A Discretization of the Hybrid Gradient Algorithm for Linear Regression with Sampled Hybrid Signals

Nathan Wu, Ryan S. Johnson, and Ricardo G. Sanfelice

***Abstract*— We consider the problem of estimating a vector of unknown constant parameters for a linear regression model whose inputs and outputs are discretized hybrid signals – that is, they are samples of hybrid signals that exhibit both continuous (flow) and discrete (jump) evolution. Using a hybrid systems framework, we propose a hybrid gradient descent algorithm that operates during both flows and jumps. We show that this algorithm guarantees exponential convergence of the parameter estimate to the unknown parameter under a new notion of discretized hybrid persistence of excitation that relaxes the classical discrete-time persistence of excitation condition. Simulation results validate the properties guaranteed by the new algorithm.**

I. Introduction

Hybrid systems are a class of dynamical systems with state variables that can exhibit both continuous and discrete evolution. Such systems provide new and promising modeling frameworks for a wide range of applications, including switching systems [1] and systems with event-triggered control [2]. In such applications, it is often necessary to estimate the unknown parameters of the system in order to achieve the desired control objective [3]. However, the hybrid nature of these systems stymies the applicability of classical continuous-time or discrete-time parameter estimation algorithms.

Several approaches exist in the literature for parameter estimation for certain classes of hybrid dynamical systems [4], [5], [6], such as by interpreting hybrid systems as a part of the piecewise affine framework (PWA) [7], [8]. However, these works all assume that measurements are available continuously during flows. This assumption is often violated in practice since, due to the need for analog to digital conversion, measurements are typically only available at discrete time instants during flows. In addition, the PWA description is broad, and a focused attempt at the identification of specifically hybrid systems may lead to more optimized approaches that would not apply to the PWA class as a whole.

Motivated by the need for an estimation algorithm that is capable of operating with discretized hybrid signals, in this paper, we propose a discretized hybrid gradient descent algorithm to estimate the unknown parameters of discretized hybrid linear regression models. We begin with a literature review of the continuous-time and discrete-time gradient descent algorithms in Section II, followed by a motivational example in Section III. In Section IV, we show that our proposed algorithm ensures exponential convergence of the parameter estimation error to zero under a new discretized hybrid persistence of excitation condition. Examples are given in Section V and concluding remarks are in Section VI. Due to space constraints, some proofs are sketched or omitted and will be published elsewhere.

N. Wu, R. S. Johnson, and R. G. Sanfelice are with the Department of Electrical and Computer Engineering, University of California, 1156 High Street, Santa Cruz, CA 95064, USA. naawu@ucsc.edu, rsjohnson@ucsc.edu, ricardo@ucsc.edu. This research has been partially supported by the National Science Foundation under Grant no. ECS-1710621, Grant no. CNS-2039054, and Grant no. CNS-2111688, by the Air Force Office of Scientific Research under Grant nos. FA9550-19-1-0169, FA9550-20-1-0238, and FA9550-23-1-0145, by the Air Force Research Laboratory under Grant nos. FA8651-22-1-0017 and FA8651-23-1-0004, and by the Army Research Office under Grant no. W911NF-20-1-0253.

II. Preliminaries

A. Notation

We use the following notation and definitions. The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$ denote the sets of all nonnegative integers, real numbers, and nonnegative real numbers, respectively. The Euclidean norm of vectors and the associated induced matrix norm are denoted by $|\cdot|$. Given nonempty sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, $X \backslash Y$ denotes set subtraction. For a function $\phi : \mathbb{R}^n \to \mathbb{R}^m$, $\operatorname{dom} \phi$ denotes the domain of ϕ. The symbol 0 denotes either the scalar zero or the zero matrix of appropriate dimension.

B. Discretized Hybrid Systems

Consider a hybrid system defined as in [9]

$$\begin{aligned} \dot{x} &= f(x) \quad x \in C \\ x^+ &= g(x) \quad x \in D \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $f : C \to \mathbb{R}^n$ is the flow map which models continuous dynamics on the flow set $C \subset \mathbb{R}^n$, and $g : D \to \mathbb{R}^n$ is the jump set which models discrete dynamics on the jump set $D \subset \mathbb{R}^n$. A discretization of the hybrid system in (1) is denoted $\mathcal{H}_s$ and is defined by [10]

$$\begin{aligned} x^+ &= f_s(x) \quad x \in C_s \\ x^+ &= g_s(x) \quad x \in D_s \end{aligned}$$

where $s \in (0, s^*]$, for some $s^* > 0$, is the step size of the discretization of the continuous-time dynamics, $f_s : C_s \to \mathbb{R}^n$ is the flow map modeling the discretized dynamics of $\mathcal{H}$ on the discretized flow set $C_s \in \mathbb{R}^n$, and $g_s : D_s \to \mathbb{R}^n$ is the jump map modeling the discretized dynamics on the discretized jump set $D_s \in \mathbb{R}^n$. Inspired by the forward Euler's method [10], one way to discretize a hybrid system is to define f_s and g_s as[1]

$$\begin{array}{rclr} f_s(x) & := & x + sf(s) & x \in C_s := C \\ g_s(x) & := & g(x) & x \in D_s := D. \end{array} \qquad (2)$$

Solutions to discretized hybrid systems are given on discretized hybrid time domains.

Definition 1: (Discretized hybrid time domain) A set $E_s \subset \mathbb{N} \times \mathbb{N}$ is a *discretized hybrid time domain* if for each $(K, J) \in E_s$, there exists a unique finite nondecreasing sequence $\{k_j\}_{j=0}^{J+1}$ such that $k_0 = 0$, $k_{j+1} \in \mathbb{N} \backslash \{0\}$ for each $j \in \{0, 1, \ldots, J\}$, and

$$E_s \cap (\{0, 1, \ldots, K\} \times \{0, 1, \ldots, J\}) = \bigcup_{j=0}^{J} \bigcup_{k=k_j}^{k_{j+1}} (k, j).$$

The operations $\sup_k E_s$ and $\sup_j E_s$ denote the supremum in the k and j coordinates, respectively, in E_s.

Definition 2: (Global pre-exponential stability) Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is *globally pre-exponentially stable*[2] for the hybrid system $\mathcal{H}_s$ if there exist strictly positive real numbers κ and λ such that each solution ϕ_s to $\mathcal{H}_s$ satisfies

$$|\phi_s(k, j)|_{\mathcal{A}} \leq \kappa e^{-\lambda(k+j)} |\phi_s(0, 0)|_{\mathcal{A}} \quad \forall (k, j) \in \operatorname{dom} \phi_s.$$

C. Linear Regression

In preparation for our proposed discreitzed hybrid gradient descent (GD) algorithm, we review GD algorithms in continuous time and in discrete time [3]. Consider the continuous time linear regression problem

$$y_c(t) = \theta^\top \psi_c(t) \quad \forall t \geq 0 \qquad (3)$$

where $t \mapsto y_c(t) \in \mathbb{R}$ is a measured output, $\theta \in \mathbb{R}^n$ is a vector of constant unknown parameters, and $t \mapsto \psi_c(t) \in \mathbb{R}^n$ is a measured input. To estimate θ, an estimator of the output can be constructed as follows:

$$\hat{y}_c(t) = \hat{\theta}_c(t)^\top \psi_c(t)$$

where $t \mapsto \hat{y}_c(t) \in \mathbb{R}$ is the estimated output and $t \mapsto \hat{\theta}_c(t) \in \mathbb{R}^n$ is an estimate of θ. The error between the estimated and true outputs is then

$$e_c(t) := \hat{y}_c(t) - y_c(t) = \tilde{\theta}_c(t)^\top \psi_c(t)$$

where $\tilde{\theta}_c := \hat{\theta}_c - \theta$ is the parameter estimation error.

In order to minimize the cost function $J_c(e_c) := \frac{e_c^2}{2}$, the GD algorithm estimates $\hat{\theta}_c$ using the following update law:

$$\dot{\hat{\theta}}_c(t) = -\gamma_c \psi_c(t) e_c(t) \qquad (4)$$

where $\gamma_c > 0$ is a design parameter.

In a discrete-time setting, consider the linear regression problem

$$y_d(j) = \theta^\top \psi_d(j) \quad \forall j \in \mathbb{N}. \qquad (5)$$

To estimate θ, an estimator of the output can be constructed as follows:

$$\hat{y}_d(j) = \hat{\theta}_d(j)^\top \psi_d(j)$$

The error between the estimated and true outputs is then

$$e_d(j) := \hat{y}_d(j) - y_d(j) = \tilde{\theta}_d^\top \psi_d(j).$$

In order to minimize the cost function $J_d(e_d) := \frac{e_d^2}{2}$, a discrete-time GD algorithm is suggested such that $\hat{\theta}_d$ evolves according to the following update law:

$$\hat{\theta}_d(j+1) = \hat{\theta}_d(j) - \frac{\gamma_d \psi_d(j) e_d(j)}{1 + \gamma_d \psi_d^\top(j) \psi_d(j)}. \qquad (6)$$

The following persistence of excitation conditions are sufficient and necessary for the convergence of $\hat{\theta}_c$ and $\hat{\theta}_d$ to θ for the continuous-time and discrete-time GD algorithms, respectively; see, e.g., [3].

Definition 3: (Persistence of excitation)

(C0) A signal $\mathbb{R}_{\geq 0} \ni t \mapsto \psi(t) \in \mathbb{R}^n$ is *persistently exciting* (PE) if there exist $T, \mu > 0$ such that, for each $t_o \geq 0$,

$$\int_{t_o}^{t_o+T} \psi(t)\psi(t)^\top dt \geq \mu I.$$

(C1) A signal $\mathbb{N} \ni j \mapsto \psi(j) \in \mathbb{R}^n$ is PE if there exist $J \in \mathbb{N} \setminus \{0\}$ and $\mu > 0$ such that, for each $j_o \in \mathbb{N}$,

$$\sum_{j=j_o}^{j_o+J} \psi(j)\psi(j)^\top \geq \mu I.$$

III. Motivation

Consider the linear regression models in (3) and (5), but now with a hybrid signal ψ as a regressor, defined on a hybrid time domain E. A hybrid gradient algorithm was introduced in [4] to solve this hybrid linear regression problem, such that the estimate $\hat{\theta}$ converges to θ. The parameter estimate $\hat{\theta}$ behaves according to the update law in (4) during periods of flow, and according to the update law in (6) at jumps. However, in practice, the hybrid signal $(t, j) \mapsto \psi(t, j)$ is only available at discrete-time instants during flows, as measurements cannot truly be taken continuously. Given discretized hybrid signals $(k, j) \mapsto y_s(k, j)$ and $(k, j) \mapsto \psi_s(k, j)$ defined on a discretized hybrid time domain $E_s \subset \mathbb{N} \times \mathbb{N}$, we instead seek to solve a discretized hybrid linear regression model of the form

$$y_s(k, j) := \theta^\top \psi_s(k, j) \quad \forall (k, j) \in E_s \qquad (7)$$

where

$$\psi_s(k, j) := h(k, \psi(t, j)) \quad \forall (k, t, j) \in \mathbb{N} \times \operatorname{dom} \psi$$

for some $h : \mathbb{N} \times \operatorname{rge} \psi \to E_s$, where rge ψ is the range of ψ, that samples ψ at time instances $(ks, j) \in \operatorname{dom} \psi$ [10].

For systems of this type, we cannot directly apply the hybrid gradient algorithm in [4] to estimate the unknown

[1] It is possible that the state x of a system $\mathcal{H}_s$ reaches a point outside of a set $C \cup D$, where it would normally remain in if not due to discretization. In [10], the jump map and jump set are modified to "catch" these points and ensure that solutions of hybrid systems are better approximated by their discretizations. In this paper, discretized measurement data is assumed to account for these modifications, although they are not presented or elaborated on.

[2] The term "pre-exponential," differentiates itself from the term "exponential," in the case where a solution exists that is maximal but not complete. In this way, the conditions for completeness can be separated from those for stability and attractivity – see [9] for details.

parameter θ, as it requires continuous measurements during flows, which are often not feasible in practice. Thus, in this paper, we propose a hybrid algorithm for estimating unknown parameters in discretized hybrid linear regression models.

As further motivation for the proposed algorithm, consider a model as in (7), with discretized hybrid time domain

$$E_s = \bigcup_{i=0}^{\infty} [(\{(\alpha+1)i,\, (\alpha+1)i+1,\, \ldots,\, (\alpha+1)(i+1)\} \times \{2i\}) \cup (\{(\alpha+1)(i+1)\} \times \{2i+1\})],$$

step size $s = \frac{2\pi}{\alpha}$ for some $\alpha \in \mathbb{N} \setminus \{0\}$, known output signal $(k,j) \mapsto y_s(k,j) \in \mathbb{R}$, unknown parameter $\theta = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, and known regressor signal $(k,j) \mapsto \psi_s(k,j) \in \mathbb{R}^2$. During flows, the value of ψ_s is

$$\psi_s(k,j) = \begin{bmatrix} \sin(sk) & 0 \end{bmatrix}^\top$$

and, each time ψ_s jumps, the value of ψ_s after the jump is

$$\psi_s(k,j+1) = \begin{cases} \begin{bmatrix} 0.5 & 1 \end{bmatrix}^\top & j \in \{0,2,4,\ldots\} \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^\top & j \in \{1,3,5,\ldots\}. \end{cases}$$

Suppose our goal is to estimate θ. We first employ the discretized continuous-time GD algorithm in (4) and the discrete-time GD algorithm in (6). The discretized continuous GD algorithm utilizes measurements only during flows, and the discrete-time GD algorithm utilizes measurements immediately after each jump. Both algorithms fail to estimate the unknown parameter θ, as shown in Figure 1. To see why they fail, note that the discrete-time signal that is obtained by neglecting the evolution of ψ_s at jumps does not satisfy the discrete-time PE condition in (C1). Similarly, the discrete-time signal that is obtained by neglecting the evolution of ψ_s during flows also does not satisfy (C1). On the other hand, the discretized hybrid GD algorithm proposed in this paper successfully estimates θ by leveraging information during both flows and jumps, as shown in Figure 1.[3]

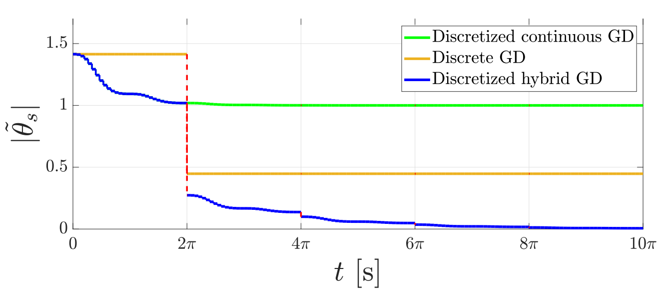

Fig. 1: The projection onto t of the norm of the estimation error $\tilde{\theta}_s$ given sampled data, simulated using the HyEQ Toolbox [11].

[3]The simulation files for this example can be found here: https://github.com/HybridSystemsLab/HybridGradientDiscrete-Motivation

IV. Discrete Hybrid GD Algorithm

A. Linear Regression for Discretized Hybrid Systems

Inspired by [4], we propose a discretized hybrid GD algorithm to update the estimate $\hat{\theta}_s$ according to the update law in (4), discretized based on (2) such that

$$\begin{aligned} \hat{\theta}_s(k+1,j) &= \hat{\theta}_s(k,j) - s\gamma_c \psi_s(k,j) \\ &\quad \times (\hat{\theta}_s^\top(k,j)\psi_s(k,j) - y_s(k,j)) \end{aligned}$$

during intervals of flow, where $s > 0$ is the step size due to discretization and $\gamma_c > 0$ is a design parameter. In addition, based on the update law in (6), the value of $\hat{\theta}_s$ after each time ψ_s jumps is given by

$$\begin{aligned} \hat{\theta}_s(k,j+1) &= \hat{\theta}_s(k,j) - \frac{\gamma_d \psi_s(k,j+1)}{1+\gamma_d \psi_s(k,j+1)^\top \psi_s(k,j+1)} \\ &\quad \times (\hat{\theta}_s^\top(k,j)\psi_s(k,j+1) - y_s(k,j+1)). \end{aligned}$$

where $\gamma_d > 0$ is a design parameter. Then, the dynamics of the estimation error $\tilde{\theta}_s := \hat{\theta}_s - \theta$ are

$$\begin{aligned} \tilde{\theta}_s(k+1,j) &= \tilde{\theta}_s(k,j) - s\gamma_c \psi_s(k,j)\psi_s(k,j)^\top \tilde{\theta}_s(k,j) \\ \tilde{\theta}_s(k,j+1) &= \tilde{\theta}_s(k,j) - \frac{\gamma_d \psi_s(k,j+1)\psi_s(k,j+1)^\top}{1+\gamma_d \psi_s(k,j+1)^\top \psi_s(k,j+1)} \tilde{\theta}_s(k,j) \end{aligned}$$

during flows and at jumps, respectively.

Given a regressor $\psi_s : E_s \to \mathbb{R}^n$ and an output $\psi_s : y \to \mathbb{R}$ satisfying (7), where E_s is a discretized hybrid time domain, the dynamics of $\hat{\theta}_s$ are captured by the system $\mathcal{H}_s^g$, with state $x := (\hat{\theta}_s, k, j) \in \mathcal{X} := \mathbb{R}^n \times E_s$ and data

$$\mathcal{H}_s^g : \begin{cases} \dot{x} = F_s^g(x) & x \in C_s^g \\ x^+ = G_s^g(x) & x \in D_s^g \end{cases}$$

where

$$\begin{aligned} F_s^g(x) &:= \begin{bmatrix} \hat{\theta}_s - s\gamma_c \psi_s(k,j)(\hat{\theta}_s^\top \psi_s(k,j) - y_s(k,j)) \\ k+1 \\ j \end{bmatrix} \\ G_s^g(x) &:= \begin{bmatrix} \hat{\theta}_s - \frac{\gamma_d \psi_s(k,j+1)(\hat{\theta}_s^\top \psi_s(k,j+1) - y_s(k,j+1))}{1+\gamma_d \psi_s(k,j+1)^\top \psi_s(k,j+1)} \\ k \\ j+1 \end{bmatrix} \\ C_s^g &:= \{x \in \mathcal{X} : (k+1,j) \in E_s\} \\ D_s^g &:= \{x \in \mathcal{X} : (k,j+1) \in E_s\} \end{aligned}$$

Note that this construction is such that our proposed algorithm flows and jumps in tandem with ψ_s.

Remark 1: After each jump, the output y_s satisfies $y_s(k,j+1) = \theta_s^\top \psi_s(k,j+1)$ for each $(k,j) \in \operatorname{dom} \psi_s$ such that $(k,j+1) \in \operatorname{dom} \psi_s$. This system is not causal, and in practice, since post-jump measurements of ψ_s and y_s are not available until after they jump, each jump in the estimator state will occur at the discrete time instant right after the corresponding jump in ψ_s occurs.

B. Stability Analysis

Convergence of $\tilde{\theta}_s$ to zero implies that $\hat{\theta}_s$ converges to θ. We define a hybrid system $\tilde{\mathcal{H}}_s^g$ with state $\tilde{x} = (\tilde{\theta}_s, k, j) \in \mathcal{X}$

that captures the dynamics of the error $\tilde{\theta}_s$, as follows:

$$\begin{bmatrix}\tilde{\theta}_s^+\\ k^+\\ j^+\end{bmatrix} = \begin{bmatrix}\tilde{\theta}_s - s\gamma_c\psi_s(k,j)\psi_s(k,j)^\top\tilde{\theta}_s\\ k+1\\ j\end{bmatrix} \qquad \tilde{x}\in C_s^g$$
$$\begin{bmatrix}\tilde{\theta}_s^+\\ k^+\\ j^+\end{bmatrix} = \begin{bmatrix}\tilde{\theta}_s - \frac{\gamma_d\psi_s(k,j+1)\psi_s(k,j+1)^\top}{1+\gamma_d\psi_s(k,j+1)^\top\psi_s(k,j+1)}\tilde{\theta}_s\\ k\\ j+1\end{bmatrix} \qquad \tilde{x}\in D_s^g. \tag{8}$$

Inspired by [4], we propose the following notion of persistence of excitation to enable us to guarantee global pre-exponential stability of the set

$$\mathcal{A} := \{\tilde{x}\in\mathcal{X} : \tilde{\theta} = 0\} \tag{9}$$

for the discretized hybrid system $\tilde{\mathcal{H}}_s^g$. Global pre-exponential stability of $\mathcal{A}$ implies that, for each solution $\tilde{x}$ to $\tilde{\mathcal{H}}_s^g$, the distance from $\tilde{x}$ to the set $\mathcal{A}$ – or, equivalently, the distance from $\tilde{\theta}_s$ to the origin – is bounded above by an exponentially decreasing function of the initial condition (see Definition 2). As a consequence, for each complete solution $\tilde{x}$ to $\tilde{\mathcal{H}}_s^g$, the parameter estimation error converges exponentially to zero.

Assumption 1: (Discretized hybrid persistence of excitation) Given a discretized hybrid signal $\psi_s : E_s \to \mathbb{R}^n$, as well as parameters $\gamma_c, \gamma_d, s > 0$, there exist $\Gamma, \mu > 0$ such that for each (k', j'), $(k^*, j^*) \in E_s$ satisfying $\Gamma + 1 \geq k^* - k' + j^* - j' \geq \Gamma$, the following holds:

$$\sum_{j=j'}^{j^*}\sum_{k=\max\{k',k_j\}}^{\min\{k^*,k_{j+1}\}} s\gamma_c\psi_s(k,j)\psi_s(k,j)^\top + \sum_{j=j'}^{j^*-1}\frac{\gamma_d\psi_s(k_{j+1},j+1)\psi_s(k_{j+1},j+1)^\top}{1+\gamma_d\psi_s(k_{j+1},j+1)^\top\psi_s(k_{j+1},j+1)} \geq \mu I \tag{10}$$

where $\{k_j\}_{j=0}^J$ is the sequence defining E_s as in Definition 1, $k_{J+1} = K$, $J := \sup_j E_s$, and $K := \sup_k E_s$.

Remark 2: The hybrid PE condition in Assumption 1 relaxes the discrete-time PE condition (C1). Indeed, it is possible that ψ_s satisfies Assumption 1 when (C1) is not satisfied by the discrete-time signal that is obtained by neglecting the evolution of ψ_s during flows.

We now establish our main result stating conditions to ensure the set $\mathcal{A}$ in (9) is globally pre-exponentially stable for $\tilde{\mathcal{H}}_s^g$.

Theorem 1: Given a hybrid system $\tilde{\mathcal{H}}_s^g$, a discretized hybrid signal $\psi_s(k,j) : E_s \to \mathbb{R}^n$, and design parameters $\gamma_c, \gamma_d, s > 0$, suppose Assumption 1 holds and there exists ψ_M such that $|\psi_s(k,j)| \leq \psi_M$ for all $(k,j)\in E_s$ and $\gamma_c \in (0, \frac{1}{s\psi_M^2}]$. Then, the $\tilde{\theta}_s$ component of each solution $\tilde{x}$ to $\tilde{\mathcal{H}}_s^g$ satisfies

$$|\tilde{\theta}_s(k,j)| \leq \kappa e^{-\lambda(k+j)}|\tilde{\theta}_s(0,0)|$$

for all $(k,j)\in E_s$, with κ and λ given by

$$\kappa := \sqrt{\frac{1}{1-\sigma}}, \quad \lambda := \frac{1}{2(\Gamma+1)}\ln\left(\frac{1}{1-\sigma}\right)$$

where $\sigma := \frac{\mu}{(1+\sqrt{2(\Gamma+2)^3})^2}$ with Γ, μ from Assumption 1.

Proof: This proof is in Appendix A. ■

V. Examples

A. Clock Skew Estimation

Consider a clock used to time periodic events, which evolves according to the differential equation $\dot{\tau} = 1 + \epsilon$, where $\tau \in \mathbb{R}_{\geq 0}$ is a timer variable, and $\epsilon \in \mathbb{R}$ is an unknown parameter representing the skew between it and a reference clock. The dynamics of the clock can be written as a hybrid system (1) with an added piecewise constant input $u \in \{0,1\}$, where $u = 1$ during events to be timed, and $u = 0$ otherwise.[4] The clock has state $z := (\tau, q) \in \mathbb{R}_{\geq 0}\times\{0,1\}$, where q is a logic variable, and dynamics

$$\begin{bmatrix}\dot{\tau}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}(1+\epsilon)q\\ 0\end{bmatrix} \qquad (\tau,q,u)\in C_c$$
$$\begin{bmatrix}\tau^+\\ q^+\end{bmatrix} = \begin{bmatrix}0\\ 1-q\end{bmatrix} \qquad (\tau,q,u)\in D_c \tag{11}$$

where

$$C_c := \{(\tau,q,u)\in\mathbb{R}_{\geq 0}\times\{0,1\}\times\{0,1\} : q = u\}$$
$$D_c := \{(\tau,q,u)\in\mathbb{R}_{\geq 0}\times\{0,1\}\times\{0,1\} : q \in \{0,1\}\setminus\{u\}\}.$$

Given $\tau : E \to \mathbb{R}_{\geq 0}$ and $q : E \to \{0,1\}$ satisfying (11), where $E = \operatorname{dom}\tau = \operatorname{dom} q$ is a hybrid time domain, we define

$$y(t,j) := \tau(t,j), \quad \psi(t,j) := \begin{cases}0 & \tau(t,j) = 0\\ t - t_j & \tau(t,j) > 0\end{cases}$$

for all $(t,j)\in\operatorname{dom} E$. The signals y and ψ satisfy $y(t,j) = \theta^\top\psi(t,j)$ for all $(t,j)\in E$, with $\theta = 1+\epsilon$. We sample y and ψ during flows and jumps, with a sample period of $s > 0$ during flows, and at jumps, at the times when y and ψ jump. The resulting signals, denoted as y_s and ψ_s, respectively, are defined on a discretized hybrid time domain $E_s \subset \mathbb{N}\times\mathbb{N}$ as in Definition 1. Then, y_s and ψ_s satisfy (7) for all $(k,j)\in E_s$. First, let events to be timed occur every 1 second and have a duration of 0.5 seconds. Given measurement data ψ_s with $\epsilon = 0.1$ and a step size of $s = 0.05$, such that and $\psi_M = 0.5$, we can set $\gamma_c = 20$ and $\gamma_d = 2$, and it can be shown that ψ_s satisfies Assumption 1 with with $\Gamma = 22$ and $\mu = 20.59$. Then, the conditions of Theorem 1 hold, and we can apply $\mathcal{H}_s^g$ to estimate θ. The results of simulating this system with initial conditions $x_c(0,0) = (0,1)$, $\hat{\theta}_s = 0$ are shown in Figure 2[5] and Figure 3, showing that $\tilde{\theta}_s$ converges to 0 in accordance with Theorem 1.

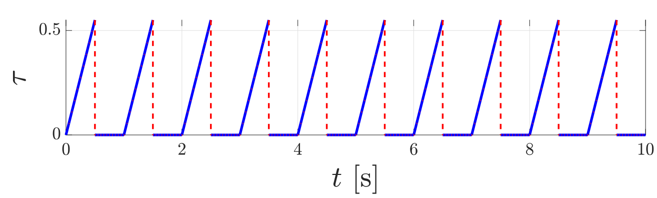

Fig. 2: Projection onto t of τ for the clock model.

[4] See [9] for details on hybrid systems with inputs.

[5] The simulation files for this example can be found here: https://github.com/HybridSystemsLab/HybridGradientDiscrete-ClockSkew

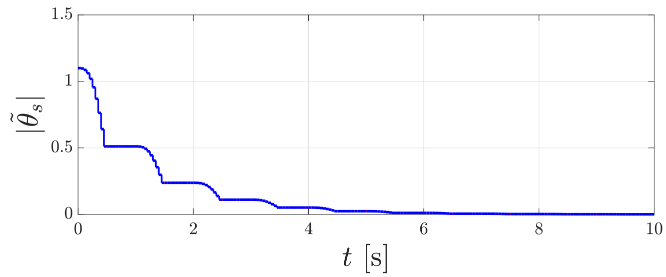

Fig. 3: The projection onto t of the norm of $\tilde{\theta}_s$ given measurement data of the clock model using $\mathcal{H}_s^g$.

B. Controller Gain Matrix Estimation

Consider a model of a pressure mounter machine that captures the dynamics of its main shaft. Its dynamics resemble those of an closed-loop mass-spring-damper system with an appropriate controller. Note that a similar problem was studied in [5]. Let $z_1 \in \mathbb{R}$ denote the vertical position of the shaft ($z_1 = 0$ at rest, $z_1 = z_{\max} > 0$ while in contact with the workbench), and $z_2 \in \mathbb{R}$ denote the vertical velocity of the machine. During flows, the dynamics are

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = -\frac{k}{m}z_1 - \frac{c}{m}z_2 + \frac{1}{m}u$$

where $m > 0$ is the mass of the machine, $k > 0$ is the spring constant, and $c > 0$ is the friction coefficient. The input $u \in \mathbb{R}$ is provided by a full-state feedback controller of the form

$$u = -K_P z + v \tag{12}$$

where $K_P \in \mathbb{R}^2$ is the unknown controller gain matrix and $v \in \mathbb{R}$ is a reference command. During jumps, the machine will impact a plate at position $z_{\max}$, and then rebound from the plate at a velocity scaling with the restitution coefficient, $\lambda \in (0, 1)$, as

$$z_1^+ = z_1, \qquad z_2^+ = -\lambda z_2.$$

Combining the expressions above, we write the closed-loop dynamics of the pressure mounter machine as a hybrid system as in (1) with an added input $(t, j) \mapsto v(t, j)$ as

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} K_P \right) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} v(t,j) \quad && z \in C_P \\ \begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -\lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} && z \in D_P \end{aligned}$$

where $C_P := \{z \in \mathbb{R}^2 : z_1 \leq z_{\max}\}$ and $D_P := \{z \in \mathbb{R}^2 : z_1 = z_{\max}, z_2 \geq 0\}$, with $z_{\max} > 0$.

Given $z : E \to \mathbb{R}^2$, $v : E \to \mathbb{R}$, and $u : E \to \mathbb{R}$ satisfying (12), where $E := \operatorname{dom} z = \operatorname{dom} v = \operatorname{dom} u$ is a hybrid time domain, we sample these signals during flows and jumps, with a sample period of $s > 0$ during flows. The resulting signals, denoted as z_s, v_s, and u_s, respectively, are defined on a discretized hybrid time domain, denoted by $E_s \subset \mathbb{N} \times \mathbb{N}$ as in Definition 1. Then, it follows from (12) that, for all $(k, j) \in E_s$,

$$u_s(k, j) = -K_P z_s(k, j) + v_s(k, j). \tag{13}$$

By defining $y_s(k, j) := u_s(k, j) - v_s(k, j)$ and $\psi_s(k, j) := -z_s(k, j)$ for all $(k, j) \in E_s$, we rewrite (13) into the form of the discretized hybrid linear regression model in (7) as $y_s(k, j) = \theta^\top \psi_s(k, j)$ for all $(k, j) \in E_s$, where $\theta := K_P^\top$.

We sample y and ψ during flows and jumps, with a sample period of $s > 0$ during flows, and at jumps, at the times when y and ψ jump. Then, we employ our proposed estimation algorithm $\mathcal{H}_g^s$ to estimate the unknown controller gain matrix K_P. The pressure mounter machine has parameters $m = 0.5$, $k = 25$, $c = 1.5$, $\lambda = 0.95$, $z_{\max} = 3$, and $K_P = [0.495, 0.678]^\top$. Our proposed estimator has parameters $\gamma_c = 0.138$ and $\gamma_d = 1$, with a sample period of $s = 0.02$ seconds. So that Theorem 1 holds numerically, we choose v such that the machine's trajectories achieve a limit cycle in steady-state. The simulation has initial conditions $z(0, 0) = (0, 0)$ and $\hat{\theta}(0, 0) = (0, 0)$. The trajectory of the pressure mounter state is shown in the plots[6] of Figure 4. The parameter estimation error for our proposed algorithm is shown in Figure 5. The estimation error converges exponentially to zero in accordance with Theorem 1.

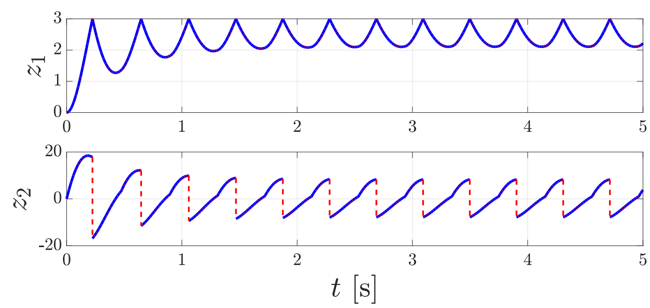

Fig. 4: The projection onto t of z_1 and z_2.

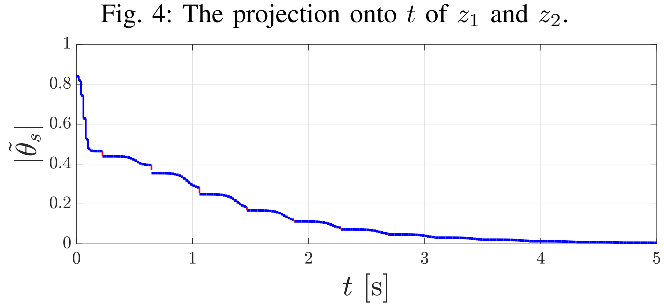

Fig. 5: The projection onto t of the parameter estimation error for $\mathcal{H}_s^g$.

VI. Conclusion

In this paper, we proposed an algorithm for estimating unknown parameters in hybrid linear regression models, given samples of the hybrid signals. The algorithm applies a discretized version of continuous GD during flows, and discrete GD at jumps. It was shown that a discretized hybrid persistence of excitation condition is sufficient to guarantee convergence of the estimation error to zero. In future work, we will consider identification of flow and jump maps and sets for a similar class of discretized hybrid systems, as well as identification of flow and jump maps for hybrid systems with nonlinear dynamics.

[6] The simulation files for this example can be found here: https://github.com/HybridSystemsLab/HybridGradientDiscrete-GainID

Appendix

A stability analysis of the set $\mathcal{A}$ for $\tilde{\mathcal{H}}_s^g$ is presented. Given $A_c, A_d : E_s \to \mathbb{R}^{n\times n}$, where $E_s := \operatorname{dom} A_c = \operatorname{dom} A_d$ is a discretized hybrid time domain, note that $\tilde{\mathcal{H}}_s^g$ belongs to a class of discretized hybrid systems $\tilde{\mathcal{H}}_s$ with state $x = (\tilde{\theta}_s, k, j) \in \mathcal{X} = \mathbb{R}^n \times E_s$ and dynamics

$$\begin{aligned}\begin{bmatrix}\tilde{\theta}_s^+\\ k^+\\ j^+\end{bmatrix} &= \begin{bmatrix}\tilde{\theta}_s - A_c(k,j)\tilde{\theta}_s\\ k+1\\ j\end{bmatrix} =: F_s(x) \quad x \in C_s\\ \begin{bmatrix}\tilde{\theta}_s^+\\ k^+\\ j^+\end{bmatrix} &= \begin{bmatrix}\tilde{\theta}_s - A_d(k,j)\tilde{\theta}_s\\ k\\ j+1\end{bmatrix} =: G_s(x) \quad x \in D_s\end{aligned} \tag{14}$$

where $C_s := C_s^g$ and $D_s := D_s^g$.

Remark 3: System $\tilde{\mathcal{H}}_s^g$ in (8) reduces to $\tilde{\mathcal{H}}_s$ in (14) when

$$A_c(k,j) = s\gamma_c\psi_s(k,j)\psi_s(k,j)^\top \tag{15}$$

for all $(k,j) \in E_s$ and

$$A_d(k,j) = \frac{\gamma_d\psi_s(k,j+1)\psi_s(k,j+1)^\top}{1+\gamma_d\psi_s(k,j+1)^\top\psi_s(k,j+1)} \tag{16}$$

for all $(k,j) \in E_s$ such that $(k,j+1) \in E_s$.

To establish pre-exponential stability of $\mathcal{A}$ for $\tilde{\mathcal{H}}_s^g$, we formulate results for $\tilde{\mathcal{H}}_s$. To this end, we assume the following regarding A_c and A_d in $\tilde{\mathcal{H}}_s$.

Assumption 2: Given $A_c, A_d : E_s \to \mathbb{R}^{n\times n}$, where $E_s := \operatorname{dom} A_c = \operatorname{dom} A_d$ is a discretized hybrid time domain, the following conditions hold:

(B0) $A_c(k,j) = A_c(k,j)^\top \geq 0$ for all $(k,j) \in E_s$ and $A_d(k,j) = A_d(k,j)^\top \geq 0$ for all $(k,j) \in E_s$ such that $(k,j+1) \in E_s$,

(B1) $|A_c(k,j)| \leq 1$ for all $(k,j) \in E_s$ and $|A_d(k,j)| \leq 1$ for all $(k,j) \in E_s$ such that $(k,j+1) \in E_s$.

Assumption 3: Given $A_c, A_d : E_s \to \mathbb{R}^{n\times n}$, where $E_s := \operatorname{dom} A_c = \operatorname{dom} A_d$, there exist $\Gamma, \mu > 0$ such that for each (k',j'), $(k^*,j^*) \in E_s$ satisfying $\Gamma + 1 \geq k^* - k' + j^* - j' \geq \Gamma$, the following holds:

$$\sum_{j=j'}^{j^*}\sum_{k=\max\{k',k_j\}}^{\min\{k^*,k_{j+1}\}} A_c(k,j) + \sum_{j=j'}^{j^*-1} A_d(k_{j+1},j) \geq \mu I \tag{17}$$

where $\{k_j\}_{j=0}^J$ is the sequence defining E_s as in Definition 1, $k_{J+1} = K$, $J := \sup_j E_s$, and $K := \sup_k E_s$.

To prove Theorem 1, we establish the following auxiliary result:

Theorem 2: Given $A_c, A_d : E_s \to \mathbb{R}^{n\times n}$, where $E_s := \operatorname{dom} A_c = \operatorname{dom} A_d$, suppose Assumption 2 and Assumption 3 hold. Then, the $\tilde{\theta}_s$ component of each solution x to $\tilde{\mathcal{H}}_s$ in (14) satisfies

$$|\tilde{\theta}_s(k,j)| \leq \kappa e^{-\lambda(k+j)}|\tilde{\theta}_s(0,0)| \tag{18}$$

for all $(k,j) \in E_s$, with κ and λ given by

$$\kappa := \sqrt{\frac{1}{1-\sigma}}, \quad \lambda := \frac{1}{2(\Gamma+1)}\ln\left(\frac{1}{1-\sigma}\right) \tag{19}$$

where $\sigma := \frac{\mu}{(1+\sqrt{2(\Gamma+2)^3})^2}$ with Γ, μ from Assumption 3.

Sketch of Proof: Consider the Lyapunov function

$$V(x) := |x|_{\mathcal{A}}^2 = \tilde{\theta}_s^\top\tilde{\theta}_s \quad \forall x \in C_s \cup D_s \cup G_s(D_s).$$

Since $(k,j) \mapsto \psi_s(k,j) \in \mathbb{R}^n$ satisfies the hybrid persistency of excitation condition in Assumption 2, it can be shown that

$$V(x(k,j)) \leq \frac{1}{1-\sigma}(1-\sigma)^{\frac{k+j}{\Gamma+1}}V(x(0,0))$$

for all $(k,j) \in E_s$, with κ and λ given in (19). Hence, (18) follows from the definition of V.

A. Proof of Theorem 1

Given the construction of A_c and A_d in (15) and (16), item (B0) of Assumption 2 holds. In addition, by substituting A_c and A_d in (15) and (16) into (10), we obtain (17), and thus Assumption 3 follows from Assumption 1. Given that $\gamma_c \in (0, \frac{1}{s\psi_M^2}]$ and $\gamma_d > 0$, we have from (15) and (16) that $|A_c(k,j)| \leq |\frac{1}{s\psi_M^2}s\psi_s(k,j)\psi_s(k,j)^\top| \leq 1$ for all $(k,j) \in E_s$ and $|A_d(k,j)| = |\frac{\psi_s(k,j+1)\psi_s(k,j+1)^\top}{\frac{1}{\gamma_d}+\psi_s(k,j+1)^\top\psi_s(k,j+1)}| \leq 1$ for all $(k,j) \in E_s$ such that $(k,j+1) \in E_s$. Thus, item (B1) of Assumption 2 holds.

Since Assumption 2 and Assumption 3 hold, from the equivalence between the data of $\tilde{\mathcal{H}}_s^g$ in (8) and $\tilde{\mathcal{H}}_s$ in (14), with A_c and A_d as in (15) and (16), Theorem 1 holds.

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