(x, (\kappa_{C1}(x), u_{C2})) + \langle \nabla V(x), F(x, (\kappa_{C1}(x), u_{C2})) \rangle \leq 0 \\ \forall (x, u_{C2}) : \ (x(\kappa_{C1}(x), u_{C2})) \in C, \end{aligned} \tag{24}$$

$$L_D(x, \kappa_D(x)) + V(G(x, \kappa_D(x))) = V(x) \qquad \forall x \in D_\kappa, \tag{25}$$

$$\begin{aligned} L_D(x, (u_{D1}, \kappa_{D2}(x))) + V(G(x, (u_{D1}, \kappa_{D2}(x)))) \geq V(x) \\ \forall (x, u_{D1}) : \ (x, (u_{D1}, \kappa_{D2}(x))) \in D, \end{aligned} \tag{26}$$

$$\begin{aligned} L_D(x, (\kappa_{D1}(x), u_{D2})) + V(G(x, (\kappa_{D1}(x), u_{D2}))) \leq V(x) \\ \forall (x, u_{D2}) : \ (x, (\kappa_{D1}(x), u_{D2})) \in D. \end{aligned} \tag{27}$$

Corollary 3.15. (Saddle-point equilibrium under the existence of a Lyapunov function) *Consider a two-player zero-sum hybrid game with closed-loop dynamics $\mathcal{H}_\kappa$ as in (3) described by (C, F, D, G) satisfying Assumption 3.5, and $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ such that $C_\kappa = \Pi(C), D_\kappa = \Pi(D)$, and every maximal solution to $\mathcal{H}_\kappa$ from $\overline{C_\kappa} \cup D_\kappa$ is complete. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, continuous functions $L_C : C \to \mathbb{R}_{\geq 0}$ and $L_D : D \to \mathbb{R}_{\geq 0}$ defining the stage costs for flows and jumps, respectively, and $q : \mathbb{R}^n \to \mathbb{R}$ defining the terminal cost, suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{C_\kappa}$, satisfying (22)-(27), and such that for each $\xi \in \overline{C_\kappa} \cup D_\kappa$, each $\phi \in \mathcal{S}^\infty_{\mathcal{H}_\kappa}(\xi)$ satisfies (10). If there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \overline{C_\kappa} \cup D_\kappa \tag{28}$$

and one of the following conditions holds

1) *$L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ and $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$;*
2) *$L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$ and there exists a continuous function $\eta \in \mathcal{PD}$ such that $L_C(x, \kappa_D(x)) \geq \eta(|x|_{\mathcal{A}})$ for all $x \in C_\kappa$;*
3) *$L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ and there exists a continuous function $\eta \in \mathcal{PD}$ such that $L_D(x, \kappa_D(x)) \geq \eta(|x|_{\mathcal{A}})$ for all $x \in D_\kappa$;*

then

$$\mathcal{J}^*(\xi) = V(\xi) \quad \forall \xi \in \overline{C_\kappa} \cup D_\kappa \tag{29}$$

Furthermore, the feedback law κ is the saddle-point equilibrium (see Definition 3.2) and it renders $\mathcal{A}$ uniformly globally asymptotically stable [15] for $\mathcal{H}_\kappa$.

In the next example, notice that we do not necessarily compute the value function but, similar to the application of a Lyapunov theorem, we propose a candidate with the needed regularity and then check if the conditions in Corollary 3.15 hold.

Example 3.16. (Hybrid game with nonunique solutions) From Example 3.11, recall that $\kappa_C(x) = (-R_{C1}^{-1} b_1 Px, -R_{C2}^{-1} b_2 Px)$ for every $x \in \Pi(\overline{C})$. Let $\mathcal{A} = \{0\}$ and given that $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$, (22)-(27) hold, and the function $s \mapsto \eta(s) =: P\frac{s^2}{2}$ is such that $L_D(x, \kappa_D(x)) \geq \eta(|x|_{\mathcal{A}})$ for all $x \in D_\kappa$, by setting $\alpha_1(|x|_{\mathcal{A}}) = \underline{\lambda}(P)|x|^2$ and $\alpha_2(|x|_{\mathcal{A}}) = \overline{\lambda}(P)|x|^2$, from Corollary 3.15 we have that κ_C is the saddle-point equilibrium and renders $\mathcal{A}$ uniformly globally asymptotically stable for $\mathcal{H}$ as in (19). □

4 APPLICATIONS

We illustrate in the following applications with hybrid dynamics and quadratic costs how Theorem 3.8 provides conditions to solve the disturbance rejection and security problems introduced above by addressing them as zero-sum hybrid games.

4.1 Application 1: Robust Hybrid LQR

We study a special case of Application 1 and apply Theorem 3.8 in this section. Consider a hybrid system with state $x \in \mathbb{R}^n$, input $u = (u_C, u_D) = ((u_{C1}, u_{C2}), (u_{D1}, u_{D2})) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (1), described by

$$\begin{array}{rcl} C & \subset & \mathbb{R}^n \times \mathbb{R}^{m_C} \\ F(x, u_C) & = & A_C x + B_C u_C \\ & =: & \begin{bmatrix} A_{C1} & 0 \\ 0 & A_{C2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{C1} & B_{C2} \end{bmatrix} \begin{bmatrix} u_{C1} \\ u_{C2} \end{bmatrix} \\ D & \subset & \mathbb{R}^n \times \mathbb{R}^{m_D} \\ G(x, u_D) & = & A_D x + B_D u_D \\ & =: & \begin{bmatrix} A_{D1} & 0 \\ 0 & A_{D2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{D1} & B_{D2} \end{bmatrix} \begin{bmatrix} u_{D1} \\ u_{D2} \end{bmatrix} \end{array} \tag{30}$$

where $C \cup D$ is nonempty. Following Application 1, the input $u_1 := (u_{C1}, u_{D1})$ plays the role of the control and $u_2 := (u_{C2}, u_{D2})$ is the disturbance input. The problem of upper bounding the effect of the disturbance u_2 in the cost of complete solutions to $\mathcal{H}$ is formulated

[2] The conditions $C_\kappa = \Pi(C), D_\kappa = \Pi(D)$ hold when $\kappa_C(x) \in \Pi_u(x, C)$ for all $x \in \Pi(C)$ and $\kappa_D(x) \in \Pi_u(x, D)$ for all $x \in \Pi(D)$. In words, the feedback law κ defining the closed-loop system $\mathcal{H}_\kappa$ does not render input actions outside of $C \cup D$.

as a two-player zero-sum game. Thus, by solving Problem ($\diamond$) for every $\xi \in \Pi(C \cup D)$, the control objective is achieved.

With the aim of pursuing minimum energy and distance to the origin, consider the cost functions $L_C(x, u_C) := x^\top Q_C x + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2}$, $L_D(x, u_D) := x^\top Q_D x + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, and terminal cost $q(x) := x^\top P x$, where $Q_C, Q_D \in \mathbb{S}_+^n$, $R_{C1} \in \mathbb{S}_+^{m_{C_1}}$, $-R_{C2} \in \mathbb{S}_+^{m_{C_2}}$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$ and $P \in \mathbb{S}_+^n$. These functions define $\mathcal{J}$ as in (5). The following result presents a tool for the solution of the optimal control problem for hybrid systems with linear maps under the presence of disturbances.

Corollary 4.1. (Hybrid Riccati equation for disturbance rejection) *Given $A_C, A_D \in \mathbb{R}^{n\times n}$, $B_C = [B_{C1}\ B_{C2}] \in \mathbb{R}^{n\times m_C}$, $B_D = [B_{d1}\ B_{D2}] \in \mathbb{R}^{n\times m_D}$, $Q_C, Q_D \in \mathbb{S}_+^n$, $R_{C1} \in \mathbb{S}_+^{m_{C_1}}$, $-R_{C2} \in \mathbb{S}_+^{m_{C_2}}$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$, suppose there exists a matrix $P \in \mathbb{S}_+^n$ such that*

$$0 = -P(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P + Q_C + PA_C + A_C^\top P, \quad (31)$$

$$\begin{array}{rl} -R_{D2} - B_{D2}^\top P B_{D2} & \in \mathbb{S}_{0+}^{m_D}, \\ R_{D1} + B_{D1}^\top P B_{D1} & \in \mathbb{S}_{0+}^{m_D}, \end{array} \quad (32)$$

the matrix $R_v = \begin{bmatrix} R_{D1}+B_{D1}^\top PB_{D1} & B_{D1}^\top PB_{D2} \\ B_{D2}^\top PB_{D1} & R_{D2}+B_{D2}^\top PB_{D2} \end{bmatrix}$ *is invertible, and*

$$0 = -P + Q_D + A_D^\top P A_D - \begin{bmatrix} A_D^\top P B_{D1} & A_D^\top P B_{D2} \end{bmatrix} R_v^{-1} \begin{bmatrix} B_{D1}^\top P A_D \\ B_{D2}^\top P A_D \end{bmatrix} \quad (33)$$

Then, for the feedback law $\kappa_1 := (\kappa_{C1}, \kappa_{D1})$ with values[3]

$$\kappa_{C1}(x) = -R_{C1}^{-1} B_{C1}^\top P x \qquad \forall x \in \Pi(C), \quad (34)$$

$$\kappa_{D1}(x) = -[R_v^{-1}(1,1)\ R_v^{-1}(1,2)] \begin{bmatrix} B_{D1}^\top P A_D \\ B_{D2}^\top P A_D \end{bmatrix} x \quad \forall x \in \Pi(D), \quad (35)$$

the cost of complete solutions to $\mathcal{H}$ from ξ in the presence of any disturbance u_2 is upper bounded by $\xi^\top P \xi$. In addition, for each $x \in \Pi(\overline{C} \cup D)$, the value function is equal to $V(x) := x^\top P x$ and the worst-case disturbance is given by $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$, with values

$$\kappa_{C2}(x) = -R_{C2}^{-1} B_{C2}^\top P x \qquad \forall x \in \Pi(C), \quad (36)$$

$$\kappa_{D2}(x) = -[R_v^{-1}(2,1)\ R_v^{-1}(2,2)] \begin{bmatrix} B_{D1}^\top P A_D \\ B_{D2}^\top P A_D \end{bmatrix} x \quad \forall x \in \Pi(D). \quad (37)$$

4.2 Application 2: Security jumps-actuated hybrid game

We study a special case of Application 2 and apply Theorem 3.8 in this section. Consider a hybrid system with state $x \in \mathbb{R}^n$, input $u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (1), described by

$$\begin{array}{rll} \dot{x} = & F(x) & x \in C \\ x^+ = & A_D x + \begin{bmatrix} B_{D1} & B_{D2} \end{bmatrix} \begin{bmatrix} u_{D1} \\ u_{D2} \end{bmatrix} & (x, u_D) \in D \end{array} \quad (38)$$

with Lipschitz continuous $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $A_D \in \mathbb{R}^{n\times n}$, and $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$, such that $C \cup D$ is nonempty . The input u_{D1} plays the role of the control and u_{D2} is the disturbance input. Following Application 2, the problem of minimizing a cost functional $\mathcal{J}$ in the presence of the worst-case attack u_2 is formulated as a two-player zero-sum game. Thus, by solving Problem ($\diamond$) for every $\xi \in \Pi(C \cup D)$, the control objective is achieved.

With the aim of pursuing minimum energy and distance to the origin during jumps, consider the cost functions $L_C(x, u_C) := 0$, $L_D(x, u_D) := x^\top Q_D x + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, and terminal cost $q(x) := x^\top P x$, where $Q_D \in \mathbb{S}_+^n$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$ and $P \in \mathbb{S}_+^n$. These functions define $\mathcal{J}$ as in (5). The following result presents a tool for the solution of the optimal control problem for jumps-actuated hybrid systems with state-affine flow maps under a malicious input attack designed to cause as much damage as possible.

Corollary 4.2. (Hybrid Riccati equation for security) *Given $F : \mathbb{R}^n \to \mathbb{R}^n$, $A_D \in \mathbb{R}^{n\times n}$, $B_D := [B_{D1}\ B_{D2}] \in \mathbb{R}^{n\times m_D}$, $Q_D \in \mathbb{S}_+^n$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$, $\zeta \in \mathbb{R}^n$, suppose there exists a matrix $P \in \mathbb{S}_+^n$ such that*

$$0 = 2x^\top P F(x) \qquad \forall x \in \Pi(C), \quad (39)$$

$$\begin{array}{rl} -R_{D2} - B_{D2}^\top P B_{D2} & \in \mathbb{S}_{0+}^{m_D}, \\ R_{D1} + B_{D1}^\top P B_{D1} & \in \mathbb{S}_{0+}^{m_D}, \end{array} \quad (40)$$

the matrix $R_v = \begin{bmatrix} R_{D1}+B_{D1}^\top PB_{D1} & B_{D1}^\top PB_{D2} \\ B_{D2}^\top PB_{D1} & R_{D2}+B_{D2}^\top PB_{D2} \end{bmatrix}$ *is invertible, and*

$$0 = -P + Q_D + A_D^\top P A_D - \begin{bmatrix} A_D^\top P B_{D1} & A_D^\top P B_{D2} \end{bmatrix} R_v^{-1} \begin{bmatrix} B_{D1}^\top P A_D \\ B_{D2}^\top P A_D \end{bmatrix} \quad (41)$$

Then, with the feedback law

$$\kappa_{D1}(x) = -[R_v^{-1}(1,1)\ R_v^{-1}(1,2)] \begin{bmatrix} B_{D1}^\top P A_D \\ B_{D2}^\top P A_D \end{bmatrix} x \quad \forall x \in \Pi(D) \quad (42)$$

the cost functional $\mathcal{J}$ is minimized in the presence of the worst-case attack u_2, given by

$$\kappa_{D2}(x) = -[R_v^{-1}(2,1)\ R_v^{-1}(2,2)] \begin{bmatrix} B_{D1}^\top P A_D \\ B_{D2}^\top P A_D \end{bmatrix} x \quad \forall x \in \Pi(D) \quad (43)$$

In addition, for each $x \in \Pi(\overline{C} \cup D)$, the value function is equal to $V(x) := x^\top P x$.

Example 4.3. (Bouncing ball) Inspired by the problem in [31], consider a simplified model of a juggling system as in [23], with state $x \in \mathbb{R}^2$, input $u_D := (u_{D1}, u_{D2}) \in \mathbb{R}^2$, and dynamics $\mathcal{H}$ as in (1) , described by

$$\begin{array}{rcl} C & = & \mathbb{R}_{\geq 0} \times \mathbb{R} \\ F(x) & = & \begin{bmatrix} x_2 \\ -1 \end{bmatrix} \\ D & = & \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2 \\ G(x, u_D) & = & \begin{bmatrix} 0 \\ -\lambda x_2 + u_{D1} + u_{D2} \end{bmatrix} \end{array} \quad (44)$$

where u_{D1} is the control input, u_{D2} is the action of an attacker, and $\lambda \in (0,1)$ is the coefficient of restitution of the ball. Let $\mathcal{A} = \{0\}$. As an instance of Application 2, the scenario in which u_{D1} is designed to minimize a cost functional $\mathcal{J}$ under the presence of the worst-case attack u_{D2} is formulated as a two-player zero-sum game. With the aim of pursuing minimum energy and distance to the origin during jumps, consider the cost functions $L_C(x, u_C) := 0$, $L_D(x, u_D) := x_2^2 Q_D + u_D^\top R_D u_D$, and terminal cost $q(x) := \frac{1}{2}x_2^2 + x_1$

[3] The notation $R_v^{-1}(p,q)$ denotes the (p,q) entry of the matrix R_v^{-1}.

defining $\mathcal{J}$ as in (5), with $R_D := \begin{bmatrix} R_{D1} & 0 \\ 0 & R_{D2} \end{bmatrix}$ and $Q_D, R_{D1}, -R_{D2} > 0$. Here, u_{D1} is designed by player P_1 which aims to minimize $\mathcal{J}$ while player P_2 seeks to maximize it by means of choosing u_{D2}. The function $V(x) := \frac{1}{2}x_2^2 + x_1$ is such that $\langle \nabla V(x), F(x)\rangle = 0$ for all $x \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, making V a solution to (8). In addition, the function V is such that

$$\begin{aligned}
&\min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\mathbb{R}^2}} \max_{u_{D2}} \quad \{L_D(x,u_D) + V(G(x,u_D))\} \\
&= \min_{u_{D1}\in\mathbb{R}} \max_{u_{D2}\in\mathbb{R}} \left\{ x_2^2 Q_D + u_D^\top R_D u_D + \frac{(-\lambda x_2 + u_{D1} + u_{D2})^2}{2} \right\} \\
&= \frac{1}{2}x_2^2
\end{aligned} \tag{45}$$

for all $(x, u_D) \in D$, and attained by $\kappa_D(x) = (\kappa_{D1}(x), \kappa_{D2}(x))$ with $\kappa_{D1}(x) = \frac{R_{D2}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ and $\kappa_{D2}(x) = \frac{R_{D1}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ when

$$Q_D = \frac{-2R_{D1}R_{D2}\lambda^2 + R_{D1} + R_{D2} + 2R_{D1}R_{D2}}{2R_{D1} + 2R_{D2} + 4R_{D1}R_{D2}}, \tag{46}$$

which makes V a solution to (9). Thus, given that V is continuously differentiable on $\mathbb{R}^2$, and that (8) and (9) hold thanks to (45) and (46), from Theorem 3.8 , the value function is $\mathcal{J}^*(\xi_1, \xi_2) := \frac{\xi_2^2}{2} + \xi_1$. Figure 2 displays this behavior. Given that $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$, and (22)-(27)

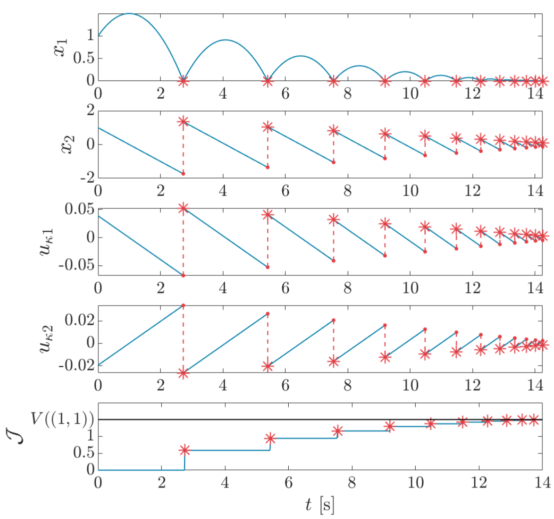

Figure 2: Bouncing ball solutions attaining minimum cost under worst-case u_2, with $\lambda = 0.8, R_{D1} = 10, R_{D2} = -20$, and $Q_D = 0.189$.

hold, by setting $\alpha_1(s) = \min\left\{ \frac{1}{2}\left(\frac{s}{\sqrt{2}}\right)^2, \frac{s}{\sqrt{2}} \right\}$ and $\alpha_2(s) = \frac{1}{2}s^2 + s$, from Corollary 3.15, we have that κ_D is the saddle-point equilibrium and renders $\mathcal{A}$ uniformly globally asymptotically stable for $\mathcal{H}$.

In Figure 3, we let the players select feedback laws close to the Nash equilibrium and calculate the cost associated to the new laws. The variation of the cost along the changes in the feedback laws makes evident the saddle-point geometry. This example illustrates how our results apply to Zeno systems. □

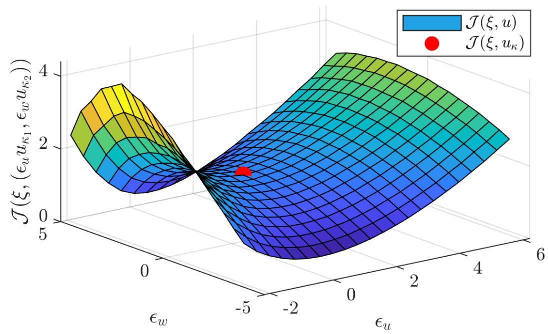

Figure 3: Saddle point behavior in the cost of solutions to bouncing ball from $\xi = (1, 1)$ when varying the feedback gains around the optimal value. The cost is evaluated on solutions $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$ with feedback law variations specified by ϵ_u and ϵ_w in $u = (\epsilon_u \kappa_1(\phi), \epsilon_w \kappa_2(\phi))$.

5 CONCLUSION AND FUTURE WORK

In this paper, we formulate a two-player zero-sum game under dynamic constraints given in terms of hybrid dynamical systems as in [15]. Scenarios in which the control action is selected by a player P_1 to accomplish an objective and countereffect the damage of an adversarial player P_2 are studied. By encoding the objectives of the players in the optimization of a cost functional, sufficient conditions in Hamilton-Jacobi-Isaacs form are provided to upper bound the cost for any disturbance. The main result allows the optimal strategy of P_1 to minimize the cost under the worst-case scenario attack in security applications. Additional conditions are proposed to allow the saddle-point strategy to render a set of interest asymptotically stable by letting the value function take the role of a Lyapunov function.

Future work includes the extension of the results to the finite-horizon optimal control problems under adversarial scenarios by framing them as zero-sum hybrid games, and to settings where the uniqueness of solutions assumption can be relaxed, as in Example 3.11. Structural conditions on the system that do not involve V and guarantee the existence of a solution to Problem $(\diamond)$ based on the smoothness and regularity of the data of the system, similar to those in [13] will also be studied. We expect the results can be generalized to randomized strategies, in particular, through the connection between set-valued dynamics and nonuniqueness of solutions, which captures nondeterminism.

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