Coupling Flow and Jump Observers for Hybrid Systems with Known Jump Times (Full Version)

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Abstract: This work presents a novel observer design methodology for general hybrid systems with known jump times. We propose to combine a high-gain *flow-based* observer estimating the part of the state that is instantaneously observable from the flow output, with a *jump-based* observer estimating the rest of the state from the knowledge of the jump output as well as some additional *fictitious* outputs, describing how the instantaneously observable part of the state impacts the non-observable one at jumps. We give general Lyapunov-based sufficient conditions for coupling such observers and propose an observer design for a large class of hybrid systems with nonlinear maps covering mechanical systems with uncertain impact models. Application to a bouncing ball with an unknown restitution coefficient illustrates the proposed approach.

Keywords: Hybrid system, high-gain observer, observability, detectability, Lyapunov analysis.

1. INTRODUCTION

Observer design for hybrid systems, namely systems combining both continuous and discrete behavior, is widely investigated in the literature (Bernard and Sanfelice, 2022). When the times of the discrete events in the solutions, i.e., the *jump times*, are known or detected, the observer jumps can be triggered at the same time as those of the system, making their solutions have the same (hybrid) time domain. This allows us to compare them at the same hybrid time and thus facilitates the convergence analysis of the observer estimate to the system state. Such hybrid systems with known jump times include in particular impulsive (possibly switched) systems (Medina and Lawrence, 2008, 2009; Tanwani et al., 2013) and continuous-time systems with sampled measurements (Raff and Allgöwer, 2007; Ferrante et al., 2016). As surveyed in (Bernard and Sanfelice, 2022), most observer designs for those hybrid systems assume either: 1) Observability of the flow or jump dynamics and output, and Lyapunov/LMI-based sufficient conditions; or 2) Observability of the full state during flows from the flow output only, leading to (high-gain) *flow-based* observers, relying on high-gain continuous observers (Gauthier et al., 1992) and persistent flowing; or 3) Observability of the full state from the jump output only, leading to *jump-based* observers, relying on persistent jumping and discrete observers designed for an equivalent discrete system modeling the flow-jump combination.

However, state components may exhibit different kinds of observability properties, associated with the flow and/or jump output(s), or hidden inside the flow-jump coupling. In output regulation, (Cox et al., 2016) exploits these ideas for hybrid systems with linear maps, periodic jumps, and flow output only, where part of the dynamics is instantaneously observable during flows from the flow output, while part of the non-observable dynamics becomes visible in the observable part at jumps. A similar phenomenon is exploited in (Tanwani et al., 2013; Shim and Tanwani, 2014) for (non)linear switched systems where observability is brought by switching among different non-observable modes; and in (Tran et al., 2022) for hybrid systems with linear maps by the interaction of non-observable flows and jumps. Observer design then relies on a state decomposition, where state components with different observability properties are separated. It typically couples a high-gain flow-based observer estimating the instantaneously observable states during flows, with back-and-forth algorithms or jump-based observers, estimating the rest of the states from the jump output and *fictitious* outputs describing how those states impact the instantaneously observable ones at jumps.

In this paper, we generalize the approach in (Tran et al., 2022) to hybrid systems with *nonlinear* maps. Assuming (after a change of coordinates) that a part of the state is instantaneously observable during flows, we propose to combine a *nonlinear* high-gain observer during flows with a jump-based one estimating the rest of the state and implicitly exploiting observability conditions from an *extended* output at jumps made of the system's jump output as well as some additional *fictitious* measurements. We provide general Lyapunov-based sufficient conditions to perform this coupling, including observability, dwell-time, and decoupling conditions in the dynamics. For that, we rely on Input-to-State Stability (ISS) properties in each observer with respect to the errors coming from the other observer, similar to (Liberzon et al., 2014). Then, we propose a constructive observer design achieving this coupling for a class of hybrid systems with nonlinear maps, including mechanical systems with uncertain impact models. Illustrations are provided considering state and restitution coefficient estimation in a bouncing ball. Our method differs from (Shim and Tanwani, 2014) mainly in the way the fictitious output is treated and in the resulting jump-based observer: instead of estimating the fictitious output through back-and-forth high-gain observers and resetting the estimate via asynchronous parallel computations, we design a synchronized observer with correction terms implicitly exploiting this *hidden* information.

Notations. Let $\mathbb{R}$ (resp. $\mathbb{N}$) be the set of real numbers (resp. natural numbers, i.e., $\{0,1,2,\ldots\}$). For a given input $t \mapsto u(t)$ to $\dot{x} = f(x,u)$, $\Psi_{f(\cdot,u)}(x_0,t,\tau)$ is the associated flow operator from initial value x_0 at initial time t evaluated *after* τ time unit(s). Let sat be a Lipschitz element-wise saturation function with levels to be picked depending on the context. For a hybrid arc $(t,j) \mapsto x(t,j)$ (see (Goebel et al., 2012)), we denote $\operatorname{dom} x$ its domain, $\operatorname{dom}_t x$ (resp. $\operatorname{dom}_j x$) the domain's projection on $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{N}$), and for $j \in \mathbb{N}$, $t_j(x)$ the unique time such that $(t_j(x),j) \in \operatorname{dom} x$ and $(t_j(x),j-1) \in \operatorname{dom} x$ (for hybrid systems with inputs, see (Sanfelice, 2021)) A solution x to a hybrid system is *complete* if $\operatorname{dom} x$ is unbounded. The notions of class-$\mathcal{K}$ and class-$\mathcal{KL}$ functions used are from (Khalil, 1996, Definitions 4.2 and 4.3). For a function $V : \mathbb{R}^{n_\eta} \to \mathbb{R}$ and a hybrid system with state $\eta \in \mathbb{R}^{n_\eta}$ and input u, we denote $\dot{V}(\eta,u)$ the derivative of V along the flows and $V^+(\eta,u)$ the value of V after a jump.

2. PROBLEM FORMULATION

Consider a hybrid system

$$\begin{cases} \dot{x} = f(x,u_c) & (x,u_c) \in C & y_c = h_c(x,u_c) \\ x^+ = g(x,u_d) & (x,u_d) \in D & y_d = h_d(x,u_d) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state; $y_c \in \mathbb{R}^{n_{y,c}}$ (resp. $y_d \in \mathbb{R}^{n_{y,d}}$) is the output known during the flow intervals (resp. at the jump times); $u_c \in \mathbb{R}^{n_{u,c}}$ and $u_d \in \mathbb{R}^{n_{u,d}}$ are known exogenous signals; the maps f and g are the flow and jump maps; h_c and h_d are the flow and jump output maps; C and D are the flow and jump sets, respectively. The jump times of each solution are assumed to be exactly detected.

Remark 1. The model (1) covers: 1) Time-varying systems, by including the times t, j in the inputs or the state; and 2) Impulsive and switched systems with state jumps as in (Shim and Tanwani, 2014), by treating the switching signal as a known input; and 3) Continuous-time systems with sampled sporadic outputs, by treating the sampling events as jumps triggering the availability of y_d.

Definition 1. For a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$, we say that a hybrid arc $(t,j) \mapsto x(t,j)$ has *flow lengths within* $\mathcal{I}$ if:

- $0 \leq t - t_j(x) \leq \sup \mathcal{I}$ for all $(t,j) \in \operatorname{dom} x$;
- $t_{j+1}(x) - t_j(x) \in \mathcal{I}$ holds for all $j \in \mathbb{N}_{>0}$ if $\sup \operatorname{dom}_j x = +\infty$, and for all $j \in \{1,2,\ldots,\sup \operatorname{dom}_j x - 1\}$ otherwise.

Assumption 1. There exist compact sets $\mathcal{X} \subset \mathbb{R}^{n_x}$ and $\mathcal{I} \subset \mathbb{R}_{>0}$ such that each maximal solution x to (1) initialized in $\mathcal{X}_0 \subset \mathbb{R}^{n_x}$ with inputs $(u_c,u_d) \in \mathcal{U}_c \times \mathcal{U}_d$ is complete, remains in $\mathcal{X}$, and has flow lengths within $\mathcal{I}$.

As the jump times of the system are known, it is natural to strive for a *synchronized* asymptotic observer of the form

$$\begin{cases} \dot{\hat{\zeta}} = \mathcal{F}(\hat{\zeta}, y_c, u_c) & \text{when (1) flows} \\ \hat{\zeta}^+ = \mathcal{G}(\hat{\zeta}, y_d, u_d) & \text{when (1) jumps} \\ \hat{x} = \mathcal{T}(\hat{\zeta}, y_c, y_d, u_c, u_d) \end{cases} \tag{2}$$

where $\hat{\zeta} \in \mathbb{R}^{n_\zeta}$ is the observer state; $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{T}$ are the observer dynamics and output maps designed such that each maximal solution $(x,\hat{\zeta})$ to the cascade (1)-(2) initialized in $\mathcal{X}_0 \times \mathbb{R}^{n_\zeta}$ and with inputs $(u_c,u_d) \in \mathcal{U}_c \times \mathcal{U}_d$ is complete and verifies

$$\lim_{t+j \to +\infty} |x(t,j) - \hat{x}(t,j)| = 0. \tag{3}$$

To that end, we assume that there exist integers n_o and n_{no} and a uniformly left-invertible transformation $T : \mathbb{R}^{n_x} \times [0,+\infty) \times \mathbb{N} \to \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}}$ of the form

$$T(x,t,j) = (T_o(x,t,j), T_{no}(x,t,j))$$

such that, along each solution x of (1) initialized in $\mathcal{X}_0$ with inputs $(u_c,u_d) \in \mathcal{U}_c \times \mathcal{U}_d$, $(t,j) \mapsto z(t,j)$ defined by the image of T along x, i.e., for each $(t,j) \in \operatorname{dom} x$,

$$z(t,j) = \begin{pmatrix} z_o(t,j) \\ z_{no}(t,j) \end{pmatrix} = \begin{pmatrix} T_o(x(t,j),t,j) \\ T_{no}(x(t,j),t,j) \end{pmatrix} \in \mathbb{R}^{n_z}, \tag{4}$$

with $n_z = n_o + n_{no}$, is a solution to

$$\begin{cases} \left.\begin{aligned} \dot{z}_o &= f_o(z_o,u_c) \\ \dot{z}_{no} &= f_{no}(z_o,z_{no},u_c) \end{aligned}\right\} (z,u_c) \in C_z \\ \left.\begin{aligned} z_o^+ &= g_o(z_o,z_{no},u_d) \\ z_{no}^+ &= g_{no}(z_o,z_{no},u_d) \end{aligned}\right\} (z,u_d) \in D_z \end{cases} \tag{5a}$$

for some maps (f_o,f_{no},g_o,g_{no}), sets C_z and D_z obtained from C and D via T, with the outputs

$$y_c = h_o(z_o,u_c), \qquad y_d = h_{no}(z_o,z_{no},u_d), \tag{5b}$$

and the given inputs, such that Assumption 2 below holds.

Assumption 2. The flow pair (f_o,h_o) is independent of z_{no} and is *instantaneously observable* on C_z for any input $u_c \in \mathcal{U}_c$, namely, the knowledge of $t \mapsto h_o(z_o(t),u_c(t))$ for an arbitrarily short time determines the solution $t \mapsto z_o(t)$ to $\dot{z}_o = f_o(z_o,u_c)$ uniquely as long as $(z_o,u_c) \in C_z$.

This observability condition allows us to consider a *high-gain* observer of the pair (f_o,h_o) during flows, which estimates z_o arbitrarily fast from the knowledge of y_c, and is ISS with respect to errors in z_{no} affecting the estimate of z_o at jumps. Then, we propose to estimate z_{no} via a jump-based observer from the knowledge of y_d as well as the estimate of z_o. More precisely, our observer is

$$\begin{cases} \left.\begin{aligned} \dot{\hat{z}}_o &= \hat{f}_o(\hat{z}_o,p,\tau,y_c,u_c) \\ \dot{\hat{z}}_{no} &= \hat{f}_{no}(\hat{z}_o,\hat{z}_{no},p,\tau,y_c,u_c) \\ \dot{p} &= \varphi_c(\hat{z}_o,\hat{z}_{no},p,\tau,y_c,u_c) \\ \dot{\tau} &= 1 \end{aligned}\right\} \text{when (5) flows} \\ \left.\begin{aligned} \hat{z}_o^+ &= \hat{g}_o(\hat{z}_o,\hat{z}_{no},p,\tau,y_d,u_d) \\ \hat{z}_{no}^+ &= \hat{g}_{no}(\hat{z}_o,\hat{z}_{no},p,\tau,y_d,u_d) \\ p^+ &= \varphi_d(\hat{z}_o,\hat{z}_{no},p,\tau,y_d,u_d) \\ \tau^+ &= 0 \end{aligned}\right\} \text{when (5) jumps} \end{cases} \tag{6}$$

where $p \in \mathbb{R}^{n_p}$ might contain additional observer states (see Example 1) and τ is a timer keeping track of the time elapsed since the previous jump. This timer evolves in $[0,\max \mathcal{I}]$ during flows and is in $\mathcal{I}$ at jumps according to Assumption 1. The maps $\hat{f}_o$, $\hat{f}_{no}$, φ_c, $\hat{g}_o$, $\hat{g}_{no}$, and φ_d are to be designed such that (6) is an asymptotically stable observer for (5), namely, given $\mathcal{X}$ from Assumption 1,

(z-AS) There exist a class-$\mathcal{KL}$ function β_z and $\mathcal{P}_0 \subseteq \mathbb{R}^{n_p}$ such that for any solution $(z,\hat{z},p,\tau)$ of the cascade

(5)-(6) initialized in $\mathcal{Z}_0 \times \mathbb{R}^{n_z} \times \mathcal{P}_0 \times \{0\}$ where $\mathcal{Z}_0 := T(\mathcal{X}, 0, 0)$, with inputs $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$,

$$|z(t,j) - \hat{z}(t,j)| \leq \beta_z(|z(0,0) - \hat{z}(0,0)|, t+j), \quad \forall (t,j) \in \operatorname{dom} z. \quad (7)$$

This asymptotic stability can be brought back to the x-coordinates if $x \mapsto T(x,0,0)$ is *continuous* on $\mathcal{X}$ and T is *injective* with respect to x in $\mathcal{X}$, uniformly in (t,j) (achieved at least after a certain time). In Section 3, we start by deriving general Lyapunov conditions to combine a high-gain flow-based observer (estimating z_o during flows from y_c) with a jump-based observer (estimating z_{no} from y_d and the estimate of z_o). Then, in Section 4, we propose a constructive observer design for a certain class of hybrid systems. Since z_o is estimated arbitrarily fast during flows, the key idea of this design is to exploit the detectability of z_{no} from an *extended* output at jumps, made of y_d, but also a *fictitious* output characterizing the way z_{no} impacts z_o at jumps, namely $g_o(z_o, z_{no}, u_d)$ in (5a).

3. GENERAL LYAPUNOV-BASED SUFFICIENT CONDITIONS FOR COUPLING OBSERVERS

In this section, we present intermediary technical results, namely sufficient Lyapunov-based conditions for coupling an arbitrarily fast high-gain flow-based observer with a jump-based one. These are general conditions that will later be applied for, but not limited to, a certain system class in Section 4. We typically require some ISS properties from each observer with respect to the error coming from the other one. Note that Assumption 2 is required throughout, to satisfy the high-gain flow-based conditions.

3.1 Conditions for Exponential Stability

Theorem 1. Suppose Assumption 1 holds and define $\tau_M := \max \mathcal{I}$. Consider the cascade (5)-(6) and sets $\mathcal{P}_c, \mathcal{P}_d \subseteq \mathbb{R}^{n_p}$ such that each solution $(z, \hat{z}, p, \tau)$ initialized in $\mathcal{Z}_0 \times \mathbb{R}^{n_z} \times \mathcal{P}_0 \times \{0\}$ with inputs $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$ is such that $p(t,j) \in \mathcal{P}_c$ during flows and $p(t,j) \in \mathcal{P}_d$ at jumps. Assume that there exist a function $V_{no} : \mathbb{R}^{n_{no}} \times \mathbb{R}^{n_{no}} \times \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$, scalars $\ell_0 > 0$, $\underline{b}_{no} > 0$, $\overline{b}_{no} > 0$, $\lambda_c > 0$, a_c, $c_{noo} \geq 0$, $d_{ono} \geq 0$, a_d and rational functions $\underline{b}_o > 0$, $\overline{b}_o > 0$, $d_o \geq 0$, $d_{noo} \geq 0$ such that $a_c \tau_M + a_d < 0$ and, for all $\ell > \ell_0$, there exists function $V_{o,\ell} : \mathbb{R}^{n_o} \times \mathbb{R}^{n_o} \times \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$ such that:

(1) *(Uniform boundedness)* For all $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$, $z = (z_o, z_{no}) \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$ or $(z, u_d) \in D_z$, $\hat{z} = (\hat{z}_o, \hat{z}_{no}) \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c \cup \mathcal{P}_d$, and $\tau \in [0, \tau_M]$,
$$\underline{b}_o(\ell)|z_o - \hat{z}_o|^2 \leq V_{o,\ell}(z_o, \hat{z}_o, p, \tau) \leq \overline{b}_o(\ell)|z_o - \hat{z}_o|^2,$$
$$\underline{b}_{no}|z_{no} - \hat{z}_{no}|^2 \leq V_{no}(z_{no}, \hat{z}_{no}, p, \tau) \leq \overline{b}_{no}|z_{no} - \hat{z}_{no}|^2;$$
(2) *(Flow-based conditions)* For all $u_c \in \mathcal{U}_c$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c$, and $\tau \in [0, \tau_M]$,
$$\dot{V}_{o,\ell}(z, \hat{z}, p, \tau, u_c) \leq -\ell \lambda_c V_{o,\ell}(z_o, \hat{z}_o, p, \tau),$$
$$\dot{V}_{no}(z, \hat{z}, p, \tau, u_c) \leq a_c V_{no}(z_{no}, \hat{z}_{no}, p, \tau) + c_{noo} V_{o,\ell}(z_o, \hat{z}_o, p, \tau);$$
(3) *(Jump-based conditions)* For all $u_d \in \mathcal{U}_d$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_d) \in D_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_d$, and $\tau \in \mathcal{I}$,
$$V_{o,\ell}^+(z, \hat{z}, p, \tau, u_d) \leq d_o(\ell) V_{o,\ell}(z_o, \hat{z}_o, p, \tau) + d_{ono}|z_{no} - \hat{z}_{no}|^2,$$
$$V_{no}^+(z, \hat{z}, p, \tau, u_d) \leq e^{a_d} V_{no}(z_{no}, \hat{z}_{no}, p, \tau) + d_{noo}(\ell) V_{o,\ell}(z_o, \hat{z}_o, p, \tau).$$

Then, there exists $\ell^\star \geq \ell_0$ such that if $\ell > \ell^\star$, (z-AS) holds with $\beta_z(\delta, s) = \rho \delta e^{-\lambda s}$ for some $\rho > 0$ and $\lambda > 0$.

Notice that $\hat{f}_o$, $\hat{g}_o$, φ_c, and φ_d can be chosen first to guarantee the existence of $V_{o,\ell}$ satisfying the inequalities involving it in Theorem 1 for some $\ell_0 > 0$, $\underline{b}_o > 0$, $\overline{b}_o > 0$, $\lambda_c > 0$, $d_o > 0$, and $d_{ono} \geq 0$, independently of $\hat{f}_{no}$, $\hat{g}_{no}$, and V_{no}, which can be designed in a second step.

Proof. Consider the Lyapunov function $W_\ell : \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$ defined as
$$W_\ell(z, \hat{z}, p, \tau) = e^{\frac{\lambda_c \ell \tau}{2}} V_{o,\ell}(z_o, \hat{z}_o, p, \tau) + r e^{-\varepsilon \tau} e^{-a_c(\tau - \tau_M)} V_{no}(z_{no}, \hat{z}_{no}, p, \tau),$$
where $r > 0$ and $\varepsilon > 0$ are analysis parameters. The role of $e^{\frac{\lambda_c \ell \tau}{2}}$ is to bring convergence from flows to jumps, while that of $e^{-\varepsilon \tau}$ is indeed to bring contraction from jumps to flows; r is tuned to ensure negativity at jumps despite the interactions between these components. First, we have for all $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$ or $(z, u_d) \in D_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c \cup \mathcal{P}_d$, and $\tau \in [0, \tau_M]$,
$$\underline{\rho}_o(\ell)|z_o - \hat{z}_o|^2 + \underline{\rho}_{no}|z_{no} - \hat{z}_{no}|^2 \leq W_\ell(z, \hat{z}, \tau) \leq \overline{\rho}_o(\ell)|z_o - \hat{z}_o|^2 + \overline{\rho}_{no}|z_{no} - \hat{z}_{no}|^2, \quad (8)$$
where $\underline{\rho}_o(\ell) = \underline{b}_o(\ell)$, $\underline{\rho}_{no} = \underline{b}_{no} r e^{-\varepsilon \tau_M}$, $\overline{\rho}_o(\ell) = \overline{b}_o(\ell) e^{\frac{\lambda_c \ell \tau_M}{2}}$, and $\overline{\rho}_{no} = \overline{b}_{no} r e^{a_c \tau_M}$.
During flows, for all $u_c \in \mathcal{U}_c$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c$, and $\tau \in [0, \tau_M]$,
$$\dot{W}_\ell(z, \hat{z}, p, \tau, u_c) \leq -\min\left\{\frac{\lambda_c \ell}{2} - r c_{noo}, \varepsilon\right\} W_\ell(z, \hat{z}, p, \tau).$$
At jumps, for all $u_d \in \mathcal{U}_d$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_d) \in D_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_d$, and $\tau \in \mathcal{I}$,
$$W_\ell^+(z, \hat{z}, p, \tau, u_d) - W_\ell(z, \hat{z}, p, \tau) \leq - \left(e^{\frac{\lambda_c \ell \tau_m}{2}} \underline{b}_o(\ell) - d_o(\ell)\overline{b}_o(\ell) - r d_{noo}(\ell)\overline{b}_o(\ell)\right)|z_o - \hat{z}_o|^2 - \left(r\overline{b}_{no}(e^{-\varepsilon \tau_M} - e^{a_c \tau_M + a_d}) - d_{ono}\right)|z_{no} - \hat{z}_{no}|^2$$
where $\tau_m := \min \mathcal{I} > 0$. Then, we choose successively
$$0 < \varepsilon < -\frac{a_c \tau_M + a_d}{\tau_M}, \qquad r > \frac{d_{ono}}{\overline{b}_{no}(e^{-\varepsilon \tau_M} - e^{a_c \tau_M + a_d})},$$
and finally ℓ sufficiently large to have both
$$e^{\frac{\lambda_c \ell \tau_m}{2}} \underline{b}_o(\ell) > d_o(\ell)\overline{b}_o(\ell) + r d_{noo}(\ell)\overline{b}_o(\ell), \qquad \frac{\lambda_c \ell}{2} > r c_{noo},$$
which is possible because exponential growth wins over a rational one. Then, (z-AS), with $\beta_z(\delta, s) = \rho \delta e^{-\lambda s}$ for some $\rho > 0$ and $\lambda > 0$, directly follows from (Sanfelice, 2021). ■

Remark 2. Note that while similar results might be obtained via a small-gain methodology as in (Liberzon et al., 2014), we choose here to follow an explicit Lyapunov proof.

Example 1. Assume that (5) is such that f_o and h_o take the following triangular form *decoupled* from z_{no}:
$$\dot{z}_o = A z_o + \Phi(z_o, u_c), \qquad y_c = z_{o,1} = H z_o, \quad (9)$$
$$\text{with } A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \Phi(z_o, u_c) = \begin{pmatrix} \Phi_1(z_{o,1}, u_c) \\ \Phi_2(z_{o,1}, z_{o,2}, u_c) \\ \dots \\ \Phi_{n_o}(z_o, u_c) \end{pmatrix},$$
and $H = (1 \; 0 \; \dots \; 0)$, where Φ and g_o are Lipschitz with

respect to z, uniformly in their respective input. The instantaneous observability of the pair (f_o, h_o) in Assumption 2 is then automatically guaranteed by (9). When f and h_c in (1) are both autonomous, such a form can be obtained using the autonomous transformation

$$T_o(x) = (h_c(x), L_f h_c(x), \ldots, L_f^{n_o-1} h_c(x)), \tag{10}$$

but here we additionally require the decoupling of $\dot{z}_o$ from z_{no}. A high-gain flow-based observer (Gauthier et al., 1992) defines $(\hat{f}_o, \hat{g}_o)$ as

$$\begin{cases} \dot{\hat{z}}_o = A\hat{z}_o + \text{sat}(\Phi(\hat{z}_o, u_c)) + \ell\mathcal{L}(\ell)K(y_c - H\hat{z}_o) \\ \hat{z}_o^+ = \text{sat}(g_o(\hat{z}_o, \hat{z}_{no}, u_d)), \end{cases} \tag{11}$$

where $\ell > 0$, $\mathcal{L}(\ell) = \text{diag}(1, \ell, \ldots, \ell^{n_o-1})$, and $K = (k_1, k_2, \ldots, k_{n_o})$ chosen independently of ℓ such that $A - KH$ is Hurwitz. It can then be checked that the Lyapunov function

$$V_o(z_o, \hat{z}_o) = (z_o - \hat{z}_o)^\top \mathcal{L}^{-1}(\ell) P \mathcal{L}^{-1}(\ell)(z_o - \hat{z}_o), \tag{12}$$

where the constant matrix $P = P^\top > 0$ is a solution to

$$P(A - KH) + (A - KH)^\top P \leq -aP, \tag{13}$$

for some $a > 0$, verifies the assumptions of Theorem 1. In another case, if f_o and h_o are such that

$$\dot{z}_o = A(u_c, y_c)z_o + \Phi(z_o, u_c), \qquad y_c = H(u_c)z_o, \tag{14}$$

still with A, H, Φ of the same shape as (9) (but varying in u_c, y_c) and Φ, g_o Lipschitz, a Kalman-like high-gain flow-based observer (Besancon, 1999) defines $(\hat{f}_o, \varphi_c, \hat{g}_o, \varphi_d)$ as

$$\begin{cases} \dot{\hat{z}}_o = A(u_c, y_c)\hat{z}_o + \text{sat}(\Phi(\hat{z}_o, u_c)) \\ \qquad + \ell\mathcal{L}(\ell)P^{-1}H^\top(u_c)(y_c - H(u_c)\hat{z}_o) \\ \dot{P} = -\ell(\mu P - A^\top(u_c, y_c)P - PA(u_c, y_c) \\ \qquad + H^\top(u_c)H(u_c)) \\ \hat{z}_o^+ = \text{sat}(g_o(\hat{z}_o, \hat{z}_{no}, u_d)) \\ P^+ = P_0, \end{cases} \tag{15}$$

initialized with $P_0 > 0$, for μ large enough. In this case, the extra observer state in (6) is $p = P$. The observability in Assumption 2 is linked to the existence of $\ell^\star > 0$ and $\alpha > 0$ such that for all $\ell > \ell^\star$ and for all $t > \frac{1}{\ell}$,

$$\int_{t-\frac{1}{\ell}}^t \star^\top \underbrace{H(u_c(s))\psi_{A(u_c,y_c)}(s,t)}_{\star} ds \geq \frac{\alpha}{\ell}(\mathcal{L}^{-1}(\ell))^2,$$

where $\psi_{A(u_c,y_c)}(s,t)$ is the transition matrix of the linear continuous dynamics $\dot{v} = A(u_c, y_c)v$ from time t to time s. It can then be checked that the Lyapunov function

$$V_o(z_o, \hat{z}_o, P) = (z_o - \hat{z}_o)^\top \mathcal{L}^{-1}(\ell) P \mathcal{L}^{-1}(\ell)(z_o - \hat{z}_o), \tag{16}$$

with the varying gain P admitting a strictly positive uniform lower bound thanks to observability, verifies the assumptions of Theorem 1.

While the flow-based part of the observer characterized by $V_{o,\ell}$ is standard via well-known high-gain constructions as shown in Example 1, the choice of the jump-based part related to V_{no} is more intricate. A constructive design is presented in Section 4 for a certain class of hybrid systems.

Remark 3. With the high-gain designs in Example 1, an error $z_o - \hat{z}_o$ appearing in $\hat{f}_{no}$ would typically make c_{noo} in Theorem 1 depend on ℓ, which is not allowed. Therefore, we may need to require f_{no} and $\hat{f}_{no}$ to be independent of z_o and $\hat{z}_o$. This can always be done by taking $z_{no} = x$ or any left-invertible function of x, meaning that z_o is somehow estimated twice, through both z_o and z_{no}. This obstruction to handle coupling during flows with a high-gain design was similarly noticed in the linear output regulation context (Cox et al., 2016, Proposition 6). Note also that with the existing high-gain designs in Example 1, $\overline{b}_o$ does not depend on ℓ (and the same for Theorem 2 below).

3.2 Conditions for Arbitrarily Fast Exponential Stability

Under certain conditions, as shown in Theorem 2 below, the coupling of an arbitrarily fast high-gain flow-based observer and an arbitrarily fast jump-based one that has an ISS property can actually result in arbitrarily fast exponential stability of the error (in the z-coordinates).

Theorem 2. Suppose Assumption 1 holds and define $\tau_M := \max \mathcal{I}$. Consider the cascade (5)-(6) and sets $\mathcal{P}_c, \mathcal{P}_d \subseteq \mathbb{R}^{n_p}$ such that each solution $(z, \hat{z}, p, \tau)$ initialized in $\mathcal{Z}_0 \times \mathbb{R}^{n_z} \times \mathcal{P}_0 \times \{0\}$ with inputs $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$ is such that $p(t,j) \in \mathcal{P}_c$ during flows and $p(t,j) \in \mathcal{P}_d$ at jumps. Assume there exist scalars $\ell_0 > 0$, $\gamma_0 > 0$, $\lambda_c > 0$, $c_{no} \geq 0$, and $\lambda_d \geq 0$, rational functions $\underline{b}_o > 0$ and $\overline{b}_o > 0$, functions $\underline{b}_{no} > 0$ and $\overline{b}_{no} > 0$, functions $c_{noo} \geq 0$ and $d_{ono} \geq 0$, and functions $d_o \geq 0$ and $d_{noo} \geq 0$ rational in their first argument such that, for all $\ell > \ell_0$ and $0 < \gamma < \gamma_0$, there exist functions $V_{o,\ell} : \mathbb{R}^{n_o} \times \mathbb{R}^{n_o} \times \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$ and $V_{no,\gamma} : \mathbb{R}^{n_{no}} \times \mathbb{R}^{n_{no}} \times \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$ such that:

(1) *(Uniform boundedness)* For all $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$, $z = (z_o, z_{no}) \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$ or $(z, u_d) \in D_z$, $\hat{z} = (\hat{z}_o, \hat{z}_{no}) \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c \cup \mathcal{P}_d$, and $\tau \in [0, \tau_M]$,

$$\underline{b}_o(\ell)|z_o - \hat{z}_o|^2 \leq V_{o,\ell}(z_o, \hat{z}_o, p, \tau) \leq \overline{b}_o(\ell)|z_o - \hat{z}_o|^2,$$
$$\underline{b}_{no}(\gamma)|z_{no} - \hat{z}_{no}|^2 \leq V_{no,\gamma}(z_{no}, \hat{z}_{no}, p, \tau) \leq \overline{b}_{no}(\gamma)|z_{no} - \hat{z}_{no}|^2;$$

(2) *(Flow-based conditions)* For all $u_c \in \mathcal{U}_c$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c$, and $\tau \in [0, \tau_M]$,

$$\dot{V}_{o,\ell}(z, \hat{z}, p, \tau, u_c) \leq -\ell\lambda_c V_{o,\ell}(z_o, \hat{z}_o, p, \tau),$$
$$\dot{V}_{no,\gamma}(z, \hat{z}, p, \tau, u_c) \leq c_{no}V_{no,\gamma}(z_{no}, \hat{z}_{no}, p, \tau) + c_{noo}(\gamma)V_{o,\ell}(z_o, \hat{z}_o, p, \tau);$$

(3) *(Jump-based conditions)* For all $u_d \in \mathcal{U}_d$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_d) \in D_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_d$, and $\tau \in \mathcal{I}$,

$$V_{o,\ell}^+(z, \hat{z}, p, \tau, u_d) \leq d_o(\ell, \gamma)V_{o,\ell}(z_o, \hat{z}_o, p, \tau) + d_{ono}(\gamma)|z_{no} - \hat{z}_{no}|^2,$$
$$V_{no,\gamma}^+(z, \hat{z}, p, \tau, u_d) \leq \gamma e^{-\lambda_d}V_{no,\gamma}(z_{no}, \hat{z}_{no}, p, \tau) + d_{noo}(\ell, \gamma)V_{o,\ell}(z_o, \hat{z}_o, p, \tau).$$

Then, for any $\lambda > 0$, there exists $0 < \gamma^\star \leq \gamma_0$ such that there exists $\ell^\star \geq \ell_0$ such that if $0 < \gamma < \gamma^\star$ and $\ell > \ell^\star$, (z-AS) holds with $\beta_z(\delta, s) = \rho\delta e^{-\lambda s}$ for some $\rho > 0$.

Note that the dependence of the functions on γ is arbitrary.

Proof. Consider the following Lyapunov function $W_{\ell,\gamma} : \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$ (with $r > 0$ and $\varepsilon > 0$):

$$W_{\ell,\gamma}(z, \hat{z}, p, \tau) = e^{\frac{\lambda_c\ell\tau}{2}}V_{o,\ell}(z_o, \hat{z}_o, p, \tau) + re^{-\varepsilon\tau}V_{no,\gamma}(z_{no}, \hat{z}_{no}, p, \tau).$$

First, we have for all $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$ or $(z, u_d) \in D_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c \cup \mathcal{P}_d$, and $\tau \in [0, \tau_M]$,

$$\underline{\rho}_o(\ell)|z_o - \hat{z}_o|^2 + \underline{\rho}_{no}(\gamma)|z_{no} - \hat{z}_{no}|^2 \leq W_{\ell,\gamma}(z, \hat{z}, p, \tau) \leq \overline{\rho}_o(\ell)|z_o - \hat{z}_o|^2 + \overline{\rho}_{no}(\gamma)|z_{no} - \hat{z}_{no}|^2, \tag{17}$$

where $\underline{\rho}_o(\ell) = \underline{b}_o(\ell)$, $\underline{\rho}_{no}(\gamma) = \underline{b}_{no}(\gamma) r e^{-\varepsilon\tau_M}$, $\overline{\rho}_o(\ell) = \overline{b}_o(\ell) e^{\frac{\lambda_c \ell \tau_M}{2}}$, and $\overline{\rho}_{no}(\gamma) = \overline{b}_{no}(\gamma) r$.
During flows, for all $u_c \in \mathcal{U}_c$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_c) \in C_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_c$, and $\tau \in [0, \tau_M]$,

$$\dot{W}_{\ell,\gamma}(z, \hat{z}, p, \tau, u_c) \leq -\min\left\{\frac{\lambda_c \ell}{2} - r c_{noo}(\gamma), \right. \\ \left. \varepsilon - c_{no}\right\} W_{\ell,\gamma}(z, \hat{z}, p, \tau).$$

At jumps, for all $u_d \in \mathcal{U}_d$, $z \in \mathbb{R}^{n_z}$ such that $(z, u_d) \in D_z$, $\hat{z} \in \mathbb{R}^{n_z}$, $p \in \mathcal{P}_d$, and $\tau \in \mathcal{I}$,

$$\begin{aligned} &W^+_{\ell,\gamma}(z, \hat{z}, p, \tau, u_d) - W_{\ell,\gamma}(z, \hat{z}, p, \tau) \leq \\ &-\left(e^{\frac{\lambda_c \ell \tau_m}{2}} \underline{b}_o(\ell) - d_o(\ell,\gamma)\overline{b}_o(\ell) - r d_{noo}(\ell,\gamma)\overline{b}_o(\ell)\right)|z_o - \hat{z}_o|^2 \\ &\quad - \left(r\overline{b}_{no}(\gamma)(e^{-\varepsilon\tau_M} - \gamma e^{-\lambda_d}) - d_{ono}(\gamma)\right)|z_{no} - \hat{z}_{no}|^2, \end{aligned}$$

with $\tau_m := \min \mathcal{I} > 0$. Then, given any $\lambda > 0$, we choose successively

$$\varepsilon > c_{no} + 2\lambda\left(\frac{1}{\tau_m} + 1\right), \qquad 0 < \gamma < e^{\lambda_d - \varepsilon\tau_m},$$
$$r > \frac{d_{ono}(\gamma)}{\overline{b}_{no}(\gamma)(e^{-\varepsilon\tau_M} - \gamma e^{-\lambda_d})},$$

and finally ℓ sufficiently large to have both

$$e^{\frac{\lambda_c \ell \tau_m}{2}} \underline{b}_o(\ell) > d_o(\ell,\gamma)\overline{b}_o(\ell) + r d_{noo}(\ell,\gamma)\overline{b}_o(\ell),$$
$$\frac{\lambda_c \ell}{2} > r c_{noo}(\gamma) + 2\lambda\left(\frac{1}{\tau_m} + 1\right),$$

which is possible because exponential growth wins over a rational one. We then obtain during flows and at jumps respectively,

$$\begin{aligned} \dot{W}_{\ell,\gamma}(z, \hat{z}, p, \tau) &\leq -2\lambda\left(\frac{1}{\tau_m} + 1\right) W_{\ell,\gamma}(z, \hat{z}, p, \tau), \\ W^+_{\ell,\gamma}(z, \hat{z}, p, \tau) - W_{\ell,\gamma}(z, \hat{z}, p, \tau) &\leq -a_{d,o}|z_o - \hat{z}_o|^2 \\ &\quad - a_{d,no}|z_{no} - \hat{z}_{no}|^2, \end{aligned}$$

for some $a_{d,o} > 0$ and $a_{d,no} > 0$. That implies that

$$\begin{aligned} &W_{\ell,\gamma}(z(t,j), \hat{z}(t,j), p(t,j), \tau(t,j)) \leq \\ &\quad e^{-2\lambda(\frac{1}{\tau_m}+1)t} W_{\ell,\gamma}(z(0,0), \hat{z}(0,0), p(0,0), \tau(0,0)), \end{aligned}$$

for all $(t,j) \in \operatorname{dom} z = \operatorname{dom} \hat{z}$. From Assumption 1, we have $j \leq \frac{t}{\tau_m} + 1$ or $t \geq \frac{t+j-1}{\frac{1}{\tau_m}+1}$ for all $(t,j) \in \operatorname{dom} z = \operatorname{dom} \hat{z}$. Therefore, we have

$$\begin{aligned} &W_{\ell,\gamma}(z(t,j), \hat{z}(t,j), p(t,j), \tau(t,j)) \leq \\ &\quad e^{2\lambda} e^{-2\lambda(t+j)} W_{\ell,\gamma}(z(0,0), \hat{z}(0,0), p(0,0), \tau(0,0)), \end{aligned}$$

for all $(t,j) \in \operatorname{dom} z = \operatorname{dom} \hat{z}$. Then ($z$-AS), with $\beta_z(\delta, s) = \rho\delta e^{-\lambda s}$ for some $\rho > 0$, directly follows from (Sanfelice, 2021). ■

Remark 4. It is seen that ρ in β_z increases as λ increases, characterizing the *peaking* phenomenon typically encountered in high-gain designs when we push convergence arbitrarily fast. Discrete observers satisfying the conditions in Theorem 2 include (Ţiclea and Besançon, 2013) and (Tran and Bernard, 2023), where the former requires linearity in the dynamics and output. Note that we can only recover, under the uniform injectivity of T, asymptotic stability in the x-coordinates. To recover arbitrarily fast exponential stability, we must require a stronger injectivity from T.

4. OBSERVER DESIGN FOR A CLASS OF SYSTEMS

In this section, we use Theorem 1 to design a full observer for the class of hybrid systems (1) that can be put, via some uniformly left-invertible transformation, into the form

$$\begin{cases} \left.\begin{aligned} \dot{z}_o &= f_o(z_o, u_c) \\ \dot{\xi}_{no} &= A_{no}\xi_{no} + B_{no} \end{aligned}\right\} (z_o, \xi_{no}, u_c) \in C_{z,\xi} \\ \left.\begin{aligned} z_o^+ &= J_o(z_o, u_d) + J_{ono}(z_o, u_d)\xi_{no} \\ \xi_{no}^+ &= J_{no}(z_o, u_d)\xi_{no} + J_{noo}(z_o, u_d) \end{aligned}\right\} (z_o, \xi_{no}, u_d) \in D_{z,\xi} \end{cases} \tag{18a}$$

with the outputs

$$y_c = h_o(z_o, u_c), \qquad y_d = H_{d,noo}(z_o, u_d) + H_{d,no}(z_o, u_d)\xi_{no}. \tag{18b}$$

The model (18) covers a wide class of hybrid systems, including the case of (mechanical) systems with uncertain impacts, where we need to estimate the state z_o observable during flows and some impact parameters contained in ξ_{no}, which affect z_o at jumps in an affine way but not during flows. These parameters then typically become detectable from y_c through the way they affect z_o at jumps, namely from the *fictitious* output $J_{ono}(z_o, u_d)\xi_{no}$. Examples of hybrid systems that fit into the form (18) range from a bouncing ball (Example 2) to walking robots (Short and Sanfelice, 2018) and spiking neurons (Izhikevich, 2003).

Remark 5. The results in this section can be seen as the nonlinear version of those in (Tran et al., 2022) for hybrid systems with linear maps, where we have $f_o = 0$, $A_{no} = 0$, and $B_{no} = 0$ thanks to an appropriate time-varying linear transformation. Note that for the case of nonlinear maps treated in this paper, n_z can be larger than n_x.

Following Assumption 1, solutions to (18) of interest, initialized in some set $\mathcal{Z}_0$, are complete. We also define a compact set $\mathcal{Z}_o \times \Xi_{no}$ where the solutions of (18) of interest remain. For simplicity, we take $C_{z,\xi} \subseteq \mathcal{Z}_o \times \Xi_{no} \times \mathcal{U}_c$ and $D_{z,\xi} \subseteq \mathcal{Z}_o \times \Xi_{no} \times \mathcal{U}_d$. Denote $D_o := \{z_o \in \mathbb{R}^{n_o} : \exists \xi_{no} \in \mathbb{R}^{n_{no}}, u_d \in \mathbb{R}^{n_{u,d}} : (z_o, \xi_{no}, u_d) \in D_{z,\xi}\}$.

Assumption 3. The maps f_o, J_o, J_{ono}, J_{no}, J_{noo}, $H_{d,noo}$, and $H_{d,no}$ are Lipschitz with respect to z_o on $\mathcal{Z}_o$, uniformly in $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$. The maps J_{no}, J_{ono}, and $H_{d,no}$ are bounded on $\mathcal{Z}_o \times \mathcal{U}_d$.

The observer we propose for (18) takes the form

$$\begin{cases} \dot{\hat{z}}_o = \hat{f}_o(\hat{z}_o, p, \tau, y_c, u_c) \\ \dot{\hat{\xi}}_{no} = A_{no}\hat{\xi}_{no} + B_{no} + e^{A_{no}\tau} K_d \frac{d}{dt}\Psi_{f_{o,\mathrm{sat}}(\cdot,u_c)}(\hat{z}_o, t, -\tau) \\ \dot{p} = \varphi_c(p, \tau, y_c, u_c) \\ \dot{\tau} = 1 \\ \hat{z}_o^+ = J_o(\mathrm{sat}(\hat{z}_o), u_d) + J_{ono}(\mathrm{sat}(\hat{z}_o), u_d)\hat{\xi}_{no} \\ \hat{\xi}_{no}^+ = J_{no}(\mathrm{sat}(\hat{z}_o), u_d)\hat{\xi}_{no} + J_{noo}(\mathrm{sat}(\hat{z}_o), u_d) \\ \qquad + L_d(\mathrm{sat}(\hat{z}_o), u_d, \tau)(y_d - H_{d,noo}(\mathrm{sat}(\hat{z}_o), u_d) \\ \qquad\qquad - H_{d,no}(\mathrm{sat}(\hat{z}_o), u_d)\hat{\xi}_{no}) \\ p^+ = \varphi_d(p, \tau, y_d, u_d) \\ \tau^+ = 0, \end{cases} \tag{19}$$

where $f_{o,\mathrm{sat}}$ is globally Lipschitz with respect to z_o, uniformly in $u_c \in \mathcal{U}_c$, and equal to f_o on $\mathcal{Z}_o \times \mathcal{U}_c$; $\mathrm{sat}(\hat{z}_o) = \hat{z}_o$ on $\mathcal{Z}_o$; and K_d and L_d are the gains to design.

Assumption 4. Under Assumption 2, assume that $z_o \in \mathbb{R}^{n_o}$ can be estimated with a high-gain flow-based observer. Thus, the maps $\hat{f}_o$, φ_c, and φ_d in (19) are such that there exist $\mathcal{P}_0$, $\mathcal{P}_c$, $\mathcal{P}_d$, $V_{o,\ell}$, ℓ_0, $\underline{b}_o$, $\overline{b}_o$, λ_c, d_o, and d_{ono} satisfying the conditions in Theorem 1 with ξ_{no} replacing z_{no}.

To estimate ξ_{no}, we propose to design the gains K_d and L_d exploiting the detectability brought by an extended output made of $H_{d,no}(z_o,u_d)\xi_{no}$ in y_d as well as $J_{ono}(z_o,u_d)\xi_{no}$ affecting z_o at jumps. For this, we assume the following.

Assumption 5. There exist a symmetric positive definite matrix $Q \in \mathbb{R}^{n_{no}\times n_{no}}$, gains $K_d \in \mathbb{R}^{n_{no}\times n_o}$ and $(z_o,u_d,\tau) \mapsto L_d(z_o,u_d,\tau) \in \mathbb{R}^{n_{no}\times n_{y,d}}$ bounded on $\mathcal{Z}_o \times \mathcal{U}_d \times \mathcal{I}$ such that for all $z_o \in \mathcal{Z}_o \cap D_o$, $u_d \in \mathcal{U}_d$, and $\tau \in \mathcal{I}$:

$$\Phi^\top(z_o,u_d,\tau)Q\Phi(z_o,u_d,\tau) - Q < 0, \qquad (20)$$

where

$$\Phi(z_o,u_d,\tau) = \Bigg(J_{no}(z_o,u_d) - (K_d \;\; L_d(z_o,u_d,\tau))\begin{pmatrix} J_{ono}(z_o,u_d) \\ H_{d,no}(z_o,u_d)\end{pmatrix}\Bigg)e^{A_{no}\tau}.$$

Contrary to L_d, which may depend on z_o, u_d, and the timer τ, K_d is required to be constant to perform the analysis (in the proof of Theorem 3). This extra requirement is similar to the one we made in (Tran et al., 2022) for hybrid systems with linear maps. Assumption 5 is thus a nonlinear version of the LMI-based one in (Tran et al., 2022), which is stronger than *quadratic detectability* (Wu, 1995) by the constant nature of K_d. To solve (20), we can use *gridding* (grid-based solving) assuming a particular structure of L_d, possibly with checking of the obtained gains using a much denser grid (Wu, 1995). If the form of Φ allows it, we may also solve (20) using the *polytopic* method thanks to residue matrix expansion as in (Ferrante et al., 2016), or still *gridding* but with a theoretical proof of stability extended from (Sferlazza et al., 2019).

Theorem 3. Under Assumptions 1, 3, 4, and 5, the cascade (18)-(19) verifies (z-AS) (with $(\xi_{no},\hat{\xi}_{no})$ replacing $(z_{no},\hat{z}_{no})$).

Proof. By Assumption 3 and (McShane, 1934, Corollary 1), there exists an extension $f_{o,\text{sat}}$ of f_o that is defined and globally Lipschitz on $\mathbb{R}^{n_o}\times\mathbb{R}^{n_{u_c}}$. To use the fictitious output, we use the transformation $(z_o,\xi_{no},\hat{z}_o,\hat{\xi}_{no},p,\tau,t,j) \mapsto (z_o,z_{no},\hat{z}_o,\hat{z}_{no},p,\tau)$, with

$$\begin{aligned} z_{no} &= e^{-A_{no}\tau}\xi_{no} - K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau) + \int_0^{-\tau} e^{A_{no}(-\tau-s)}B_{no}ds, \\ \hat{z}_{no} &= e^{-A_{no}\tau}\hat{\xi}_{no} - K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau) + \int_0^{-\tau} e^{A_{no}(-\tau-s)}B_{no}ds. \end{aligned}$$

This change of coordinates is well-defined by the global Lipschitzness of $f_{o,\text{sat}}$ and there exists a compact set $\mathcal{Z}_{no}$ such that for any $(z_o,\xi_{no},\hat{z}_o,\hat{\xi}_{no},p,\tau,t,j) \in \mathcal{Z}_o\times\Xi_{no}\times\mathbb{R}^{n_o}\times\mathbb{R}^{n_{no}}\times\mathbb{R}^{n_p}\times[0,\tau_M]\times\mathbb{R}_{\geq 0}\times\mathbb{N}$, we have $z_{no}\in\mathcal{Z}_{no}$. In the new coordinates, the dynamics of $(z_o,\hat{z}_o,p,\tau)$ are obtained by replacing ξ_{no} and $\hat{\xi}_{no}$ with

$$\begin{aligned} \xi_{no} &= e^{A_{no}\tau}(z_{no} + K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau)) + \int_0^{\tau} e^{A_{no}(\tau-s)}B_{no}ds, \\ \hat{\xi}_{no} &= e^{A_{no}\tau}(\hat{z}_{no} + K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau)) + \int_0^{\tau} e^{A_{no}(\tau-s)}B_{no}ds, \end{aligned} \qquad (21)$$

and considering the extended inputs $u_{c,\text{ext}} = (u_c,t) \in \mathcal{U}_{c,\text{ext}}$ and $u_{d,\text{ext}} = (u_d,t)\in\mathcal{U}_{d,\text{ext}}$, with $\mathcal{U}_{c,\text{ext}} = \mathcal{U}_c\times\mathbb{R}_{\geq 0}$ and $\mathcal{U}_{d,\text{ext}} = \mathcal{U}_d\times\mathbb{R}_{\geq 0}$. Concerning the dynamics of z_{no} and $\hat{z}_{no}$, we start by showing that $\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau))$ is constant along solutions. To do that, pick a solution resulting from the interconnection (18)-(19) initialized in $\mathcal{Z}_0\times\mathbb{R}^{n_z}\times\mathcal{P}_0\times\{0\}$. Since the solution component z_o flows according to f_o with input u_c, for each $j\in\text{dom}_j\, z_o$, and each $s\in[0,t-t_j]$, we have

$$\Psi_{f_o(\cdot,u_c)}(z_o(t,j),t,-s) = z_o(t-s,j),$$

and since the trajectory $t\mapsto z_o(t,j)$ remains in $\mathcal{Z}_o$ and τ is initialized as $\tau(0,0)=0$, we have for all $(t,j)\in\text{dom}\, z_o$,

$$\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o(t,j),t,-\tau(t,j)) = \Psi_{f_o(\cdot,u_c)}(z_o(t,j),t,-\tau(t,j)).$$

Besides, by definition of the dynamics, τ initialized as $\tau(0,0)=0$ is the time elapsed since the previous jump, namely $\tau(t,j) = t-t_j$ for all $j\in\text{dom}_j\, z_o$. Therefore, exploiting again that z_o evolves according to f_o with input u_c, we have $\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o(t,j),t,-\tau(t,j)) = \Psi_{f_o(\cdot,u_c)}(z_o(t,j),t,-(t-t_j)) = z_o(t_j,j)$ for all $j\in\text{dom}_j\, z_o$ and $t\in[t_j,t_{j+1}]$, so that $t\mapsto\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o(t,j),t,-\tau(t,j))$ is constant. Therefore, solutions of the interconnection (18)-(19) that are initialized in $\mathcal{Z}_0\times\mathbb{R}^{n_z}\times\mathcal{P}_0\times\{0\}$ are such that the variable z_{no} takes the dynamics

$$\begin{aligned} \dot{z}_{no} &= -A_{no}e^{-A_{no}\tau}\xi_{no} + e^{-A_{no}\tau}\dot{\xi}_{no} - e^{-A_{no}\tau}B_{no} \\ &= -A_{no}e^{-A_{no}\tau}\xi_{no} + e^{-A_{no}\tau}(A_{no}\xi_{no}+B_{no}) - e^{-A_{no}\tau}B_{no} \\ &= 0 \\ z_{no}^+ &= \xi_{no}^+ - K_d z_o^+ \\ &= J_{no}(z_o,u_d)\xi_{no} + J_{noo}(z_o,u_d) \\ &\quad - K_d(J_o(z_o,u_d) + J_{ono}(z_o,u_d)\xi_{no}) \\ &= (J_{no}(z_o,u_d) - K_dJ_{ono}(z_o,u_d))\xi_{no} \\ &\quad + J_{noo}(z_o,u_d) - K_dJ_o(z_o,u_d) \\ &= \phi(z_o,u_d,\tau)(z_{no} + K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau) \\ &\quad + \textstyle\int_0^\tau e^{-A_{no}s}B_{no}ds) + J_{noo}(z_o,u_d) - K_dJ_o(z_o,u_d), \end{aligned}$$

where $\phi(z_o,u_d,\tau) = (J_{no}(z_o,u_d) - K_dJ_{ono}(z_o,u_d))e^{A_{no}\tau}$, and the variable $\hat{z}_{no}$ takes the dynamics

$$\begin{aligned} \dot{\hat{z}}_{no} &= -A_{no}e^{-A_{no}\tau}\hat{\xi}_{no} + e^{-A_{no}\tau}\dot{\hat{\xi}}_{no} \\ &\quad - K_d\tfrac{d}{dt}\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau) - e^{-A_{no}\tau}B_{no} \\ &= -A_{no}e^{-A_{no}\tau}\hat{\xi}_{no} + e^{-A_{no}\tau}(A_{no}\hat{\xi}_{no} + B_{no} \\ &\quad + e^{A_{no}\tau}K_d\tfrac{d}{dt}\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau)) \\ &\quad - K_d\tfrac{d}{dt}\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau) - e^{-A_{no}\tau}B_{no} \\ &= 0 \\ \hat{z}_{no}^+ &= \hat{\xi}_{no}^+ - K_d\hat{z}_o^+ \\ &= J_{no}(\text{sat}(\hat{z}_o),u_d)\hat{\xi}_{no} + J_{noo}(\text{sat}(\hat{z}_o),u_d) \\ &\quad + L_d(\text{sat}(\hat{z}_o),u_d,\tau)(y_d - H_{d,noo}(\text{sat}(\hat{z}_o),u_d) \\ &\quad - H_{d,no}(\text{sat}(\hat{z}_o),u_d)\hat{\xi}_{no}) - K_d(J_o(\text{sat}(\hat{z}_o),u_d) \\ &\quad + J_{ono}(\text{sat}(\hat{z}_o),u_d)\hat{\xi}_{no}) \\ &= (J_{no}(\text{sat}(\hat{z}_o),u_d) - L_d(\text{sat}(\hat{z}_o),u_d,\tau)H_{d,no}(\text{sat}(\hat{z}_o),u_d) \\ &\quad - K_dJ_{ono}(\text{sat}(\hat{z}_o),u_d))\hat{\xi}_{no} \\ &\quad + J_{noo}(\text{sat}(\hat{z}_o),u_d) - K_dJ_o(\text{sat}(\hat{z}_o),u_d) \\ &\quad + L_d(\text{sat}(\hat{z}_o),u_d,\tau)(y_d - H_{d,noo}(\text{sat}(\hat{z}_o),u_d) \\ &= \Phi(\text{sat}(\hat{z}_o),u_d,\tau)(\hat{z}_{no} + K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau) \\ &\quad + \textstyle\int_0^\tau e^{-A_{no}s}B_{no}ds) \\ &\quad + J_{noo}(\text{sat}(\hat{z}_o),u_d) - K_dJ_o(\text{sat}(\hat{z}_o),u_d) \\ &\quad + L_d(\text{sat}(\hat{z}_o),u_d,\tau)(y_d - H_{d,noo}(\text{sat}(\hat{z}_o),u_d)), \end{aligned}$$

where Φ is defined in Assumption 5. The flow and jump sets are subsets of $\mathcal{Z}_o\times\mathcal{Z}_{no}\times\mathcal{U}_c$ and $\mathcal{Z}_o\times\mathcal{Z}_{no}\times\mathcal{U}_d$, respectively. Now we deduce the error dynamics. For brevity, let us denote

$$\begin{aligned} \Upsilon &= z_{no} + K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau) + \textstyle\int_0^\tau e^{-A_{no}s}B_{no}ds, \\ \hat{\Upsilon} &= \hat{z}_{no} + K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau) + \textstyle\int_0^\tau e^{-A_{no}s}B_{no}ds. \end{aligned}$$

We then see that

$$
\begin{aligned}
y_d &= H_{d,noo}(z_o,u_d) + H_{d,no}(z_o,u_d)e^{A_{no}\tau}\Upsilon,\\
z_{no}^+ &= \phi(z_o,u_d,\tau)\Upsilon + J_{noo}(z_o,u_d) - K_d J_o(z_o,u_d),\\
\hat{z}_{no}^+ &= \Phi(\text{sat}(\hat{z}_o),u_d,\tau)\hat{\Upsilon}\\
&\quad + J_{noo}(\text{sat}(\hat{z}_o),u_d) - K_d J_o(\text{sat}(\hat{z}_o),u_d)\\
&\quad + L_d(\text{sat}(\hat{z}_o),u_d,\tau)(H_{d,noo}(z_o,u_d)\\
&\quad - H_{d,noo}(\text{sat}(\hat{z}_o),u_d) + H_{d,no}(z_o,u_d)e^{A_{no}\tau}\Upsilon).
\end{aligned}
$$

Define the error $\tilde{z}_{no} := z_{no} - \hat{z}_{no}$. We then get

$$
\begin{aligned}
\tilde{z}_{no}^+ &= \phi(z_o,u_d,\tau)\Upsilon - \Phi(\text{sat}(\hat{z}_o),u_d,\tau)\hat{\Upsilon}\\
&\quad - L_d(\text{sat}(\hat{z}_o),u_d,\tau)H_{d,no}(z_o,u_d)e^{A_{no}\tau}\Upsilon\\
&\quad + (J_{noo}(z_o,u_d) - J_{noo}(\text{sat}(\hat{z}_o),u_d))\\
&\quad - K_d(J_o(z_o,u_d) - J_o(\text{sat}(\hat{z}_o),u_d))\\
&\quad - L_d(\text{sat}(\hat{z}_o),u_d,\tau)(H_{d,noo}(z_o,u_d)\\
&\qquad - H_{d,noo}(\text{sat}(\hat{z}_o),u_d)).
\end{aligned}
$$

Now add and subtract both the terms $\phi(\text{sat}(\hat{z}_o),u_d,\tau)\Upsilon$ and $L_d(\text{sat}(\hat{z}_o),u_d,\tau)H_{d,no}(\text{sat}(\hat{z}_o),u_d)e^{A_{no}\tau}\Upsilon$ to get

$$
\begin{aligned}
\tilde{z}_{no}^+ &= \Phi(\text{sat}(\hat{z}_o),u_d,\tau)(\Upsilon - \hat{\Upsilon})\\
&\quad + (\phi(z_o,u_d,\tau) - \phi(\text{sat}(\hat{z}_o),u_d,\tau) - L_d(\text{sat}(\hat{z}_o),u_d,\tau)\\
&\quad \times (H_{d,no}(z_o,u_d) - H_{d,no}(\text{sat}(\hat{z}_o),u_d))e^{A_{no}\tau})\Upsilon\\
&\quad + (J_{noo}(z_o,u_d) - J_{noo}(\text{sat}(\hat{z}_o),u_d))\\
&\quad - K_d(J_o(z_o,u_d) - J_o(\text{sat}(\hat{z}_o),u_d))\\
&\quad - L_d(\text{sat}(\hat{z}_o),u_d,\tau)\\
&\quad \times (H_{d,noo}(z_o,u_d) - H_{d,noo}(\text{sat}(\hat{z}_o),u_d)).
\end{aligned}
$$

Now see that $\Upsilon - \hat{\Upsilon} = \tilde{z}_{no} + K_d(\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau) - \Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau))$ and use the expression of Υ. The error $\tilde{z}_{no}$ takes the dynamics

$$
\begin{aligned}
\dot{\tilde{z}}_{no} &= 0\\
\tilde{z}_{no}^+ &= \Phi(\text{sat}(\hat{z}_o),u_d,\tau)\tilde{z}_{no} + \Phi(\text{sat}(\hat{z}_o),u_d,\tau)K_d\\
&\quad \times (\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau) - \Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau))\\
&\quad + (\phi(z_o,u_d,\tau) - \phi(\text{sat}(\hat{z}_o),u_d,\tau) - L_d(\text{sat}(\hat{z}_o),u_d,\tau)\\
&\quad \times (H_{d,no}(z_o,u_d) - H_{d,no}(\text{sat}(z_o),u_d))e^{A_{no}\tau})\times\\
&\quad (z_{no} + K_d\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau) + \textstyle\int_0^\tau e^{-A_{no}s}B_{no}ds)\\
&\quad + (J_{noo}(z_o,u_d) - J_{noo}(\text{sat}(\hat{z}_o),u_d)) - K_d\\
&\quad \times (J_o(z_o,u_d) - J_o(\text{sat}(\hat{z}_o),u_d)) - L_d(\text{sat}(\hat{z}_o),u_d,\tau)\\
&\quad \times (H_{d,noo}(z_o,u_d) - H_{d,noo}(\text{sat}(\hat{z}_o),u_d)).
\end{aligned}
$$

Consider the Lyapunov function $V_{no}(z_{no},\hat{z}_{no}) = (z_{no} - \hat{z}_{no})^\top Q(z_{no} - \hat{z}_{no})$, with Q in (20). By the global Lipschitzness of $f_{o,\text{sat}}$ with respect to z_o, uniformly with respect to u_c, Grönwall's inequality allows to show that $\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\cdot,t,-\tau))$ is Lipschitz, uniformly with respect to $(t,\tau) \in \mathbb{R}_{\geq 0} \times [0,\tau_M]$. Indeed, we have for any $(z_{o,a},z_{o,b},u_c,t,\tau) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_o} \times \mathcal{U}_c \times \mathbb{R}_{\geq 0} \times [0,\tau_M]$,

$$
\begin{aligned}
\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,a},t,-\tau) &= z_{o,a}\\
&\quad + \textstyle\int_t^{t-\tau} f_{o,\text{sat}}(\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,a},t,s-t),u_c(s))ds,\\
\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,b},t,-\tau) &= z_{o,b}\\
&\quad + \textstyle\int_t^{t-\tau} f_{o,\text{sat}}(\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,b},t,s-t),u_c(s))ds.
\end{aligned}
$$

By subtracting both sides and using the triangle inequality,

$$
\begin{aligned}
&|\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,a},t,-\tau) - \Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,b},t,-\tau)|\\
&\leq |z_{o,a} - z_{o,b}| + \textstyle\int_{t-\tau}^t |f_{o,\text{sat}}(\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,a},t,s-t),u_c(s))\\
&\qquad - f_{o,\text{sat}}(\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,b},t,s-t),u_c(s))|ds.
\end{aligned}
$$

Denoting L as the Lipschitz constant of $f_{o,\text{sat}}$ with respect to z_o, we get

$$
\begin{aligned}
&|\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,a},t,-\tau) - \Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,b},t,-\tau)|\\
&\leq |z_{o,a} - z_{o,b}| + \textstyle\int_{t-\tau}^t L|\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,a},t,s-t)\\
&\qquad - \Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,b},t,s-t)|ds.
\end{aligned}
$$

Using Grönwall's inequality, we get

$$
\begin{aligned}
&|\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,a},t,-\tau) - \Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_{o,b},t,-\tau)|\\
&\qquad \leq |z_{o,a} - z_{o,b}|e^{L\tau} \leq |z_{o,a} - z_{o,b}|e^{L\tau_M},
\end{aligned}
$$

which means $\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\cdot,t,-\tau))$ is Lipschitz, uniformly with respect to $(t,\tau) \in \mathbb{R}_{\geq 0} \times [0,\tau_M]$. Thanks to Assumptions 3 and 5 and Young's inequality on the cross terms, there exist $c_1 \in [0,1)$ and $c_2,c_3,c_4 > 0$ such that for any $\kappa > 0$, $z \in \mathcal{Z}_o \times \mathcal{Z}_{no}$, $\hat{z} \in \mathbb{R}^{n_z}$, and $(\tau,t,u_c,u_d) \in [0,\tau_M] \times \mathbb{R} \times \mathcal{U}_c \times \mathcal{U}_d$,

$$
\begin{aligned}
\dot{V}_{no}(z,\hat{z},\tau,t,u_c) &= 0,\\
V_{no}^+(z,\hat{z},\tau,t,u_d) &\leq \left(c_1 + \tfrac{c_2}{\kappa}\right)V_{no}(z_{no},\hat{z}_{no},\tau)\\
&\qquad + (c_3\kappa + c_4)|z_o - \hat{z}_o|^2.
\end{aligned}
$$

Picking κ large enough so that $c_1 + \frac{c_2}{\kappa} \in [0,1)$, V_{no} satisfies the second item of each condition of Theorem 1. Besides,

$$
\begin{aligned}
\xi_{no} - \hat{\xi}_{no} &= e^{A_{no}\tau}(z_{no} - \hat{z}_{no})\\
&\quad + K_d(\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(z_o,t,-\tau) - \Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\hat{z}_o,t,-\tau)),
\end{aligned}
$$

and since τ remains in $[0,\tau_M]$ and $\Psi_{f_{o,\text{sat}}(\cdot,u_c)}(\cdot,t,-\tau))$ is Lipschitz, uniformly with respect to $(t,\tau) \in \mathbb{R}_{\geq 0} \times [0,\tau_M]$, using Young's inequality, there exist $c_5 > 0$ and $c_6 > 0$ such that for all $(z_o,z_{no},\hat{z}_o,\hat{z}_{no},\tau,u_c,t) \in \mathcal{Z}_o \times \mathcal{Z}_{no} \times \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times [0,\tau_M] \times \mathcal{U}_c \times \mathbb{R}_{\geq 0}$, $(\xi_{no},\hat{\xi}_{no})$ defined in (21) verifies

$$
|\xi_{no} - \hat{\xi}_{no}| \leq c_5|z_{no} - \hat{z}_{no}| + c_6|z_o - \hat{z}_o|, \qquad (22)
$$

so that $V_{o,\ell}$ satisfies the inequalities involving it in Theorem 1 in the new coordinates. Since φ_c and φ_d are independent of $(\hat{z}_o,\hat{z}_{no})$, solutions $(z_o,z_{no},\hat{z}_o,\hat{z}_{no},p,\tau)$ to the cascade initialized in $\mathcal{Z}_0 \times \mathbb{R}^{n_z} \times \mathcal{P}_0 \times \{0\}$ with inputs $(u_{c,\text{ext}},u_{d,\text{ext}}) \in \mathcal{U}_{c,\text{ext}} \times \mathcal{U}_{d,\text{ext}}$ are such that $p(t,j) \in \mathcal{P}_c$ during flows and $p(t,j) \in \mathcal{P}_d$ at jumps. Applying Theorem 1, the result follows from the uniform invertibility of the change of coordinates (deduced from (22)). ■

Remark 6. When f_o and h_o are linear, for example if $\dot{z}_o = A_o z_o + B_o$ and $y_c = H_o z_o$, with the pair (A_o,H_o) observable, the observer (19) takes the much simpler form

$$
\begin{cases}
\dot{\hat{z}}_o = A_o\hat{z}_o + B_o + \Gamma(\ell)(y_c - H_o\hat{z}_o)\\
\dot{\hat{\xi}}_{no} = A_{no}\hat{\xi}_{no} + B_{no} + e^{A_{no}\tau}K_d e^{-A_o\tau}\Gamma(\ell)(y_c - H_o\hat{z}_o)\\
\dot{\tau} = 1\\
\hat{z}_o^+ = J_o(\text{sat}(\hat{z}_o),u_d) + J_{ono}(\text{sat}(\hat{z}_o),u_d)\hat{\xi}_{no}\\
\hat{\xi}_{no}^+ = J_{no}(\text{sat}(\hat{z}_o),u_d)\hat{\xi}_{no} + J_{noo}(\text{sat}(\hat{z}_o),u_d)\\
\qquad + L_d(\text{sat}(\hat{z}_o),u_d,\tau)(y_d - H_{d,noo}(\text{sat}(\hat{z}_o),u_d)\\
\qquad\qquad - H_{d,no}(\text{sat}(\hat{z}_o),u_d)\hat{\xi}_{no})\\
\tau^+ = 0,
\end{cases}
\qquad (23)
$$

where $\ell \mapsto \Gamma(\ell) \in \mathbb{R}^{n_o \times n_{y,c}}$ is an appropriate high-gain (see (Bernard and Sanfelice, 2022, Example 4.2)). This form is similar to (Tran et al., 2022) for linear maps.

Example 2. Consider a bouncing ball with an unknown restitution coefficient, described by

$$
\begin{cases}
\dot{x} = (x_2, -c_f x_2 + u_c, 0), & \text{when } x_1 \geq 0\\
x^+ = (x_1, -x_2 x_3 + u_d, x_3), & \text{when } x_1 = 0 \text{ and } x_2 \leq 0
\end{cases}
$$

with flow output $y_c = x_1$ and no jump output, where x_1 and x_2 are the ball's height and velocity, instantaneously observable through y_c; $x_3 = 0.5$ is the restitution coefficient at the impact that we also want to estimate using its interaction with x_2 at jumps; $c_f = 0.1$ is the friction coefficient; $u_c = -9.8$ is the gravitational acceleration; and $u_d = 5$ is a pulse at the impact which serves to: 1) Uniformly bound (x_1,x_2) in a compact set $\mathcal{Z}_o$ not containing 0 (otherwise x_3 would not be uniformly detectable); and 2) Uniformly bound the flow lengths away from 0 and providing enough flowing to estimate z_o.

By choosing $z_o = (z_{o,1},z_{o,2}) = (x_1,x_2)$ and $\xi_{no} = x_3$, we obtain the form (18) with the maps $f_o(z_o, u_c) = (z_{o,2}, -c_f z_{o,2} + u_c)$, $A_{no} = 0$, $B_{no} = 0$, $J_o(z_o, u_d) = (z_{o,1}, u_d)$, $J_{ono}(z_o, u_d) = (0, -z_{o,2})$, $J_{no}(z_o, u_d) = 1$, $J_{noo}(z_o, u_d) = 0$, and $H_{d,noo}(z_o, u_d) = 0$. Notice that here these maps are linear and hence Lipschitz with respect to z_o, uniformly in (u_c, u_d), so Assumption 3 is satisfied. Since $y_d = 0$, we take $L_d = 0$ and estimate ξ_{no} using the fictitious output $J_{ono}(z_o, u_d)\xi_{no}$. Assumption 5 now becomes $(1 - K_d(0, -z_{o,2}))^\top Q (1 - K_d(0, -z_{o,2})) - Q < 0$ for $z_o \in \mathcal{Z}_o \cap D_o$, with Q a positive scalar and $D_o := \{z_o \in \mathbb{R}^2 : z_{o,1} = 0, z_{o,2} \leq 0\}$. Supposing for instance that $z_{o,2} \in [-10, -0.5]$ at jumps, Assumption 5 is satisfied by taking $K_d = (0\ 0.1)$. Because the maps f_o and h_o are linear time-invariant in this application, Remark 6 applies. Simulation results are in Figure 1. Here, ξ_{no} is estimated without using y_d, but only the fictitious output $-x_2 x_3$.

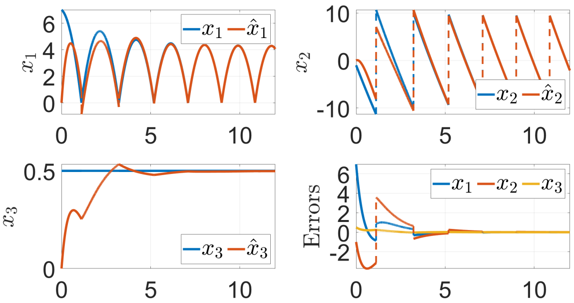

Fig. 1. State and parameter estimation in a bouncing ball.

5. CONCLUSION

This paper presents detectability necessary conditions and Lyapunov-based sufficient conditions for coupling flow- and jump-based observers for hybrid systems with known jump times. We also apply these to design an observer for a wide class of hybrid systems, followed by an application to state and parameter estimation in a bouncing ball. Future work includes finding implementable *approximations* of the innovation term in $\dot{\hat{\xi}}_{no}$ in (19) (which is possible since this term vanishes asymptotically) and building a nonlinear jump-based observer to estimate z_{no} (or ξ_{no}) with a fully nonlinear jump map. A candidate for this is the nonlinear Luenberger observer (Brivadis et al., 2019), which can have both an arbitrarily fast convergence rate and an ISS property, hence coherent with the results developed in Theorem 2.

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