Regularity of optimal solutions and the optimal cost for hybrid dynamical systems via reachability analysis

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Abstract

For a general optimal control problem for dynamical systems with hybrid dynamics, we study the dependency of the optimal cost on perturbations to problem constraints and mappings. The former limits the allowable values for the state, input, initial and terminal condition, and the terminal time, while the latter is given by the right-hand sides of the differential/difference inclusions defining the dynamics, as well as the functions defining the cost functional. We show that upper and lower semicontinuous dependence of solutions on initial conditions – properties that are captured by outer and inner well-posedness, respectively – lead to the existence of a solution to the optimal control problem and upper/lower semicontinuity of the optimal cost. In particular, by exploiting properties of finite horizon reachable sets for hybrid systems, we show that the optimal cost varies upper semicontinuously when the hybrid system is outer well-posed, and lower semicontinuously when it is inner well-posed and an additional assumption requiring partial knowledge of solutions holds. Consequently, when the system is both inner and outer well-posed and the aforementioned assumption holds, the optimal cost varies continuously, and optimal solutions vary outer/upper semicontinuously. We further show that even in the absence of this solution-based assumption, the optimal cost (respectively, solutions) can be continuously (respectively, outer/upper semicontinuously) approximated. Results are demonstrated by examples, theoretically and numerically.

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1. Introduction

Models and algorithms characterized by the interplay of continuous-time dynamics and instantaneous changes have become prevalent due to their capabilities of leading to solutions to control problems that classical techniques cannot solve, or simply do not apply. Examples of outstanding control problems that such hybrid techniques have been able to solve include the design of event-triggered control algorithms (Chai, Casau, & Sanfelice, 2020; Postoyan, Tabuada, Nešić, & Anta, 2015; Tabuada, 2007), stabilization over networks (Hespanha, Naghshtabrizi, & Xu, 2007; Nešić & Teel, 2004), observers and synchronization strategies under intermittent information (Ferrante, Gouaisbaut, Sanfelice, & Tarbouriech, 2016; Phillips & Sanfelice, 2019), and control of mechanical systems exhibiting impacts (Ronsse, Lefèvre, & Sepulchre, 2007; Tian, Koessler, & Sanfelice, 2013). These advances have been enabled by the modeling, analysis, and design techniques for hybrid dynamical systems. A hybrid dynamical system, or simply a hybrid system, is a dynamical system that exhibits characteristics of both continuous-time and discrete-time dynamical systems.

Numerous tools are available in the literature for the study of hybrid systems, in particular, for hybrid systems modeled as hybrid automata (Branicky, Borkar, & Mitter, 1998; Lygeros, Johansson, Simić, Zhang, & Sastry, 2003; van der Schaft & Schumacher, 2000), impulsive systems (Aubin, Chellaboina, & Nersesov, 2002), and hybrid inclusions (Goebel, Sanfelice, & Teel, 2012; Sanfelice, 2021). The literature is rich in tools for the analysis of reachability (Altın & Sanfelice, 2020b; Collins, 2011; Lygeros, Tomlin, & Sastry, 1999), asymptotic stability (Goebel et al., 2012; Haddad et al., 2006; Lygeros et al., 2003), forward invariance (Aubin et al., 2002; Chai & Sanfelice, 2019, 2021), control design (Sanfelice, 2021), and robustness (Goebel et al., 2012; Sanfelice, 2021). On the other hand, optimality for hybrid systems is much less mature. Initial results on optimality of trajectories over finite horizons were developed in Sussmann (1999), including a maximum principle for optimality for a class of switched systems. This result was extended in Pakniyat and Caines (2020) and Shaikh and Caines (2007) to a broader class of systems, one allowing
for state resets — the models considered therein are in the spirit of hybrid automata. More recently, linear–quadratic control for a class of hybrid systems with a sample-and-hold structure was considered in Cristofaro, Possieri, and Sassano (2018) and Possieri and Teel (2016). In particular, the development in Possieri and Teel (2016) is within the hybrid inclusions framework of Goebel et al. (2012) and Sanfelice (2021), for the special case when the continuous (respectively, discrete) dynamics are governed by a linear differential (respectively, difference) equation. The problem of guaranteeing existence of optimal control inputs for hybrid systems in the hybrid inclusion framework was studied in Goebel (2019). In that work, conditions for existence of optimal control inputs require the corresponding continuous-time optimal control problem to have a solution and have lower semicontinuous optimal cost with respect to the initial/terminal condition and terminal time. The latter is shown to hold under mild regularity when the differential equation has a single-valued right-hand side that is affine in control, with a linear growth condition. Optimality of state-state-feedback laws for hybrid inclusions with continuous and discrete dynamics modeled by (single-valued) nonlinear maps was studied in Ferrante and Sanfelice (2019). Infinitesimal conditions involving a Lyapunov-like function are presented in Ferrante and Sanfelice (2019) to guarantee optimality over the infinite (hybrid) horizon. The finite horizon optimization problem for the same broad class of hybrid systems was formulated and developed in a sequence of papers leading to a model predictive control framework; see Altın, Ojaghi, and Sanfelice (2018), Altın and Sanfelice (2019, 2020a) and Ojaghi, Altın, and Sanfelice (2019).

Though the advances cited above have contributed to optimal control for hybrid systems, some of the key properties of the optimal control problem associated with general hybrid systems, wherein trajectories are constrained to evolve continuously (flow) in certain regions of the state space and exhibit instantaneous changes (jump) under certain conditions, have not been yet revealed in the literature. Specifically, regularity properties of the optimal cost, in particular, (semi)continuous dependence of the optimal cost and trajectories on perturbations to the problem data, which comprises constraints (state-input constraints, initial/terminal condition constraints, terminal time constraints) and mappings (right-hand sides of differential/difference inclusions, functions defining the cost functional), have not yet been investigated. Very importantly, conditions enabling the approximations of the optimal cost in a continuous manner are not available in the literature. Indeed, results that permit relating the effect of varying parameters when they approach nominal values, the expectation being that the optimal cost also approaches its nominal value, are missing. Understanding such a dependency is critical due to the fact that numerical computation of (optimal) trajectories without error is not possible (Sanfelice & Teel, 2010; Stuart & Humphries, 1996).

### 1.1. Problem formulation and contributions

Motivated by the need to understand the dependency of the optimal cost on problem parameters, we formulate a hybrid optimal control problem for hybrid inclusions and reveal key properties about its regularity and existence of solutions. Specifically, we consider hybrid systems described by constrained differential and difference inclusions as in Goebel et al. (2012) and Sanfelice (2021), which are given as

\[
\mathcal{H} \begin{cases} 
  \dot{x} \in F(x) & x \in C \\
  x^+ \in G(x) & x \in D.
\end{cases}
\]  

(1)

Above, the flow map \( F : \mathbb{R}^n \to \mathbb{R}^n \) defines the continuous-time evolution (flows) of the state \( x \in \mathbb{R}^n \) on the flow set \( C \subset \text{dom} F \).

The jump map \( G : \mathbb{R}^n \to \mathbb{R}^n \) defines the discrete transitions (jumps) of \( x \) on the jump set \( D \subset \text{dom} G \). Informally, a solution of \( \mathcal{H} \) is a function \( (t, j) \mapsto x(t, j) \), where \( t \) defines the flow time and \( j \) defines the number of jumps. Given a constraint set \( J \subset \mathbb{R}^n \to \mathbb{R}_\geq 0 \times \mathbb{N} \times \mathbb{R}^n \) and a cost function \( \mathcal{J} : \mathbb{R}^n \times \mathbb{R}_\geq 0 \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_\geq 0 \), the corresponding hybrid optimal control problem we consider is given as follows:

\[
\begin{align*}
\text{minimize} & \quad \mathcal{J}(x(0), 0, (T, J), x(T, J)) \\
\text{subject to} & \quad (x(0, 0), (T, J), x(T, J)) \in \tilde{\mathcal{S}}_H,
\end{align*}
\]  

(2)

where \( \tilde{\mathcal{S}}_H \) denotes the set of solutions of \( \mathcal{H} \) with compact domains (see Section 2), and \( (T, J) \) denotes the terminal (hybrid) time of \( x \). These notions are made precise in the next section. When the cost function \( \mathcal{J} \) depends only on the terminal point \( x(T, J) \) and the constraint set \( \Omega \) is of the form \( \{ x^n_j \} \times \{ (T, J)^j \} \times X \), this is a standard initial value problem in Mayer form with terminal constraints defined by the set \( X \). Problems similar to (2) (e.g. variable time with boundary constraints) are considered in Clarke (2013); see, for example, Problem (OC1) therein.

Our choice of the relatively simplistic structure of optimization problem in (2) is motivated by the possibility of passing from more general problems to the one in (2).\(^1\) For example, given the Bolza cost functional in Altın and Sanfelice (2019) for controlled hybrid equations, which includes stage costs for flows and jumps, one can pass to a Mayer cost functional as in (2) by augmenting the dynamics with an additional state representing the running cost. The continuous/discrete-time analogues of this trick are well known in the literature and can be found in standard references on optimal control, such as Liberon (2012). For the control inputs, we refer to Filippov’s lemma (e.g. Clarke (2013, Corollary 23.4)), which establishes equivalence between solutions of a controlled differential equation and the corresponding differential inclusion. Finally, we observe that state constraints aside from endpoint constraints are omitted in (2), since these can be embedded in the flow set \( C \) and jump set \( D \), as noted in Altın and Sanfelice (2020a). A similar approach has been taken in Wolenski (1990), where the author studies the continuous-time counterpart of (2) to characterize the value function.

This paper reveals the following key properties of the hybrid optimal control problem in (2), using recently developed notions of well-posedness for hybrid systems (Altın & Sanfelice, 2023) and their applications to reachable sets:

1) existence of optimal solutions;

2) upper semicontinuous dependence of the optimal cost on the data of the optimal control problem;

3) lower semicontinuous dependence of the optimal cost on the data of the optimal control problem;

4) continuous dependence of the optimal cost on the data of the optimal control problem;

5) outer/upper semicontinuous dependence of optimal solutions the data of the optimal control problem.

We emphasize here that aforementioned problem data is comprehensive, as it consists of state-input constraints, initial/terminal condition constraints, terminal time constraints, right-hand sides of differential/difference inclusions of the hybrid system, and the functions defining the cost functional of the optimal control problem.

The results are illustrated in multiple examples in Section 6. The first two results require the hybrid system in question to have the so-called “outer well-posedness” property, which is guar-
anted under mild regularity conditions. Lower semicontinuity of the optimal cost requires “inner well-posedness”, guaranteed under a combination of regularity, tangent cone, and geometric conditions, and also necessitate some assumptions on the structure of solutions. Consequently, combining inner/outside well-posedness properties with this assumption lead to continuity of the optimal cost and upper/outside semicontinuity of optimal solutions. Importantly, as shown in Section 5, when the assumption on the solution set cannot be satisfied, continuous (respectively, outer/upper semicontinuous) approximations of the optimal cost (respectively, solutions) are still possible.

1.2. Organization of the paper

Section 2 recalls basic concepts from hybrid inclusions and set-valued analysis. Section 3 presents an overview of well-posed hybrid systems. Section 4 presents the main results on continuity of the optimal cost and upper/outside semicontinuity of optimal solutions. Section 5 makes remarks about the assumptions involved in the continuity properties established in Section 4, and shows these properties are still possible when the assumption on the set of solutions fail. Section 6 presents two examples to which the main results are applied.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ denotes real numbers, $\mathbb{R}_{\geq 0}$ nonnegative reals, and $\mathbb{N}$ nonnegative integers. The 2-norm is denoted $|.|$. Given a pair of sets $S_1, S_2$, $S_1 \subseteq S_2$ indicates $S_1$ is a subset of $S_2$, not necessarily proper. Let $A \subseteq \mathbb{R}^n$ be nonempty. At the distance of a vector $x \in \mathbb{R}^n$ to the set $A$ is $|x|_A := \inf_{a \in A} |x - a|$. The closed unit ball in $\mathbb{R}^n$ centered at the origin is denoted $\mathbb{B}$, $r\mathbb{B}$ is the closed ball of radius $r$ centered at the origin, and $A + B$ is the set of all $x$ such that $|x - a| \leq r$ for some $a \in A$. The closure, interior, and boundary of a set $S \subseteq \mathbb{R}^n$ are denoted $\overline{S}, \text{int}S,$ and $\partial S$. The domain of a set-valued mapping $M : S :\to \mathbb{R}^m,$ denoted dom $M$, is the set of all $x \in S$ such that $M(x)$ is nonempty. Given a set $S' \subseteq S, M|_{S'}$ denotes the restriction of $M$ to $S'$.

2.1. Hybrid inclusions: Solutions and reachable sets

We introduce the concept of solution to the hybrid system in (1), whose data is the 4-tuple $(C, F, D, G)$. At times, we refer to the system using the notation $\mathcal{H} = (C, F, D, G)$. Solutions of the hybrid system $\mathcal{H}$ belong to a class of functions called hybrid arcs. Hybrid arcs are parametrized by hybrid time $(t,j)$, where $t \in \mathbb{R}_{\geq 0}$ denotes the ordinary time and $j \in \mathbb{N}$ denotes the number of jumps. A function $x$ mapping a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ to $\mathbb{R}^n$ is a hybrid arc if (1) its domain, denoted dom$x$, is a hybrid time domain, and (2) it is locally absolutely continuous on each connected component of dom$x$. Formally, a set $E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T,j) \in E$, there exists a nondecreasing sequence $(t_j)_{j=0}^{\infty}$ with $t_0 = 0$ such that $E \cap \{(t,0) \times [0,1, \ldots \} = \cup_{j=0}^{\infty} \{(t, t_{j+1}) \times [j] \}$. Then, a function $x : \text{dom} x \to \mathbb{R}^n$ is a hybrid arc if dom $x$ is a hybrid time domain and for every $j \geq 0$, the function $t \mapsto x(t,j)$ is locally absolutely continuous on the interval $t^j := [t : (t,j) \in \text{dom} x]$.

**Definition 1 (Goebel et al., 2012, Definition 26).** A hybrid arc $x$ is a solution of the hybrid system in (1) if $x(0,0) \in \text{cl}(C) \cup D$ and the following hold:

- for every $j \geq 0$ with nonempty $t^j := [t : (t,j) \in \text{dom} x], x(t,j) \in C$ for all $t \in \text{int} t^j$ and $x(t,j) \in F(x(t,j))$ for almost all $t \in t^j$;
- for all $(t,j) \in \text{dom} x$ such that $(t,j+1) \in \text{dom} x, x(t,j) \in D$ and $x(t,j+1) \in G(x(t,j))$.

A hybrid arc $x$ is called complete if its domain is unbounded, and bounded if its range is bounded. It is said to have finite escape time if $x(t,j)$ tends to infinity as $t$ tends to $T$ from the left. If the domain of $x$ is compact, we say that $(T,j) \in \text{dom} x$ is the terminal (hybrid) time of $x$ if $t \leq T$ and $j \leq 1$ for all $(t,j) \in \text{dom} x$. Similarly, $T$ is referred to as the terminal ordinary time of $x$. The same terminology is used for hybrid arcs that are solutions of the hybrid system $\mathcal{H}$; e.g., a solution $x$ of $\mathcal{H}$ is bounded if its range is bounded.

A solution $x$ of the hybrid system $\mathcal{H}$ is maximal if it cannot be extended to another solution. The notation $S_\mathcal{H}(S)$ refers to the set of all maximal solutions of $x$ of $\mathcal{H}$ originating from $S$ (i.e., $x(0,0) \in S$ for every $x \in S_\mathcal{H}(S)$), and $S_\mathcal{H} := S_\mathcal{H}(\mathbb{R}^m)$. If every $x \in S_\mathcal{H}(\mathbb{R}^m)$ is bounded or complete, we say that $\mathcal{H}$ is pre-forward complete from $S$. We say that $t$ is a jump time of $x$ if there exists $j$ such that $(t,j), (t,j+1) \in \text{dom} x$. The notation $S_\mathcal{H}$ in (2) denotes the set of all solutions of $\mathcal{H}$ (not necessarily maximal) with compact hybrid domains; i.e., $\text{dom} x$ is compact for every $x \in S_\mathcal{H}$. Note that every such $x$ has a terminal hybrid time $(T,j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$.

Given an initial condition $x_0$ and hybrid time $(T,j)$, we define the reachable set of the hybrid system $\mathcal{H}$ as the set of points reached by solutions originating from $x_0$ at hybrid time $(T,j)$.

**Definition 2 (Reachable SetMappings).** Given a hybrid system $\mathcal{H} = (C, F, D, G)$, the reachable set mapping $R_\mathcal{H} : (\text{cl}(C) \cup D) \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}^m$ of $\mathcal{H}$ is the set-valued mapping that associates with every $x_0, T,$ and $j$, the reachable set of $\mathcal{H}$ from $x_0$ at time $(T,j)$, i.e., $R_\mathcal{H}(x_0, T,j) := \{x(T,j) : x \in S_\mathcal{H}(x_0), (T,j) \in \text{dom} x\}$.

2.2. Limits, semicontinuity, and boundedness of set-valued maps

Let $S \subseteq \mathbb{R}^n, x \in \text{cl} S$, and consider a set-valued mapping $M : S \to \mathbb{R}^m$. The inner limit of $M$ as $x$ tends to $x$, lim inf$_{x \to x}$, $M(x)$, is the set of all $y$ such that for any sequence $(x_i)_{i=0}^{\infty}$ in $S$ convergent to $x$, there exist $i \geq 0$ and a sequence $(y_i)_{i=0}^{\infty}$ convergent to $y$ with $y_i \in M(x_i)$ for all $i \geq 0$. The outer limit of $M$ as $x$ tends to $x$, lim sup$_{x \to x}$, $M(x)$, is the set of all $y$ for which there exist a sequence $(x_i)_{i=0}^{\infty}$ in $S$ convergent to $y$ and a sequence $(y_i)_{i=0}^{\infty}$ convergent to $y$ with $y_i \in M(x_i)$ for all $i \geq 0$. If the inner and outer limits (as $x$ tends to $x$) are equal, the limit of $M$ as $x$ tends to $x$, denoted lim$_{x \to x}$, $M(x)$, is defined to be equal to them. Limits of sequences of sets are defined in the same manner. Let $X \subseteq S$ and $x \in \text{cl} X$. Then, the mapping $M$ is inner semicontinuous (respectively, outer semicontinuous) at $x$ relative to $X$ if the inner (respectively, outer) limit of $M(x)$ as $x$ tends to $x$ contains (respectively, is contained in) $M(x)$. It is continuous at $x$ relative to $X$ if it is both inner and outer semicontinuous at $x$ relative to $X$. In addition, $M$ is locally bounded at $x \in X$ relative to $X$ if there exists $\varepsilon > 0$ such that the set $M(x + \varepsilon B) \cap X$ is bounded. When these properties hold for all $x \in X$, we drop the qualifier “at $x$”, and if $X = S$, we drop the qualifier “relative to $X$”. These definitions follow Tyrrell Rockafellar and Wets (2009, Definitions 4.1, 5.4, and 5.14).

2.3. Graphical convergence and closeness of hybrid arcs

A sequence $(x_i)_{i=0}^{\infty}$ of hybrid arcs is said to be locally eventually bounded if for any $t \geq 0$, there exist $i \geq 0$ and a compact set $K$ such that $x_i(t,j) \in K$ for every $i \geq i$ and $(t,j) \in \text{dom} x_i$.

2 Outer semicontinuity is equivalent to upper semicontinuity (Aubin & Frankowska, 2009, Definition 1.4.1) for locally bounded set-valued maps with closed values (Goebel et al., 2012, Lemma 5.15). Inner semicontinuity coincides with lower semicontinuity (Aubin & Frankowska, 2009, Definition 1.4.2).
with \( t + j \leq \tau \). It is said to converge graphically to a mapping \( M : R_\geq 0 \times N \rightarrow R^n \), called the **graphical limit** of \( \{x_i\}_{i=0}^\infty \), if the sequence \( \{gph x_i\}_{i=0}^\infty \) converges to \( gph M \) (in the set convergence sense), where \( gph \) denotes the graph of a set-valued mapping. Graphical convergence is motivated by the fact that solutions of a hybrid system can have different time domains, which renders the uniform norm an insufficient metric to analyze convergence; see Goebel et al. (2012, Chapter 5). To measure closeness, in lieu of the uniform norm, we use a concept called \((\tau, \varepsilon)\)-closeness, given in Appendix A.

### 3. Background on well-posed hybrid systems

Fundamental in our analysis are the various notions of well posedness for hybrid systems. We provide a brief overview of these to keep the paper self contained.

#### 3.1. Nominal well-posedness

Roughly speaking, **nominally outer well-posed** hybrid systems are those hybrid systems whose solutions depend outer semi-continuously on initial conditions: for a hybrid system \( H \) that is nominally outer well-posed, the graphical limit \( x \) of a locally eventually bounded graphically convergent sequence \( \{x_i\}_{i=0}^\infty \) of solutions is itself a solution. The precise definition is recalled below.

**Definition 3 (Altın & Sanfelice, 2020b, Definition 3.2).** A hybrid system \( H \) is said to be **nominally outer well-posed** on a set \( S \subset R^n \) if for every graphically convergent sequence of solutions \( \{x_i\}_{i=0}^\infty \) of \( H \) satisfying \( \lim_{i \to \infty} x_i(0,0) := x_0 \in S \), the following holds:

(a) if the sequence \( \{x_i\}_{i=0}^\infty \) is locally eventually bounded, then the graphical limit \( x \) is a solution of \( H \) originating from \( x_0 \);
(b) if the sequence \( \{x_i\}_{i=0}^\infty \) is not locally eventually bounded, then there exists \( (T, J) \in R_\geq 0 \times N \) such that

\[
x = M_{|dom M \cap ((0,T) \times (0,1...J))} \text{ is a solution of } H \text{ originating from } x_0 \text{ that escapes to infinity at time } (T, J), \text{ where } M \text{ is the graphical limit of } \{x_i\}_{i=0}^\infty.
\]

See also Goebel et al. (2012, Definition 6.2). In simple words, this property guarantees that small variations in the initial condition do not lead to large changes in the behavior of solutions. Importantly, nominal well-posedness is implied when the data of the system satisfies mild regularity conditions called the **hybrid basic conditions** (Goebel et al., 2012, Assumption 6.5), see Theorem 23 in Appendix B.

The natural counterpart to nominal outer well-posedness is called **nominal inner well-posedness**. For hybrid system \( H = (C, F, D, G) \) nominally inner well-posed on \( S \), given a bounded or complete solution \( x \) originating from \( S \) and a sequence of initial conditions \( \{\xi_i\}_{i=0}^\infty \in cl(C) \cup D \) convergent to \( x(0,0) \), one can find a locally bounded sequence of solutions \( \{x_i\}_{i=0}^\infty \) graphically convergent to \( x \), where each \( x_i \) originates from \( \xi_i \). This is a constructive property, in the sense that it guarantees that a given solution can be approximated by other solutions with small variations in their initial conditions. Sufficent conditions for nominal inner well-posedness are provided in Theorem 24 in Appendix B.

1. Previously referred to in the literature simply as nominal well-posedness; e.g. Goebel et al. (2012, Definition 6.2). The new terminology was introduced in Altın and Sanfelice (2020b) to accommodate the then novel notion of nominal inner well-posedness.

2. For all notions of well-posedness, for simplicity, we omit the qualifier “on S” when \( S = R^n \). Also, we say “at \( x_0 \)” instead of “on S” if \( S = \{x_0\} \) for some \( x_0 \).

3. Roughly speaking, **nominally outer well-posed** hybrid systems are those hybrid systems whose solutions depend outer semi-continuously on initial conditions: for a hybrid system \( H \) that is nominally outer well-posed, the graphical limit \( x \) of a locally eventually bounded graphically convergent sequence \( \{x_i\}_{i=0}^\infty \) of solutions is itself a solution. The precise definition is recalled below.

### 3.2. Well-posedness

Consider a hybrid system \( H_\delta = (C_\delta, F_\delta, D_\delta, G_\delta) \) parametrized by a scalar \( \delta \in (0,1) \). The notions of outer and inner well-posedness are concerned with the behavior of solutions as the parameter \( \delta \) tends to zero. Roughly speaking, given a hybrid system \( H \), a family of hybrid systems \( \{H_\delta\} \subset H \cup D \) is said to be an inner well-posed perturbation of \( H \) if given a bounded or complete solution \( x \) of \( H \) originating from \( S \), one can find a locally bounded sequence of solutions \( \{x_\delta\}_{\delta=0}^\infty \) of this family that is graphically convergent to \( x \). Altın and Sanfelice (2023, Definition 8). Just like nominal inner well-posedness, this property guarantees that a given solution of the nominal system can be approximated with small variations in their initial condition, provided the perturbation parameter \( \delta \in (0,1) \) is also small. For sufficient conditions for inner well-posedness, see Altın and Sanfelice (2023).

**Definition 5 (Inner Well-Posed Perturbations).** A family of hybrid systems \( \{H_\delta\} \) is said to be an inner well-posed perturbation of a hybrid system \( H \) on a set \( S \subset (cl(C) \cup D) \) such that \( (0,0) \) is an equilibrium point, and for every solution \( x \) of \( H \) originating from \( S \), the following holds:

- (○) given any sequence \( \{\xi_i\}_{i=0}^\infty \in cl(C) \cup D \) convergent to \( x(0,0) \) with \( \xi_i \in cl(C) \cup D \) for all \( i \geq 0 \), for every \( i \geq 0 \), there exists a solution \( x_i \) of \( H_\delta \) originating from \( \xi_i \) such that (a) and (b) in Definition 4 hold.

Outer well-posedness, being a property tailored towards robustness, considers specific families of hybrid systems, namely, those given by \( \rho \)-perturbations defined below, and requires the analogue of the graphical convergence property for nominal outer well-posedness to hold for all \( \rho \)-perturbations with continuous function \( \rho \). Every outer well-posed system is nominally outer well-posed, and hybrid basic conditions guarantee outer well-posedness as well as nominal outer well-posedness; see Theorem 23 in Appendix B. The relevant definitions (Goebel et al., 2012, Definitions 6.27 and 6.29) are recalled below for completeness.

**Definition 6 (\( \rho \)-Perturbation).** Given a hybrid system \( H = (C, F, D, G) \) and a function \( \rho : R^m \to R_{\geq 0} \), the \( \rho \)-perturbation of \( H \) is the hybrid system \( H^\rho \) with data \((C^\rho, F^\rho, D^\rho, G^\rho)\), where \( C^\rho = C + \rho \), \( F^\rho = F - \rho \), \( D^\rho = D \), and \( G^\rho = G \).
\[ C^o = \{ x : x + \rho(x)B \cap C \neq \emptyset \}, \quad D^o = \{ x : x + \rho(x)B \cap D \neq \emptyset \}, \]
\[ F^o(x) = c(\text{con} F(x + \rho(x)B \cap C)) + \rho(x)B, \]
\[ G^o(x) = \{ z : z \in y + \rho(y)B, \quad y \in G(x + \rho(x)B) \cap D \}, \]
for all \( x \in \mathbb{R}^n \), where \( \text{con} \) denotes the convex hull. Moreover, given any \( \delta \in (0, 1) \), \( \mathcal{H}^{\rho} \) denotes the \( \rho \)-perturbation of \( \mathcal{H} \).

The relationship between general families of hybrid systems \( \mathcal{H}_\delta \) and \( \rho \)-perturbations are made concrete by the notion of domination by a \( \rho \)-perturbation. Essentially, given a function \( \rho \), a family \( \mathcal{H}_\rho \) is dominated by the \( \rho \)-perturbation of \( \mathcal{H} \) if it can be overapproximated by the \( \rho \)-perturbation.

Definition 7 (Outer Well-Posedness). A hybrid system \( \mathcal{H} \) is said to be outer well-posed on a set \( S \subset \mathbb{R}^n \) if for every continuous function \( \rho \), every positive sequence \( \{ \rho_i \}_{i=0}^\infty \) convergent to zero, and every graphically convergent sequence of hybrid arcs \( \{ x_i \}_{i=0}^\infty \) such that \( x_i \) is a solution of \( \mathcal{H}^{\rho_i} \) and \( \lim_{i \to \infty} x_i(0, 0) =: x_0 \in S \), (a)–(b) in Definition 3 hold.

Theorem 9 (Existence of Optimal Solutions). Let \( \mathcal{H} = (C, F, D, G) \) be a hybrid system. Given a compact set \( K \), suppose that \( \mathcal{H} \) is nominally outer well-posed on \( K \) and pre-forward complete from \( K \). Then, given a closed constraint set \( \Omega \) and a cost function \( J \) that is lower semicontinuous on \( \Omega \), there exists an optimal solution of the optimal control problem (2) if it is feasible and \( \Omega \subset K \times \mathbb{T} \times \mathbb{X} \) for a compact set \( \mathbb{T} \subset \mathbb{R}_{>0} \times \mathbb{N} \) and some set \( \mathbb{X} \subset \mathbb{R}^m \).

Proof. If the set \( \Omega \subset K \times \mathbb{T} \times \mathbb{X} \) is closed and the set \( K \times \mathbb{T} \) is compact, the projection of \( \Omega \) onto \( \mathbb{R}^n \times (\mathbb{R}_{>0} \times \mathbb{N}) \), denoted \( c \), is compact. Observe that given a feasible solution \( x \), \( (x(0, 0), (T, J)) \in \mathcal{C} \), where \( (T, J) \) is the terminal time of \( x \). Construct an augmented hybrid system \( \mathcal{H}^\eta \) with state \( z := (\eta, s, i, x) \), where \( \eta \) represents the initial condition, \( (s, i) \) represents hybrid time, and \( x \) evolves according to the dynamics of \( \mathcal{H} \), given by
\[
\begin{align*}
J^\eta(z) := & \inf_{(x(0, 0), (T, J)) \in \mathcal{C}} J(x(0, 0), (T, J), x(T, J)) \quad z \in \mathbb{R}^n \\
\end{align*}
\]
with \( \mathcal{C} := \mathbb{R}^n \times \mathbb{R}_{>0} \times \mathbb{N} \times \mathbb{C} \) and \( D^\eta := \mathbb{R}^n \times \mathbb{R}_{>0} \times \mathbb{N} \times \mathbb{C} \). Since \( \mathcal{H} \) is nominally outer well-posed on \( K \), it is straightforward to show nominal outer well-posedness of \( \mathcal{H}^\eta \) on the compact set \( K^\eta := \{ z : \eta = x \in K, \ s = i = 0 \} \). Similarly, since \( \mathcal{H} \) is pre-forward complete from \( K \), one can show that \( \mathcal{H}^\eta \) is pre-forward complete from \( K^\eta \). From these two facts and Altın and Sanfelice (2020b, Proposition 4.2), the reachable set \( \mathbb{R}^n (c) \) is compact, where \( c := \{(z, T, J) : z \in K^\eta, (\eta, T, J) \in \mathcal{C} \} \). Consequently, the intersection of \( \mathbb{R}^n (c) \) and \( \Omega \) is compact. The optimal control problem (2) can then be recast as the minimization of the cost function \( J \) on this intersection. Since the optimal control problem (2) is feasible, the intersection must be nonempty, and the minimum of \( J \) on the intersection exists due to lower semicontinuity of \( J \). Hence, there exists an optimal solution.

4. Key properties of the hybrid optimal control problem: Existence and continuous dependency

We present our main results on existence of optimal solutions, along with regularity of the optimal cost and the set of optimal solutions to the hybrid optimal control problem in (2). The proof of existence reveals that the regularity properties of reachable set mappings are closely related to the aforementioned regularity properties of the optimal costs, since (2) can equivalently be represented as a finite-dimensional minimization problem over an appropriate reachable set of an augmented hybrid system. Our approach relies on exploiting this link using Berge’s maximum theorem (Aubin & Frankowska, 2009, Theorem 1.4.16). A stronger version of the theorem further enables us to conclude regularity, more precisely, upper/outer semicontinuity of the set of optimal solutions.

Before introducing our results, we note that the terminology concerning (2) is standard: the hybrid optimal control problem (2) is said to be feasible if there exists a solution \( x \) of \( \mathcal{H} \) that respects the constraint in (2), with \( x \) referred to as a feasible solution of (2). The optimal cost of the problem, denoted \( J^*_\rho (\Omega) \), is the infimum of \( J \) over all feasible solutions, with \( J^*_\rho (\Omega) := \inf_{\Omega} J(x(0, 0), (T, J), x(T, J)) \) if (2) is not feasible, i.e.,
\[
J^*_\rho (\Omega) := \inf_{x(0, 0) \in \mathbb{T}, (T, J) \in \Omega} J(x(0, 0), (T, J), x(T, J)),
\]
where \( (T, J) \) denotes the terminal time of \( x \). A feasible solution of (2) that attains the infimum \( J^*_\rho (\Omega) \) is said to be an optimal solution of (2).

4.1. Existence of optimal solutions and upper semicontinuity of the optimal cost

Within the setting of nominally outer well-posed hybrid systems, proving existence of optimal solutions is fairly straightforward under standard regularity conditions. This approach differs from the one in Goebel (2019), in that it requires no assumptions on the corresponding optimal control problem of the underlying continuous-time system.

---

\[ \text{Proof.} \]

The value function corresponding to \( \mathcal{H} \) is the function \( \mathcal{H} \) maps \( x_0 \) to \( J^*_\rho (\Omega) \) in the specific case of \( \Omega = \{ x_0 \} \times c \) for some \( c \subset \mathbb{R}_{>0} \times \mathbb{N} \times \mathbb{R}^n \).
is also locally bounded and outer semicontinuous at the origin. Now, construct the augmented hybrid system \( H' \) in (3), which is nominally outer well-posed on the set \( K'(0) \), and note that the reachable set mapping \( \mathcal{R}_{H'} \) is outer semicontinuous and locally bounded at \( c'(0) \) by Altın and Sanfelice (2020b, Theorem 4.1). Then, the mapping from \( \vartheta \) to \( \mathcal{R}_{H'}(c'(_{\vartheta})) \cap M(\vartheta) \) is also outer semicontinuous and locally bounded at the origin. Equivalently, it is upper semicontinuous (Goebel et al., 2012, Lemma 5.15). Recasting the optimal control problem as the minimization of \( J \) on this intersection (for each \( \vartheta \), Berge’s maximum theorem (Aubin & Frankowska, 2009, Theorem 1.4.16) is applicable, and lower semicontinuity of \( J \), combined with the upper semicontinuity of the aforementioned intersection, leads to upper semicontinuity of \( \vartheta \mapsto J^*_H(M(\vartheta)) \).

For a fixed-time initial value problem without terminal constraints (i.e., the set \( \Omega \) in (2) is of the form \( [x_0] \times \{(T,J)\} \times \mathbb{R}^n \)), one can simply take \( M(x_0, T, J) = [x_0] \times \{(T, J)\} \times \mathbb{S} \) and invoke Theorem 10 to conclude upper semicontinuity of the value function, where \( S \) is an arbitrary compact set that contains the reachable set \( \mathcal{R}_{H}(x_0, T, J) \), as the reachable set is compact by Altın and Sanfelice (2020b, Proposition 4.2). Moreover, upper semicontinuous dependence of the value function on initial conditions can easily be extended to show upper semicontinuous dependence of the optimal cost on the magnitude of perturbations on the constraints (flow/jump set of the hybrid system and the mixed constraint set \( \Omega \)) and mappings (flow/jump maps and the cost function) defining the optimal control problem.

Theorem 11. Let \( H \) be a hybrid system and given a compact set of initial conditions \( K \), suppose that \( H \) is outer well-posed on \( K \) and pre-forward complete from \( K \). Consider a closed constraint set \( \Omega \) such that \( \Omega \subset K \times T \times \mathbb{R}^n \) for a compact set \( T \subset \mathbb{R}_{\geq 0} \times \mathbb{N} \). Let \( \{H_\theta\} \) be a family of hybrid systems dominated by the \( \rho \)-perturbation of \( H \) for some continuous function \( \rho \). Moreover, let \( S \subset \mathbb{R}^n \) be a set containing the origin and \( M : S \to \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{H}^n \) be a set-valued mapping that is locally bounded and outer semicontinuous at the origin, with \( M(0) = \emptyset \). For each \( \theta \in \Theta \), let \( \mathcal{J}(\cdot;\theta) \) be a cost function. Suppose that the function \( (\xi(\cdot), \theta) \mapsto J(\xi(\cdot);\theta) \) is lower semicontinuous at \( (\xi^*, 0) \) for all \( \xi^* \in \Omega \). Then, the function \( (\delta, \theta) \mapsto J^*_H(M(\theta); \vartheta) \) with \( J^*_H(M(\theta); \vartheta) := J^*_H(M(\theta); \vartheta) \) for all \( \theta \in \mathbb{R}^n \), is upper semicontinuous at the origin.

The proof of Theorem 11 is almost the same as that of Theorem 10. It is omitted for brevity. The required outer semicontinuity and local boundedness properties for the reachable set are proved in Altın and Sanfelice (2023, Theorem 29).

4.2. Continuity of the optimal cost and outer/upper semicontinuity of optimal solutions

Lower semicontinuous, and more strongly, continuous dependence of the optimal cost on perturbations on constraint sets (flow/jump set and the mixed constraint set \( \Omega \)) and mappings (flow/jump map and the cost function \( \mathcal{J} \)) can similarly be established within the setting of inner well-posedness. Remarkably, the assumptions we use to prove these properties elegantly lead to outer/upper semicontinuous dependence of the set of optimal solutions on constraints and mappings; see Theorem 16. In establishing these stronger results, for simplicity and brevity, we focus on fixed-time initial value problems without terminal constraints. That is, we develop our results by focusing on (2) with the constraint set \( \Omega = [x_0] \times \{(T, J)\} \times \mathbb{R}^n \) (fixed initial value, fixed time, no terminal constraints).

The main results in this subsection consider perturbations to both the hybrid system and the cost function, i.e., they consider parametrized families of hybrid systems and cost functions. Results concerning the nominal case are recovered as immediate corollaries. One key assumption that we make is that the points belonging to the reachable set \( \mathcal{R}_{H}(x_0, T, J) \) correspond to maximal solutions of the hybrid system \( H \) that originate from \( x_0 \) and do not jump or terminate at ordinary time \( T \). This allows us to conclude lower semicontinuous, and where appropriate, continuous dependence of the reachable set mappings on their arguments and parameters. As shown below, for lower semicontinuity of the optimal cost, it suffices to assume inner well-posedness.

Theorem 12. Let \( H \) be a hybrid system. Given an initial condition \( x_0 \) and \( (T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \), suppose that the reachable set \( \mathcal{R}_{H}(x_0, T, J) \) is nonempty. Then, for every \( \xi \in \mathcal{R}_{H}(x_0, T, J) \), there exists \( x \in \mathcal{S}_{H}(x_0) \) such that \( \xi = x(T, J) \) and \( T \) is not a jump time or the terminal ordinary time of \( x \). Let \( \{H_{\delta} = (c_{\delta}, F_{\delta}, D_{\delta}, G_{\delta})\} \) be an inner well-posed perturbation of \( H \) at \( x_0 \). Given a set \( S \subset \mathbb{R}^n \) containing the origin, for each \( \theta \in \Theta \), consider a cost function \( \mathcal{J}(\cdot; \theta) \), and suppose that the perturbation \( (\xi(\cdot), \theta) \mapsto \mathcal{J}(\xi(\cdot); \theta) \) is upper semicontinuous at \( (\xi^*, 0) \) for all \( \xi^* \in \Omega \) defined by \( H \).

Proof. We go through a reachability analysis as in Theorems 9 and 10. However, augmentation of the system is not necessary since there are no terminal constraints and the constraints are not mixed. That is, we are interested in instances of problem (2), for which the constraint \( x(0,0), (T, J), x(T, J) \in \Omega \) can be rewritten in the form \( x(0,0) = \xi, (T, J) = (s, i) \). Consequently, given \( \delta, \theta \), and \( (x_0^*, T, J^*) \), the scalar \( h_{\delta, \theta}(x_0^*, T, J^*) \) is the minimum of \( \mathcal{J}(\cdot; \theta) \) on the reachable set \( \mathcal{R}_{H}(x_0^*, T, J^*) \). Noting that a) the family of reachable set mappings depend inner/lower semicontinuously on its arguments and the parameter \( \delta \) by Altın and Sanfelice (2023, Theorem 30), i.e.,

\[
\mathcal{R}_{\mathcal{H}}(x_0, T, J) \subset \bigcap_{\delta > 0} \bigcap_{\theta > 0} \mathcal{R}_{\mathcal{H}}(x_0^*, T, J^*)
\]

and b) the reachable set \( \mathcal{R}_{H}(x_0, T, J) \) is nonempty, it follows that upper semicontinuity of \( \mathcal{J} \) implies (5), c.f. Berge’s maximum theorem (Aubin & Frankowska, 2009, Theorem 1.4.16).

Corollary 13. Let \( H = (C, F, D, G) \) be a hybrid system. Given an initial condition \( x_0 \) and \( (T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \), suppose that the reachable set \( \mathcal{R}_{H}(x_0, T, J) \) is nonempty and \( H \) is nominally inner well-posed at \( x_0 \). Moreover, suppose that for every \( \xi \in \mathcal{R}_{H}(x_0, T, J) \), there exists \( x \in \mathcal{S}_{H}(x_0) \) such that \( \xi = x(T, J) \).

Proof. By invoking Theorem 11, one can consider a “terminal constraint set” \( S \) containing the reachable set \( \mathcal{R}_{H}(x_0, T, J) \) in its interior. Its immediate corollary, Corollary 15, can also be concluded by combining Theorem 10 and Corollary 13.
Theorem 14 (Continuity of the Optimal Cost). Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system, and given an initial condition $x_0$, suppose that $\mathcal{H}$ is outer well-posed at $x_0$ and pre-forward complete from $x_0$. Let $\{\mathcal{H}_t\} = (C_t, F_t, D_t, G_t)$ be an inner well-posed perturbation of $\mathcal{H}$ at $x_0$ that is dominated by a $\rho$-perturbation of $\mathcal{H}$ for some continuous function $\rho$. Moreover, given $(T, J) \in \mathbb{R}_+ \times \mathbb{N}$, suppose that the reachable set $\mathcal{R}_{\mathcal{H}_t}(x_0, T, J)$ is nonempty and for every $\xi \in \mathcal{R}_{\mathcal{H}_t}(x_0, T, J)$, there exists $x \in \mathcal{S}_p(x_0)$ such that $\xi = x(T, J)$ and $T$ is not a jump time or the terminal time of $x$. Given a set $S \subset \mathbb{R}^n$ containing the origin, for each $\theta \in S$, consider a cost function $J(\cdot; \theta)$, and suppose that the function $(\xi, \theta) \mapsto J(\xi; \theta)$ is continuous at $(\xi^*, 0)$ for all $\xi^* \in \Omega := \{x_0 \times \{(T, J)\} \times \mathbb{R}^n$. Then, the scalar $h(x_0, T, J)$ in (5) and the family of functions $\{h_\theta\}$ in (4) satisfy

$$h(x_0, T, J) = \lim_{\delta \to 0, \theta \to 0} h_\theta(x_0, T^*, J; \theta).$$

Corollary 15 (Continuity of the Optimal Cost). Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system, and given an initial condition $x_0$, suppose that $\mathcal{H}$ is nominally outer and inner well-posed at $x_0$ and pre-forward complete from $x_0$. Given $(T, J) \in \mathbb{R}_+ \times \mathbb{N}$, suppose that the reachable set $\mathcal{R}_{\mathcal{H}_t}(x_0, T, J)$ is nonempty and for every $\xi \in \mathcal{R}_{\mathcal{H}_t}(x_0, T, J)$, there exists $x \in \mathcal{S}_p(x_0)$ such that $\xi = x(T, J)$ and $T$ is not a jump time or the terminal time of $x$. Consider a cost function $J(\cdot; \theta)$ that is continuous at all $\xi^* \in \Omega := \{x_0 \times \{(T, J)\} \times \mathbb{R}^n$. Then, the function $h$ in (6) is continuous at $(x_0, T, J)$.

Moreover, under the conditions of Theorem 14, the set of optimal solutions depend on perturbations in an outer/inner semi-continuous manner, as shown below. In Theorem 16, for fixed $\delta > 0$ and $\theta \in S$, $O(x_0, T^*, J^*; \theta)$ denotes the set of optimal solutions of the optimal control problem with hybrid system $\mathcal{H}_t$, constraint set $\{x_0 \times \{(T, J)\} \times \mathbb{R}^n$, and cost function $J(\cdot; \theta)$ in the same fashion. $O(x_0, T, J)$ denotes the set of optimal solutions of the optimal control problem with hybrid system $\mathcal{H}$, constraint set $\{x_0 \times \{(T, J)\} \times \mathbb{R}^n$, and cost function $J(\cdot; 0)$.

Theorem 16 (Optimal Solutions). Under the conditions of Theorem 14, the following statements are true.

Local Boundness: There exist $\varepsilon > 0$ and a compact set $K$ such that

$$\delta \in (0, \varepsilon], \theta \in \mathbb{R}, x' \in x_0 + \varepsilon B, (T', J') \in (T, J) + \varepsilon B \implies x(T', J') \in K$$

for all $(t, j) \in \text{dom} \chi$ and $x \in O_{\mathcal{H}_t}(x_0, T^*, J^*; \theta)$.

Outer Semi-continuity: Let $\{\delta_i\}_{i=0}^{\infty}$ be a positive sequence convergent to zero, $[\theta_i]_{i=0}^{\infty} \subset S$ be a sequence convergent to the origin, and $[x_i]_{i=0}^{\infty}$ be a graphically convergent sequence of optimal solutions such that $x_i \in O_{\mathcal{H}_t}(x_0, T, J; \theta_i)$ for all $i \geq 0$. Then, sequences $\{\delta_i\}_{i=0}^{\infty}$ and $[x_i]_{i=0}^{\infty}$ converge to $x_0$ and $(T, J)$, respectively, $x \in O_{\mathcal{H}_t}(x_0, T, J)$ is such that $x$ is $\varepsilon$-close. As shown earlier, optimal solutions are locally bounded, hence the sequence $[x_i]_{i=0}^{\infty}$ is locally eventually bounded. Using (Goebel et al., 2012, Theorem 6.1) and without relabeling, we pass to a graphically convergent subsequence. The limit of the sequence, say $x^*$, is then optimal by our prior conclusion. That is, $x^* \in O_{\mathcal{H}_t}(x_0, T, J)$. However, by Goebel et al. (2012, Theorem 5.25), for large enough $i$, $x^*$ is $\varepsilon$-close, which is a contradiction.

Similarly, in the following, $O(x_0, T', J')$ denotes the set of optimal solutions of the optimal control problem with hybrid system $\mathcal{H}$, constraint set $\{x_0 \times \{(T', J')\} \times \mathbb{R}^n$, and cost function $J$.

Corollary 17 (Optimal Solutions). Under the conditions of Corollary 15, the following statements are true.

Local Boundness: There exist $\varepsilon > 0$ and a compact set $K$ such that

$$x' \in x_0 + \varepsilon B, (T', J') \in (T, J) + \varepsilon B \implies x(T', J') \in K$$

for all $(t, j) \in \text{dom} \chi$ and $x \in O_{\mathcal{H}_t}(x_0, T', J')$.

Outer Semi-continuity: Let $[x_i]_{i=0}^{\infty}$ be a graphically convergent sequence of optimal solutions such that $x_i \in O_{\mathcal{H}_t}(\xi_i, T_i, J_i)$ for all $i \geq 0$. Then, if the sequences $[\xi_i]_{i=0}^{\infty}$ and $[(T_i, J_i)]_{i=0}^{\infty}$ converge to $\xi_0$ and $(T, J)$, respectively, $x \in O_{\mathcal{H}_t}(x_0, T, J)$, where $x$ is the graphical limit of $[x_i]_{i=0}^{\infty}$.
Upper Semicontinuity: For all $\tau \geq 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that the following holds: for every $x_0 \in x_0 + \eta B$, $(T', J') \in (T, J) + \eta B$, and $x' \in C_p(x_0, T', J')$, there exists $x \in C_p(x_0, T, J)$ such that $x$ and $x'$ are $(\tau, \varepsilon)$-close.

5. Remarks on continuous approximation of the optimal cost and upper semicontinuous approximation of optimal solutions

As seen so far, the regularity properties of reachable set mappings are closely related to those of the optimal control problem, since the optimal control problem in (2) is equivalent to a finite-dimensional minimization problem over an appropriate reachable set. The downside to this approach is that since reachable set mappings are not continuous in general (Altın & Sanfelice, 2023; Collins, 2011), it might be difficult to argue continuous or lower semicontinuous dependence of the optimal cost with respect to the constraints of the optimal control problem. Indeed, vaguely speaking, the results in Section 4.2 establishing continuity of the optimal cost and semicontinuity properties of optimal solutions require that the parameter $T$ therein is not a jump time or the terminal ordinary time of a solution; see, e.g., the second sentence in the statement of Theorem 12. Aside from requiring partial knowledge of solutions, these results cannot guarantee continuity properties of the optimal control problem globally (for example continuity of the optimal cost may not be achievable at certain combinations of initial condition and hybrid time). However, it is easy to show that if the reachable set mappings corresponding to a parametric optimal control problem vary continuously at points related to the problem parameters, the optimal cost (respectively, optimal solutions) varies continuously (respectively, outer/upper semicontinuously) when there are no terminal constraints (i.e., the constraint set $\Omega$ in (2) is of the form $C \times \mathbb{R}^n$).

Fortunately, the results reported in Altın and Sanfelice (2023) show that when the solutions to hybrid systems depend upper and lower semicontinuously on initial conditions and perturbations, the reachable sets can be varied/approximated continuously, provided certain class-$\mathcal{K}_\infty$ bounds are respected, see Theorems 32 and 34 therein. For completeness, we include Altın and Sanfelice (2023, Theorem 32), restated to omit perturbations to $\mathcal{H}$ and only consider a single initial condition and hybrid time.

Theorem 18 (Altın & Sanfelice, 2023, Theorem 32). Let $\mathcal{H}$ be a hybrid system, and given an initial condition $x_0$, suppose that $\mathcal{H}$ is nominally inner and outer well-posed at $x_0$ and pre-forward complete from $x_0$. Then, for any hybrid time $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, there exists a class-$\mathcal{K}$ function $\alpha$ such that for every $x_0 \in K$ and $(T, J) \in T$,

$$
\lim_{\varepsilon \to 0^+} \mathcal{R}_\mathcal{H}(x_0, [\max\{0, T - \varepsilon\}, T + \varepsilon], J) = \mathcal{R}_\mathcal{H}(x_0, T, J).
$$

Continuity of the optimal cost and outer/upper semicontinuity of the optimal solutions of the corresponding problem can then be concluded as in the proofs of the main results, via reachability analysis. Accordingly, we state the former result regarding continuity of the optimal cost, without proof.

Theorem 19. Let $\mathcal{H}$ be a hybrid system, and given an initial condition $x_0$, suppose that $\mathcal{H}$ is nominally inner and outer well-posed at $x_0$ and pre-forward complete from $x_0$. Given a hybrid time $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, suppose that the reachable set $\mathcal{R}_\mathcal{H}(x_0, T, J)$ is nonempty. Consider a cost function $J$ that is continuous at all $t \in T := \{x_0\} \times \{T, J\} \times \mathbb{R}^n$, and let

$$
h(x_0, \varepsilon) = J''(x_0, [\max\{0, T - \varepsilon\}, T + \varepsilon], J) \times \mathbb{R}^n.
$$

Then, there exists a class-$\mathcal{K}$ function $\alpha$ such that

$$
\lim_{\varepsilon \to 0^+} h(x_0, \varepsilon) = h(x_0, 0),
$$

where $\forall \mathcal{H}_0 \in \mathcal{R}(C \cup D, \varepsilon \geq 0)$.

6. Examples

In this section, we consider concrete finite horizon optimization problems for hybrid plants given by

$$
\mathcal{H}_P \left\{ \begin{array}{l}
\dot{x}_P \in F_P(x_P, u) \quad (x_P, u) \in C_P \\
\dot{x}'_P \in G_P(x_P, u) \quad (x'_P, u) \in D_P,
\end{array} \right.
$$

where $C_P$ is the flow set, $F_P$ is the flow map, $D_P$ is the jump set, and $G_P$ is the jump map. A solution of $\mathcal{H}_P$ is defined by a pair (called a solution pair $(x, J)$) on a hybrid time domain $(x_P, u)$ satisfying the dynamics of $\mathcal{H}_P$, in a similar manner as the way a solution of the (closed-loop) hybrid system $\mathcal{H}$ in (1) is defined in Section 2.1. Given a solution pair $(x_P, u)$ with compact domain, the associated cost is defined by

$$
\sum_{j=0}^{T_{j+1}} L_{C_P}(x_P(t, j), u(t, j)) dt + \sum_{j=0}^{T_{j+1}} L_{D_P}(x_P(t_{j+1}, j), u(t_{j+1}, j)) + V(x_P(T, J)).
$$

where $T_{j+1}$ is the $j$th jump time and $(T, J) \in \text{dom}(x_P, u)$ is the terminal time, i.e.,

$$
\text{dom}(x_P, u) = \bigcup_{j=0}^{T_{j+1}} (\{t_j\} \times \{j\})
$$

and $T = T_{j+1}$. In (9), the first term $L_{C_P}$ is the stage cost capturing the cost over intervals of flows, $L_{D_P}$ is the stage cost capturing the cost to jump, and $V$ is the terminal cost. This leads to the following finite horizon hybrid optimization problem.

Problem 6.1. Given a hybrid system $\mathcal{H}_P$ as in (9), a stage cost for flows $L_{C_P}$, a stage cost for jumps $L_{D_P}$, a terminal cost $V$, a terminal constraint set $X_P$, a hybrid time $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, and an initial condition $\xi$, find a solution pair $(x_P, J)$ minimizing (10) subject to

- the initial condition constraint $x_P(0, 0) = \xi$, and
- the terminal condition constraint $x_P(T, J) \in X_P$.

Note that the flow and jump sets of $\mathcal{H}_P$ impose constraints that the solution pair needs to satisfy during flows and jumps, respectively. In fact, for the solution pair to exist up to hybrid time $(T, J)$ it has to reside in $C_P$ and $D_P$ as (Sanfelice, 2021, Definition 2.29) indicates, $(x_P, u)$ is a solution of $\mathcal{H}_P$ if $(x_P(0, 0), u(0, 0)) \in \mathcal{R}(C_P \cup D_P)
$ and

- for each $j \geq 0$, $(x_P(t, j), u(t, j)) \in C_P$ for all $t \in I^j$ and $(x'_P(t, j), u(t, j)) \in C_P$ for all $t \in I^j$, where $I^j := \{t_j \in \text{dom}(x_P, u)\};$
- for each $(t, j) \in \text{dom}(x_P, u)$ such that $(t, j+1) \in \text{dom}(x_P, u)$, $(x_P(t, j+1), u(t, j+1)) \in D_P$ and $x_P(t, j+1) \in X_P(x_P(t, j), u(t, j)).$

Given $\xi \in \mathcal{R}(C_P \cup D_P)$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, $h(\xi, T, J)$ denotes the value of (10) given a minimizing solution pair $(x_P, u)$ of $\mathcal{H}_P$ subject to the constraints $x_P(0, 0) = \xi$ and $x_P(T, J) \in X_P.$
6.1. Thermostat

A model capturing the evolution of the temperature of a room controlled by a heater that can either be on or off is given by

\[ \dot{z} = -z + z_q + z_q \Delta, \]

where \( z \in \mathbb{R} \) is the temperature of the room, \( z_q \in \mathbb{R} \) denotes the effective temperature outside of the room, \( z_\Delta > 0 \) represents the capacity of the heater, and \( q \in \{0, 1\} \) represents whether the heater is on (\( q = 1 \)) or off (\( q = 0 \)). Using this model, we are interested in designing a control algorithm that fulfills the following:

(a) steer the temperature to a desired range \([z_{\min}, z_{\max}]\), where \( z_0 < z_{\min} < z_{\max} < z_0 + z_\Delta ; \) and

(b) minimize the number of on/off switches of the heater.

To meet these specifications, we properly define the elements in Problem 6.1. The desired steering property can be guaranteed by selecting the flow cost \( L_{C_F} \) as a smooth indicator of the set \([z_{\min}, z_{\max}]\). The jump cost \( L_{D_P} \) can be used to penalize switches from on to off as well as from off to on. Recall that (10), which defines a cost functional, evaluates the jump cost at the current value of the solution pair. For the particular case of controlling the temperature, the jump cost should only depend on the current value of \( q \). Furthermore, since \( q \) can only change its value at the switches, it needs to be forced to remain constant in between switches. To facilitate the formulation of the optimization problem, we treat \( q \) as an additional logical state and incorporate an input, denoted \( u \in \{0, 1\} \), playing the role of the decision variable for the optimization problem. The resulting system is given as in (9), with state \( x_0 = (z, q) \in \mathbb{R} \times \{0, 1\} \), input \( u \in \{0, 1\} \), and data \((C_F, F, D_P, G)\) given by

\[
C_F = \{(x, u) : q \in \{0, 1\}, u = 0\},
F_F(x_0, u) = (-z + z_q + z_\Delta, 0) \quad \forall (x, u) \in \mathbb{R}^2,
D_P = \{(x, u) : q \in \{0, 1\}, u = 1\},
G_P(x_0, u) = (z, 1 - q) \quad \forall (x, u) \in \mathbb{R}^2.
\]

With this data, flows of the plant are allowed when \( u = 0 \). In this regime, the temperature \( z \) evolves according to its continuous-time model and \( q \) remains constant due \( F_F \) leading to \( q = 0 \). At jumps, which are triggered when \( u \) is equal to one, the update law \( 1 - q \) toggles the value of \( q \) from 0 to 1 or from 1 to 0.

With this hybrid model, we specify the stage cost for flows \( L_{C_F} \), the stage cost for jumps \( L_{D_P} \), the terminal cost \( V \), and the terminal constraint set \( X_F \) associated with Problem 6.1. The flow cost can be defined as an indicator of the set \( A^T := [z_{\min}, z_{\max}] \) that is smooth enough. One suitable choice is the globally Lipschitz function

\[ L_{C_F}(x_0, u) := L_{C_F}(z) := \|z\|_A^T \quad \forall (x_0, u) \in C_F, \]

which depends only on the temperature \( z \). The jump cost is defined as a continuous function that penalizes switches. Exploiting the fact that \( q \) is a state variable, a suitable choice capturing the cost of either transition is

\[ L_{D_P}(x_0, u) := L_{D_P}(q) := c_{1 \rightarrow 0} q + c_{0 \rightarrow 1} (1 - q) \quad \forall (x_0, u) \in D_P, \]

where \( c_{1 \rightarrow 0} \) and \( c_{0 \rightarrow 1} \) are nonnegative constants that quantify the cost of switching the heater from on to off and from off to on, respectively. The terminal cost \( V \) is chosen to be equal to \( L_F \), so as to quantify the distance to the desired temperature range, and the terminal constraint set \( X_F \) is simply chosen to be equal to the closed set \( \mathbb{R} \times \{0, 1\} \).

6.1.1. Reformulation of the optimal control problem

Next, we pass to the Mayer form in (2) by defining the associated hybrid system \( \mathcal{H} \) with state \( x := (x_0, \ell) = (z, q, \ell) \), where \( \ell \) is the running cost, with the data \((C, F, D, G)\) defined as

\[ C := \{x : q \in \{0, 1\}\}, \]

\[ F(x) := (-z + z_q + z_\Delta, 0, L_C(z)) \quad \forall x \in C, \]

\[ D := C, \]

\[ G(x) := (z, 1 - \ell, q + L_D(q)) \quad \forall x \in D. \]

Note that in this closed-loop formulation, since \( C = D \), jumps can occur at any time.

Given an initial condition \( \xi \) for \((z, q) \) and hybrid time \((T, J) \in \mathbb{R} \times \mathbb{N} \), the constraint set \( \Omega \) is chosen as

\[ \Omega = \{x_0\} \times [(T, J)] \times X \]

with \( x_0 := (\xi, 0) \) and \( X := X_F \times \mathbb{R} \times \{0, 1\} \times \mathbb{R} \), and the cost function

\[ J(x_0, (T, J), x_f) = \ell + V(z_f) = \ell + L_C(z_f), \]

where \( x_f := (z_f, q_f, \ell_f) \).

6.1.2. Existence of solutions and upper semicontinuity of the optimal cost

To answer the question of existence of solutions to Problem 6.1 with the choices above, we apply Theorem 9 to the Mayer formulation of this problem. We first note that the hybrid system \( \mathcal{H} \) constructed above is nominally outer well-posed according to Theorem 23. Indeed, the sets \( C \) and \( D \) are closed, which implies that \((A1)\) therein holds. Moreover, since \( L_{C_F} \) and \( L_{D_P} \) are continuous (in fact, globally Lipschitz) by definition, the flow map \( F \) and the jump map \( G \) are (globally Lipschitz) continuous single-valued maps. Hence, items \((A2)\) and \((A3)\) hold. Furthermore, for the given initial condition \( \xi \), the set \( K := \{x_0\} \) is compact. In addition, the cost function \( J \) is globally Lipschitz since \( V = L_C \) is globally Lipschitz. Finally, we observe that the hybrid system \( \mathcal{H} \) is such that every maximal solution is complete. In fact, due to the form of \( C \) and \( D \) combined with the regularity properties of \( F \), (Goebel et al., 2012, Proposition 6.10) implies that there exists a nontrivial (i.e., consisting of more than a single point) solution from each initial condition in \( C \cup D \) that every maximal solution is complete. More strongly, from every initial condition in \( C \cup D \), one can find a complete solution that is continuous (only flows), and similarly, a complete solution that is discrete (only jumps).

Using the properties established above, by Theorem 9, there exists an optimal solution to the optimal control problem (2) if it is feasible for the given hybrid time \((T, J) \) defining the compact set \( \tau := [(T, J)] \), which implies that Problem 6.1 has a solution. Moreover, by Corollary 13, the optimal cost varies upper semicontinuously. Clearly, if \( J = 0 \), then, due to existence of complete continuous solutions from every point in \( C = C \cup D \), there is always a feasible solution. For the case of arbitrary \( J \), one can than extend this feasible solution by trivial jumps; that is, one can extend the aforementioned continuous solution from hybrid time \((T, 0) \) to \((T, J) \) with \( J \) consecutive jumps. Since the terminal constraint set does not impose any constraint on \( z \), it turns out that any choice of \((T, J) \in \mathbb{R} \times \mathbb{N} \) leads to feasibility. These findings are summarized as follows.

Theorem 6.2. Given the cost functions \( L_{C_F} \) and \( L_{D_P} \) defined in (11)–(12), suppose that the terminal cost function \( V(x_f) = L_C(x_0, u) \) for every \((x_0, u) \in C = C \cup D \) and the terminal constraint set \( X_F = \mathbb{R} \times \{0, 1\} \). Then, Problem 6.1 can be solved, and the optimal cost function \( h \) is upper semicontinuous.
6.2. Bouncing ball

Consider a ball bouncing vertically on a horizontal flat surface, whose motion is modeled by the controlled hybrid system $\mathcal{H}_p = (\mathcal{C}_p, \mathcal{F}_p, \mathcal{D}_p, \mathcal{G}_p)$, where:

$$\mathcal{C}_p = \{(x_p, u) : \rho \geq 0, u \in [u_{\min}, u_{\max}]\}$$

$$\mathcal{F}_p(x_p, u) = (v, -\gamma) \quad \forall (x_p, u) \in \mathbb{R}^2$$

$$\mathcal{D}_p = \{(x_p, u) : p = 0, v \leq 0, u \in [u_{\min}, u_{\max}]\}$$

$$\mathcal{G}_p(x_p, u) = (0, -\lambda u + v) \quad \forall (x_p, u) \in \mathbb{R}^2$$

for some $u_{\max} \geq u_{\min} \geq 0$, $x_p = (p, v)$ is the state with $p \geq 0$ representing the position (height), $u$ the velocity of the ball, $\gamma > 0$ is the gravitational acceleration, and $\lambda \in [0, 1)$ is the coefficient of restitution. This system augments the canonical (autonomous) bouncing ball model with an input $u$ that affects the post-jump velocity.

As with the thermostat, Problem 6.1 is equivalent with (2). Given a particular instance of Problem 6.1 for the bouncing ball, the autonomous hybrid system $\mathcal{H} = (C, F, D, G)$ arising in this conversion to Mayer form has state $x = (p, v, \ell)$ (with $\ell \in \mathbb{R}$ representing the running cost), where $C = \{x : p \geq 0\}$, $D = \{x : p = 0, v \leq 0\}$, and for every $x \in \mathbb{R}$,

$$F(x) = (v, -\gamma, 0, L_c(p, v))$$

$$G(x) = \{(0, -\lambda u + v, \ell + V(p, v)) : u \in [u_{\min}, u_{\max}]\}$$

(13)

The cost function $L_c$ above is introduced to simplify the problem and replace $L_p$, as the flow map $F$ does not depend on $u$. Similarly, the cost function $L_v$ is introduced as $p = 0$ at jumps. The constraint set of the problem is given as $\mathcal{C} = \{x_0 \times [(T, J)] \times X \times X_0 = (\xi, 0) \quad \mathcal{X} = X \times \mathbb{R}$, and the cost function $J(x, (T, J), \eta) = \ell + V(p, v)$, where $x_0$ and $v$ are the terminal constraint set and terminal cost function in Problem 6.1.

6.2.2. Continuity of the optimal cost and outer/upper semicontinuity of optimal solutions

To conclude stronger properties about Problem 6.1, we assume that the terminal cost $V$ is continuous on the set $C$, which would imply that the resulting closed-loop cost function $V$ is continuous on $c(C) \cup D$, and the terminal constraint set $x_0 = \mathbb{R}^n$. In the sequel, we rely on Corollaries 15 and 17. To invoke these corollaries, it is necessary and sufficient to also show that the reachable set $R_{\mathcal{H}}(x_0, T, J)$ is nonempty, and for every $\xi \in R_{\mathcal{H}}(x_0, T, J)$, there exists $x \in S_{\mathcal{H}}(x_0)$ such that $\xi = x(T, J)$ and $T$ is not a jump time or the terminal ordinary time of $x$.

Given the initial condition $\xi = (\xi_1, \xi_2)$ and a parameter $v \in [u_{\min}, u_{\max}]$, let $(x, u)$ be the unique solution pair with $x(0, 0) \in \xi$ and $\text{dom}(x, u)$ unbounded, satisfying $u(t, j) = u(s, j) = v$ for all $(t, j), (s, j) \in \text{dom}(x, u)$. Existence of such a pair is easy to show following an analysis similar to the one in the previous subsection. Regardless of the choice of the input $v$, the first impact with the ground occurs at ordinary time $(\xi_2 + \sqrt{\xi_2^2 + 2\gamma \xi_1})/\gamma$ (Goebel et al., 2012, Example 2.12) with velocity $-\sqrt{\xi_2^2 + 2\gamma \xi_1}$ on the latter can be derived using conservation of energy during flows. Given $j \geq 1$, let $t_j$ be the ordinary time of jump $j$ and let $v_j \geq 0$ be the velocity of the ball immediately after jump $j$, i.e., $v_j := x(t_j, J)$. Then, $v_j = \lambda \sqrt{\xi_2^2 + 2\gamma \xi_1} + v$. Moreover, $t_{j+1} = t_j + t_j/\gamma$ by Goebel et al. (2012, Example 2.12) and $v_{j+1} = \lambda v_j + v \geq 0$ (again due to conservation of energy, which implies that the velocity right after jump $j$ and right before jump $j + 1$ differ only in their sign) for all $j \geq 1$. From these equations, one can then derive

$$t_1 = \left(\xi_2 + \sqrt{\xi_2^2 + 2\gamma \xi_1}\right)/\gamma$$

(14)

$$t_v = t_1 + \frac{2}{\gamma(1-\lambda)}\psi(v, J) \quad \forall v \geq 0,$$

where the superscript $v$ is included to indicate dependency on the input parameter $v$, and

$$\psi(v, J) := (j - 1)v + \left(\lambda \sqrt{\xi_2^2 + 2\gamma \xi_1} + \frac{v}{1-\lambda}\right)(1-\lambda^{j-1})$$

(15)
with \( v_1 \) indicating the (post-impact) velocity after the first jump, i.e. \( x(t_1, 1) \). Since \( t_j^* \) is an increasing function of \( v \) for fixed \( j \), one can then infer the following: (a) the reachable set \( R_{\xi^*}(x_0, T, J) \) of the augmented autonomous system is nonempty if \( t_j^* \leq T \leq t_j^{max} \), (b) for every \( \xi \in R_{\xi^*}(x_0, T, J) \) there exists \( x \in S_H(x_0) \) such that \( \xi = x(T, J) \) and \( T \) is not a jump time or the terminal ordinary time of \( x \) if \( t_j^{max} \leq T \leq t_j^{min} \).

Using these two deductions, we reach the following result, where \( \mathcal{O}_{\xi}(\xi, T, J) \) denotes the set of optimal solution pairs of Problem 6.1; that is, the set of solution pairs \( (\mathcal{X}_P(t, J), u(t, J)) \) subject to the constraints \( x_0(0) = \xi \) and \( x(T, J) \in X_P \) minimizing (10).

Theorem 21. Suppose that the cost functions \( L_C \) and \( L_D \) are continuous on the sets \( C = \{ (p, v) : h \geq 0 \} \) and \( D = \{ (v, u) : v \leq 0, u \in [u_{min}, u_{max}] \} \), respectively. Moreover, suppose that the terminal constraint set \( \mathcal{X}_P = \mathbb{R}^n \), the terminal cost \( V \) is continuous on \( C \), and \( t_j^{max} \leq T \leq t_j^{min} \), where \( t_j^* \) is defined in (14)-(15). Then, Problem 6.1 can be solved, and the function \( h \) is continuous at \((\xi, T, J)\). Moreover, the following hold.

Local Boundedness There exist \( \varepsilon > 0 \) and a compact set \( K \) such that
\[
\mathcal{O}_{\xi}(\xi, T, J) \times \mathcal{O}_{\xi}(\xi, T, J) \rightarrow (x_0(t, J), u(t, J)) \subseteq K
\]
for all \((t, J) \in \text{dom}(x, u)\) and \((x_0, u) \in \mathcal{O}_{\xi}(\xi', T', J')\).

Outer Semicontinuity Let \( \{(x_0', u_0')\}^\infty_{i=0} \) be a sequence of optimal solution pairs such that \( (x_0', u_0') \in \mathcal{O}_{\xi}(\xi, T, J) \) for all \( i \geq 0 \) and \( (x_0', u_0') \in \mathcal{O}_{\xi}(\xi, T, J) \) is graphically convergent. Then, if the sequences \( (\xi_i) \rightarrow \xi \) and \( (T_i, J_i) \rightarrow (T, J) \), respectively, there exists \( u \) such that \( (x_0, u) \in \mathcal{O}_{\xi}(\xi, T, J) \), where \( x_0 \) is the graphical limit of \( (x_0') \rightarrow \xi \). In addition, given \( j \geq 0 \), the limit \( u(t_{j+1}, J) \) of \( u(t_{j+1}, J) \in \mathcal{O}_{\xi}(\xi, T, J) \) and \( (x_0', u_0') \) for large \( i \).

Upper Semicontinuity For all \( \tau \geq 0 \) and \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that the following holds: for every \( x_0' \in x_0 + \eta\mathbb{B} \), \((T', J') \in (T, J) + \eta\mathbb{B} \), and \((x_0', u_0') \in \mathcal{O}_{\xi}(x_0', T', J') \), there exists \( (x_0, u) \in \mathcal{O}_{\xi}(x_0, T, J) \) such that \( x = x_0 + \eta\mathbb{B} \) close, and the holds for every \( j \geq 0 \): if \( t_{j+1}^* \) and \( t_j^* \) are the \( (j+1) \) jump times of \((x, u)\) and \((x', u')\), respectively, then \( \|u(t_{j+1}, J) - u(t_j^*, J)\| < \varepsilon \).

Proof. Existence of an optimal solution pair and continuity of \( h \) follows directly from Corollary 15. Local boundedness is due to the local boundedness result in Corollary 17 and the fact that the inputs are constrained to the compact set \([u_{min}, u_{max}]\). For the second item, graphical convergence of the state trajectories to \( x \) is proved in Corollary 17, and the statement regarding the inputs at jump times follows from graphical convergence of the state trajectories, Altm and Sanfelice (2020c, Lemma 2), continuity of \( G_p \), and the fact that the mapping \( G_p(x_0, \cdot) \) is one-to-one. The final statement can then be proven by contradiction as in the proof of Theorem 16.

6.2.3. Numerical example

To numerically illustrate the results, we consider the control problem in Altm and Sanfelice (2020a) of ensuring that the ball reaches a desired peak height \( p_{des} \) after every impact, asymptotically as the number of jumps tends to infinity. Equivalently, due to conservation of energy during flows, the control objective can be viewed as asymptotically stabilizing the set \( \mathcal{A} := \{x_P : W(x_P) = W(p_{des})\} \), where \( W(x_P) = \gamma p + v^2/2 \) is the total energy function, which is continuous. To achieve this objective, we let the terminal cost function \( V = W \), and select \( L_C \) as the zero function. For the jump cost, let \( L_D(v, u) = \gamma p_{des}(v + \sqrt{2}p_{des})^2/2 \) if \( v \geq -\sqrt{2}p_{des}/\lambda \), otherwise, let
\[
L_D(v, u) = \min \left\{ \gamma p_{des}(v + \sqrt{2}p_{des})^2/2, (v^2/2 - \gamma p_{des}^2) - (\lambda^2v^2/2 - \gamma p_{des}^2) \right\}
\]
which is continuous.

Simulation results9 corresponding to the parameters \( \gamma = 9.81 \text{ m/s}^2 \), \( \lambda = 0.8 \), \( u_{min} = 1 \text{ m/s} \), \( u_{max} = 10 \text{ m/s} \), and \( p_{des} = 2 \) m can be seen in Figs. 1 and 2. The optimal control problem is solved by casting it as a nonlinear program with linear inequalities, using the closed-form analytical solutions of the system and the fact that the flow cost function \( L_C \) is zero. Although the condition regarding jump times in Theorem 21 are not verified explicitly (to avoid conservative results, which can happen with large \( u_{max} - u_{min} \)), the findings of the theorem regarding continuity of the optimal cost, graphical convergence of the trajectories, and convergence of the inputs at jump times are still observed. In particular, Fig. 1 shows that the optimal cost depends continuously on the initial height and the ordinary time horizon in a neighborhood of their nominal values, and similarly, the optimal input (at jump times) depends continuously on the initial height and the ordinary time horizon at their nominal values. In Fig. 2, outer semicontinuous dependence of the optimal trajectories on the initial height and the ordinary time horizon can be observed: as the initial height and the time horizon parameter converge to their nominal values, the corresponding optimal trajectories similarly converge (graphically) to the optimal trajectory corresponding to these nominal values.

7. Conclusion

Existence of optimal solutions and their dependency on the problem data, as well as properties revealing the dependency of the optimal cost on the problem data are established for a general hybrid optimal control problem. It is shown that nominal outer well-posedness of the hybrid system is instrumental in guaranteeing not only the existence of an optimal solution, but also upper semicontinuous dependence of the optimal cost on initial conditions, terminal time, and terminal constraints. When nominal outer well-posedness is strengthened to outer well-posedness, upper semicontinuous dependence of the optimal cost extends to perturbations to all problem constraints, cost functions, and dynamics of the hybrid system. If, in addition, the system is inner well-posed, then the optimal cost is either continuous or can be continuously approximated, and the set of optimal solutions are outer/upper semicontinuous or can be outer/upper semicontinuously approximated, again with respect to perturbations to constraints, cost functions, and the dynamics.

With sufficient conditions for outer and inner well-posedness for the considered class of hybrid systems already available in the literature, the results in this paper pave the road to the understanding of the effect of computation and approximation in emerging tools for hybrid dynamical systems, such as numerical simulation, model predictive control, and parameter estimation. Future work includes exploiting the continuous approximation of the optimal cost to the model predictive control framework proposed in Altm and Sanfelice (2018) and Altm and Sanfelice (2019, 2020a) to analyze the effects of discretization on the associated optimal control problem, and extending some of the stronger continuity results in this paper to problems with terminal constraints.

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9 Code can be found at https://github.com/HybridSystemsLab/HybridOptimalControlBouncingBall.
• for every $(t, j) \in \text{dom} x$ satisfying $t + j \leq \tau$, there exists $(t', j') \in \text{dom} x'$ such that $|t - t'| < \varepsilon$ and $|x(t, j) - x'(t', j)| < \varepsilon$;

• for every $(t', j') \in \text{dom} x$ satisfying $t' + j' \leq \tau$, there exists $(t, j) \in \text{dom} x$ such that $|t - t'| < \varepsilon$ and $|x'(t', j') - x(t, j)| < \varepsilon$.

Appendix B. Sufficient conditions for nominal well-posedness

Although (nominal) outer well-posedness of a hybrid system can be difficult to check, it is guaranteed when the data of the system satisfies the so-called hybrid basic conditions (Goebel et al., 2012, Theorem 6.8).

Theorem 23. A hybrid system $\mathcal{H} = (C, F, D, G)$ is (nominally) outer well-posed if the following hold:

(A1) The sets $C$ and $D$ are closed.

(A2) The flow map $F$ is locally bounded and outer semicontinuous relative to $C$, and $C \subset \text{dom} F$. Furthermore, for every $x \in C$, the set $F(x)$ is convex.

(A3) The jump map $G$ is locally bounded and outer semicontinuous relative to $D$, and $D \subset \text{dom} G$.

Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, denote by $T_j(x)$ the Bouligand tangent cone to $S$ at $x$ (Goebel et al., 2012, Definition 5.12) and by $M(x)$ the Dubovitsky–Miliutin tangent cone to $S$ at $x$ (Aubin, 2009, Definition 4.3.1). A set of sufficient conditions for nominal inner well-posedness, which use these tangent cones, are given below (Altın and Sanfelice, 2020b, Theorem 1.1). For a proof of this result, see the discussion at the end of Altın and Sanfelice (2023, Section 5.2).

Theorem 24. Given a hybrid system $\mathcal{H} = (C, F, D, G)$, suppose that the flow set $C$ is closed and (A2) holds. Then, $\mathcal{H}$ is nominally inner well-posed if the following hold:

(B1) For every $x \in C$, there exists an extension of $F_x$ that is closed valued and Lipschitz,\footnote{A set-valued mapping $M$ is Lipschitz on $X$ if it has nonempty values on $X$ and there exists $L \geq 0$ such that $M(x) \subset M(x') + L|x - x'|B$ for every $x, x' \in X$.} on a neighborhood of $x$.

(B2) For every $x \in \partial C$ such that $F(x) \cap T_j(x)$ is nonempty, there exists $r > 0$ such that $F(x') \subset M_{\text{int} C}(x')$ for all $x' \in (x + rB) \cap \partial C$, and $(x + rB) \cap D \subset C$.

(B3) For every $x \in \text{int} C \cap \partial D$, $F(x) \cap M_{\text{int} D}(x)$ is nonempty.

(B4) For every $x \in \partial C \cap \partial D$, either of the following hold:

- there exists $r > 0$ such that $(x + rB) \cap C \subset D$;
- $F(x) \cap M_{\text{int} C}(x) \cap M_{\text{int} D}(x)$ is nonempty;
- $F(x) \cap T_j(x)$ is empty and there exists $r > 0$ such that $(x + rB) \cap \partial C \subset D$.

(B5) The jump map $G$ is inner semicontinuous relative to $D$.

(B6) The mapping $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where

$$
\tilde{G}(x) := G(x) \cap (\tilde{C} \cup D), \quad \forall x \in \mathbb{R}^n,
$$

$$
\tilde{C} := \text{int}(C) \cup \{x \in \partial C : F(x) \cap T_j(x) \neq \emptyset\},
$$

is inner semicontinuous relative to $D$.

References


Appendix A. Closeness of hybrid arcs

The following recalls (Goebel et al., 2012, Definition 5.23).

Definition 22. Given $\tau \geq 0$ and $\varepsilon > 0$, two hybrid arcs $x$ and $x'$ are said to be $(\tau, \varepsilon)$-close if

Fig. 1. Simulation results for the bouncing ball with $J = 2$ and initial velocity $v = 0$, as the ordinary time horizon parameter $T$ and the initial height $p$ are varied. (a) Optimal cost as $T$ and $p$ are varied. (b) Convergence of the optimal input as $T$ and $p$ tend to the nominal values of $T = 4$ and $p = 1$; the vector $u \in \mathbb{R}^2$ represents the values of the optimal input at jump times, $u^* \in \mathbb{R}$ corresponds to the case $T = 4$ and $p = 1$.

Fig. 2. Graphical convergence of the state trajectories with $J = 1$ and initial velocity $v = 0$, as the ordinary time horizon parameter $T$ and the initial height $p$ tend to nominal values of $T = 2$ and $p = 2$. 


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