Certifying the LTL Formula p Until q in Hybrid Systems

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***Abstract*—In this paper, we propose sufficient conditions to guarantee that a linear temporal logic formula of the form p Until q, denoted by $p\mathcal{U}q$, is satisfied for a hybrid system. Roughly speaking, the formula $p\mathcal{U}q$ is satisfied means that the solutions, initially satisfying proposition p, keep satisfying this proposition until proposition q is satisfied. To certify such a formula, connections to invariance notions – specifically, conditional invariance and eventual conditional invariance – as well as finite-time convergence properties are established. As a result, sufficient conditions involving the data of the hybrid system and an appropriate choice of Lyapunov-like functions, such as barrier functions, are derived. Examples illustrate the results throughout the paper.**

I. Introduction

Linear temporal logic (LTL) is a language used to express complex temporal properties of dynamical systems in terms of formulas. Each LTL formula is composed of a set of propositions related by temporal and logical operators. A required temporal property, also called *specification*, is guaranteed for a dynamical system if and only if the corresponding LTL formula is true along the solutions of the considered system. Hence, LTL provides a framework to formulate complex dynamical properties, that go beyond stability, convergence, or safety [1]–[4]. For example, in [3], LTL is employed to express the safety-plus-stability specification; see also [5].

A widely used approach to certify formulas along solutions to dynamical systems consists in using model-checking approaches [6]–[8]. In such approaches, the system is modeled as a finite- (or infinite-) state automaton and existing model-checking algorithms are able to answer about the satisfaction of the formula. The disadvantage of these approaches lies in their decidability; namely, whether the satisfaction of the formula for the automaton is equivalent to its satisfaction for the actual system [7]. Furthermore, when the system exhibits hybrid phenomena, this problem remains mostly unsolved. In other works, such as in [9], theorem provers are developed to analyze LTL formulas by simulating a discretized version of the system. As in every numerical tool, the sensitivity to discretization and the dimension of the system is a drawback.

Other works use analytical approaches inspired by Lyapunov-like techniques. For example, in [10], a class of LTL formulas is modeled as a finite-state transition system, which combined to the actual continuous-time control system form a hybrid control system. The considered formula is verified by guaranteeing a recurrence property. In [11], an approach combining automata-based tools with barrier functions is introduced. This approach provides a collection of barrier functions to certify the considered LTL formulas. In [12], sufficient Lyapunov-like conditions to certify the *always p* and the *eventually p* formulas are proposed.

Along the lines of [12], one can consider basic formulas involving *until* operators. More precisely, we distinguish the strong until (denoted $\mathcal{U}_s$) and the weak until (denoted $\mathcal{U}_w$) operators; see [13]–[15]. That is, given two propositions p and q, the satisfaction of the formula $p\mathcal{U}_s q$ implies that proposition p is true until q happens to be true, and q must become true eventually. For the weak version, the satisfaction of the formula $p\mathcal{U}_w q$ implies that proposition p is true until q happens to be true; however, q is not required to become true as long as p remains true. Until operators are among the most useful operators in LTL, and are a building block for more complex formulae.

In this paper, sufficient conditions for the satisfaction of the LTL formulas $p\mathcal{U}_w q$ and $p\mathcal{U}_s q$ along the solutions to hybrid systems are proposed. The proposed sufficient conditions are infinitesimal, involve only the data of the hybrid system and appropriate choice of Lyapunov-like functions, without requiring the computation of the solutions nor the discretization of the right-hand side. With such tools, more complex formulae can be certified through decomposition by building a finite state automaton; see [12, Section 6.5] and the references therein. A key intermediate step to deduce the proposed sufficient conditions consists in establishing sufficient and equivalent relationships between the satisfaction of the considered formulas and the following dynamical properties. 1) *Conditional Invariance* (CI), which suggests that the solutions to the hybrid system remain in a set if they start from a (likely different) set [16]. This property coincides with safety, as defined in [17]. 2) *Eventual Conditional Invariance* (ECI), which suggests that the solutions reach a given set in finite time and remain in it, provided that they start from a (likely different) given set [18]. 3) *Finite-Time Attractivity* (FTA), which suggests that the solutions reach a set in finite time [19].

In Section III, given a hybrid system denoted $\mathcal{H}$ [20], we establish that CI for an auxiliary version of $\mathcal{H}$, denoted $\mathcal{H}_w$, is sufficient for the satisfaction of the formula $p\mathcal{U}_w q$ for $\mathcal{H}$. Moreover, we show that the formula $p\mathcal{U}_s q$ is verified for $\mathcal{H}$ if $p\mathcal{U}_w q$ is satisfied and an auxiliary system, denoted $\mathcal{H}_s$, exhibits an ECI property. Finally, we show that the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}$ is equivalent to the satisfaction of $p\mathcal{U}_w q$ for $\mathcal{H}$ plus $\mathcal{H}_s$ exhibiting an FTA property. In Section IV, sufficient Lyapunov-like conditions, to guarantee the satisfaction of the formulas $p\mathcal{U}_w q$ and $p\mathcal{U}_s q$, are deduced by exploiting the relationships established in Section III and

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the characterizations of ECI and FTA notions in [21]; see also [22]. Finally, academic examples are provided all along the paper to illustrate concepts and results.

A preliminary version of this work is in [19], where detailed proofs and examples have been omitted.

Notation. Let $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. For x, $y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the Euclidean norm of x, $|x|_K := \inf_{y \in K} |x - y|$ defines the distance between x and the nonempty set K, and $\langle x, y\rangle = x^\top y$ denotes the inner product between x and y. For a set $K \subset \mathbb{R}^n$, we use $\mathrm{int}(K)$ to denote its interior, ∂K to denote its boundary, $\mathrm{cl}(K)$ to denote its closure, and $U(K)$ to denote any open neighborhood of K. The contingent cone at a point $x \in \mathbb{R}^n$ of a set K is given by $T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{|x + hv|_K}{h} = 0 \right\}$. For a set $Q \subset \mathbb{R}^n$, $K \backslash Q$ denotes the subset of elements of K that are not in Q. By $\mathcal{C}^1$, we denote the set of continuously differentiable functions. By $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, we denote a set-valued map associating each element $x \in \mathbb{R}^m$ to a subset $F(x) \subset \mathbb{R}^n$. For a scalar function $V : \mathrm{dom}\, V \to \mathbb{R}$, $L_V(r) := \{x \in \mathrm{dom}\, V : V(x) \leq r\}$, for some $r \in [0, \infty]$, is the r-sublevel set of V.

II. Preliminaries

Following the modeling framework proposed in [20], we consider hybrid systems modeled as

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D, \end{cases} \tag{1}$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$, the jump set $D \subset \mathbb{R}^n$, and the flow and jump maps, respectively, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

A solution ϕ to $\mathcal{H}$ is parameterized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a discrete jump variable $j \in \mathbb{N}$; see [22, Definition 2.1]. The hybrid time domain of ϕ, $\mathrm{dom}\, \phi$, is such that for each $(T, J) \in \mathrm{dom}\, \phi$, $\mathrm{dom}\, \phi \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\})$ for a sequence $\{t_j\}_{j=0}^{J+1}$, such that $t_{j+1} \geq t_j$ and $t_0 = 0$. A solution ϕ to $\mathcal{H}$ is said to be maximal if there is no solution ϕ' to $\mathcal{H}$ such that $\phi(t, j) = \phi'(t, j)$ for all $(t, j) \in \mathrm{dom}\, \phi$ with $\mathrm{dom}\, \phi$ a proper subset of $\mathrm{dom}\, \phi'$. It is said to be nontrivial if $\mathrm{dom}\, \phi$ contains at least two elements. A solution ϕ is said to be complete if its domain is unbounded. It is eventually discrete if $T = \sup_t \mathrm{dom}\, \phi < \infty$ and $\mathrm{dom}\, \phi \cap (\{T\} \times \mathbb{N})$ contains at least two elements. See [20] for more details about hybrid dynamical systems.

For convenience, we define the range of a solution ϕ to $\mathcal{H}$ as $\mathrm{rge}\, \phi := \{\phi(t, j) : (t, j) \in \mathrm{dom}\, \phi\}$. We use $\mathcal{S}_{\mathcal{H}}(x)$ to denote the set of maximal solutions to $\mathcal{H}$ starting from $x \in \mathrm{cl}(C) \cup D$. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $\mathcal{R}(\mathcal{A})$ denotes the (infinite-horizon) reachable set from $\mathcal{A}$; i.e., $\mathcal{R}(\mathcal{A}) := \{\phi(t, j) : \phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{A}), (t, j) \in \mathrm{dom}\, \phi\}$.

Definition 2.1 (Settling-time function)**.** *Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a solution ϕ to $\mathcal{H}$ starting from $\mathrm{cl}(C) \cup D$, the settling-time function $\mathcal{T}_{\mathcal{A}} : \mathcal{S}_{\mathcal{H}}(\mathrm{cl}(C) \cup D) \mapsto \mathbb{R}_{\geq 0}$ is given by*

$$\mathcal{T}_{\mathcal{A}}(\phi) := \begin{cases} \infty & \text{if } \mathcal{R}(\phi(0,0)) \cap \mathcal{A} = \emptyset \\ \min\limits_{\substack{\phi(t,j) \in \mathcal{A} \\ (t,j) \in \mathrm{dom}\phi}} t + j & \text{otherwise.} \end{cases} \tag{2}$$

Given a solution ϕ to $\mathcal{H}$ starting from $\mathbb{R}^n \backslash \mathcal{A}$, the function $\mathcal{T}_{\mathcal{A}}$ provides (when finite) the first hybrid time at which the solution ϕ reaches the set $\mathcal{A}$.

A. LTL and Until Operators

An atomic proposition p is a statement on the system state x that is either `True` ($\equiv 1$) or `False` ($\equiv 0$). A proposition p is treated as a (single-valued) function of x, that is, as the function $x \mapsto p(x) \in \{0, 1\}$. The set of all possible atomic propositions is denoted by $\mathcal{P}$.

In the following, given two atomic propositions $p, q \in \mathcal{P}$, we introduce the LTL formulas studied in this paper:

- $p\mathcal{U}_s q$: A solution ϕ to $\mathcal{H}$ satisfies $p\mathcal{U}_s\, q$ if either $q(\phi(0, 0)) = 1$; or,
  - there exists $(t^\star, j^\star) \in \mathrm{dom}\, \phi$ such that $t^\star + j^\star > 0$, $q(\phi(t^\star, j^\star)) = 1$, and $p(\phi(s, k)) = 1$ for all $(s, k) \in \mathrm{dom}\, \phi$ such that $0 \leq s + k < t^\star + j^\star$.
- $p\mathcal{U}_w q$: A solution ϕ to $\mathcal{H}$ satisfies $p\mathcal{U}_w\, q$ if either ϕ satisfies $p\mathcal{U}_s\, q$; or $p(\phi(s, k)) = 1$ for all $(s, k) \in \mathrm{dom}\, \phi$ such that $s + k \geq 0$.

Remark 2.2. *The formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if, for each maximal solution ϕ to $\mathcal{H}$ with $p(\phi(0, 0)) + q(\phi(0, 0)) \geq 1$,*[1] *$p\mathcal{U}_s q$ is satisfied. Similarly, the formula $p\mathcal{U}_w q$ is said to be satisfied for $\mathcal{H}$ if, for each maximal solution ϕ to $\mathcal{H}$ with $p(\phi(0, 0)) + q(\phi(0, 0)) \geq 1$, $p\mathcal{U}_w q$ is satisfied.*

B. Set Invariance and Attractivity Notions

Given the sets $K \subset \mathbb{R}^n$, $\mathcal{X}_o \subset K$, $\mathcal{O} \subset \mathrm{cl}(C) \cup D$, and $\mathcal{A} \subset \mathbb{R}^n$, we introduce the following notions:[2]

- **Forward (pre-)invariance.** A set K is *forward pre-invariant* for $\mathcal{H}$ if, for each solution ϕ starting from K, $\mathrm{rge}\, \phi \subset K$. It is *forward invariant* for $\mathcal{H}$ if it is forward pre-invariant for $\mathcal{H}$ and every maximal solution ϕ starting from K is complete.
- **Conditional invariance.** The set K is CI with respect to the set $\mathcal{X}_o$ for $\mathcal{H}$ if, for each solution ϕ starting from $\mathcal{X}_o$, $\mathrm{rge}\, \phi \subset K$.
- **ECI.** The set $\mathcal{A}$ is ECI with respect to $\mathcal{O}$ for $\mathcal{H}$ if, for each maximal solution ϕ starting from $\mathcal{O}$, there exists a hybrid time $(t^\star, j^\star) \in \mathrm{dom}\, \phi$ such that $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \mathrm{dom}\, \phi$ such that $t + j \geq t^\star + j^\star$.
- **Pre-ECI.** The set $\mathcal{A}$ is pre-ECI with respect to $\mathcal{O}$ for $\mathcal{H}$ if, for each complete solution ϕ starting from $\mathcal{O}$, there exists a hybrid time $(t^\star, j^\star) \in \mathrm{dom}\, \phi$ such that $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \mathrm{dom}\, \phi$ such that $t + j \geq t^\star + j^\star$.
- **FTA.** The closed set $\mathcal{A}$ is *finite-time attractive (FTA)* with respect to $\mathcal{O}$ for $\mathcal{H}$ if, for each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, $\mathcal{T}_{\mathcal{A}}(\phi) < \infty$.
- **Pre-FTA.** The closed set $\mathcal{A}$ is *pre-finite-time attractive (pre-FTA)* with respect to $\mathcal{O}$ for $\mathcal{H}$ if, for each complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, $\mathcal{T}_{\mathcal{A}}(\phi) < \infty$.

[1] Since the functions p and q map to $\{0, 1\}$, $p(\phi(0, 0)) + q(\phi(0, 0)) \geq 1$ implies that $p(\phi(0, 0)) = 1$ or $q(\phi(0, 0)) = 1$.

[2] Since $\mathcal{H}$ can have maximal solutions that are not complete, the pre-ECI notion requires the ECI property for complete solutions only. A similar comment applies to pre-FTA. Moreover, unlike CI, ECI of $\mathcal{A}$ with respect to $\mathcal{O}$ for $\mathcal{H}$, with $\mathcal{A} \subset \mathcal{O}$, does not imply that solutions remain in $\mathcal{O} \cup \mathcal{A}$.

Remark 2.3. *Due to space constraints, sufficient infinitesimal conditions for CI, ECI, and FTA can be found in [22, Section IV]; see also the references therein.*

III. Formulating $p\mathcal{U}q$ in terms of CI, ECI, and FTA

First, we illustrate our approach on the constrained differential inclusion, modeling the continuous-time dynamics of $\mathcal{H}$, given by

$$\dot{x} \in F(x) \qquad\qquad x \in C. \tag{3}$$

To this end, we introduce the sets

$$P := \{x \in \mathbb{R}^n : p(x) = 1\},\ \ Q := \{x \in \mathbb{R}^n : q(x) = 1\}. \tag{4}$$

Note that P and Q collect the set of points where the atomic propositions p and q are satisfied, respectively. For the purposes of this discussion, we impose that the sets C, P, and Q are closed and $P \subset C$.

By definition, when $p\mathcal{U}_w q$ is satisfied for (3), it follows that, for each solution ϕ to (3) starting from $P \cup Q$, at least one of the following properties holds:

1) The solution ϕ remains in the set P for all time.
2) The solution ϕ starts and remains in the set P up to when it reaches the set Q.
3) The solution ϕ starts from the set Q.

Based on items 1)-3), each solution to (3) starting from the set $P \backslash Q$ needs to either remain in P for all time or remain in P until reaching Q (if it happens); namely, solutions starting from $P \backslash Q$ should satisfy either item 1) or item 2). Interestingly, the satisfaction of either item 1) or item 2) can be guaranteed, in an equivalent way, via CI of $P \cup Q$ with respect to $P \backslash Q$ for the following auxiliary system:

$$\begin{cases} \dot{x} \in F(x) & x \in C \backslash Q \\ x^+ = x & x \in Q. \end{cases} \tag{5}$$

System (5) is used to characterize the behavior of system (3) outside the set Q. Furthermore, when $p\mathcal{U}_s q$ is satisfied for (3), it follows that, for each solution ϕ to (3) starting from $P \cup Q$, at least one of the following properties holds:

1) The solution ϕ starts and remains in the set P until it reaches the set Q in finite time.
2) The solution ϕ starts from the set Q.

The satisfaction of $p\mathcal{U}_s q$ requires, additionally to $p\mathcal{U}_w q$ being satisfied, that every maximal solution ϕ to (3) starting from $P \backslash Q$ actually reaches Q in finite time. When the set $P \cup Q$ is CI with respect to $P \backslash Q$ for (5), $p\mathcal{U}_s q$ is guaranteed if and only if Q is ECI with respect to the set $P \cup Q$ for the system

$$\begin{cases} \dot{x} \in F(x) & x \in (C \backslash Q) \cap P \\ x^+ = x & x \in Q. \end{cases} \tag{6}$$

Note that system (6) can be viewed as the restriction of (5) to $P \cup Q$.

Now, we extend the proposed approach to hybrid systems. To do so, we introduce the following assumption.

Assumption 3.1. *The sets C, P, and Q are closed, and $P \subset C \cup D$.*

Following (5), we introduce the auxiliary hybrid system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ given by

$$\begin{aligned} &F_w(x) := F(x) && x \in C_w := C \backslash Q, \\ &G_w(x) := \begin{cases} x & \text{if } x \in Q \\ G(x) & \text{if } x \in D \backslash Q \end{cases} && x \in D_w := D \cup Q, \end{aligned} \tag{7}$$

which is used to characterize the behavior of $\mathcal{H}$ outside the set Q. Indeed, the solutions to $\mathcal{H}$ are the solutions to $\mathcal{H}_w$ (and vice versa) up to when they reach (if they do) the set Q. We will show that having $P \cup Q$ being CI with respect to $P \backslash Q$ for $\mathcal{H}_w$ is sufficient for $p\mathcal{U}_w q$. Furthermore, when the set $P \cup Q$ is CI with respect to $P \backslash Q$ for $\mathcal{H}_w$, we show that $p\mathcal{U}_s q$ is guaranteed if Q is ECI with respect to $P \cup Q$ for the auxiliary hybrid system $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ given by

$$\begin{aligned} &F_s(x) := F(x) && x \in C_s := (C \backslash Q) \cap P \\ &G_s(x) := \begin{cases} x & \text{if } x \in Q \\ G(x) & \text{otherwise} \end{cases} && x \in D_s := (D \cap P) \cup Q. \end{aligned} \tag{8}$$

As opposed to the continuous-time case, the equivalences for constrained differential inclusions stated above Assumption 3.1 do not hold in the hybrid case.

Remark 3.2. *The hybrid system $\mathcal{H}_s$, similar to the system in* (6)*, is just the restriction of $\mathcal{H}_w$ in* (7) *to $P \cup Q$. It is easy to see that $C_s = C_w \cap (P \cup Q)$ and $D_s = D_w \cap (P \cup Q)$.*

The satisfaction of $p\mathcal{U}_w q$ for $\mathcal{H}$ can also be guaranteed by showing CI of $P \cup Q$ with respect to $P \cup Q$ (namely, forward invariance of $P \cup Q$) for $\mathcal{H}_w$. Furthermore, the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}_w$ can be guaranteed (in an equivalent way) by showing FTA of Q with respect to $P \cup Q$ for $\mathcal{H}$, instead of ECI of Q with respect to $P \cup Q$ for $\mathcal{H}_s$.

Example 3.3 (Timer)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a constantly evolving timer with the state $x \in \mathbb{R}$ and*

$$\begin{aligned} F(x) &:= 1 \qquad && \forall x \in C := [0, 1], \\ G(x) &:= 0 \qquad && \forall x \in D := [1, \infty). \end{aligned}$$

Define the atomic propositions p and q as

$$p(x) := \begin{cases} 1 & \text{if } x \in [1/2, 1] \\ 0 & \text{otherwise,} \end{cases} \qquad q(x) := \begin{cases} 1 & \text{if } x \in [1, \infty) \\ 0 & \text{otherwise,} \end{cases}$$

for each $x \in \mathbb{R}^n$. The sets P and Q in (4) *and the system $\mathcal{H}_w$ in* (7) *are given by $Q = D$, $P = [1/2, 1]$, and*

$$\begin{aligned} F_w(x) &:= 1 \qquad && \forall x \in C_w := [0, 1), \\ G_w(x) &:= x \qquad && \forall x \in D_w := D = Q. \end{aligned}$$

We notice that each solution to $\mathcal{H}_w$ starting from $P \backslash Q = [1/2, 1)$ flows in P and reaches $x = 1 \in Q$. Once a solution reaches $x = 1$, it jumps according to the jump map $G_w(x) = x$ and stays at $\{1\} \in Q$ by jumping since it cannot flow back to $P \backslash Q$. Hence, the solutions to $\mathcal{H}_w$ starting from $P \backslash Q$ never leave the set $P \cup Q$, which implies that the set $P \cup Q$ is CI with respect to $P \backslash Q$ for $\mathcal{H}_w$. Note that CI of $P \cup Q$ with respect to $P \backslash Q$ does not hold for $\mathcal{H}$ since once a solution to $\mathcal{H}$ reaches Q, it jumps outside $P \cup Q$. Therefore, the formula $f = p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$ since the solutions to $\mathcal{H}$ starting from $P \backslash Q$ remain in P until reaching the jump set $D = Q$. △

A. Sufficient Conditions for $p\mathcal{U}_w q$ using CI

The following result characterizes the satisfaction of $p\mathcal{U}_w q$ using CI for hybrid systems.

Theorem 3.4 ($p\mathcal{U}_w q$ via CI)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be given as in* (4)*, and let the system $\mathcal{H}_w$ be as in* (7)*. The formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$ if $P \cup Q$ is CI with respect to $P\backslash Q$ for $\mathcal{H}_w$.*

Proof. Suppose that $P \cup Q$ is CI with respect to $P\backslash Q$ for $\mathcal{H}_w$. We show that, for each solution ϕ to $\mathcal{H}$ such that $\phi(0,0) \in P\backslash Q$, ϕ stays in $P \cup Q$ up to when Q is reached. Indeed, let ψ be a maximal solution to $\mathcal{H}_w$ such that $\psi(t,j) = \phi(t,j)$ for all $(t,j) \in \operatorname{dom}\phi$ up to when Q is reached; such a solution ψ to $\mathcal{H}_w$ always exists since the systems $\mathcal{H}$ and $\mathcal{H}_w$ share the same data outside the set Q. Furthermore, since $P \cup Q$ is CI with respect to $P\backslash Q$ for $\mathcal{H}_w$, we conclude that $\psi(t,j) \in P \cup Q$ for all $(t,j) \in \operatorname{dom}\psi$. Therefore, $\phi(t,j) \in P \cup Q$ for all $(t,j) \in \operatorname{dom}\phi$ up to when it reaches Q, which completes the proof. □

Note that having $p\mathcal{U}_w q$ satisfied for $\mathcal{H}$ does not necessarily imply that $P \cup Q$ is CI with respect to $P\backslash Q$ for $\mathcal{H}_w$.

Example 3.5. *Let $\mathcal{H} = (C, F, D, G)$ with*

$$\begin{aligned} \dot{x} &= F(x) := 1 \qquad && x \in C := [0, \infty) \\ x^+ &= G(x) := 0 \qquad && x \in D := [1, \infty). \end{aligned}$$

Define atomic propositions p and q such that

$$p(x) := \begin{cases} 1 & \text{if } x \in [1/2, \infty) \\ 0 & \text{otherwise,} \end{cases} \quad q(x) := \begin{cases} 1 & \text{if } x \in [-1, 0] \\ 0 & \text{otherwise.} \end{cases}$$

The sets P and Q in (4) *and the data of $\mathcal{H}_w$ in* (7) *are given by $P = [1/2, \infty), Q = [-1, 0]$, and*

$$\begin{aligned} &F_w(x) := F(x) && \forall x \in C_w = (0, \infty) \\ &G_w(x) := \begin{cases} 0 & \text{if } x \in [1, \infty) \\ x & \text{if } x \in [-1, 0] \end{cases} && \forall x \in D_w = [-1, 0] \cup [1, \infty). \end{aligned}$$

Each solution to $\mathcal{H}$ starting from $P\backslash Q (= P)$ either flows in P or reaches $\{0\} \in Q$ after a jump. Hence, the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. Now, each solution to $\mathcal{H}_w$ starting from $P\backslash Q (= P)$ is also a solution to $\mathcal{H}$ up to when it reaches Q, by reaching $\{0\}$ after a jump. Once a solution to $\mathcal{H}_w$ lands on $\{0\}$, both jumps according to $x^+ = G_w(x) = x$ and flows according to $\dot{x} = F_w(x) = 1$ are allowed by the concept of solution; see [22, Definition 2.1]. In particular, the solution to $\mathcal{H}_w$ flowing from $\{0\}$ is nontrivial and leaves the set $P \cup Q$. Hence, we conclude that the satisfaction of $p\mathcal{U}_w q$ for $\mathcal{H}$ does not imply that $P \cup Q$ is CI with respect to $P\backslash Q$ for $\mathcal{H}_w$. △

B. Sufficient Conditions for $p\mathcal{U}_s q$ using $p\mathcal{U}_w q$ plus ECI

The following result characterizes the satisfaction of $p\mathcal{U}_s q$ using ECI in addition to the satisfaction of $p\mathcal{U}_w q$.

Theorem 3.6 ($p\mathcal{U}_s q$ via $p\mathcal{U}_w q$ + ECI)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be given as in* (4) *such that Assumption 3.1 holds, and let the data of $\mathcal{H}_s$ be given as in* (8)*. The formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if*

1) *the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$; and*
2) *the set Q is ECI with respect to $P \cup Q$ for $\mathcal{H}_s$.*

Proof. By definition of $\mathcal{H}_s$, if the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$ by item 1), each solution to $\mathcal{H}_s$ starting from $P\backslash Q$ remains in $P \cup Q$. Furthermore, when additionally Q is ECI with respect to $P \cup Q$ for $\mathcal{H}_s$, each maximal solution to $\mathcal{H}_s$ starting from $P\backslash Q$ remains in the set $P \cup Q$ and reaches the set Q in finite hybrid time. The proof is completed if we show that each maximal solution ϕ to $\mathcal{H}$ starting from $P\backslash Q$ stays in $P \cup Q$ for all $(t,j) \in \operatorname{dom}\phi$ such that $t + j \le \mathcal{T}_Q(\phi)$, and $\mathcal{T}_Q(\phi) < \infty$. To this end, let ϕ be a maximal solution to $\mathcal{H}$ starting from $P\backslash Q$. By item 1), ϕ remains in $P\backslash Q$ up to when it reaches Q (if that ever happens). Next, since both $\mathcal{H}$ and $\mathcal{H}_s$ share the same data on $P\backslash Q$, there always exists a solution ψ to $\mathcal{H}_s$ such that $\psi(t,j) = \phi(t,j)$ for all $(t,j) \in \operatorname{dom}\phi$ provided that $t + j \le \mathcal{T}_Q(\phi) = \mathcal{T}_Q(\psi)$. Furthermore, by item 2), we know that $\mathcal{T}_Q(\psi) = \mathcal{T}_Q(\phi) < \infty$. Then, since we already know that $\psi(t,j) \in P \cup Q$ for all $(t,j) \in \operatorname{dom}\psi$ by item 1), we conclude that $\phi(t,j) = \psi(t,j) \in P \cup Q$ for all $(t,j) \in \operatorname{dom}\phi$ provided that $t + j \le \mathcal{T}_Q(\phi) = \mathcal{T}_Q(\psi)$; and thus, the proof is completed. □

The following example shows that having $p\mathcal{U}_s q$ satisfied for $\mathcal{H}$ does not necessarily imply that Q is ECI with respect to $P \cup Q$ for $\mathcal{H}_s$.

Example 3.7. *Consider the hybrid system $\mathcal{H}$ in Example 3.5 with p and q therein replaced by $\tilde{p}$ and $\tilde{q}$, respectively,*

$$\tilde{p}(x) := \begin{cases} 1 & \text{if } x \in [0, 1+\varepsilon] \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{q}(x) := \begin{cases} 1 & \text{if } x \in [-1, 0] \cup [1+\varepsilon, \infty) \\ 0 & \text{otherwise,} \end{cases}$$

with $0 < \varepsilon < 1$. Let P and Q be as in (4) *with $\tilde{p}$ and $\tilde{q}$ instead of p and q, respectively. The system $\mathcal{H}_s$ in* (8) *is given by*

$$\begin{aligned} &F_s(x) := F(x) && \forall x \in C_s = (0, 1+\varepsilon) \\ &G_s(x) := \begin{cases} x & \text{if } x \notin [1, 1+\varepsilon) \\ 0 & \text{otherwise} \end{cases} && \forall x \in D_s = [-1, 0] \cup [1, \infty). \end{aligned}$$

Each solution to $\mathcal{H}$ starting from $P\backslash Q = (0, 1+\varepsilon)$ either flows in P and reaches $[1+\varepsilon, \infty) \subset Q$ or reaches $\{0\} \in Q$ after a jump from $(1, 1+\varepsilon) \subset D$. Hence, the formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$. Now, we consider a solution to $\mathcal{H}$ starting from $P\backslash Q$ that reaches Q for the first time by jumping from $[1, 1+\varepsilon) \subset D$ to $\{0\}$. Such a solution is also a solution to $\mathcal{H}_s$ up to when it reaches $\{0\} \in Q$ for the first time, from where, both the jump according to $x^+ = G_s(x) = x$ and the flow according to $\dot{x} = F_s(x) = 1$ are allowed by the concept of solution; see [22, Definition 2.1]. In particular, the solution flowing from $\{0\}$ is nontrivial and leaves the set Q. Hence, we conclude that the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}$ does not necessarily imply that Q is ECI with respect to $P \cup Q$ for $\mathcal{H}_s$. △

C. Equivalence Between $p\mathcal{U}_s q$ and $p\mathcal{U}_w q$ plus FTA

The following result characterizes the satisfaction of $p\mathcal{U}_s q$ using FTA in addition to the satisfaction of $p\mathcal{U}_w q$.

Theorem 3.8 ($p\mathcal{U}_s q$ via $p\mathcal{U}_w q$ + FTA)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q,* *let the sets P and Q be as in* (4) *such that the set Q is closed and let the data of $\mathcal{H}_s$ be given in* (8)*. The formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if and only if*

1) *the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$; and*
2) *the set Q is FTA with respect to $P \cup Q$ for $\mathcal{H}_s$.*

Proof. Suppose that $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$. By definition of $p\mathcal{U}_s q$, $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. Next, we show that Q is FTA with respect to $P \cup Q$ for $\mathcal{H}_s$. To do so, we consider a maximal solution ϕ to $\mathcal{H}_s$ starting from $P \setminus Q$. Since $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$, the solution ϕ either reaches Q in finite time or remains in $P \setminus Q$. To exclude the latter case, we show that when ϕ remains in $P \setminus Q$, then ϕ is a maximal solution to $\mathcal{H}$. Indeed, assume the existence of a solution ψ to $\mathcal{H}$ that is a nontrivial extension of ϕ; namely, there exists $I \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $I \neq \emptyset$ and $\operatorname{dom} \psi = \operatorname{dom} \phi \cup I$. Note that $\psi(\operatorname{dom} \phi) = \phi(\operatorname{dom} \phi) \subset P \setminus Q$. Also, since ψ must remain in $P \setminus Q$ up to when it reaches Q, we can choose I such that $\psi(\operatorname{dom} \phi \cup I) \subset P \setminus Q$. Hence, ψ is a solution to $\mathcal{H}_s$, which contradicts the fact that ϕ is a maximal solution to $\mathcal{H}_s$. Furthermore, since $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$, we conclude that ϕ, being a maximal solution to $\mathcal{H}$, must reach Q in finite hybrid time.

Now, suppose that the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. This implies that each maximal solution ϕ to $\mathcal{H}$ remains in $P \setminus Q$ for all hybrid time; otherwise, ϕ remains in $P \setminus Q$ up to when it reaches Q in finite hybrid time. To exclude the first scenario, we note that when ϕ remains in $P \setminus Q$ for all hybrid time, it follows that ϕ is also a maximal solution to $\mathcal{H}_s$. However, by item 2), the maximal solutions to $\mathcal{H}_s$ must reach Q. □

IV. Main Results

In this section, we combine the results from Section III and the Lyapunov-like conditions for CI, ECI, and FTA that are reported in [22, Sections IV and V] to propose sufficient infinitesimal conditions certifying the formulas $p\mathcal{U}_w q$ and $p\mathcal{U}_s q$.

Our results hold under the following mild assumption[3] on the data of the hybrid system $\mathcal{H}$.

Assumption 4.1. *The map F is outer semicontinuous and locally bounded with nonempty and convex values on C, and the map G has nonempty images on D.*

A. Certifying $p\mathcal{U}_w q$ using Sufficient Conditions for CI

First, we present sufficient conditions that guarantee the satisfaction of the formula $p\mathcal{U}_w q$ by employing the sufficient conditions for CI in [22, Proposition 4.2].

Theorem 4.2 ($p\mathcal{U}_w q$ via CI)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be as in* (4) *such that Assumptions 3.1 and 4.1 hold. Then, the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$ if there exists a $\mathcal{C}^1$ function $B : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\begin{cases} B(x) \leq 0 & \forall x \in P \setminus Q \\ B(x) > 0 & \forall x \in (C \cup D) \cap (\mathbb{R}^n \setminus (P \cup Q)), \end{cases} \tag{9}$$

the set $K := \{x \in C \cup D \cup Q : B(x) \leq 0\}$ is closed, and the following hold:

1) *$\langle \nabla B(x), \eta \rangle \leq 0$ for all $x \in (C \setminus Q) \cap (U(\partial K) \setminus K)$ and all $\eta \in F(x) \cap T_{C \setminus Q}(x)$.*
2) *$B(\eta) \leq 0$ for all $x \in K \cap (D \setminus Q)$ and all $\eta \in G(x)$.*
3) *$G(x) \subset C \cup D \cup Q$ for all $x \in K \cap (D \setminus Q)$.*

Proof. Let the system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ be as in (7). Since $K = \{x \in C \cup D \cup Q : B(x) \leq 0\}$ and B satisfies $B(x) \leq 0$ for all $x \in P \setminus Q$ and $B(x) > 0$ for all $x \in (C \cup D) \setminus (P \cup Q) = (C \cup D \cup Q) \setminus (P \cup Q)$, we conclude that B is a barrier function candidate with respect to $(P \setminus Q, \mathbb{R}^n \setminus (P \cup Q))$ for $\mathcal{H}_w$ in (7); see [22, Definition 4.1]. Furthermore, item 1) implies that $\langle \nabla B(x), \eta \rangle \leq 0$ for all $x \in (U(\partial K) \setminus K) \cap C_w$ and all $\eta \in F(x) \cap T_{C_w}(x)$. Item 2) implies that $B(\eta) \leq 0$ for all $x \in K \cap (D \setminus Q)$ and all $\eta \in G_w(x)$. Furthermore, when $x \in K \cap Q$, $G_w(x) = x$ and $B(x) \leq 0$. Hence, $B(\eta) \leq 0$ for all $x \in K \cap D_w$ and all $\eta \in G_w(x)$. Item 3) implies that $G_w(K \cap (D \setminus Q)) \subset C_w \cup D_w$. Furthermore, $G_w(K \cap Q) \subset K \cap Q \subset C_w \cup D_w$. Hence, $G_w(K \cap D_w) \subset C_w \cup D_w$. Thus, using item 1) in [22, Proposition 4.2] with $\mathcal{O}$ and $\mathcal{X}_u$ therein replaced by $P \setminus Q$ and $\mathbb{R}^n \setminus (P \cup Q)$, respectively, we conclude that $P \cup Q$ is CI with respect to $P \setminus Q$ for $\mathcal{H}_w$. Hence, using Theorem 3.4, we conclude that $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. □

B. Certifying $p\mathcal{U}_s q$ using ECI via Flows and Jumps

In this section, we present sufficient conditions to guarantee the satisfaction of the formula $p\mathcal{U}_s q$ by using the sufficient conditions for ECI in [22, Theorem 4.4].

Theorem 4.3 ($p\mathcal{U}_s q$ using ECI)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Let the system $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ be as in* (8)*. Given atomic propositions p and q, let the sets P and Q be as in* (4) *such that Assumptions 3.1 and 4.1 hold. Then, the formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if the following hold:*

1) *The formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$.*
2) *There exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, and a constant $r_1 > 0$ such that the following hold:*
   - 2.1) *$\langle \nabla v(x), \eta \rangle \leq f_c(v(x))$ for all $x \in \operatorname{cl}(C_s)$ and for all $\eta \in F(x) \cap T_{\operatorname{cl}(C_s)}(x)$;*
   - 2.2) *$v(\eta) \leq v(x)$ for all $x \in D \cap P$ and for all $\eta \in G(x)$;*
   - 2.3) *The solutions to $\dot{y} = f_c(y)$, starting from $v(P \setminus Q)$, converge to $(-\infty, r_1)$ in finite time.*
3) *There exist a $\mathcal{C}^1$ function $w : \mathbb{R}^n \to \mathbb{R}$, a nondecreasing function $f_d : \mathbb{R} \to \mathbb{R}$, and a constant $r_2 > 0$ such that the following hold:*
   - 3.1) *$\langle \nabla w(x), \eta \rangle \leq 0$ for all $x \in \operatorname{cl}(C_s)$ and for all $\eta \in F(x) \cap T_{\operatorname{cl}(C_s)}(x)$;*
   - 3.2) *$w(\eta) \leq f_d(w(x))$ for all $x \in D \cap P$ and for all $\eta \in G(x)$;*
   - 3.3) *The solutions to $z^+ = f_d(z)$, starting from $w(P \setminus Q)$, converge to $(-\infty, r_2)$ in finite time.*
4) *One of the following conditions holds:*
   - 4a) *Each complete solution to $\mathcal{H}$ starting from $P \setminus Q$ is eventually continuous and, with r_1 coming from item 2),*

$$S_1 := \{x \in \operatorname{cl}(C_s) : v(x) < r_1\} \subset Q. \tag{10}$$

[3] The properties of F in Assumption 4.1 reduce to just continuity when F is single valued, see [23], [24]. Note that, unlike [20], our results do not require the jump map to be outer semicontinuous and locally bounded.

4b) *Each complete solution to $\mathcal{H}$ starting from $P\backslash Q$ is eventually discrete and, with r_2 coming from item 3),*

$$S_2 := \{x \in D_s : w(x) < r_2\} \subset Q. \tag{11}$$

4c) *Each complete solution to $\mathcal{H}$ starting from $P\backslash Q$ is eventually continuous, eventually discrete, or has a hybrid time domain that is unbounded in both the t and the j direction and, with r_1 and r_2 coming from item 2) and item 3) respectively,* (10) *and* (11) *hold.*

4d) *With r_1 and r_2 coming from item 2) and item 3) respectively,* (10) *and* (11) *hold, and $G(S_2) \cap \text{cl}(C_s) \subset S_1$.*

5) *No maximal solution to $\mathcal{H}$ has a finite escape time in $(P\backslash Q) \cap C$.*

6) *Every maximal solution to $\mathcal{H}$ from $((P\backslash Q) \cap \partial C)\backslash D$ is nontrivial.*

Proof. By item 1), every maximal solution to $\mathcal{H}$ starting from $P \cup Q$ satisfies $p\mathcal{U}_w q$. Hence, it remains to show that every maximal solution to $\mathcal{H}$ starting from $P\backslash Q$ reaches Q. To this end, note that each maximal solution ϕ to $\mathcal{H}$ from $P\backslash Q$ must satisfy one of the following conditions:

a) ϕ reaches Q in finite hybrid time;
b) ϕ is not complete and does not reach Q in finite hybrid time; or
c) ϕ is complete and does not reach Q in finite hybrid time.

In the rest of the proof, we show that ϕ can only satisfy case a). To do so, we first show that case b) is not possible, due to items 5) and 6), using contradiction. That is, suppose ϕ is not complete and never reaches Q; in particular, $\text{dom}\,\phi$ is bounded. Let $(T, J) = \sup \text{dom}\,\phi$. Due to the fact that ϕ never reaches Q and since ϕ satisfies $p\mathcal{U}_w q$, we conclude that ϕ remains in $P\backslash Q$. Moreover, under item 5), the maximal solution ϕ does not have a finite escape time inside $(P\backslash Q)\cap C$, which implies that $(T, J) \in \text{dom}\,\phi$. Now, by the definition of solutions to $\mathcal{H}$, $\phi(T, J) \in \text{cl}(C) \cup D$. First, let $\phi(T, J) \in D$. In this case, ϕ can be extended via a jump. Next, let $\phi(T, J) \in \text{cl}(C)\backslash D$. In this case, when $\phi(T, J) \in \text{int}(C)\backslash D$, we use Assumption 4.1 to conclude that ϕ can be extended via flow; and for the case when $\phi(T, J) \in \partial C\backslash D$, we use item 6) to conclude that ϕ can be extended via flow. Therefore, if $(T, J) \in \text{dom}\,\phi$, then ϕ can be extended via flow or a jump. This contradicts maximality of ϕ; and thus, case b) is not possible.

Next, we show that case c) is not possible due to items 2)-4) using contradiction. Suppose that items 2), 3), and 4a) hold. Suppose that there exists a complete solution ϕ to $\mathcal{H}$ that does not reach Q in finite hybrid time. By [22, Lemma 6.4], ϕ is also a maximal solution to $\mathcal{H}_s$. However, using the arguments in a) in the proof of [22, Theorem 4.4], there must exist $(t^\star, j^\star) \in \text{dom}\,\phi$ such that $\phi(t, j) \in S_1 \subset Q$ for all $(t, j) \in \text{dom}\,\phi$ and $t + j \geq t^\star + j^\star$. This implies that ϕ must reach Q in finite hybrid time via flow. Next, suppose that items 2), 3), and 4b) hold. Proceeding as when 4a) holds, we use [22, Lemma 6.4] to conclude that ϕ is a maximal solution to $\mathcal{H}_s$. Furthermore, using the arguments in b) in the proof of [22, Theorem 4.4], we conclude the existence of $(t^\star, j^\star) \in \text{dom}\,\phi$ such that $\phi(t, j) \in S_2 \subset Q$ for all $(t, j) \in \text{dom}\,\phi$ and $t + j \geq t^\star + j^\star$. This implies that ϕ must reach Q in finite hybrid time by jumps. Similarly, suppose that items 2) and 3) hold and either item 4c) or item 4d) holds. Using [22, Lemma 6.4] and the arguments in the proof of [22, Theorem 4.4], we conclude that there exists $(t^\star, j^\star) \in \text{dom}\,\phi$ such that $\phi(t, j) \in S_1 \cup S_2 \subset Q$ for all $(t, j) \in \text{dom}\,\phi$ and $t + j \geq t^\star + j^\star$. This implies that ϕ must reach Q in finite hybrid time via flow or jumps. Therefore, we conclude that case c) is not possible. □

Remark 4.4. *We note that sufficient conditions for the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}$ do not require solutions to $\mathcal{H}$ to stay in the target set Q after reaching it. Hence, when sufficient conditions that guarantee ECI are employed to derive sufficient conditions for strong until, item 4 in Theorem 4.3 can be relaxed since item 4 is for guaranteeing solutions to $\mathcal{H}$ to stay in the target set Q after reaching it. In particular, $G(S_2) \cap \text{cl}(C_s) \subset S_1$, as in item 4b in Theorem 4.3, is not really needed since it requires solutions to $\mathcal{H}$ to stay in Q after reaching $S_2 \subset Q$. Note that the properties of solutions in items 4c, 5, and 6 can be checked without solution information using results in [20] and [25].*

The following example illustrates Theorem 4.3.

Example 4.5 (Thermostat). *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a controlled thermostat system. The variable h denotes the state of the heater, i.e., $h=1$ implies the heater is on and $h=0$ implies the heater is off. The variable z is the room temperature, z_o denotes the temperature outside the room, and $z_\triangle$ denotes the capacity of the heater to raise the temperature such that $z_o < z_{\min} < z_{\max} < z_o + z_\triangle$. The system $\mathcal{H}$ with the state $x := (h, z) \in \{0, 1\} \times \mathbb{R}$ is given by*

$$F(x) := \begin{bmatrix} 0 & -z + z_o + z_\triangle h \end{bmatrix}^\top \quad \forall x \in C := C_0 \cup C_1$$
$$G(x) := \begin{bmatrix} 1 - h & z \end{bmatrix}^\top \quad \forall x \in D := D_0 \cup D_1,$$

where $C_0 := \{x \in \mathbb{R}^2 : h = 0, z \geq z_{\min}\}$, $C_1 := \{x \in \mathbb{R}^2 : h = 1, z \leq z_{\max}\}$, $D_0 := \{x \in \mathbb{R}^2 : h = 0, z \leq z_{\min}\}$, and $D_1 := \{x \in \mathbb{R}^2 : h = 1, z \geq z_{\max}\}$. Define, for each $x \in \mathbb{R}^2$, the atomic propositions p and q as

$$p(x) := \begin{cases} 1 & \text{if } x \in \{1\} \times (-\infty, z_{\max}] \\ 0 & \text{otherwise,} \end{cases}$$
$$q(x) := \begin{cases} 1 & \text{if } x \in \{0\} \times [z_{\min}, +\infty) \\ 0 & \text{otherwise.} \end{cases}$$

The sets P and Q in (4) *are given by $P = \{1\} \times (-\infty, z_{\max}]$ with $z_{\max} > 0$ and $Q = \{0\} \times [z_{\min}, \infty)$. To show the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}$, we apply Theorem 4.3. First, consider the barrier function candidate $B(x) := (-1)^h(z_{\max} - z)$. Indeed, B is a barrier function candidate with respect to $(P\backslash Q, (\{0, 1\} \times \mathbb{R})\backslash(P \cup Q))$ for $\mathcal{H}$ since for all $x \in P\backslash Q = P$, $B(x) = z - z_{\max} \leq 0$; and for all $x \in (C \cup D)\backslash(P \cup Q) = (\{0\} \times (-\infty, z_{\min})) \cup (\{1\} \times (z_{\max}, \infty))$, $B(x) > 0$. Moreover, for all $x \in C\backslash Q = \{1\} \times (-\infty, z_{\max}]$, $B(x) = z - z_{\max} \leq 0$; and thus, $(C\backslash Q) \cap (U(\partial K)\backslash K) = \emptyset$. Furthermore, for all $x \in K \cap (D\backslash Q) = \{(1, z_{\max})\}$, $G(x) = \{(0, z_{\max})\} \subset C \cup D \cup Q$ and $B(G(x)) = 0$; hence, items 1) - 3) in Theorem 4.2 hold. It follows that the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$; and thus, item 1) holds. Next, consider the functions $v(x) = -z + z_o + z_\triangle$ and $f_c(y) = -y$. Recall that $z_{\max} < z_o + z_\triangle$. For all $x \in \text{cl}(C_s) = \{1\} \times (-\infty, z_{\max}]$, $\langle \nabla v(x), F(x) \rangle = z - z_o - z_\triangle \leq$*

$f_c(v(x)) = z - z_o - z_\triangle$; hence, item 2.1) holds. Moreover, for all $x \in D \cap P = \{(1, z_{\max})\}$, $v(G(x)) - v(x) = 0$; hence, item 2.2) holds. Furthermore, the solutions to $\dot{y} = f_c(y)$ from $v(P\backslash Q) = [z_o + z_\triangle - z_{\max}, \infty)$ reach $(-\infty, z_o + z_\triangle - z_{\max})$; and thus, item 2.3) holds for $r_1 = z_o + z_\triangle - z_{\max}$. Moreover, $S_1 = \{x \in \mathrm{cl}(C_s) : v(x) < z_o + z_\triangle - z_{\max}\}$ is empty. On the other hand, for the functions $w(x) = -z + z_{\max}$ and $f_d(z) = z/2$, for all $x \in \mathrm{cl}(C_s)$, $\langle \nabla w(x), F(x)\rangle = z - z_o - z_\triangle < 0$ since $z_{\max} < z_o + z_\triangle$; and for all $x \in D \cap P = \{(1, z_{\max})\}$, $w(G(x)) = 0 = f_c(w(x))$. Hence, we conclude that items 3.1) and 3.2) hold. Moreover, the solutions to $z^+ = f_d(z)$ starting from $w(P\backslash Q) = (0, \infty)$ reach $(-\infty, z_{\max} - z_{\min})$; and thus, item 3.3) holds for $r_2 = z_{\max} - z_{\min}$. Moreover, every complete solution to $\mathcal{H}_s$ is eventually discrete due to the jump map G_s and $S_2 = \{0\} \times (z_{\min}, \infty) \subset Q$. Hence, with S_1 and S_2 satisfying (10) *and* (11)*, item 4c) holds. Furthermore, since the flow map F has global linear growth on $(P\backslash Q) \cap C$, the solutions to $\mathcal{H}$ do not have a finite escape time inside $(P\backslash Q) \cap C$; hence, item 5) holds. Finally, since $(P\backslash Q) \cap \partial C = \emptyset$, item 6) holds. Thus, we conclude that $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$.* △

C. Certifying $p\mathcal{U}_s q$ using ECI via Flows

The following result follows from [22, Proposition 4.9], and considers the case where the set Q is reached by flows only.

Theorem 4.6 ($p\mathcal{U}_s q$ using ECI via flows)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be as in* (4) *such that Assumptions 3.1 and 4.1 hold. Let C_s and D_s be as in* (8)*. Then, the formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if the following hold:*

1) *The formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$.*
2) *There exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, and a constant $r_1 > 0$ such that the following hold:*
   - 2.1) *$\langle \nabla v(x), \eta\rangle \le f_c(v(x))$ for all $x \in \mathrm{cl}(C_s)$ and for all $\eta \in F(x) \cap T_{\mathrm{cl}(C_s)}(x)$;*
   - 2.2) *$v(\eta) \le v(x)$ for all $x \in D \cap P$ and for all $\eta \in G(x)$;*
   - 2.3) *The solutions $\dot{y} = f_c(y)$ starting from $v(P\backslash Q)$ converge to $(-\infty, r_1)$ in finite time, and $S_1 := \{x \in \mathrm{cl}(C_s) : v(x) < r_1\} \subset Q$.*
3) *For each solution $\phi \in \mathcal{S}_{\mathcal{H}}(P\backslash Q)$, there exists a solution y to $\dot{y} = f_c(y)$ starting from $v(\phi(0,0))$ such that there exists $t^\star \ge 0$ satisfying:*

$$t^\star \le \sup\{t : (t,j) \in \mathrm{dom}\,\phi\},\ y(t) \in (-\infty, r_1)\ \forall t \ge t^\star. \quad (12)$$

Proof. Consider the system $\mathcal{H}_s$ introduced in (8). Using [22, Theorem 3.6], we show that Q is ECI with respect to $P\backslash Q$ for $\mathcal{H}_s$ to conclude that $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$. To this end, we show that Q is ECI with respect to $P\backslash Q$ for $\mathcal{H}_s$ via [22, Theorem 4.16]. First, we show that items 1) and 3) in [22, Proposition 4.9], required in [22, Theorem 4.16], hold for $\mathcal{H}_s$. Notice that under item 2), item 1) in [22, Proposition 4.9] holds for $\mathcal{H}_s$. Moreover, item 3) in [22, Proposition 4.9] is verified for $\mathcal{H}_s$ since solutions jumping from Q remain in Q due to the definition of the jump map G_s, which is $G_s(x) = x$ for all $x \in Q$. Finally, to show that Q is ECI with respect to $P\backslash Q$ for $\mathcal{H}_s$ via [22, Theorem 4.16], we show that, for each maximal solution ϕ to $\mathcal{H}_s$ starting from $P\backslash Q$, there exists a solution y to $\dot{y} = f_c(y)$ starting from $v(\phi(0,0))$ satisfying $y(t) \in (-\infty, r_1]$ for all $t \ge t^\star$, for some nonnegative $t^\star \le \sup\{t : (t,j) \in \mathrm{dom}\,\phi\}$. To show this, we first use item 1) and the construction of $\mathcal{H}_s$, to conclude that, each maximal solution ϕ to $\mathcal{H}_s$ starting from $P\backslash Q$ remains in $P \cup Q$. Hence, either ϕ reaches Q in finite time, or ϕ remains in $P\backslash Q$. Next, by [22, Lemma 6.4], we conclude that ϕ is a maximal solution to $\mathcal{H}$. Finally, using item 3), we conclude the existence of a solution y to $\dot{y} = f_c(y)$ starting from $v(\phi(0,0))$ such that, for some $t^\star \ge 0$, (12) holds; and thus, we conclude that Q is ECI with respect to $P\backslash Q$ for $\mathcal{H}_s$ via [22, Theorem 4.16]. Therefore, via [22, Theorem 3.6], the formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$, which completes the proof. □

D. Certifying $p\mathcal{U}_s q$ using ECI via Jumps

The following result follows [22, Proposition 4.10], and considers the case where the set Q is reached by jumps only.

Theorem 4.7 ($p\mathcal{U}_s q$ using ECI via jumps)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be as in* (4) *such that Assumptions 3.1 and 4.1 hold. Let C_s and D_s be as in* (8)*. Then, the formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if the following hold:*

1) *The formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$.*
2) *There exist a $\mathcal{C}^1$ function $w : \mathbb{R}^n \to \mathbb{R}$, $f_d : \mathbb{R} \to \mathbb{R}$ which is nondecreasing, and a constant $r_2 > 0$ such that the following hold:*
   - 2.1) *$\langle \nabla w(x), \eta\rangle \le 0$ for all $x \in \mathrm{cl}(C_s)$ and for all $\eta \in F(x) \cap T_{\mathrm{cl}(C_s)}(x)$;*
   - 2.2) *$w(\eta) \le f_d(w(x))$ for all $x \in D \cap P$ and for all $\eta \in G(x)$;*
   - 2.3) *The solutions to $z^+ = f_d(z)$ starting from $w(P\backslash Q)$ converge to $(-\infty, r_2)$ in finite time, and $S_2 := \{x \in D_s \cup \mathrm{cl}(C_s) : w(x) < r_2\} \subset Q$.*
3) *For each solution $\phi \in \mathcal{S}_{\mathcal{H}}(P\backslash Q)$, there exists a solution z to $z^+ = f_d(z)$ starting from $v(\phi(0,0))$ such that there exists $j^\star \in \mathbb{N}$ satisfying:*

$$j^\star \le \sup\{j : (t,j) \in \mathrm{dom}\,\phi\},\ z(j) \in (-\infty, r_2)\ \forall j \ge j^\star.$$

Proof. The proof follows the exact same steps used to prove Theorem 4.6 while using [22, Theorem 4.17] instead of [22, Theorem 4.16]. □

Remark 4.8. *When the lengths of the flow interval between successive jumps is approximately known, we can employ the conditions for pre-ECI in [22, Theorem 4.14], that use single scalar Lyapunov-like function; see [22, Theorem 6.8].*

E. Certifying $p\mathcal{U}_s q$ using FTA via Flows and Jumps

Along the lines of Remark 4.4, we propose sufficient conditions that guarantee the satisfaction of $p\mathcal{U}_s q$ using sufficient conditions for FTA in [22, Theorem 5.1].

Theorem 4.9 ($p\mathcal{U}_s q$ using FTA)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q as in* (4) *be such that Assumptions 3.1 and 4.1 hold. For $\mathcal{N}$ an open neighborhood of Q, we suppose that there exist functions $V : \mathcal{N} \to \mathbb{R}_{\ge 0}$ and $W : \mathcal{N} \to \mathbb{R}_{\ge 0}$ that* *are positive definite with respect to Q and such that $P\backslash Q \subset L_V(r) \cap (C \cup D) \subset \mathcal{N}$ and $P\backslash Q \subset L_W(r) \cap (C \cup D) \subset \mathcal{N}$, for some $r > 0$. Then, the formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if the following hold:*

1) *The formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$.*
2) *There exist constants $c_1 > 0$ and $c_2 \in [0, 1)$ such that*

$$\langle \nabla V(x), \eta \rangle \leq -c_1 V^{c_2}(x) \quad \forall x \in (C \cap \mathcal{N} \cap P)\backslash Q, \forall \eta \in F(x) \cap T_C(x), \tag{13}$$

$$V(\eta) - V(x) \leq 0 \;\; \forall x \in (D \cap \mathcal{N} \cap P)\backslash Q, \forall \eta \in G(x).$$

3) *There exists a constant $c > 0$ such that*

$$\langle \nabla W(x), \eta \rangle \leq 0 \quad \forall x \in (C \cap \mathcal{N} \cap P)\backslash Q, \forall \eta \in F(x) \cap T_C(x),$$

$$W(\eta) - W(x) \leq -\min\{c, W(x)\} \quad \forall x \in (D \cap \mathcal{N} \cap P)\backslash Q, \quad \forall \eta \in G(x).$$

4) *No maximal solution to $\mathcal{H}$ has a finite escape time in $(P\backslash Q) \cap C$.*
5) *Every maximal solution to $\mathcal{H}$ from $((P\backslash Q) \cap \partial C)\backslash D$ is nontrivial.*

Proof. Consider the system $\mathcal{H}_s$ introduced in (8). Using [22, Theorem 5.1] for $\mathcal{H}_s$ under items 2) and 3), we conclude that Q is pre-FTA with respect to $P\backslash Q$ for $\mathcal{H}_s$. Next, we show that $P \cup Q$ is forward invariant for $\mathcal{H}_s$ exactly as we did in the proof of Theorem 4.3. Thus, using Theorem [22, Theorem 5.10], we conclude that Q is FTA with respect to $P\backslash Q$ for $\mathcal{H}_s$. Finally, the proof is completed using Theorem 3.8. □

Example 4.10 (Thermostat)**.** *Consider the hybrid system in Example 4.5 with the state $x := (h, z) \in \{0, 1\} \times \mathbb{R}$. Let the propositions p and q given as in Example 4.5. To show the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}$, we apply Theorem 4.9. We already showed in Example 4.5 that the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$; and thus, item 1) in Theorem 4.9 holds. Next, consider the functions $V(x) = W(x) = z_{\min} - z$. For all $x \in (C \cap P)\backslash Q = \{1\} \times (-\infty, z_{\max}]$, $\langle \nabla V(x), F(x) \rangle = z - z_o - z_\triangle \leq z_{\max} - z_o - z_\triangle$. Moreover, for all $x \in (D \cap P)\backslash Q = \{(1, z_{\max})\}$, $V(G(x)) - V(x) = 0$. Hence, item 2) holds for $c_1 = z_{\max} - z_o - z_\triangle$ and $c_2 = 0$. On the other hand, for all $x \in (C \cap P)\backslash Q$, $\langle \nabla W(x), F(x) \rangle = z - z_o - z_\triangle < 0$ since $z_o + z_\triangle > z_{\max}$; and for all $x \in (D \cap P)\backslash Q = \{(1, z_{\max})\}$, $W(G(x)) - W(x) = 0 < -W(x)$. Hence, we conclude that item 3) holds for $c = z_{\max} - z_{\min}$. Furthermore, since the flow map F has global linear growth on $(P\backslash Q) \cap C$, the solutions to $\mathcal{H}$ do not have a finite escape time inside $(P\backslash Q) \cap C$; hence, item 4) holds. Finally, since $(P\backslash Q) \cap \partial C = \emptyset$, item 5) holds. Thus, Theorem 4.9 implies that $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$.* △

Remark 4.11. *As in Theorems 4.6 and 4.7, we can formulate sufficient conditions for $p\mathcal{U}_s q$ using sufficient conditions for FTA via flows, or for FTA via jumps. Due to space constraints, those results are reported to [22, Theorems 6.13 and 6.14].*

V. Conclusion

Lyapunov-like techniques are introduced to certify the $p\mathcal{U}_w q$ and $p\mathcal{U}_s q$ formulas for hybrid systems. In the first place, sufficient and equivalence relationships are established between the satisfaction of the considered formulas and specific invariance and attractivity notions, such as CI, ECI, and FTA. Then, using sufficient infinitesimal conditions guaranteeing the aforementioned invariance and attractivity notions, sufficient infinitesimal conditions for the satisfaction of the $p\mathcal{U}_w q$ and $p\mathcal{U}_s q$ formulas are deduced. Future research direction includes analyzing more complex specifications, where the until operator is involved in addition to other operators. In particular, with the proposed tools, more complex formulae can be certified through decomposition by building a finite state automaton; see [12, Section 6.5] and the references therein.

References

[1] J. G. Thistle and W. Wonham, "Control problems in a temporal logic framework," *International Journal of Control*, vol. 44, no. 4, 1986.

[2] A. Girard and G. J. Pappas, "Verification using simulation," in *International Workshop on Hybrid Systems: Computation and Control*. Springer, 2006, pp. 272–286.

[3] T. Wongpiromsarn, U. Topcu, and R. M. Murray, "Receding horizon temporal logic planning," *IEEE Transactions on Automatic Control*, vol. 57, no. 11, pp. 2817–2830, 2012.

[4] Y. M. Kwon and G. Agha, "LTLC: Linear temporal logic for control," *Hybrid Systems: Computation and Control*, pp. 316–329, 2008.

[5] N. Giorgetti, G. J. Pappas, and A. Bemporad, "Bounded model checking of hybrid dynamical systems," in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 672–677.

[6] C. Baier and J.-P. Katoen, *Principles of model checking*. MIT press, 2008.

[7] P. Tabuada, *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media, 2009.

[8] C. Belta, Y. ordanov, and E. A. Gol, *Formal methods for discrete-time dynamical systems*. Springer, 2017, vol. 89.

[9] A. Platzer, *Logical analysis of hybrid systems: proving theorems for complex dynamics*. Springer Science & Business Media, 2010.

[10] A. Bisoffi and D. V. Dimarogonas, "Satisfaction of linear temporal logic specifications through recurrence tools for hybrid systems," *IEEE Transactions on Automatic Control*, 2020.

[11] T. Wongpiromsarn, U. Topcu, and A. Lamperski, "Automata theory meets barrier certificates: Temporal logic verification of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, 2015.

[12] H. Han and R. G. Sanfelice, "Linear temporal logic for hybrid dynamical systems: Characterizations and sufficient conditions," *Nonlinear Analysis: Hybrid Systems*, vol. 36, p. 100865, 2020.

[13] E. A. Emerson, "Temporal and modal logic," in *Formal Models and Semantics*. Elsevier, 1990, pp. 995–1072.

[14] C. Eisner, D. Fisman, and J. Havlicek, "A topological characterization of weakness," in *Proceedings of the twenty-fourth annual ACM symposium on Principles of distributed computing*, 2005, pp. 1–8.

[15] N. Piterman and A. Pnueli, "Temporal logic and fair discrete systems," in *Handbook of Model Checking*. Springer, 2018, pp. 27–73.

[16] M. Maghenem and R. G. Sanfelice, "Sufficient conditions for forward invariance and contractivity in hybrid inclusions using barrier functions," *Automatica*, p. 109328, 2020.

[17] S. Prajna, A. Jadbabaie, and G. J. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, 2007.

[18] G. S. Ladde and S. Leela, "Analysis of invariant sets," *Annali di Matematica Pura ed Applicata*, vol. 94, no. 1, pp. 283–289, 1972.

[19] H. Han, M. Maghenem, and R. G. Sanfelice, "Sufficient conditions for satisfaction of formulas with until operators in hybrid systems," in *Proceedings of the 23rd International Conference on Hybrid Systems: Computation and Control*, 2020, pp. 1–10.

[20] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[21] Y. Li and R. G. Sanfelice, "Finite time stability of sets for hybrid dynamical systems," *Automatica*, vol. 100, pp. 200–211, 2019.

[22] H. Han, M. Maghenem, and R. G. Sanfelice, "Certifying the LTL formula p until q in hybrid systems," *arXiv preprint arXiv:2106.06455*, 2021. [Online]. Available: https://arxiv.org/pdf/2106.06455.pdf

[23] J. P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*. Springer Science & Business Media, 2012, vol. 264.

[24] J. P. Aubin, *Viability Theory*. Birkhauser Boston Inc., 1991.

[25] R. G. Sanfelice, *Hybrid Feedback Control*. New Jersey: Princeton University Press, 2021.