On robust, global stabilization of the attitude of an underactuated rigid body using hybrid feedback

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***Abstract*— A hybrid feedback is developed for robust, global stabilization of the attitude of an underactuated rigid body. For the case where two angular velocities are considered as controls, this objective is achieved. For the case where the two controls are torques, the objective is achieved in a "practical" sense, i.e., robust, global asymptotic stability of some arbitrarily small neighborhood of the desired attitude is achieved. To assist with the exposition, a robustly, globally stabilizing hybrid controller is also developed for the case where three angular velocities are considered as controls. This solution provides an alternative to one that has appeared recently in the literature.**

I. Introduction

The main contribution of this paper is the description of a hybrid feedback for robust, global stabilization of the point $\xi^* := (0,0,0,1)^\top$ for the nonlinear control system

$$\dot{\xi} = W(\omega)\xi \qquad \xi \in C_0 := \left\{\xi \in \mathbb{R}^4 : \xi^\top \xi = 1\right\} \qquad (1)$$

$$W(\omega) = \frac{1}{2}\begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \qquad (2)$$

using feedback controls ω in the set

$$\Omega_{12} := \left\{\omega \in \mathbb{R}^3 : \omega_3 = \omega_1\omega_2 = 0\ ,\ \omega^\top\omega \le 1\ \right\}\ . \qquad (3)$$

Notice that any control choice renders the set C_0 invariant since $W(\omega) + W^\top(\omega) = 0$ for all $\omega \in \mathbb{R}^3$.

We are motivated by the work in [4] which provides what is apparently the first hybrid feedback (in that case, a sample-and-hold controller) for global stabilization of ξ^* for the system (1). The feedback in [4] uses controls in the set

$$\Omega_{123} := \left\{\omega \in \mathbb{R}^3 : \omega_1\omega_2 = \omega_2\omega_3 = \omega_3\omega_1 = 0\ ,\ \omega^\top\omega \le 1\ \right\} \qquad (4)$$

(notice that Ω_{12} is a strict subset of Ω_{123}) which facilitates extending the solution to the stabilization problem for the system (1)-(2) augmented with the dynamics

$$\left.\begin{array}{rcl} \dot{\omega}_1 & = & u_1 \\ \dot{\omega}_2 & = & u_2 \\ \dot{\omega}_3 & = & a\omega_1\omega_2 \end{array}\right\} =: f(\omega) + Bu \qquad (5)$$

using feedback controls u. It is assumed that $a \neq 0$.

Rigid body dynamics motivate the control system (1)-(2), (5). The system (1)-(2) corresponds to the kinematics of the rigid body, expressed in terms of a unit quaternion, and ξ^* corresponds to the desired attitude of the rigid body. The vector ω corresponds to angular velocities, and the equations (5) correspond to angular velocity dynamics

$$\begin{array}{rcl} \mathcal{J}_x\dot{\omega}_1 - (\mathcal{J}_y - \mathcal{J}_z)\omega_2\omega_3 & = & \mathcal{M}_x \\ \mathcal{J}_y\dot{\omega}_2 - (\mathcal{J}_z - \mathcal{J}_x)\omega_3\omega_1 & = & \mathcal{M}_y \\ \mathcal{J}_z\dot{\omega}_3 - (\mathcal{J}_x - \mathcal{J}_y)\omega_1\omega_2 & = & \mathcal{M}_z \end{array}$$

(where the moments of inertia $\mathcal{J}_x$, $\mathcal{J}_y$, and $\mathcal{J}_z$ are positive scalars) in the case of two input torques such that $\mathcal{M}_z \equiv 0$, and after an input transformation from $(\mathcal{M}_x, \mathcal{M}_y)$ to u. The condition $a \neq 0$ corresponds to the condition $\mathcal{J}_x \neq \mathcal{J}_y$.

The stabilization problem for underactuated angular velocity dynamics has been studied in several papers, notably [1], [12], [10] which consider the case of one input torque. The attitude stabilization for the case of two input torques has been studied in [6], [2], [8], [9], [13], [5], [4]. For additional discussions about the attitude control problem, see [14].

In the paper [4], the authors address the attitude stabilization problem with two input torques by first solving the kinematic stabilization problem globally using controls in the set Ω_{123} and then using the controls u in the system (5) to regulate the variables ω to their desired values. We also first address the kinematic stabilization problem globally, but we consider the problem with controls in Ω_{12} rather than Ω_{123}. Such a solution is useful for the situation where $a \neq 0$ is very small in magnitude so that it is significantly more difficult to change ω_3 than it is to change ω_1 or ω_2. Moreover, the solution that we provide will also succeed in regulating the attitude when $a = 0$ and ω_3 starts at zero.

In addition to a solution to the kinematic stabilization problem with controls in Ω_{12}, we will also give a global hybrid feedback controller using controls in Ω_{123}. This control law can serve as an alternative to the algorithm proposed in [4], which is based on using sample-and-hold control. We will present the solution to the kinematic control problem with controls in Ω_{123} first since many of the ideas in that solution appear again in the solution to the kinematic control problem with controls in the set Ω_{12}.

The paper is organized as follows: In Section II we review hybrid systems and the notion of global asymptotic stability for said systems. In Section III we solve the robust, global stabilization problem for the kinematics using controls in the set Ω_{123}. In Section IV we solve this problem using controls in the set Ω_{12}. In Section V we show how to use the solution to the kinematic control problem with controls in Ω_{12} to solve the robust, global stabilization problem for the system (1)-(2) together with the dynamical equation $\dot{\omega}_3 = a\omega_1\omega_2$ where $a \neq 0$ using controls (ω_1, ω_2). In Section VI, we discuss the application of our ideas to the problem of robust, global stabilization of an underactuated rigid body.

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II. HYBRID SYSTEMS AND ASYMPTOTIC STABILITY

A. Well-posed hybrid systems

Our treatment of hybrid systems follows that given in [7]. The emphasis in [7] is on a framework that guarantees that asymptotic stability in hybrid systems is robust to small perturbations, whether they come from parameter variations, external disturbances or measurement noise. Here, we call such systems "well-posed" hybrid systems. For the purposes of this paper, a well-posed hybrid system is written as

$$\begin{array}{rcll} \dot{x} & = & f(x) & x \in C \\ x^+ & \in & G(x) & x \in D \end{array} \tag{6}$$

where (see [7] for more details)

- the sets $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are closed,
- $f : C \to \mathbb{R}^n$ is continuous,
- the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer-semicontinuous, locally bounded, and nonempty on D.

Solutions to the hybrid system (6) are defined on hybrid time domains, which are subsets of the reals times the nonnegative integers [7]. The system (6) is said to be *forward complete and non-Zeno* from $C \cup D$ if every maximal solution to (6) starting in $C \cup D$ has a hybrid time domain whose real component is unbounded. The integer component may or may not be unbounded.

B. Asymptotic stability

For the hybrid system (6), the compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be globally asymptotically stable if the following properties hold:

1) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(0,0)|_{\mathcal{A}} \le \delta \Longrightarrow |x(t,j)|_{\mathcal{A}} \le \varepsilon \quad \forall (t,j) \in \text{dom } x$.
2) Each solution is bounded and each solution with an unbounded hybrid time domain satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$.

For well-posed hybrid systems, [7, Theorem 6.5] shows that global asymptotic stability is equivalent to the existence of a function $\beta \in \mathcal{KL}$ such that each solution x satisfies

$$|x(t,j)|_{\mathcal{A}} \le \beta(|x(0,0)|_{\mathcal{A}}, t+j) \qquad \forall (t,j) \in \text{dom } x$$

and [7, Theorem 6.6] shows that global asymptotic stability is robust to small perturbations. Moreover, global asymptotic stability implies the existence of a smooth Lyapunov function [3], and global asymptotic stability can be established using invariance principles a la LaSalle [11].

C. Control objective

Our controllers for the attitude stabilization problems will contain an internal state η taking values in a compact set $\mathcal{K}$. Our first goal is to find a hybrid controller

$$\begin{array}{rcll} \omega & = & \kappa(\xi,\eta) & \\ \dot{\eta} & = & \alpha(\xi,\eta) & (\xi,\eta) \in C \\ \eta^+ & \in & G(\xi,\eta) & (\xi,\eta) \in D \end{array} \tag{7}$$

that solves the robust, global asymptotic stabilization problem for (1)-(2) using controls in Ω_{123}, respectively in Ω_{12}. This problem is defined as follows:

Definition 1: Let $\Omega \subset \mathbb{R}^3$ be closed. The controller (7) is said to *solve the robust, global asymptotic stabilization problem for (1)-(2) using controls in* Ω if

1) C and D are closed sets and $C \cup D = C_0 \times \mathcal{K}$,
2) The functions $\kappa, \alpha : C \to \Omega$ are continuous; G is outer-semicontinuous, locally bounded and nonempty on D,
3) the closed-loop system (1),(7) is forward complete and non-Zeno from $C_0 \times \mathcal{K}$ and has the compact set $\mathcal{A} = \{\xi^*\} \times \mathcal{K}$ globally asymptotically stable.

In both of the problems below, the internal state η will contain a timer τ taking values in the interval $[0,1]$, a (discrete) logic state $p \in \mathcal{P}$ that indicates which angular velocity component is being used, and a (discrete) logic state $q \in \mathcal{Q}$ that indicates the current mode of the system. When using controls in the set Ω_{12}, η will also contain a variable $\varsigma \in \{-1,1\}$, a variable $\varphi \in [0,\pi/4]$ and a variable $\chi \in \mathcal{Q}$. Both controllers will make use of functions σ_i, ($i = 1,2,3$ for controls in Ω_{123} and $i = 1,2$ for controls in Ω_{12}) defined on the set $\{0,1,2,3\}$ and satisfying $\sigma_i(p) = 1$ when $p = i$ and $\sigma_i(p) = 0$ when $p \ne i$.

III. ATTITUDE STABILIZATION USING CONTROLS IN Ω_{123}

We define $\mathcal{P} := \{1,2,3\}$, $\mathcal{Q} := \{1,2,3\}$ and $\mathcal{K} := [0,1] \times \mathcal{P} \times \mathcal{Q}$. The controller state is $\eta := (\tau,p,q)'$. We take

$$\alpha(\xi,\eta) := \begin{bmatrix} \max\{0, 2-q\} \\ 0_{2\times 1} \end{bmatrix} , \tag{8}$$

i.e., $\alpha_1(\xi,(\tau,p,1)') = 1$ and $\alpha_1(\xi,(\tau,p,q)') = 0$ when $q \ne 1$, $\alpha_2(\xi,\eta) = \alpha_3(\xi,\eta) = 0$ for all (ξ,η). We also take

$$\begin{array}{rcl} C & := & \{(\xi,\eta) \in C_0 \times \mathcal{K} : (\xi,\tau,p) \in C_q\} \\ D & := & \{(\xi,\eta) \in C_0 \times \mathcal{K} : (\xi,\tau,p) \in D_q\} \end{array} \tag{9}$$

and then, for each $q \in \mathcal{Q}$, we specify C_q, D_q, $G(\xi,\eta) = G(\xi,(\tau,p,q)')$ and $\kappa(\xi,\eta) = \kappa(\xi,(\tau,p,q)')$ below.

- Mode $q = 1$: *Use zero controls for a short amount of time then and evaluate what mode should be used*. Let $\varepsilon \in (0,1)$. According to (8), the timer state τ evolves according to $\dot{\tau} = 1$ in this mode. The flow set is

  $$C_1 := C_0 \times [0,\varepsilon] \times \mathcal{P} \ ,$$

  the controls are $\omega_i = \kappa_i(\xi,(\tau,p,1)') := \sigma_i(0)$, i.e., $\omega_i = 0$ for $i = 1,2,3$, and the jump set is given by the closure of the complement (relative to $C_0 \times [0,1] \times \mathcal{P}$) of C_1. The jump map is defined on D_1. The first component of the jump map is zero, i.e., $G_1(\xi,(\tau,p,1)') := 0$ for all $(\xi,\tau,p) \in D_1$. The second component is

  $$G_2(\xi,(\tau,p,1)') := \left\{ \varrho \in \mathcal{P} : \xi_\varrho^2 \ge \xi_i^2 \ \forall i \in \mathcal{P} \right\} .$$

  The third component of the jump map is expressed in terms of a parameter θ that must be in the interval $(\underline{\theta}, \overline{\theta})$ where $\underline{\theta}$ and $\overline{\theta}$ are specified in subsequent modes. The third component of the jump map is given by

  $$G_3(\xi,(\tau,p,1)') := \begin{cases} 2 & \text{if } \xi_4 < \theta \\ 3 & \text{if } \xi_4 > \theta \\ \{2,3\} & \text{if } \xi_4 = \theta \ . \end{cases}$$

- Mode $q=2$: *Get ξ_4 above a threshold $\overline{\theta} \in (-1, 1/\sqrt{3})$.* The flow set for this mode is

$$C_2 := \left\{ p \in \mathcal{P} \ , \ \xi_4 \le \overline{\theta} \ , \ \xi_4^2 + \xi_p^2 \ge 1/3 \right\} \ .$$

There always exists $p \in \mathcal{P}$ such that $\xi_4^2 + \xi_p^2 \ge 1/3$ since, if not

$$1 > \xi_1^2 + \xi_2^2 + \xi_3^2 + 3\xi_4^2 = 1 + 2\xi_4^2 \ge 1$$

which is a contradiction. The controls are given by $\omega_i = \kappa_i(\xi, (\tau, p, 2)') := \sigma_i(p)$. The jump set D_2 is the closure of the complement (relative to $C_0 \times [0,1] \times \mathcal{P}$) of C_2. The jump map is $G(\xi, (\tau, p, 2)') = (0, p, 1)'$. (It is possible to add an extra controller state that takes values in the discrete set $\{-1, 1\}$ and multiplies $\sigma_i(p)$ in the definition of κ_i in order to make ξ_4 move in the "better" of the two directions that can be taken to get it above the desired threshold. This feature is not necessary for global asymptotic stability and so it is omitted to simplify the presentation.)
- Mode $q=3$: *Normal operation, which is allowed when* $\xi_4 \ge \underline{\theta} \in (-1, \overline{\theta})$. In this mode, the controller increases ξ_4 by making sure it is connected to a state ξ_1, ξ_2, or ξ_3 that is bounded away from zero in an appropriate sense. The control laws here are

$$\omega_i = -\sigma_i(p)\xi_i =: \kappa_i(\xi, \tau, p, 3)$$

(note that ξ_i be replaced by any function f having the property that $\xi_i f(\xi_i)$ is continuous and positive definite) and the flow set is given by

$$C_3 := \left\{ p \in \mathcal{P} \ , \ \xi_4 \ge \underline{\theta} \ , \ \xi_p^2 \ge \mu \left(\xi_j^2 + \xi_k^2 \right) \right\}$$

where the p, j and k are distinct elements of $\mathcal{P}$ and $\mu \in (0, 1/2)$. Notice that the flow equation for ξ_4 will satisfy $\dot{\xi}_4 = \xi_p^2$. The only value of $\xi \in C_3$ that makes this derivative zero is the value $\xi = \xi^*$. The jump set D_3 is the closure of the complement (relative to $C_0 \times [0,1] \times \mathcal{P}$) of C_3. The jump map is $G(\xi, (\tau, p, 2)') = (0, p, 1)'$.

Theorem 1: The hybrid controller specified above solves the robust, global asymptotic stabilization problem for (1)-(2) using controls in Ω_{123}.

Sketch of proof: The first two items and the first part of the third item of the robust, global asymptotic stabilization problem are satisfied by construction. Regarding global asymptotic stability, we note that the energy function $1 - \xi_4$ is strictly decreasing in mode 3 and constant in mode 1. Since mode 2 only runs when ξ_4 is bounded away from one, this establishes stability of the point ξ^*. Due to the hysteresis, after mode 2 activates once, it never activates again. Moreover, the only time it activates is perhaps at $t \in [0, \varepsilon]$ and when it activates, it is for a uniformly bounded duration. It then follows from the invariance principles in [11] that the set $\mathcal{A}$ is globally asymptotically stable. ■

IV. ATTITUDE STABILIZATION USING CONTROLS IN Ω_{12}

A. Controller description and result

The control strategy in this section is based on the strategy for Ω_{123}. However, some additional modes are used to account for the fact that it is not possible to link the states ξ_3 and ξ_4 directly since we are insisting that $\omega_3 \equiv 0$.

We define $\mathcal{P} := \{1, 2\}$, $\mathcal{Q} := \{1, 2, \ldots, 8\}$ and $\mathcal{K} := [0,1] \times \mathcal{P} \times \mathcal{Q} \times \{-1, 1\} \times [0, \pi/4] \times \mathcal{Q}$. The controller state is $\eta := (\tau, p, q, \varsigma, \varphi, \chi)'$. We take

$$\alpha(\xi, \eta) := \begin{bmatrix} \alpha_1(q) \\ 0_{5\times 1} \end{bmatrix} \qquad (10)$$

where $\alpha_1(1) = \alpha_1(6) = 1$, and $\alpha_1(q) = 0$ for $q \notin \{1, 6\}$. Also

$$\begin{aligned} C \ &:= \ \{(\xi, \tau, p, q, \varsigma, \varphi, \chi) \in C_0 \times \mathcal{K} \ , \\ &\qquad\qquad (\xi, \tau, p, \varsigma, \varphi, \chi) \in C_q\} \\ D \ &:= \ \{(\xi, \tau, p, q, \varsigma, \varphi, \chi) \in C_0 \times \mathcal{K} \ , \\ &\qquad\qquad (\xi, \tau, p, \varsigma, \varphi, \chi) \in D_q\} \ . \end{aligned} \qquad (11)$$

We specify C_q, D_q, $G(\xi, \eta) = G(\xi, (\tau, p, q, \varsigma, \varphi, \chi)')$ and $\kappa(\xi, \eta) = \kappa(\xi, (\tau, p, q, \varsigma, \varphi, \chi)')$ for each $q \in \mathcal{Q}$ below.

- Mode $q=1$: *Use zero controls for a short amount of time and then evaluate what mode should be used.* This mode is just like the $q=1$ mode for the previous controller. Let $\varepsilon \in (0,1)$. The flow set is

$$C_1 := C_0 \times [0, \varepsilon] \times \mathcal{P} \times \{-1, 1\} \times [0, \pi/4] \times \mathcal{Q}$$

and $\omega_i = \kappa_i(\xi, (\tau, p, 1, \varsigma, \varphi, \chi)') := \sigma_i(0)$, i.e., $\omega_i = 0$ for $i = 1, 2$. The jump set is the closure of the complement (relative to $\Gamma := C_0 \times [0,1] \times \mathcal{P} \times \{-1, 1\} \times [0, \pi/4] \times \mathcal{Q}$) of C_1. The jump map is defined on D_1. The first component is always zero. The fourth through sixth components are always $(\varsigma, \varphi, \chi)'$. The second and third components are given as follows. Define

$$\mathcal{I}(\xi) := \left\{ \varrho \in \mathcal{P} : \xi_\varrho^2 \ge \xi_i^2 \ \forall i \in \mathcal{P} \right\} \ .$$

The parameters $\overline{\theta}$, $\underline{\theta}$ and ν will be specified in modes 2, 3, and 4, respectively. Let $\theta \in (\underline{\theta}, \overline{\theta})$ and let $\nu_2 \in (1/3, \nu)$. If $\chi \notin \{5, 6, 7\}$, $\xi_4 \le \theta$, and $\xi_4^2 + \max\left\{\xi_1^2, \xi_2^2\right\} \ge \nu_2$ then $(p^+, q^+) \in \mathcal{I}(\xi) \times \{2\}$; if $\chi \notin \{5, 6, 7\}$, $\xi_4 \le \theta$ and $\xi_4^2 + \max\left\{\xi_1^2, \xi_2^2\right\} \le \nu_2$ then $(p^+, q^+) \in \mathcal{I}(\xi) \times \{4\}$. If both conditions hold then $(p^+, q^+) \in \mathcal{I}(\xi) \times \{2, 4\}$. The parameter μ will be specified in mode 3. Let $\widehat{\mu} > \mu/(1-\mu)$. If $\chi \notin \{5, 6, 7\}$, $\xi_4 \ge \theta$, and $\xi_1^2 + \xi_2^2 \ge \widehat{\mu}\xi_3^2$ then $(p^+, q^+) \in \mathcal{I}(\xi) \times \{3\}$; if $\chi \notin \{5, 6, 7\}$, $\xi_4 \ge \theta$, and $\xi_1^2 + \xi_2^2 \le \widehat{\mu}\xi_3^2$ then $(p^+, q^+) = (1, 5)$. If both conditions hold then $(p^+, q^+) \in (\mathcal{I}(\xi) \times \{3\}) \cup (1, 5)$. If $\chi \in \{5, 6, 7\}$ then $(p^+, q^+) = (\mathrm{mod}(\chi + 1, 2) + 1, \chi + 1)$.
- Mode $q=2$: *Get ξ_4 above a threshold $\overline{\theta} \in (1/2, 1/\sqrt{3})$.* The flow set for this mode is

$$C_2 := \left\{ p \in \mathcal{P} \ , \ \xi_4 \le \overline{\theta} \ , \ \xi_4^2 + \xi_p^2 \ge 1/3 \right\} \ ,$$

and $\omega_i = \kappa_i(\xi, (\tau, p, 2, \varsigma, \varphi, \chi)') := \sigma_i(p)$. The jump set D_2 is the closure of the complement (relative to Γ) of C_2. The jump map is $G(\xi, (\tau, p, 2, \varsigma, \varphi, \chi)') = (0, p, 1, \varsigma, \varphi, q)'$.
- Mode $q=3$: *Normal operation, which is allowed when* $\xi_4 \ge \underline{\theta} \in (1/\sqrt{5}, \overline{\theta})$. Increase ξ_4 by making sure it is connected to a state that is bounded away from zero in an appropriate sense. The control laws here are

$$\omega_i = \kappa_i(\xi(\tau, p, 3, \varsigma, \varphi, \chi)') := -\sigma_i(p)\xi_i$$

(again ξ_i can be replaced by a more general function) and the flow set is given by

$$C_3 := \left\{p \in \mathcal{P},\ \xi_4 \geq \underline{\theta},\ \xi_p^2 \geq \mu\left(\xi_j^2 + \xi_3^2\right)\right\}$$

where the j and p are distinct elements of the set $\mathcal{P}$ and $\mu \in (0, 1/2)$. The flow equation for ξ_4 will satisfy $\dot{\xi}_4 = \xi_p^2$. The jump set D_3 is the closure of the complement (relative to Γ) of C_3. The jump map is $G(\xi, (\tau, p, 3, \varsigma, \varphi, \chi)') = (0, p, 1, \varsigma, \varphi, q)'$.

- Mode $q = 4$: *Enable getting ξ_4 above threshold.* Let $\nu \in (1/3, 1/2)$. The flow set for this mode is

$$C_4 := \left\{p \in \mathcal{P},\ \xi_4 \leq \overline{\theta},\ \xi_4^2 + \max\left\{\xi_1^2, \xi_2^2\right\} \leq \nu\right\}\ ,$$

  and $\omega_i = \kappa_i(\xi, (\tau, p, 4, \varsigma, \varphi, \chi)') := \sigma_i(p)$. Notice that ξ_3 will be driven toward zero, but won't be able to reach it, as $\xi_3^2 = 0$ implies $\xi_4^2 + \max\left\{\xi_1^2, \xi_2^2\right\} \geq 1/2$. The jump set D_4 is the closure of the complement (relative to Γ) of C_4. The jump map is $G(\xi, (\tau, p, 4, \varsigma, \varphi, \chi)') = (0, p, 1, \varsigma, \varphi, \chi)'$.
- Mode $q = 5$: *Increase ξ_4 by emptying ξ_1 into ξ_4*. The flow set for this mode is

$$C_5 := \left\{p = 1;\ -\varsigma\xi_p \geq 0;\ \xi_4 \geq \underline{\theta}\sqrt{2}/2\right\}\ ,$$

  and $\omega_i = \kappa_i(\xi, (\tau, p, 5, \varsigma, \varphi, \chi)') := \varsigma\sigma_i(p)$. The jump set D_5 is the closure of the complement (relative to Γ) of C_5. The jump map is such that φ^+ is equal to $s \in (0, \pi/4]$ such that

$$\cot(2s) = \frac{\cos^2(s) - \sin^2(s)}{2\sin(s)\cos(s)} = \left|\frac{\xi_2}{\xi_3}\right|\ . \qquad (12)$$

  Thus, $\varphi^+ = 0.5\cot^{-1}(|\xi_2/\xi_3|)$ for $\xi_3 \neq 0$, $\varphi^+ = 0$ for $\xi_3 = 0, \xi_2 \neq 0$, and $\varphi^+ \in [0, \pi/4]$ for $\xi_3 = \xi_2 = 0$. Also $\varsigma^+ = \text{sgn}(\xi_2\xi_3)$ and when $\xi_2\xi_3 = 0$ we set $\varsigma^+ \in \{-1, 1\}$. Finally $\tau^+ = 0$, $p^+ = p$, $q^+ = 1$, and $\chi^+ = q$.
- Mode $q = 6$: *"open loop" maneuver to decrease the magnitude of ξ_3; ξ_1 connected to ξ_4* The controls are given by $\omega_i = \kappa_i(\xi, (\tau, p, 6, \varsigma, \varphi, \chi)') := \varsigma\sigma_i(p)$. The flow set of this mode is

$$C_6 := \{p = 1;\ \tau \in [0, \min\{\phi(\xi_4), \varphi\}]\}\ .$$

  The function $\phi : [-1, 1] \to [0, \pi/4]$ is Lipschitz continuous, nonincreasing, and satisfies $\phi(1) = 0$. The jump set D_6 is the closure of the complement (relative to Γ) of C_6. The jump map satisfies $\tau^+ = 0$, $p^+ = 2$, $q^+ = 1$, $\varsigma^+ \in \{s \in \{-1, 1\} : -s\xi_2 \geq 0\}$, $\varphi^+ = \varphi$, and $\chi^+ = q$.
- Mode $q = 7$: *increase ξ_4 by emptying ξ_2 into ξ_4*. The flow set of this mode is

$$C_7 := \left\{p = 2; -\varsigma\xi_p \geq 0; \xi_4 \geq \underline{\theta}\sqrt{2}/2\right\}$$

  and $\omega_i = \kappa_i(\xi, (\tau, p, 6, \varsigma, \varphi, \chi)') := \varsigma\sigma_i(p)$. The jump set D_7 is the closure of the complement (relative to Γ) of C_7 and the jump map satisfies $\tau^+ = 0$, $p^+ = 1$, $q^+ = 1$, $\varsigma^+ \in \{s \in \{-1, 1\} : -s\xi_1 \geq 0\}$, $\varphi^+ = \varphi$ and $\chi^+ = q$.
- Mode $q = 8$: *increase ξ_4 by emptying ξ_1 into ξ_4*. The flow set of this mode is

$$C_8 := \left\{p = 1; -\varsigma\xi_p \geq 0; \xi_4 \geq \underline{\theta}\sqrt{2}/2\right\}\ ,$$

  and $\omega_i = \kappa_i(\xi, (\tau, p, 8, \varsigma, \varphi, \chi)') := \varsigma\sigma_i(p)$. The jump set D_8 is the closure of the complement (relative to Γ) of C_8. The jump map is $G(\xi, (\tau, p, q, \varsigma, \varphi, \chi)') = (0, p, 1, \varsigma, \varphi, q)'$.

Theorem 2: The hybrid controller specified above solves the robust, global asymptotic stabilization problem for (1)-(2) using controls in Ω_{12}.

Sketch of proof: The proof is similar to that of the previous result. However, after ξ_4 is brought above its minimum threshold, there is still one mode (mode 6) where the function $1 - \xi_4$ can (and typically does) increase. In Section IV-B we explain why this function cannot increase very much if it starts close to zero, thereby ensuring stability. In Section IV-C we explain why the sequence of modes $5 \to 1 \to 6 \to 1 \to 7 \to 1 \to 8$, which will always be the sequence by which mode 6 is reached, ensures that the sequence of modes $6 \to 1 \to 7 \to 1 \to 8$ decrease the function $1 - \xi_4$.

Figure 1 shows a simulation of the closed-loop system with hybrid controller using controls in Ω_{12}. Initially, the controller takes ξ_4 above thresholds using mode 4 and 2, and shortly after enters into the sequence $5 \to 1 \to 6 \to 1 \to 7 \to 1 \to 8$, which makes $1 - \xi_4$ decrease and in turn, steers ξ asymptotically to ξ^*.

B. On the amount of time spent in Mode 6

Recall the definition of C_6, that $\dot{\tau} = 1$ in mode 6, and that the function ϕ in the definition of C_6 takes values in $[0, \pi/4]$. It follows that the maximum amount of time that can be spent in mode 6 before moving back to mode 1 is $\pi/4$ seconds. However, the time in mode 6 will be much shorter than $\pi/4$ when ξ_4 starts close to 1. This is seen as follows. Let $L > 0$ be the Lipschitz constant for the function ϕ and let $V(\xi_4) := 1 - \xi_4$. Since $\phi(1) = 0$, it follows that $\phi(\xi_4) \leq LV(\xi_4)$. In mode 6, $\dot{\xi}_4 = -\omega_1\xi_1$ and, since $\omega^\top\omega \leq 1$ and $\xi^\top\xi = 1$, we have $|\omega_1\xi_1| \leq |\xi_1| \leq \sqrt{2}\sqrt{1 - \xi_4}$. Thus

$$\langle\nabla V(\xi_4), -\omega_1\xi_1\rangle \leq \sqrt{2V(\xi_4)}\ . \qquad (13)$$

By standard comparison theorems, it follows that in mode 6

$$V(\xi_4(t)) \leq \frac{1}{2}\left(\sqrt{2V(\xi_4(0))} + t\right)^2\ . \qquad (14)$$

It follows that the amount of time that can be spent is mode 6 before leaving is upper bounded by the smaller of the two solutions t^* of the equation

$$t^* = \frac{L}{2}\left(\sqrt{2V(\xi_4(0))} + t^*\right)^2 \qquad (15)$$

and the maximum value of $V(\xi_4)$ over this interval is t^*/L. It can be seen that t^* tends to zero as $V(\xi_4(0))$ tends to zero.

C. Calculations related to Modes 5-8

After leaving mode 5 and entering mode 6 through mode 1, we will have $\xi_1 = 0$ and $\xi_2^2 \leq \widehat{\mu}\xi_3^2$ and $\xi_4 \geq \theta > 0$. We consider the effect of applying constant controls where first $\omega_1 \neq 0$, for τ_1 seconds, and then $\omega_2 \neq 0$, for τ_2 seconds, zeroing out ξ_2, and then $\omega_1 \neq 0$, for τ_3 seconds, zeroing out ξ_1. To save on notation, consider the definitions

$$\begin{array}{rclrcl} s_1 & := & \sin(\omega_1\tau_1) & c_1 & := & \cos(\omega_1\tau_1) \\ s_2 & := & \sin(\omega_2\tau_2) & c_2 & := & \cos(\omega_2\tau_2) \\ \tau_{12} & := & \tau_1 + \tau_2 & \tau_{123} & := & \tau_1 + \tau_2 + \tau_3\ . \end{array} \qquad (16)$$

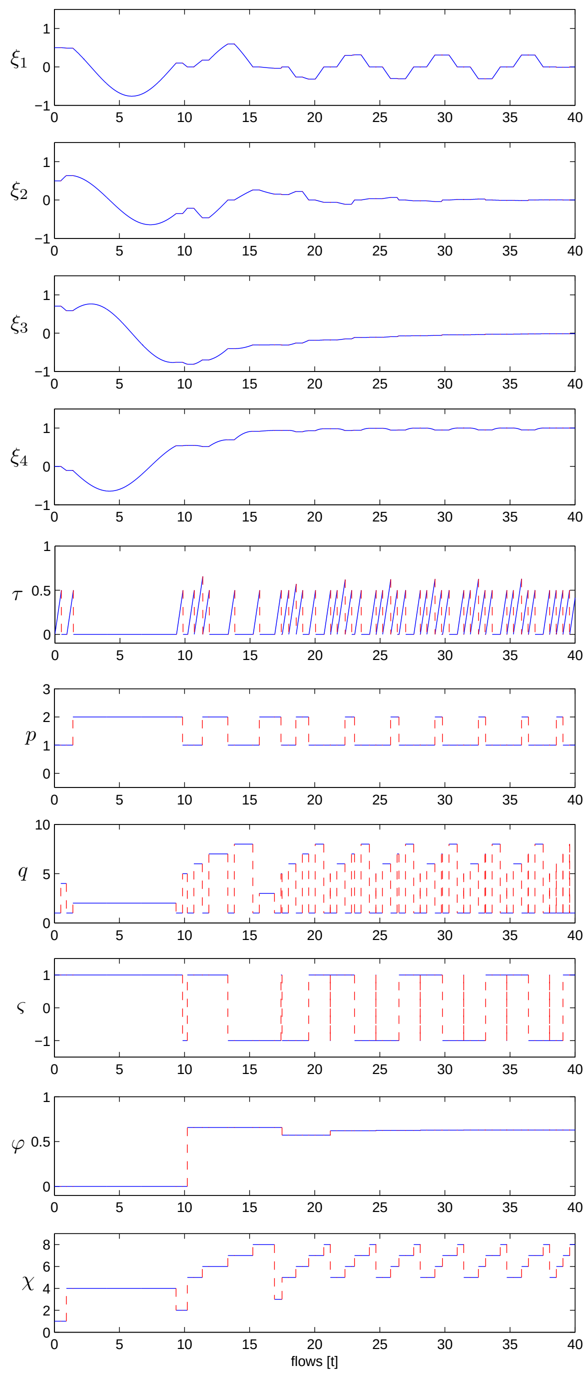

Fig. 1. Closed-loop trajectories for the attitude stabilization problem using controls in Ω_{12}. Component ξ converges to ξ^* by executing a sequence $5\to1\to6\to1\to7\to1\to8$, after mode 4 and 2 take ξ_4 above thresholds. Initial condition: $\xi^0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}, 0), \tau^0 = 0, p^0 = 1, q^0 = 1, \varsigma^0 = 1, \varphi^0 = 0, \chi^0 = 1$. Parameters: $\mu = \frac{1}{4}$, $\theta = \frac{\overline{\theta}+\underline{\theta}}{2}$;, $\overline{\theta} = \frac{1/2+1/\sqrt{3}}{2}$, $\underline{\theta} = \frac{1/\sqrt{5}+\overline{\theta}}{2}$, $\varepsilon = \frac{1}{2}$, $\nu = \frac{1/3+1/2}{2}$, $\nu_2 = \frac{1/3+\nu}{2}$, $\tilde{\mu} = \frac{2\mu}{1-\mu}$.

We claim that $|\xi_3(\tau_3)| \leq \rho(\xi_4)|\xi_3|$ where ρ is a continuous function that is less than one except when $\xi_4 = 1$. Then

$$\begin{aligned} 1 - \xi_4^2(\tau_{123}) &= \xi_1^2(\tau_{123}) + \xi_2^2(\tau_{123}) + \xi_3^2(\tau_{123}) \\ &= \xi_2^2(\tau_{123}) + \xi_3^2(\tau_{123}) \\ &= \xi_2^2(\tau_{12}) + \xi_3^2(\tau_{12}) = \xi_3^2(\tau_{12}) \end{aligned} \quad (17)$$

$$\begin{aligned} \Rightarrow 1 - \xi_4^2(\tau_{123}) &\leq \rho(\xi_4)^2 \xi_3^2 \\ &\leq \rho(\xi_4)^2 \left(\xi_3^2 + \xi_1^2 + \xi_2^2\right) \\ &= \rho(\xi_4)^2 \left(1 - \xi_4^2\right) \end{aligned} \quad (18)$$

which shows that the net effect of the maneuvers is to increase ξ_4 uniformly toward one. To establish the claim, first consider the effect of $\omega_1\tau_1$ and the general form of the solutions from the initial condition ξ. We get

$$\begin{aligned} \xi_1(\tau_1) &= s_1\xi_4 + c_1\xi_1 \ , \quad & \xi_2(\tau_1) &= s_1\xi_3 + c_1\xi_2 \\ \xi_3(\tau_1) &= c_1\xi_3 - s_1\xi_2 \ , \quad & \xi_4(\tau_1) &= c_1\xi_4 - s_1\xi_1 \end{aligned}$$

and

$$\begin{aligned} \xi_1(\tau_{12}) &= c_2\left(s_1\xi_4 + c_1\xi_1\right) - s_2\left(c_1\xi_3 - s_1\xi_2\right) \\ \xi_2(\tau_{12}) &= c_2\left(s_1\xi_3 + c_1\xi_2\right) + s_2\left(c_1\xi_4 - s_1\xi_1\right) \\ \xi_3(\tau_{12}) &= s_2\left(s_1\xi_4 + c_1\xi_1\right) + c_2\left(c_1\xi_3 - s_1\xi_2\right) \\ \xi_4(\tau_{12}) &= -s_2\left(s_1\xi_3 + c_1\xi_2\right) + c_2\left(c_1\xi_4 - s_1\xi_1\right) \end{aligned} \ .$$

Knowing that $\xi_2(\tau_{12})$ will be zero, we get

$$s_2 = -\frac{c_2\left(s_1\xi_3 + c_1\xi_2\right)}{c_1\xi_4 - s_1\xi_1}$$

and thus

$$\begin{aligned} \xi_3(\tau_{12}) &= -\frac{c_2\left(s_1\xi_3 + c_1\xi_2\right)}{c_1\xi_4 - s_1\xi_1}\left(s_1\xi_4 + c_1\xi_1\right) \\ &\quad + c_2\left(c_1\xi_3 - s_1\xi_2\right) \\ &= \frac{c_2}{c_1\xi_4 - s_1\xi_1}\left[\left(c_1^2 - s_1^2\right)\left(\xi_3\xi_4 - \xi_1\xi_2\right)\right. \\ &\quad \left. - 2c_1s_1\left(\xi_2\xi_4 + \xi_1\xi_3\right)\right] \ . \end{aligned}$$

Now, using $\xi_1 = 0$ and $\xi_4 > 0$, we get

$$\xi_3(\tau_{12}) = \frac{c_2}{c_1}\left[(c_1^2 - s_1^2)\xi_3 - 2c_1s_1\xi_2\right] \ .$$

Now, we have chosen ω_1 so that the sign of $s_1\xi_2$ matches the sign of ξ_3, and we have chosen τ_1 so that $|(c_1^2 - s_1^2)\xi_3| \geq |2c_1s_1\xi_2|$ Using these facts, it follows that

$$|\xi_3(\tau_{12})| \leq \frac{c_1^2 - s_1^2}{c_1}|\xi_3| \ .$$

We note that the coefficient is less than one, except when $c_1 = 1$, and its size will depend on ξ_4 because of the description of the flow set in mode 6.

V. Robust, global stabilization of kinematics plus non-actuated angular velocity

We now consider the robust, global asymptotic stabilization problem using controls (ω_1, ω_2) for the system (1)-(2) combined with the dynamic equation

$$\dot{\omega}_3 = a\omega_1\omega_2 \qquad a \neq 0 \ . \quad (19)$$

Let $\lambda \in (0,1)$, let $\sigma : \mathbb{R} \to [-\lambda, \lambda]$ be a continuously differentiable function so that the function $\omega_3 \mapsto a\omega_3\sigma(\omega_3)$ is negative definite, and define $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ as

$$\left.\begin{aligned} \widetilde{\omega}_1 &:= \frac{1}{1-\sigma^2(\omega_3)}\left(\omega_1 - \sigma(\omega_3)\omega_2\right) \\ \widetilde{\omega}_2 &:= \frac{1}{1-\sigma^2(\omega_3)}\left(\omega_2 - \sigma(\omega_3)\omega_1\right) \end{aligned}\right\} =: \Phi(\omega) \quad (20)$$

These relationships can be inverted since $\sigma(\omega_3)^2 \leq \lambda^2 < 1$ for all ω_3 to give

$$\begin{array}{rcl} \omega_1 & = & \widetilde{\omega}_1 + \sigma(\omega_3)\widetilde{\omega}_2 \\ \omega_2 & = & \widetilde{\omega}_2 + \sigma(\omega_3)\widetilde{\omega}_1 \end{array} \qquad (21)$$

and, in turn,

$$\begin{array}{rcl} \dot{\omega}_3 & = & a\left(\widetilde{\omega}_1 + \sigma(\omega_3)\widetilde{\omega}_2\right)\left(\widetilde{\omega}_2 + \sigma(\omega_3)\widetilde{\omega}_1\right) \\ & = & a\sigma(\omega_3)\left(\widetilde{\omega}_1^2 + \widetilde{\omega}_2^2\right) + a\left(1 + \sigma^2(\omega_3)\right)\widetilde{\omega}_1\widetilde{\omega}_2 \ . \end{array} \qquad (22)$$

Thus, we can consider the new control system, with controls $\widetilde{\omega}_1$, $\widetilde{\omega}_2$ given by (22) together with

$$\dot{\xi} = W\left(\left[\begin{array}{c} \widetilde{\omega}_1 + \sigma(\omega_3)\widetilde{\omega}_2 \\ \widetilde{\omega}_2 + \sigma(\omega_3)\widetilde{\omega}_1 \\ \omega_3 \end{array}\right]\right)\xi \ , \qquad \xi^\top\xi = 1 \ . \qquad (23)$$

We will take the controls $\widetilde{\omega} \in \Omega_{12}$. Let $(C, D, \kappa, \alpha, G)$ solve the robust, global asymptotic stabilization problem using controls in Ω_{12}, and pick $\widetilde{\omega} = \kappa(\xi, \eta)$, i.e.,

$$\begin{array}{rcll} \widetilde{\omega} & = & \kappa(\xi,\eta) & \\ \dot{\eta} & = & \alpha(\xi,\eta) & (\xi,\eta) \in C \\ \eta^+ & \in & G(\xi,\eta) & (\xi,\eta) \in D \ . \end{array} \qquad (24)$$

Theorem 3: If the controller (7) solves the robust, global asymptotic stabilization problem using controls in Ω_{12} for the system (1)-(2) then the controller (21),(24) solves the robust, global asymptotic stabilization problem (using controls in $\mathbb{R}^2$) for the system (1)-(2), (19).

Sketch of proof: Since $\widetilde{\omega} \in \Omega_{12}$, it follows that

$$\dot{\omega}_3 = a\sigma(\omega_3)\left(\widetilde{\omega}_1^2 + \widetilde{\omega}_2^2\right) \ . \qquad (25)$$

Thus, the function ω_3^2 is monotonically nonincreasing along solutions. By local robustness properties established in [7], it follows that the set $\mathcal{A} \times \{0\}$ is stable. Next, we note that solutions are non-Zeno by virtue of the assumption of the properties of the controller (7) for the system (1)-(2). Now, according to the invariance principle [11], trajectories will converge to an invariant set where, during flows, either $\widetilde{\omega}_1 = \widetilde{\omega}_2 = 0$ and ω_3 is constant or $\omega_3 = 0$. In the latter case, (ξ, η) converges to $\mathcal{A}$ by assumption. It remains to rule out the possibility that ω_3 remains at a nonzero constant. Indeed, if ω_3 is not zero then ξ will not remain at ξ^* and then $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ cannot remain at zero. This contradiction leads to the conclusion that all trajectories converge to $\mathcal{A} \times \{0\}$.

VI. Robust, global practical stabilization of an underactuated rigid body

We now discuss extending the solution of the previous subsection to the control problem where two torques are the controls. Through the globally invertible input transformation

$$v := \langle \nabla\Phi(\omega), f(\omega)\rangle + Bu \qquad (26)$$

where Φ is defined in (20) and f and B are defined in (5), we get the control system $\dot{\widetilde{w}} = v$ together with (22)-(23). The inverse of the input transformation is given by

$$\begin{array}{rcl} u_1 & = & v_1 + \sigma(\omega_3)v_2 + \nabla\sigma(x_3)a\omega_1\omega_2\widetilde{\omega}_2 \\ u_2 & = & v_2 + \sigma(\omega_3)v_1 + \nabla\sigma(x_3)a\omega_1\omega_2\widetilde{\omega}_1 \ . \end{array} \qquad (27)$$

Given the controller (24) for the system (22)-(23) and given a continuously differentiable, positive definite function $\psi : C_0 \times \mathbb{R} \to \mathbb{R}_{\geq 0}$, we propose the controller

$$\begin{array}{rcll} \widetilde{\kappa}(\xi,\eta,\omega_3) & = & \psi(\xi,\omega_3)\kappa(\xi,\eta) & \\ e & = & \widetilde{\omega} - \widetilde{\kappa}(\xi,\eta,\omega_3) & \\ v & = & -ke + \left\langle \nabla\widetilde{\kappa}(\xi,\eta), \left[\begin{array}{c} W(\omega)\xi \\ \alpha(\xi,\eta) \\ a\omega_1\omega_2 \end{array}\right]\right\rangle & \\ \dot{\eta} & = & \alpha(\xi,\eta) & (\xi,\eta) \in C \\ \eta^+ & = & G(\xi,\eta) & (\xi,\eta) \in D \end{array} \qquad (28)$$

where $k > 0$.

The proof of the final result is beyond this paper's scope.

Theorem 4: The system (1)-(2), (5) together with the controller (27), (28), (20) is such that $\mathcal{A} \times \{0\}$ is robustly, globally, practically asymptotically stable in the parameter $k > 0$, i.e., for each neighborhood of $\mathcal{A} \times \{0\}$ there exists k^* such that for each $k \geq k^*$ there is a compact set in the neighborhood that is robustly, globally asymptotically stable.

We note that, instead of making k large, an alternative is to slow down the evolution of the kinematics through a scaling of the angular velocities.

We conjecture that the function ψ can be chosen in such a way that robust, global asymptotic stability results for k above a certain threshold. Establishing such a result is the topic of ongoing research. Through simulations, we are also currently studying how the choice of different parameters in the control algorithm affects performance.

References

[1] D. Aeyels and M. Szafranski. "Comments on the stabilizability of the angular velocity of a rigid body". *Sys. & Cont. Let.*, 10 (1988), 35-39.

[2] C. Byrnes and A. Isidori. "On the attitude stabilization of rigid spacecraft". *Automatica*, vol. 27, no. 1, pp. 87-95, 1991.

[3] C. Cai, A.R. Teel and R. Goebel. "Results on the existence of smooth Lyapunov functions for (pre-)asymptotically stable hybrid systems with non-open basins of attraction". *Proceedings of the 2007 American Control Conference*, pp. 3456–3461.

[4] D. Casagrande, A. Astolfi, and T. Parsini. "A globally stabilizing time-switching control strategy for an underactuated rigid body". *Proc. of the 2007 American Control Conference*, pp. 2078–2083.

[5] J.-M. Coron and E. Kerai. "Explicit feedbacks stabilizing the attitude of a rigid spacecraft with two control torques". *Automatica*, vol. 32, no. 5, pp 669-677, 1996.

[6] P. Crouch. "Spacecraft attitude control and stabilization: applications of geometric control theory to rigid body models". *IEEE Transactions on Automatic Control*, vol. 29, no. 4, pp. 321-331, 1984.

[7] R. Goebel and A.R. Teel. "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications". *Automatica*, 42 (2006) 573–587.

[8] H. Krishnan, M Reynahoglu and H. McClamroch. "Attitude stabilization of a rigid spacecraft using gas jet actuators operating in a failure mode", *Proc. of the 31st IEEE CDC*, Dec. 1992, pp. 1612–1617.

[9] P. Morin, C. Samson, J. Pomet, and Z. Jiang. "Time-varying feedback stabilization of the attitude of a rigid spacecraft with two controls". *Systems & Control Letters*, 25 (1995) 375-385.

[10] R. Outbib and G. Sallet. "Stabilizability of the angular velocity of a rigid body revisited". *Systems & Control Letters*, 18 (1992) 93–98.

[11] R.G. Sanfelice, R. Goebel, A.R. Teel. "Invariance principles for hybrid systems with connections to detectability and asymptotic stability". *IEEE Trans. Auto. Cont.*, Vol. 52, Dec. 2007, pp. 2282–2297.

[12] E.D. Sontag and H.J. Sussmann. "Further comments on the stabilizability of the angular velocity of a rigid body". *Systems & Control Letters*, 12 (1988) 213–217.

[13] P. Tsiotras. "On the optimal regulation of an axi-symmetric rigid body with two controls". *AIAA Guidance, Navigation, and Control Conference*, San Diego, CA, July, 1996.

[14] J.T.-Y. Wen and K. Kreutz-Delgado. "The attitude control problem". *IEEE Trans. on Auto. Control* Vol. 36, Oct. 1991, pp. 1148 - 1162.