Notions, Stability, Existence, and Robustness of Limit Cycles in Hybrid Dynamical Systems

Xuyang Lou, Yuchun Li, and Ricardo G. Sanfelice, *Fellow, IEEE*

***Abstract*—This paper deals with existence and robust stability of hybrid limit cycles for a class of hybrid systems given by the combination of continuous dynamics on a flow set and discrete dynamics on a jump set. For this purpose, the notion of Zhukovskii stability, typically stated for continuous-time systems, is extended to the hybrid systems. Necessary conditions, particularly, a condition using a forward invariance notion, for existence of hybrid limit cycles are first presented. In addition, a sufficient condition, related to Zhukovskii stability, for the existence of (or lack of) hybrid limit cycles is established. Furthermore, under mild assumptions, we show that asymptotic stability of such hybrid limit cycles is not only equivalent to asymptotic stability of a fixed point of the associated Poincaré map but also robust to perturbations. Specifically, robustness to generic perturbations, which capture state noise and unmodeled dynamics, and to inflations of the flow and jump sets are established in terms of $\mathcal{KL}$ bounds. Furthermore, results establishing relationships between the properties of a computed Poincaré map, which is necessarily affected by computational error, and the actual asymptotic stability properties of a hybrid limit cycle are proposed. In particular, it is shown that asymptotic stability of the exact Poincaré map is preserved when computed with enough precision. Two examples, including a congestion control system, are presented to illustrate the notions and results throughout the paper.**

I. Introduction

A. Motivation and Related Work

Nonlinear dynamical systems with periodic solutions are found in many areas, including biological dynamics [1], neuronal systems [2], and population dynamics [3], to name just a few. In recent years, the study of limit cycles in hybrid systems has received renewed attention, mainly due to the existence of hybrid limit cycles in many engineering applications, such as walking robots [5], genetic networks [6], holonomic mechanical systems subject to impacts [7], among others. Theory for the study of such periodic behavior dates back to the work Andronov et al. in 1966 [4], where self-oscillations (limit cycles) and discontinuous oscillations were studied. Limit cycles has been studied within the impulsive differential equations framework [8], [9], [10], for example in strongly nonlinear impulsive systems [11], [12], in slowly impulsive systems [13], in the Van der Pol equation [14], in a holonomic mechanical system subject to impacts [7], and in a weakly nonlinear two-dimensional impulsive system [15]. These early developments pertain to nominal systems given in the form of impulsive differential equations, leaving the question of whether it is possible to handle more general models, such as hybrid system models, and guarantee robustness to generic perturbations wide open.

As a difference to general continuous-time systems, for which the Poincaré-Bendixson theorem uses the topology of $\mathbb{R}^2$ to rule out chaos and offers criteria for existence of limit cycles/periodic orbits, the problem of identifying the existence of limit cycles for hybrid systems has been studied for specific classes of hybrid systems. Specific results for existence of hybrid limit cycles include [5]-[23]. In particular, Grizzle et al. establish the existence and stability properties of a periodic orbit of nonlinear systems with impulsive effects via the method of Poincaré sections [5]. Using the transverse contraction framework, the existence and orbital stability of nonlinear hybrid limit cycles are analyzed for a class of autonomous hybrid dynamical systems with impulse in [18]. In [19], the existence and stability of limit cycles in reset control systems are investigated via techniques that rely on the linearization of the Poincaré map about its fixed point. In [20], we analyze the existence of hybrid limit cycles in hybrid dynamical systems and establish necessary conditions for the existence of hybrid limit cycles. Clark et al. prove a version of the Poincaré-Bendixson theorem for planar hybrid dynamical systems with empty intersection between the flow set and the jump set [21], and extend the results to the case of an arbitrary number of state spaces (each of which is a subset of $\mathbb{R}^2$) and impacts in [22]. More recently, Goodman and Colombo propose necessary conditions for existence of a periodic orbit related to the Poincaré map and sufficient conditions for local conjugacy between two Poincaré maps in systems with prespecified jump times evolving on a differentiable manifold [23]. We believe that conditions for existence of hybrid limit cycles in general hybrid systems should play a more prominent role in analysis and control of hybrid limit cycles. To the best of our knowledge, tools for the analysis of existence or nonexistence of hybrid limit cycles for the class of hybrid systems in [16], [24] are still not available in the literature.

Stability issues of hybrid limit cycles are currently a major focus in studying hybrid systems for their practical value in applications. Due to the complicated behavior caused by interaction between continuous change and instantaneous change, the study of stability of limit cycles in hybrid systems is more difficult than the study in continuous systems or discrete systems, and so becomes a challenging issue. In this respect, the Poincaré map and its variations or generalizations still play a dominating role; see, e.g., [25]-[31]. For instance, Nersesov et al. generalize the Poincaré method to analyze limit cycles for left-continuous hybrid impulsive dynamical systems [25]. Gonçalves analytically develops the local stability of limit

X. Lou is with the Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi 214122, China. Email: `Louxy@jiangnan.edu.cn` Y. Li and R. G. Sanfelice are with the Department of Computer Engineering, University of California, Santa Cruz, CA 95064, USA. Email: `yuchunli,ricardo@ucsc.edu.` Research by X. Lou has been supported by Jiangsu Provincial Natural Science Foundation of China Grants no. BK20201340, Postdoctoral Science Foundation of China Grants no. 2018M642160 and the 111 Project Grants no. B23008. Research by R. G. Sanfelice partially supported by NSF Grants no. ECS-1710621 and CNS-1544396, by AFOSR Grants no. FA9550-16-1-0015 and FA9550-19-1-0053, and by CITRIS and the Banatao Institute at the University of California.

cycles in a class of switched linear systems when a limit cycle exists [26]. The authors in [27] analyze local stability of a predefined limit cycle for switched affine systems and design switching surfaces by computing eigenvalues of the Jacobian of the Poincaré map. Motivated by robotics applications, the authors in [28]-[31] analyze the stabilization of periodic orbits in systems with impulsive effects using the Jacobian linearization of the Poincaré return map and the relationship between the stability of the return map and the stability of the hybrid zero dynamics. To the best of our knowledge, all of the aforementioned results about limit cycles are only suitable for hybrid systems that have jumps on switching surfaces and under nominal/noise-free conditions. In fact, the results therein do not characterize the robustness properties to perturbations of stable hybrid limit cycles, which is a very challenging problem due to the impulsive behavior in such systems.

Besides our preliminary results in [20], [32], [33], results for the study of existence and robustness of limit cycles in hybrid systems are currently missing from the literature, being perhaps the main reason that a robust stability theory for such systems has only been developed in [16], [24]. In fact, all of the aforementioned results about limit cycles are formulated for hybrid systems operating in nominal/noise-free conditions. The development of tools that characterize the existence of hybrid limit cycles and the robustness properties to perturbations of stable hybrid limit cycles is very challenging and demands a modeling framework that properly handles time and the complex combination of continuous and discrete dynamics.

B. Contributions

Tools for the analysis of existence of limit cycles and robustness of asymptotic stability of limit cycles in hybrid systems are not yet available in the literature. In this paper, we propose such tools for hybrid systems given as hybrid inclusions [16], which is a broad modeling framework for hybrid systems as it subsumes hybrid automata, impulsive systems, reset systems, among others; see [16], [24] for more details. We introduce a notion of hybrid limit cycle for hybrid systems modeled as hybrid equations, which are given by

$$\mathcal{H}\begin{cases} \dot{x} &= f(x) \qquad x \in C, \\ x^{+} &= g(x) \qquad x \in D, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $\dot{x}$ denotes its derivative with respect to time, and x^+ denotes its value after a jump. The state x may have components that correspond to physical states, logic variables, timers, memory states, etc. The map f and the set C define the continuous dynamics (or flows), and the map g and the set D define the discrete dynamics (or jumps). In particular, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ (respectively, $g : \mathbb{R}^n \to \mathbb{R}^n$) is a single-valued map describing the continuous (respectively, discrete) evolution while $C \subset \mathbb{R}^n$ (respectively, $D \subset \mathbb{R}^n$) is the set on which the flow map f is effective (respectively, from which jumps can occur).

For this hybrid systems framework, we develop tools for characterizing existence of hybrid limit cycles and robustness properties to perturbations of stable hybrid limit cycles.[1] The contributions of this paper include the following:

- We introduce a notion of hybrid limit cycle (with one jump per period[2]) for the class of hybrid systems in (1). Also, we define the notion of flow periodic solution and asymptotic stability of the hybrid limit cycle for such hybrid systems.[3]
- We present necessary conditions for existence of hybrid limit cycles, including compactness, transversality of the limit cycle, and a continuity of the so-called time-to-impact function. Particularly, a condition using a forward invariance notion for existence of hybrid limit cycles is first presented.
- Motivated by the use of Zhukovskii stability methods for periodic orbits in continuous-time systems, as done in [35], [36], [37], we introduce this notion for the class of hybrid systems introduced in (1) and provide a sufficient condition for Zhukovskii stability that involves the incremental stability notion introduced in [38].
- By assuming that the state space contains no isolated equilibrium point for the flow dynamics, we establish a sufficient condition for the existence of hybrid limit cycles based on Zhukovskii stability. In addition, based on an incremental graphical stability notion introduced in [38], an approach to rule out existence of hybrid limit cycles in some cases is proposed.
- We establish sufficient and necessary conditions for guaranteeing (local and global) asymptotic stability of hybrid limit cycles for a class of hybrid systems. In the process of deriving these results, we construct time-to-impact functions and Poincaré maps that cope with one jump per period of a hybrid limit cycle.
- Via perturbation analysis for hybrid systems, we propose a result on robustness to generic perturbations of asymptotically stable hybrid limit cycles, which allows for state noise and unmodeled dynamics, in terms of $\mathcal{KL}$ bounds.
- Due to the wide applicability of the Poincaré section method, we present results that relate the properties of a computed Poincaré map, which is necessarily affected by computational error, to the actual asymptotic stability properties of hybrid limit cycles.

C. Notation

The set $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}_{\geqslant 0}$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geqslant 0} := [0, +\infty)$, and $\mathbb{N}$ denotes the set of natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, 2, \cdots\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a set S, S^n denotes n cross products of S, namely $S^n = S \times S \times \cdots \times S$. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the Lie derivative of h at x in the direction of f is denoted by $L_f h(x) := \langle \nabla h(x), f(x) \rangle$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$. The range of f is

[1] Preliminary version of the results in this paper appeared without proof in the conference articles [20] and [32].

[2] Here, we mainly focus on hybrid limit cycles with "one jump per period." The case of multiple jumps per period can be treated similarly; see [33].

[3] In this work, a hybrid limit cycle is given by a closed set, while the limit cycle defined in [5], [28], [34] is given by an open set due to the right continuity assumption in the definition of solutions.

denoted by rge$f$, i.e., rge$f := \{f(x) : x \in \operatorname{dom} f\}$. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $\overline{\mathcal{A}}$ (respectively, $\overline{\text{con}}\ \mathcal{A}$) denotes its closure (respectively, its closed convex hull) and $\mathcal{A}^\circ$ denotes its interior. Given an open set $\mathcal{X} \subset \mathbb{R}^n$ containing a compact set $\mathcal{A}$, a function $\omega : \mathcal{X} \to \mathbb{R}_{\geqslant 0}$ is a *proper indicator* for $\mathcal{A}$ on $\mathcal{X}$ if ω is continuous, $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and $\omega(x) \to \infty$ as x approaches the boundary of $\mathcal{X}$ or as $|x| \to \infty$. Given a sequence of set $\mathcal{X}_i$, $\limsup_{i \to \infty} \mathcal{X}_i$ denotes the outer limit of $\mathcal{X}_i$. The set $\mathbb{B}$ denotes a closed unit ball in Euclidean space (of appropriate dimension) centered at zero. Given $\delta > 0$ and $x \in \mathbb{R}^n$, $x + \delta\mathbb{B}$ denotes a closed ball centered at x with radius δ. A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing; it belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if, in addition, is unbounded. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if, for each $t \geqslant 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \to 0^+} \beta(s, t) = 0$ and, for each $s \geqslant 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \to \infty} \beta(s, t) = 0$.

II. Motivational Example

Consider the hybrid model for a congestion control mechanism in TCP proposed in [40]. The hybrid model in congestion avoidance mode can be described as follows:

- when $q \in [0, q_{\max}]$

$$\begin{bmatrix} \dot{q} \\ \dot{r} \end{bmatrix} = \begin{cases} \begin{bmatrix} \max\{0, r - B\} \\ a \end{bmatrix} & \text{if } q = 0 \\ \begin{bmatrix} r - B \\ a \end{bmatrix} & \text{if } q > 0 \end{cases} \tag{2a}$$

- when $q = q_{\max}, r \geqslant B$

$$(q^+, r^+) = (q_{\max}, mr) \tag{2b}$$

where $q \in [0, q_{\max}]$ denotes the current queue size, $q_{\max}$ is the maximum queue size, $r \geqslant 0$ is the rate of incoming data packets, and $B \geqslant 0$ is the rate of outgoing packets. The constant $a \geqslant 1$ reflects the rate of growth of incoming data packets r while $m \in (0, 1)$ reflects the factor that makes the rate of incoming packets decrease; see [40] for details. The model in (2) reduces the rate of incoming packets r by the factor m if the queue size q equals the maximum value $q_{\max}$ with rate larger than or equal to B.

We are interested in the hybrid system (2) restricted to the region

$$M_{\mathrm{T}} := \{(q, r) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} : q \leqslant q_{\max}, aq \geqslant \frac{1}{2}r^2 - Br + \frac{B^2}{2}\} \tag{3}$$

for given parameters a, m, $q_{\max}$ and B (later, the set M_{T} will be part of our analysis); see Fig. 1. From the first piece in the definition in (2) with $a > 0$, for any maximal solution with initial condition with zero q and r less than B, q remains at zero until $r > B$. Fig. 1 is shown to analyze how we get a region from which a limit cycle with one jump exists. The points in the curve $P_3 \to P_4 \to P_5$ satisfy $aq = \frac{1}{2}r^2 - Br + \frac{B^2}{2}$. Solutions from the region M_2 result in solutions such that q reaches zero and remains at zero until $r = B$ (point P_4). The open set $M_1 := \{(q_{\max}, B)\} + \varepsilon\mathbb{B}^\circ$ with $\varepsilon > 0$ small

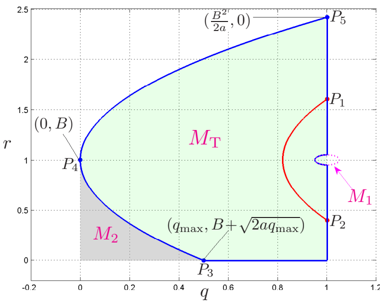


Fig. 1. Diagram of the compact set M_{T} denoted in the region with light green filled pattern. Parameters used in the plot are $B = 1, a = 1, q_{\max} = 1$, and $m = 0.25$. The points P_1 and P_2 correspond to state values in a limit cycle just before and right after each jump, respectively. The points P_1 corresponds to $(q, r) = (q_{\max}, 2B/(1 + m))$, the point P_2 corresponds to $(q, r) = (q_{\max}, 2Bm/(1 + m))$, the point P_3 corresponds to $(q, r) = (B^2/(2a), 0)$, the point P_4 corresponds to $(q, r) = (0, B)$, and the point P_5 corresponds to $(q, r) = (q_{\max}, B + \sqrt{2aq_{\max}})$.

enough, will be part of our analysis in Example 4.6 and be ruled out to ensure the transversality of the limit cycle. We are not interested in the region M_2 with gray filled pattern as it leads to a complex hybrid model which might be hard to be analyzed. The compact set M_{T} is marked by the region with light green filled pattern. Hence, the set $M_{\mathrm{T}} \setminus M_1$ (the region surrounded by blue line) is the region of the state space that we are interested in. Note that if the value of r after a jump from the point P_5 is larger than B (for instance, point P_5 jumps to point P_1), a consecutive jump will happen. Therefore, to avoid this case, we impose the condition $m(B + \sqrt{2aq_{\max}}) < B$.

From points in the set M_{T}, solutions approach a limit cycle. On M_{T} and for parameters satisfying the conditions above, the resulting system with $(q, r) \in M_{\mathrm{T}}$ can be described as a hybrid system $\mathcal{H}_{\text{TCP}}$ on M_{T} with data

$$\mathcal{H}_{\text{TCP}} \begin{cases} \dot{x} = f_{\text{TCP}}(x) := \begin{bmatrix} r - B \\ a \end{bmatrix} & x \in C_{\text{TCP}}, \\ x^+ = g_{\text{TCP}}(x) := \begin{bmatrix} q_{\max} \\ mr \end{bmatrix} & x \in D_{\text{TCP}}, \end{cases} \tag{4}$$

where $x = (q, r)$, $C_{\text{TCP}} = \{x \in \mathbb{R}^2 : q \leqslant q_{\max}\}$, $D_{\text{TCP}} = \{x \in \mathbb{R}^2 : q = q_{\max}, r \geqslant B\}$.

A limit cycle of the system in (4) with parameters $B = 1, a = 1, m = 0.25$, and $q_{\max} = 1$ is depicted in Fig. 1. This figure shows in red a limit cycle denoted as $\mathcal{O}$ and defined by the solution to the congestion control system with initial condition $P_2 = \{(1, 0.4)\}$. This solution flows to the point P_1, jumps to the point P_2, and then flows back to P_1. The interest in this paper is to find conditions under which such limit cycles may exist.

III. Definitions and Basic Properties

A. Hybrid Systems

We consider hybrid systems $\mathcal{H}$ as in [16], which can be written as in (1). The data of a hybrid system $\mathcal{H}$ is given by (C, f, D, g). The restriction of $\mathcal{H}$ on a set M is defined as $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$. A solution to $\mathcal{H}$ is parameterized by ordinary time t and a counter j for jumps. It is given by a hybrid arc[4] $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ that satisfies the dynamics of $\mathcal{H}$; see [16] for more details. A solution ϕ to $\mathcal{H}$ is said to be complete if $\operatorname{dom}\phi$ is unbounded. It is Zeno if it is complete and the projection of $\operatorname{dom}\phi$ onto $\mathbb{R}_{\geqslant 0}$ is bounded. It is discrete if $\operatorname{dom}\phi \subset \{0\} \times \mathbb{N}$. It is said to be maximal if it is not a (proper) truncated version of another solution. The set of maximal solutions to $\mathcal{H}$ from the set K is denoted as

$$\mathcal{S}_{\mathcal{H}}(K) := \{\phi : \phi \text{ is a maximal solution to } \mathcal{H} \text{ with } \phi(0,0) \in K\}.$$

We define $t \mapsto \phi^f(t, x_0)$ as a solution of the flow dynamics $\dot{x} = f(x) \quad x \in C$ from $x_0 \in \overline{C}$. A hybrid system $\mathcal{H}$ is said to be well-posed if it satisfies the *hybrid basic conditions*, namely,

A1) The sets $C, D \subset \mathbb{R}^n$ are closed.
A2) The flow map $f : C \to \mathbb{R}^n$ and the jump map $g : D \to \mathbb{R}^n$ are continuous.

The following notion of ω-limit set of a hybrid arc is used in Section V-B to formulate sufficient conditions for the existence of hybrid limit cycles.

Definition 3.1: [16, Definition 6.17] The ω-limit set of a hybrid arc $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$, denoted $\Omega(\phi)$, is the set of all points $x \in \mathbb{R}^n$ for which there exists a sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ of points $(t_i, j_i) \in \operatorname{dom}\phi$ with $\lim_{i\to\infty} t_i + j_i = \infty$ and $\lim_{i\to\infty} \phi(t_i, j_i) = x$. Every such point x is an ω-limit point of ϕ.

For more details about this hybrid systems framework, we refer the readers to [16].

B. Hybrid Limit Cycles

Before revealing their basic properties, we define hybrid limit cycles. For this purpose, we consider the following notion of flow periodic solutions.

Definition 3.2: (flow periodic solution) A complete solution ϕ^* to $\mathcal{H}$ is *flow periodic with period T^* and one jump in each period* if $T^* \in (0, \infty)$ is the smallest number such that $\phi^*(t + T^*, j + 1) = \phi^*(t, j)$ for all $(t, j) \in \operatorname{dom}\phi^*$.

The definition of a flow periodic solution ϕ^* with period $T^* > 0$ above implies that if $(t, j) \in \operatorname{dom}\phi^*$, then $(t + T^*, j+1) \in \operatorname{dom}\phi^*$. For a notion allowing for multiple jumps in a period, see [33]. A flow periodic solution to $\mathcal{H}$ as in Definition 3.2 generates a hybrid limit cycle.

Definition 3.3: (hybrid limit cycle) A flow periodic solution ϕ^* with period $T^* \in (0, \infty)$ and one jump in each period defines a *hybrid limit cycle*[5] $\mathcal{O} := \{x \in \mathbb{R}^n : x = \phi^*(t, j), (t, j) \in \operatorname{dom}\phi^*\}$.

In [39, Example 3.5], we revisit the example in Section II to further illustrate the hybrid limit cycle notion in Definition 3.3.

[4] A hybrid arc is a function ϕ defined on a hybrid time domain and for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous. A *compact hybrid time domain* is a set $\mathcal{E} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ of the form $\mathcal{E} = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_J$; the set $\mathcal{E}$ is a *hybrid time domain* if for all $(T, J) \in \mathcal{E}$, $\mathcal{E} \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain.

[5] Alternatively, the hybrid limit cycle $\mathcal{O}$ can be written as $\{x \in \mathbb{R}^n : x = \phi^*(t, j), t \in [t_s, t_s + T^*], (t, j) \in \operatorname{dom}\phi^*\}$ for some $t_s \in \mathbb{R}_{\geqslant 0}$.

IV. Necessary Conditions

A. Necessary Conditions for Existence of Hybrid Limit Cycles

In this subsection, we derive several necessary conditions for the existence of hybrid limit cycles for a class of hybrid systems $\mathcal{H}$ as in (1) satisfying the following properties.

Assumption 4.1: For a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$, there exists a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ such that

1) the flow set can be written as $C = \{x \in \mathbb{R}^n : h(x) \geqslant 0\}$ and the jump set as $D = \{x \in \mathbb{R}^n : h(x) = 0, L_f h(x) \leqslant 0\}$;
2) the flow map f is continuously differentiable on an open neighborhood of $M \cap C$, and the jump map g is continuous on $M \cap D$;
3) $L_f h(x) < 0$ for all $x \in M \cap D$, and $g(M \cap D) \cap (M \cap D) = \emptyset$.

Remark 4.2: Item 1) in Assumption 4.1 implies that flows occur when h is nonnegative while jumps only occur at points in the zero level set of h. Note that since h is continuous and f is continuously differentiable, the flow set and the jump set are closed. The state x may include logic variables, counters, timers, etc. The continuity property of f in item 2) of Assumption 4.1 is further required for the existence of solutions to $\dot{x} = f(x)$ according to [16, Proposition 2.10]. Moreover, item 2) also guarantees that solutions to $\dot{x} = f(x)$ depend continuously on initial conditions. In the upcoming results, item 3) in Assumption 4.1 allows us to establish a transversality property and restrict the analysis of a hybrid system $\mathcal{H}$ to a region of a state space $M \subset \mathbb{R}^n$, leading to the restriction of $\mathcal{H}$ given by $\mathcal{H}|_M := (M \cap C, f, M \cap D, g)$. The condition $g(M \cap D) \cap (M \cap D) = \emptyset$ is assumed to exclude discrete solutions. As we will show later, the set M is appropriately chosen for each specific system such that it guarantees completeness of maximal solutions to $\mathcal{H}|_M$ and the existence of flow periodic solutions. This is illustrated in Section II with a set M_{T}.

Remark 4.3: By items 1) and 2) of Assumption 4.1, the data of $\mathcal{H}|_M$ satisfies the hybrid basic conditions [16, Assumption 6.5]. Then, using item 3) of Assumption 4.1, [46, Lemma 2.7] implies that for any bounded and complete solution ϕ to $\mathcal{H}|_M$ there exists $r > 0$ such that $t_{j+1} - t_j \geqslant r$ for all $j \geqslant 1$, $t_j = \min I^j$, $t_{j+1} = \max I^j$; i.e., the elapsed time between two consecutive jumps is uniformly bounded below by a positive constant.

It can be shown that a hybrid limit cycle generated by periodic solutions as in Definition 3.3 is closed and bounded, as established in the following result.

Lemma 4.4: *Given a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1, suppose that $\mathcal{H}$ has a hybrid limit cycle $\mathcal{O}$. Then, $\mathcal{O}$ is compact and forward invariant*[6].

A proof can be found in [39, Lemma 4.4].

Remark 4.5: Since a hybrid limit cycle $\mathcal{O}$ to $\mathcal{H}|_M$ is compact, for any solution ϕ to $\mathcal{H}|_M$, the distance $|\phi(t, j)|_{\mathcal{O}}$ is well-defined for all $(t, j) \in \operatorname{dom}\phi$.

[6] Every $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$ is complete and satisfies $\operatorname{rge}\phi \subset \mathcal{O}$; see [41, Definition 3.3].

We revisit the previous example to illustrate the properties of a hybrid system $\mathcal{H}$ satisfying Assumption 4.1.

Example 4.6: Consider the congestion control system in Section II. By definition, the sets C_{TCP} and D_{TCP} of the model in (4) are closed. Moreover, f_{TCP} and g_{TCP} are continuously differentiable. Define the function $h : \mathbb{R}^2 \to \mathbb{R}$ as $h(x) = q_{\max} - q$. Then, C_{TCP} and D_{TCP} can be written as $C_{\text{TCP}} = \{x \in \mathbb{R}^2 : h(x) \geqslant 0\}$ and $D_{\text{TCP}} = \{x \in \mathbb{R}^2 : h(x) = 0, L_{f_{\text{TCP}}} h(x) \leqslant 0\}$, respectively. Consider the compact set $M_{\text{TCP}} := (M_{\text{T}} \cap C_{\text{TCP}}) \setminus M_1$, where M_{T} is defined in (3) and $M_1 = \{(q_{\max}, B)\} + \varepsilon\mathbb{B}^\circ$ with $\varepsilon > 0$ small enough; see Fig. 1. We obtain that $M_{\text{TCP}} \cap D_{\text{TCP}} = \{x \in \mathbb{R}^2 : q = q_{\max}, r \in [B+\varepsilon, B+\sqrt{2aq_{\max}}]\}$ and for each $x \in M_{\text{TCP}} \cap D_{\text{TCP}}$, $L_{f_{\text{TCP}}} h(x) = B - r < 0$. Moreover, due to the condition on parameters $m(B + \sqrt{2aq_{\max}}) < B$ (see Section II), it can be verified that $g_{\text{TCP}}(M_{\text{TCP}} \cap D_{\text{TCP}}) \cap (M_{\text{TCP}} \cap D_{\text{TCP}}) = \emptyset$ and $g_{\text{TCP}}(M_{\text{TCP}} \cap D_{\text{TCP}}) \subset M_{\text{TCP}} \cap C_{\text{TCP}}$. Furthermore, for any point $x \in M_{\text{TCP}} \cap C_{\text{TCP}}$, since the r component of the flow map f_{TCP}, i.e., $\dot{r} = a$, is positive, $\mathbf{T}_{M_{\text{TCP}} \cap C_{\text{TCP}}}(x) \cap \{f_{\text{TCP}}(x)\} = \{f_{\text{TCP}}(x)\} \neq \emptyset$ for each $x \in (M_{\text{TCP}} \cap C_{\text{TCP}}) \setminus D_{\text{TCP}}$.[7] When $x \in M_{\text{TCP}} \cap D_{\text{TCP}}$, we have $q = q_{\max}$ and $r \geqslant B + \varepsilon$ with $\varepsilon > 0$ small enough, which implies that $r - B > 0$ and solutions from x cannot be extended via flow. By [16, Proposition 6.10], every maximal solution to $\mathcal{H}_{\text{TCP}}|_{M_{\text{TCP}}} = (M_{\text{TCP}} \cap C_{\text{TCP}}, f_{\text{TCP}}, M_{\text{TCP}} \cap D_{\text{TCP}}, g_{\text{TCP}})$ is complete. Therefore, Assumption 4.1 holds. Moreover, a solution ϕ^* to $\mathcal{H}_{\text{TCP}}|_{M_{\text{TCP}}}$ from $\phi^*(0,0) = (q_{\max}, 2Bm/(1+m)) \in M_{\text{TCP}} \cap C_{\text{TCP}}$ is a flow periodic solution with $T^* = 2B(1-m)/(a+ma)$.

The following result establishes a transversality property of any hybrid limit cycle for $\mathcal{H}$ restricted to M.[8]

Lemma 4.7: *Given a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1, suppose that $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ has a hybrid limit cycle $\mathcal{O} \subset M \cap C$. Then, $\mathcal{O}$ is transversal to $M \cap D$.*

Proof: We proceed by contradiction. Consider the flow periodic solution ϕ^* with period T^* that generates the hybrid limit cycle $\mathcal{O}$ for $\mathcal{H}|_M$. By definition, there exists $x^* \in \mathcal{O}$ such that $x^* \in \mathcal{O} \cap (M \cap D)$ and $\phi^*(t^*, j^*) = x^*$ for some $(t^*, j^*) \in \operatorname{dom} \phi^*$. Suppose that $\mathcal{O}$ intersects $M \cap D$ at another point $x' \neq x^*$, i.e., $x' \in \mathcal{O} \cap (M \cap D)$ and $\phi^*(t', j') = x'$ for some $(t', j') \in \operatorname{dom} \phi^*$. Then, by items 1) and 3) of Assumption 4.1, it follows that $h(x') = 0$ and $L_f h(x') < 0$. Since h is continuously differentiable and f is continuous, $x \mapsto L_f h(x)$ is continuous. Then, there exists $\tilde{\delta} > 0$ such that $L_f h(x) < 0$ for all $x \in x' + \tilde{\delta}\mathbb{B}$. Therefore, the solution ϕ^* to $\mathcal{H}|_M$ cannot be extended through flow at x'. In fact, since $x' \in M \cap D$, ϕ^* will jump immediately when it reaches x'. This contradicts the fact that ϕ^* has only one jump in its period T^*. $\square$

To state our next result, let us introduce the *time-to-impact function* for hybrid systems as in $\mathcal{H}$. Alternative equivalent definitions can be found in [5] and [43, Definition 2]. In [43], a *minimal-time function* notion with respect to a closed set is presented for a constrained continuous-time system, which provides the first time that a solution starting from a given initial condition reaches that set. Following [5], for a hybrid system $\mathcal{H} = (C, f, D, g)$, the *time-to-impact function with respect to* D is defined by $T_I : \overline{C} \cup D \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, where[9]

$$T_I(x) := \inf\{t \geqslant 0 : \phi(t, j) \in D,\ \phi \in \mathcal{S}_{\mathcal{H}}(x)\} \tag{5}$$

for each $x \in \overline{C} \cup D$.

Inspired by [5, Lemma 3], we show that the function T_I is continuous on a subset of $M \cap (\overline{C} \cup D)$, as specified next.

Lemma 4.8: *Suppose a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfy Assumption 4.1. Then, T_I is continuous at points in $\mathcal{X} := \{x \in M \cap C : 0 < T_I(x) < \infty\}$.*

A proof can be found in [39, Lemma 4.12].

Next, we show that the function $x \mapsto T_I(x)$ is also continuous on a subset of $\mathcal{O}$.

Lemma 4.9: *Given a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1, suppose that $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ has a unique hybrid limit cycle $\mathcal{O} \subset M \cap C$ defined by the flow periodic solution ϕ^*. Then, T_I is continuous on $\mathcal{O} \setminus \{\phi^*(t^*, 0)\}$, where t^* is such that $(t^*, 0), (t^*, 1) \in \operatorname{dom} \phi^*$, namely, $(t^*, 0)$ is a jump time of ϕ^* and $\phi^*(t^*, 0)$ is the point in $M \cap D$ at which ϕ^* jumps.*

Proof: Consider a hybrid limit cycle $\mathcal{O} \subset M \cap C$ defined by the flow periodic solution ϕ^*. For $(t^*, 0), (t^*, 1) \in \operatorname{dom} \phi^*$, we have $\phi^*(t^*, 0) \in M \cap D$. By Lemma 4.4, since $\mathcal{O}$ is forward invariant, for all $x \in \mathcal{O} \setminus \{\phi^*(t^*, 0)\}$, there exists $t > t^*$ such that $\phi^*(t, 1)$ has a jump, which implies that $0 < T_I(x) < \infty$. By Lemma 4.8, T_I is continuous at points in $\mathcal{X} := \{x \in M \cap C : 0 < T_I(x) < \infty\}$. Then, T_I is continuous on $\mathcal{O} \setminus \{\phi^*(t, 0)\}$. $\square$

B. A Necessary Condition via Forward Invariance

Following the spirit of the necessary condition for existence of limit cycles in nonlinear continuous-time systems in [44], we have the following necessary condition for general hybrid systems with a hybrid limit cycle given by the zero-level set of a smooth enough function.

Proposition 4.10: *Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ satisfying the hybrid basic conditions with f continuously differentiable. Suppose every solution $\phi \in \mathcal{S}_{\mathcal{H}}$ is unique and there exists a hybrid limit cycle $\mathcal{O}$ for $\mathcal{H}$ with period $T^* > 0$ satisfying*

$$\mathcal{O} \subseteq \{x \in \mathbb{R}^n : p(x) = 0\},$$

where $p : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on an open neighborhood $\mathcal{U}$ of $\mathcal{O}$. Then, there exists $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ that is twice continuously differentiable on $\mathcal{U}$ and

$$W(x) \geqslant 0 \qquad \forall x \in \mathcal{O}, \tag{6}$$

$$\langle \nabla W(x), f(x) \rangle = 0 \qquad \forall x \in \mathcal{O} \cap C, \tag{7}$$

$$\langle \nabla \langle \nabla W(x), f(x) \rangle, f(x) \rangle = 0 \qquad \forall x \in \mathcal{O} \cap C, \tag{8}$$

$$W(g(x)) - W(x) = 0 \qquad \forall x \in \mathcal{O} \cap D. \tag{9}$$

[7] $\mathbf{T}_{(M \cap C)}(x)$ denotes the tangent cone to the set $M \cap C$ at x; see [16, Definition 5.12].

[8] A hybrid limit cycle $\mathcal{O}$ to a hybrid system $\mathcal{H}$ satisfying Assumption 4.1 is transversal to $M \cap D$ if $\mathcal{O}$ intersects $M \cap D$ at exactly one point $\bar{x} := \mathcal{O} \cap (M \cap D)$ with the property $L_f h(\bar{x}) \neq 0$.

[9] In particular, when there does not exist $t \geqslant 0$ such that $\phi^f(t, x) \in D$, we have $\{t \geqslant 0 : \phi^f(t, x) \in D\} = \emptyset$, which gives $T_I(x) = \infty$.

Furthermore, if p is such that $p(\bar{x}) \neq 0$ for some $\bar{x} \in C \cup D$, then W is such that (6) holds with strict inequality.

A proof can be found in [39, Proposition 4.14].

Proposition 4.10 provides a necessary condition, that by seeking for a function W with the properties therein, can be used to identify the existence of a hybrid limit cycle with period T^*. In addition, as exploited in [44, Theorem 1], it can be used to determine the stability of limit cycles for continuous-time systems.

The following example illustrates the result in Proposition 4.10.

Example 4.11: Consider the hybrid congestion control system in Example 4.6. The set defined by points (q, r) such that $q - \frac{(r-B)^2}{2a} = R$ with $R = q_{\max} - \frac{B^2(m-1)^2}{2a(m+1)^2}$ represents a hybrid limit cycle for $\mathcal{H}_{\text{TCP}}$, namely,

$$\mathcal{O} := \left\{ (q,r) \in M_{\text{TCP}} : q - \frac{(r-B)^2}{2a} = R \right\},$$

is a hybrid limit cycle. In particular, the state vector $x = (q, r)$ moves clockwise within $\mathcal{O}$ as depicted in Fig. 1. Using the flow and jump maps, it is verified that $\mathcal{O}$ is forward invariant. Note that when $\mathcal{O} \cap D_{\text{TCP}}$, $q = q_{\max}$ and $r = 2B/(m+1)$. To validate Proposition 4.10, define the continuously differentiable functions $p(x) := q - \frac{(r-B)^2}{2a} - R$, which satisfies $p(0) = -q_{\max} - \frac{4m}{(m+1)^2}\frac{B^2}{2a} \neq 0$, and $W : \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ as

$$W(x) = \left(q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)^2 > 0 \quad \forall x \in \mathcal{O}. \tag{10}$$

This function satisfies (7)-(9) since $\langle \nabla p(x), f_{\text{TCP}}(x) \rangle = [1 \ \ \frac{B-r}{a}] f_{\text{TCP}}(x) = r - B - (r - B) = 0$ for all $x \in C_{\text{TCP}}$. Then, for all $x \in \mathcal{O} \cap M_{\text{TCP}} \cap C_{\text{TCP}}$,

$$\langle \nabla W(x), f_{\text{TCP}} \rangle = 2\left(q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)(r - B - r + B) = 0$$

and $\langle \nabla \langle \nabla W(x), f_{\text{TCP}}(x) \rangle, f_{\text{TCP}}(x) \rangle = 0$. Moreover, for all $x \in \mathcal{O} \cap M_{\text{TCP}} \cap D_{\text{TCP}}$, using the fact that $q = q_{\max}$ and $r = 2B/(m+1)$, we have $W(g_{\text{TCP}}(x)) - W(x) = 0$.

V. Existence of Hybrid Limit Cycles

In this section, we introduce a stability notion that relates a solution to nearby solutions, which enables us to provide sufficient conditions for the existence of hybrid limit cycles for the class of hybrid systems in (1).

A. Zhukovskii Stability for Hybrid Systems

Zhukovskii stability for a continuous-time system consists of the property that, with a suitable reparametrization of perturbed trajectories, Lyapunov stability implies Zhukovskii stability; see, e.g., [36], [37]. We extend this notion to hybrid systems and establish links to the existence of hybrid limit cycles. To this end, inspired by [35], [36], [37], we employ the family of maps $\mathcal{T}$ defined by

$$\mathcal{T} = \{\tau(\cdot) \colon \tau : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0} \text{ is a homeomorphism}, \tau(0) = 0\}.$$

A map τ in the family $\mathcal{T}$ is employed to reparametrize ordinary time for the trajectories of the hybrid system (1) and formulate stability and attractivity notions involving the reparametrized trajectories, as formulated next.

Definition 5.1: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ as in (1). A maximal solution ϕ_1 to $\mathcal{H}$ is said to be

1) *Zhukovskii stable* (ZS) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\phi_2 \in \mathcal{S}_{\mathcal{H}}(\phi_1(0,0) + \delta\mathbb{B})$ there exists $\tau \in \mathcal{T}$ such that for each $(t,j) \in \operatorname{dom}\phi_1$ we have $(\tau(t), j) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t),j)| \leqslant \varepsilon$;
2) *Zhukovskii locally attractive* (ZLA) if there exists $\mu > 0$ such that for each $\phi_2 \in \mathcal{S}_{\mathcal{H}}(\phi_1(0,0) + \mu\mathbb{B})$ there exists $\tau \in \mathcal{T}$ such that for each $\varepsilon > 0$ there exists $T > 0$ for which we have that $(t,j) \in \operatorname{dom}\phi_1$ and $t + j \geqslant T$ imply $(\tau(t), j) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t),j)| \leqslant \varepsilon$;
3) *Zhukovskii locally asymptotically stable* (ZLAS) if it is both ZS and ZLA.

Remark 5.2: The map τ in Definition 5.1 reparameterizes the flow time of the solution ϕ_2. In particular, the ZS notion only requires that the solution ϕ_2 stays close to the solution ϕ_1 for the same value of the jump counter j but potentially at different flow times t. Note that τ in the ZS and ZLA notions may depend on the initial conditions of ϕ_1 and ϕ_2. For simplicity and for the purposes of this work, the ZLA notion is written as a uniform property, in the sense of hybrid time and over the compact set of initial conditions defined by μ. When ϕ_1 and each ϕ_2 are complete, the nonuniform version of that property would require

$$\lim_{(t,j) \in \operatorname{dom}\phi_1, t+j \to \infty} |\phi_1(t,j) - \phi_2(\tau(t),j)| = 0,$$

which resembles the notion defined in the literature of continuous-time systems; see [36, Definition 4.1] and [37, Definition 2].

The ZLAS notion will be related to existence of hybrid limit cycles by analyzing the properties of a Poincaré map in Section V-B (within the proof of Theorem 5.9) and the ω-limit set of a hybrid arc. Next, the ZLAS notion in Definition 5.1 is illustrated in an example with a hybrid limit cycle.

Example 5.3: Consider the academic system $\mathcal{H}_{\text{A}} = (C_{\text{A}}, f_{\text{A}}, D_{\text{A}}, g_{\text{A}})$ with scalar state x and data

$$\mathcal{H}_{\text{A}} \begin{cases} \dot{x} = f_{\text{A}}(x) := -ax + b & x \in C_{\text{A}}, \\ x^+ = g_{\text{A}}(x) := b_2 & x \in D_{\text{A}}, \end{cases} \tag{11}$$

where $C_{\text{A}} := [0, b_1]$ and $D_{\text{A}} := \{x \in [0, b_1] : x = b_1\}$. The parameters a, b, b_1, and b_2 satisfy $a > 0$ and $b > ab_1 > ab_2 > 0$. Define the compact set $M_{\text{A}} := [0, b_1]$ and define a continuously differentiable function $h : M_{\text{A}} \to \mathbb{R}$ as $h(x) := b_1 - x$. Then, C_{A} and D_{A} can be rewritten as $C_{\text{A}} = \{x \in M_{\text{A}} : h(x) \geqslant 0\}$ and $D_{\text{A}} = \{x \in C_{\text{A}} : h(x) = 0, L_{f_{\text{A}}} h(x) \leqslant 0\}$, respectively, where we used the property $L_{f_{\text{A}}} h(x) = -(-ax + b) = ab_1 - b < 0$ for all $x \in M_{\text{A}} \cap D_{\text{A}}$. By design, the sets C_{A} and D_{A} are closed. Moreover, the function f_{A} is continuously differentiable and the function g_{A} is continuous. Furthermore, it can be verified that $g_{\text{A}}(M_{\text{A}} \cap D_{\text{A}}) \cap (M_{\text{A}} \cap D_{\text{A}}) = \emptyset$. Therefore, Assumption 4.1 holds. Note that every maximal solution ϕ to $\mathcal{H}_{\text{A}}|_{M_{\text{A}}} = (M_{\text{A}} \cap C_{\text{A}}, f_{\text{A}}, M_{\text{A}} \cap D_{\text{A}}, g_{\text{A}})$ is unique via [16, Proposition 2.11].

To verify the ZS notion, let us consider a maximal solution ϕ_1 to $\mathcal{H}_{\text{A}}|_{M_{\text{A}}}$. For a given ε, let $0 < \delta < \min\{\varepsilon, b\varepsilon\}$. Then, for each $\phi_2 \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}(\phi_1(0,0)+\delta\mathbb{B})$, we have $T_I(\phi_1(0,0)) = \frac{1}{a}\ln\frac{a\phi_1(0,0)-b}{ab_1-b}$ and $T_I(\phi_2(0,0)) = \frac{1}{a}\ln\frac{a\phi_2(0,0)-b}{ab_1-b}$. Without loss of generality, assume $\phi_1(0,0) > \phi_2(0,0)$. Then, the solution ϕ_1 jumps before ϕ_2 since jumps occur when x reaches b_1. Denote $t_\Delta = T_I(\phi_2(0,0)) - T_I(\phi_1(0,0)) = \frac{1}{a}\ln\frac{a\phi_2(0,0)-b}{ab_1-b} - \frac{1}{a}\ln\frac{a\phi_1(0,0)-b}{ab_1-b} = \frac{1}{a}\ln\frac{a\phi_2(0,0)-b}{a\phi_1(0,0)-b} > 0$. Let us construct τ as

$$\tau(t) = \begin{cases} \frac{T_I(\phi_2(0,0))}{T_I(\phi_1(0,0))}t & t \in [0, T_I(\phi_1(0,0))], \\ t + t_\Delta & t > T_I(\phi_1(0,0)). \end{cases} \tag{12}$$

Note that τ is a homeomorphism and satisfies $\tau(0)=0$, hence it belongs to $\mathcal{T}$, and, in addition, is continuous. Then, for $j=0$, for each $t \in [0, T_I(\phi_1(0,0))]$, we have $\tau(t) = \frac{T_I(\phi_2(0,0))}{T_I(\phi_1(0,0))}t$, which satisfies $(\tau(t),0) \in \operatorname{dom}\phi_2$ and

$$\begin{aligned} &|\phi_1(t,0) - \phi_2(\tau(t),0)| \\ &= \left|((\phi_1(0,0) - \tfrac{b}{a})e^{-at} + \tfrac{b}{a}) - ((\phi_2(0,0) - \tfrac{b}{a})e^{-a\tau(t)} + \tfrac{b}{a})\right| \\ &= \left|(\phi_1(0,0) - \tfrac{b}{a})e^{-at} - (\phi_2(0,0) - \tfrac{b}{a})e^{-a\tau(t)}\right|, \end{aligned} \tag{13}$$

where $e^{-a\tau(t)} = e^{-at}\left(\frac{a\phi_1(0,0)-b}{a\phi_2(0,0)-b}\right)^{\frac{t}{T_I(\phi_1(0,0))}}$. Since $b/a > \phi_1(0,0) > \phi_2(0,0)$ and $T_I(\phi_2(0,0)) > T_I(\phi_1(0,0))$, we have that for each $t \in [0, T_I(\phi_1(0,0))]$, $0 < \left(\frac{a\phi_1(0,0)-b}{a\phi_2(0,0)-b}\right) \leqslant \left(\frac{a\phi_1(0,0)-b}{a\phi_2(0,0)-b}\right)^{t/T_I(\phi_1(0,0))} \leqslant 1$. Therefore, (13) is equivalent to

$$|\phi_1(t,0) - \phi_2(\tau(t),0)| \leqslant |\phi_1(0,0) - \phi_2(0,0)|e^{-at} \leqslant \delta < \varepsilon.$$

Note that $\phi_1(T_I(\phi_1(0,0)),0) = \phi_2(\tau(T_I(\phi_1(0,0))),0) = b_1$. In fact, for each $j \in \mathbb{N}\backslash\{0\}$ and each $t \geqslant T_I(\phi_1(0,0))$ such that $(t,j) \in \operatorname{dom}\phi_1$, we have $\tau(t) = t + t_\Delta$, which satisfies $(\tau(t),j) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t),j)| = 0 < \varepsilon$. Therefore, the solution ϕ_1 is ZS. In fact, any solution $\phi_1 \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}$ is ZS.

To verify the ZLA notion, let $\mu > 0$. Let ϕ_1 be a maximal solution to $\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}$. Then, for each $\varepsilon > 0$ and for each $\phi_2 \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}(\phi_1(0,0)+\mu\mathbb{B})$, we have $T_I(\phi_1(0,0)) = \frac{1}{a}\ln\frac{a\phi_1(0,0)-b}{ab_1-b}$ and $T_I(\phi_2(0,0)) = \frac{1}{a}\ln\frac{a\phi_2(0,0)-b}{ab_1-b}$. Similar to the above proof of the ZS notion, without loss of generality, assume $\phi_1(0,0) > \phi_2(0,0)$. Then, the solution ϕ_1 jumps before ϕ_2. Note that $\phi_1(T_I(\phi_1(0,0)),1) = \phi_2(\tau(T_I(\phi_1(0,0))),1) = b_2$. Then, for $j=1$ and for each $t \geqslant T_I(\phi_1(0,0))$, we have $\tau(t) = t + t_\Delta$, which satisfies $(\tau(t),1) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,1) - \phi_2(\tau(t),1)| = 0 < \varepsilon$. In fact, for each $j \in \mathbb{N}\backslash\{0\}$ and each $t \geqslant T_I(\phi_1(0,0))$, we have that $\tau(t) = t + t_\Delta$ and

$$(t,j) \in \operatorname{dom}\phi_1, \quad t + j \geqslant T = T_I(\phi_1(0,0)) + 1$$

imply that $(\tau(t),j) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t),j)| = 0 < \varepsilon$. Therefore, $\phi_1 \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}$ is ZLA. In fact, any solution $\phi_1 \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}$ is ZLA. Hence, every maximal solution to $\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}$ is ZLAS.

Next, we establish a link between the Zhukovskii stability notion in Definition 5.1 and incremental graphical stability as introduced in [38]. The later notion is presented next for self-containedness.

Definition 5.4: [38, Definition 3.2] Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ as in (1). The hybrid system $\mathcal{H}$ is said to be

1) *incrementally graphically stable* (δS) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any two maximal solutions ϕ_1, ϕ_2 to $\mathcal{H}$, $|\phi_1(0,0) - \phi_2(0,0)| \leqslant \delta$ implies that, for each $(t,j) \in \operatorname{dom}\phi_1$, there exists $(s,j) \in \operatorname{dom}\phi_2$ satisfying $|t-s| \leqslant \varepsilon$ and
$$|\phi_1(t,j) - \phi_2(s,j)| \leqslant \varepsilon; \tag{14}$$
2) *incrementally graphically locally attractive* (δLA) if there exists $\mu > 0$ such that for every $\varepsilon > 0$ and for any two maximal solutions ϕ_1, ϕ_2 to $\mathcal{H}$, $|\phi_1(0,0) - \phi_2(0,0)| \leqslant \mu$ implies that there exists $T > 0$ such that for each $(t,j) \in \operatorname{dom}\phi_1$ such that $t + j \geqslant T$, there exists $(s,j) \in \operatorname{dom}\phi_2$ satisfying $|t-s| \leqslant \varepsilon$ and
$$|\phi_1(t,j) - \phi_2(s,j)| \leqslant \varepsilon; \tag{15}$$
3) *incrementally graphically locally asymptotically stable* (δLAS) if it is both δS and δLA.

B. Existence of Hybrid Limit Cycles via Zhukovskii and Incremental Graphical Stability

In this section, we present conditions for the existence of a hybrid limit cycle for hybrid systems that are ZLAS. The existence of such a hybrid limit cycle is related to nonemptyness of an ω-limit set and continuity of a Poincaré map Γ on a closed set Σ near an ω-limit point.

Inspired by [42, Chapter V, Definition 2.13], the following notion is introduced in a sufficiently "short" tube $\Phi_{\bar{t}}(U) := \{\phi_x(t,0) : t \mapsto \phi_x(t,0)$ is a solution to $\dot{x} = f(x) \quad x \in \mathbb{R}^n$ from $\phi_x(0,0) \in U, t \in [0,\bar{t}], (t,0) \in \operatorname{dom}\phi_x\}$, where $U \subset \mathbb{R}^n$ and $\bar{t} \geqslant 0$.

Definition 5.5: (*forward local section*) Consider a dynamical system $\dot{x} = f(x) \quad x \in \mathbb{R}^n$. Given $U \subset \mathbb{R}^n$ and $\bar{t} \geqslant 0$, a closed set $\Sigma \subset \Phi_{\bar{t}}(U)$ is called a *local section* if for each solution ϕ_x to $\dot{x} = f(x) \quad x \in \mathbb{R}^n$ starting from $\phi_x(0) \in U$, there exists a unique $t_v \in [0,\bar{t}]$ such that $t_v \in \operatorname{dom}\phi_x$ and $\phi_x(t_v) \in \Sigma$.[10]

To guarantee the existence of a *forward local section*, inspired by [42, Chapter V, Theorem 2.14], we present the following result, which is different from [42, Chapter V, Theorem 2.14] as it only allows for forward times.

Lemma 5.6: *Consider the dynamical system $\dot{x} = f(x) \quad x \in \mathbb{R}^n$. If f is continuously differentiable and v is not an equilibrium point of the dynamical system, then, for any sufficiently small $\bar{t} > 0$, there exists $\sigma > 0$ such that there exists a forward local section $\Sigma \subset \Phi_{\bar{t}}(v + \sigma\mathbb{B})$.*

A proof can be found in [39, Lemma 5.11].

The following result reveals the behavior of the solutions to the flow dynamics of the hybrid system $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ in some neighborhood of any point in $M \cap C$ and ensures the existence of a forward local section Σ in the tube $\Phi_{\bar{t}}$.

Lemma 5.7: *Consider a hybrid system $\mathcal{H} = (C,f,D,g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that for the hybrid system $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$, $M \cap C$ has a nonempty interior and contains no equilibrium set for the flow dynamics*

$$\dot{x} = f(x) \qquad x \in M \cap C. \tag{16}$$

[10]Here, for the system $\dot{x} = f(x) \quad x \in \mathbb{R}^n$, since it is a continuous-time system, we have $\operatorname{dom}\phi_x \subset \mathbb{R}_{\geqslant 0}$. Then, we write $\phi_x(t)$ instead of $\phi_x(t,0)$.

For each $v \in (M \cap C)^\circ$ and a sufficiently small $\bar{t} > 0$, there exists $\sigma > 0$ such that each solution ϕ_x to (16) starting from $\phi_x(0,0) \in \Phi_{\bar{t}}(v+\sigma\mathbb{B})$ has the following properties: i) $\Phi_{\bar{t}}(v+\sigma\mathbb{B}) \subset (M \cap C)^\circ$; ii) there exists a forward local section $\Sigma \subset \Phi_{\bar{t}}(v+\sigma\mathbb{B})$.

A proof of Lemma 5.7 can be found in [39, Lemma 5.12].

The following result is derived via an application of the tubular flow theorem [45, Chapter 2, Theorem 1.1] to the flow dynamics (16).

Lemma 5.8: *Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that for the hybrid system $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$, $M \cap C$ has a nonempty interior and contains no critical points*[11] *of the map f. For any open set $U \subset M \cap C$ and for each point $v \in U$, there exists an open neighborhood $\mathcal{N}_v \subset U$ of v such that solutions to (16) from $\mathcal{N}_v$ are diffeomorphic to the solutions to the system*

$$\dot{\xi}_1 = 1,\ \dot{\xi}_i = 0 \quad \forall i \in \{2, 3, \cdots, n\} \tag{17}$$

on $(-1,1)^n$.

Proof: We use the tubular flow theorem [45, Chapter 2, Theorem 1.1] to prove the result. First, we verify the conditions of the tubular flow theorem. Since $M \cap C$ contains no critical points for the map f in (16), each $x \in M \cap C$ is a regular point of f. Moreover, since $M \cap C$ has a nonempty interior and f is continuously differentiable by item 2) of Assumption 4.1, f is a vector field of class $\mathcal{C}^r$,[12] $r \geqslant 1$, on any open set $U \subset M \cap C$. Therefore, all conditions in the tubular flow theorem are verified.

Now, by the tubular flow theorem, letting $v \in U$ be a regular point of f, there exists an open neighborhood $\mathcal{N}_v \subset U$ of v such that solutions to (16) from $\mathcal{N}_v$ are diffeomorphic to the solutions to the system (17) on $(-1,1)^n$. $\square$

The following result provides sufficient conditions for the existence of a hybrid limit cycle of a hybrid system.[13] In addition to technical conditions, ZLAS would serve as a sufficient condition for the existence of a hybrid limit cycle, which is motivated by the use of ZLAS for continuous-time systems in [35].

Theorem 5.9: *Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that for the hybrid system $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$, $M \cap C$ has a nonempty interior and contains no critical points for the map f, and contains no equilibrium set for the flow dynamics (16), and for each $x \in M \cap C$, each maximal solution to $\mathcal{H}|_M$ is complete with its hybrid time domain unbounded in the t direction, and each solution to (16) is not complete and ends at a point in $M \cap C$. Then, for each solution $\phi \in \mathcal{S}_{\mathcal{H}|_M}(M \cap C)$, $\mathcal{H}|_M$ has a nonempty ω-limit set $\Omega(\phi)$. In addition, if the solution ϕ is ZLAS and $\Omega(\phi) \cap (M \cap C)^\circ$ is nonempty, then $\Omega(\phi)$ is a hybrid limit cycle for $\mathcal{H}|_M$ with period given by some $T^* > 0$ and multiple jumps per period.*

Proof: First, we prove nonemptyness and forward invariance of $\Omega(\phi)$. Since the hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfy Assumption 4.1, every maximal solution to $\mathcal{H}|_M$ is unique via [16, Proposition 2.11] and $\mathcal{H}|_M$ satisfies the hybrid basic conditions. By [16, Theorem 6.8], $\mathcal{H}|_M$ is nominally well-posed. Since each solution $\phi \in \mathcal{S}_{\mathcal{H}|_M}(M \cap C)$ is unique and complete, the set $M \cap C$ is forward invariant for $\mathcal{H}|_M$. Note that completeness of each solution ϕ and the compactness of M imply that each ϕ is bounded. Then, it follows from [46, Lemma 3.3] that the ω-limit set $\Omega(\phi)$ is a nonempty, compact, and weakly invariant subset of M.

Next, we prove the existence of a forward local section Σ. By assumption, let the solution $\phi \in \mathcal{S}_{\mathcal{H}|_M}(M \cap C)$ be ZLAS. Since $\Omega(\phi) \cap (M \cap C)^\circ$ is nonempty, we can choose a point $p \in \Omega(\phi) \cap (M \cap C)^\circ$. By Definition 3.1, one can choose a sequence $\{(t_i, j_i)\}_{i=1}^\infty$ such that $\lim_{i\to\infty} t_i + j_i = \infty$ and $\lim_{i\to\infty} \phi(t_i, j_i) = p \in \Omega(\phi) \cap (M \cap C)^\circ$. Therefore, there exist positive constants $0 < \sigma < \delta$ such that $p + \sigma\mathbb{B} \subset \phi(t_l, j_l) + \delta\mathbb{B}$ and for sufficiently small $\bar{t} > 0$, $\Phi_{\bar{t}}(p+\sigma\mathbb{B}) \subset \phi(t_l, j_l) + \delta\mathbb{B}$ for some $(t_l, j_l) \in \{(t_i, j_i)\}_{i=1}^\infty$ with $(t_l, j_l) \in \operatorname{dom}\phi$. Then, with the picked constants σ and $\bar{t}$ (which can be chosen smaller if necessary), by Lemma 5.7, we have the following properties: i) $\Phi_{\bar{t}}(p+\sigma\mathbb{B}) \subset (M \cap C)^\circ$; ii) there exists a forward local section $\Sigma \subset \Phi_{\bar{t}}(p+\sigma\mathbb{B})$.

Now, to show the existence of a hybrid limit cycle, let us introduce a Poincaré map for local structure of hybrid systems. Given the forward local section $\Sigma \subset \Phi_{\bar{t}}(p+\sigma\mathbb{B})$, we denote the Poincaré map as $\Gamma : \Sigma \to \Sigma$ and define it as

$$\begin{aligned}\Gamma(x) := \big\{ \psi(t,j) \in \Sigma : \psi \in \mathcal{S}_{\mathcal{H}|_M}(x), t > 0, \\ (t,j) \in \operatorname{dom}\psi \ \big\} \quad \forall x \in \Sigma.\end{aligned} \tag{18}$$

We next prove that i) for each solution ψ to $\mathcal{H}|_M$ starting from $\psi(0,0) \in \Sigma$, there exists $(t,j) \in \operatorname{dom}\psi$ with $t > 0$ such that $\psi(t,j) \in \Sigma$, and that ii) the Poincaré map Γ has a fixed point $q \in \Sigma$.[14]

We prove the first assertion. By the definition of $\Omega(\phi)$ and from the analysis above, we have the following claim.

Claim 1: With the solution ϕ and σ above, there exists $(t_k, j_k) \in \{(t_i, j_i)\}_{i=1}^\infty$ such that $|\phi(t_k, j_k) - p| \leqslant \sigma/2$ and $|\phi(t_m, j_m) - p| \leqslant \sigma/2$ for each $t_m \geqslant t_k$ and each $j_m \geqslant j_k$.

Let $\phi_1(0,0) := \phi(t_k, j_k)$ as above and define ϕ_1 as the translation of ϕ by (t_k, j_k), which leads to a complete solution ϕ_1 due to completeness of ϕ. By assumption, the solution ϕ_1 to $\mathcal{H}|_M$ is ZLAS. Then, we have the following claim by Definition 5.1.

Claim 2: With σ above, for each $\phi_2 \in \mathcal{S}_{\mathcal{H}|_M}(\phi_1(0,0)+\delta\mathbb{B})$ there exists a function $\tau \in \mathcal{T}$ such that for $\varepsilon = \sigma/2 > 0$ there exists $T > 0$ for which we have $(t,j) \in \operatorname{dom}\phi_1$, $t + j \geqslant T$ implies that $(\tau(t), j) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t), j)| \leqslant \varepsilon = \sigma/2$.

Since $\Sigma \subset \Phi_{\bar{t}}(p+\sigma\mathbb{B}) \subset \phi_1(0,0) + \delta\mathbb{B}$, each solution ϕ_3 to $\mathcal{H}|_M$ from $\phi_3(0,0) \in \Sigma$ also satisfies **Claim 2**. In addition, by assumption, since each solution to (16) is not complete and

[11] For a differential map $f : \mathbb{R}^m \to \mathbb{R}^n$, a point x is a critical point of f if $\frac{\partial f}{\partial x}(x)$ is not full rank and is a regular point if $\frac{\partial f}{\partial x}(x)$ is full rank.

[12] $\mathcal{C}^r$ denotes the differentiability class of mappings having r continuous derivatives.

[13] Here, we establish sufficient conditions for the existence of a hybrid limit cycle *with multiple jumps* in each period. A hybrid limit cycle notion allowing for multiple jumps in a period can be defined similarly; see [33]. For specific systems with one jump as will be illustrated in next examples, the result is also applicable.

[14] A point q is a fixed point of a Poincaré map $\Gamma : \Sigma \to \Sigma$ if $q = \Gamma(q)$.

ends at a point in $M \cap C$, we have recurrent jumps. Therefore, from **Claim 1** and **Claim 2**, for each $\phi_3 \in \mathcal{S}_{\mathcal{H}|_M}(\Sigma)$, there exist $\tau \in \mathcal{T}$ and $(t_m, j_m) \in \operatorname{dom}\phi_1$ satisfying $t_m \geqslant 0$, $j_m \geqslant 1$, and $t_m + j_m \geqslant T$ such that $(\tau(t_m), j_m) \in \operatorname{dom}\phi_3$, $|\phi_1(t_m, j_m) - p| \leqslant \sigma/2$, and

$$|\phi_1(t_m, j_m) - \phi_3(\tau(t_m), j_m)| \leqslant \sigma/2,$$

which leads to $\phi_3(\tau(t_m), j_m) \in \Phi_{\bar{t}}(p + \sigma\mathbb{B})$.[15] In addition, there exists $(t_m, j_m) \in \operatorname{dom}\phi_1$ as above such that[16] $\phi_3(\tau(t_m), j_m) \in (p + \sigma\mathbb{B})\backslash D$. Let $\bar{\phi}(0,0) = \phi_3(\tau(t_m), j_m)$ and define $\bar{\phi}$ as the translation of ϕ_3 by $(\tau(t_m), j_m)$, which leads to a complete solution $\bar{\phi}$ due to completeness of ϕ_3. Then, $\bar{\phi}(0,0) \in (p + \sigma\mathbb{B})\backslash D$. By the second property of Lemma 5.7 and the definition of forward local section in Definition 5.5, we have that the solution $\bar{\phi}$ to (16) reaches the forward local section $\Sigma \subset \Phi_{\bar{t}}(p + \sigma\mathbb{B})$ at a unique time $t_p \in [0, \bar{t}]$, that is, $(t_p, 0) \in \operatorname{dom}\bar{\phi}$ and $\bar{\phi}(t_p, 0) \in \Sigma$, which implies $\phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$. Therefore, the first assertion is proved.

To prove the second assertion, that is, that the Poincaré map Γ has a fixed point $q \in \Sigma$, first we show continuity of Γ on Σ. Since $\phi_3(0,0) \in \Sigma$ and $\phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$, we have that $\Gamma(\phi_3(0,0)) = \phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$. Since $\Sigma \subset \Phi_{\bar{t}}(p + \sigma\mathbb{B})$ and $\bar{t}$ can be sufficiently small, we have $\phi_3(0,0) \in p + \sigma\mathbb{B}$ and $\Gamma(\phi_3(0,0)) \in p + \sigma\mathbb{B}$. Then, it follows that

$$|\Gamma(\phi_3(0,0)) - \phi_3(0,0)| \leqslant |\Gamma(\phi_3(0,0)) - p| + |p - \phi_3(0,0)| \leqslant 2\sigma.$$

Therefore, since the chosen σ can be small enough, we have that the map Γ in (18) is continuous on Σ.

Next, by applying the Brouwer's fixed point theorem, we show that the map Γ has a fixed point $q \in \Sigma$. Note that by assumption, each $x \in M \cap C$ is a regular point of f and all conditions in Lemma 5.8 are satisfied. Therefore, p is a regular point of f, and by using Lemma 5.8, for any open set $U \subset M \cap C$ containing p, there exists an open neighborhood $\mathcal{N}_p \subset U$ of p such that solutions to (16) from $\mathcal{N}_p$ are diffeomorphic to the solutions to the system (17) on $(-1,1)^n$. In other words, there exists a C^r diffeomorphism $H : (-1,1)^n \to \mathcal{N}_p$ such that for any solution ϕ_ξ to (17), $\phi_x = H(\phi_\xi)$ is a solution to $\dot{x} = f(x)\ x \in \mathcal{N}_p$. Note that given an initial condition $(s_1, s_2, \cdots, s_n) \in (-1,1)^n$, a solution to (17) is given by $\phi_\xi(t, 0) = (s_1 + t, s_2, \cdots, s_n)$ for all $t \in [0, 1 - s_1)$, and that the function $t \mapsto H(\phi_\xi(t, 0))$ is a solution to $\dot{x} = f(x)\ x \in \mathcal{N}_p$. Denote $\Sigma_\xi := H^{-1}(\Sigma) = \{\phi_\xi(t_p, 0) \in (-1,1)^n : t \mapsto \phi_\xi(t, 0)$ is a solution to (17) from $\phi_\xi(0,0) \in (-1,1)^n, t \in [0, t_p], (t, 0) \in \operatorname{dom}\phi_\xi\}$. Note that Σ_ξ is convex and bounded. In addition, since Σ is closed and H is a diffeomorphism, Σ_ξ is also closed and thus Σ_ξ is a convex compact set. Define a map[17] $\Gamma_\xi = H^{-1} \circ \Gamma \circ H$ as $\Gamma_\xi : \Sigma_\xi \to \Sigma_\xi$. Due to H being a diffeomorphism and continuity of Γ, Γ_ξ is continuous. Therefore, by Brouwer's fixed point theorem, Γ_ξ has a fixed point $q' \in \Sigma_\xi$, i.e., $\Gamma_\xi(q') = q'$. Since $\Gamma_\xi = H^{-1} \circ \Gamma \circ H$, we have $H^{-1} \circ \Gamma \circ H(q') = q'$, which implies that $\Gamma \circ H(q') = H(q') \in \Sigma$. Let $q = H(q')$. Therefore, we have that Γ has a fixed point $q \in \Sigma$, i.e., $\Gamma(q) = q$.

From the existence of a fixed point for Γ and the fact that $\phi_3(0,0) \in \Sigma$ and $\phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$, there is a flow periodic solution ϕ^* to $\mathcal{H}|_M$ with period $T^* = \tau(t_m) + t_p$ and j_m jumps in each period. Therefore, $\mathcal{H}|_M$ has a hybrid limit cycle $\mathcal{O}$ with j_m jumps in each period. Note that for the solution $\phi \in \mathcal{S}_{\mathcal{H}|_M}(q)$, every point ζ^* in the hybrid limit cycle $\mathcal{O}$ is in $\Omega(\phi)$ since there exists a sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ of points $(t_i, j_i) \in \operatorname{dom}\phi$ such that $\lim_{i\to\infty}\phi(t_i, j_i) = \zeta^*$ with $\lim_{i\to\infty} t_i + j_i = \infty$. To prove that every point in $\Omega(\phi)$ is also in the hybrid limit cycle $\mathcal{O}$, we proceed by contradiction. Suppose that $q \in \Omega(\phi)$ and $q \notin \mathcal{O}$. Since from the analysis above, there is a fixed point $q \in \Omega(\phi) \cap \Sigma$ for any chosen $\sigma \in (0, \delta)$ we have $\phi_3(0,0) \in \Sigma$ and $\phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$. Then, $\Gamma(q) = q$, which leads to a contradiction with $q \notin \mathcal{O}$. Thus, every point in $\Omega(\phi)$ is also in the hybrid limit cycle $\mathcal{O}$. Therefore, we have that $\Omega(\phi)$ is a hybrid limit cycle. $\square$

Remark 5.10: In Theorem 5.9, there are several ways to guarantee that $M \cap C$ contains no equilibrium set for the flow dynamics (16). One way to assure that is to check if for each $x \in M \cap C$, $f^\top(x)f(x) > 0$.

The following example illustrates Theorem 5.9.

Example 5.11: Consider the academic system $\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}$ in Example 5.3. We will verify the existence of a hybrid limit cycle via Theorem 5.9. First, items 1)-3) of Assumption 4.1 have been illustrated in Example 5.3. Since the Jacobian of the map f_{A} given by $\mathbb{J}_{f_{\mathrm{A}}}(x) = -a$ with $a > 0$ has the maximal rank 1, $M_{\mathrm{A}} \cap C_{\mathrm{A}}$ contains no critical points for the map f_{A}. By the definition of M_{A}, for all $x \in M_{\mathrm{A}} \cap C_{\mathrm{A}}$, $x \leqslant b_1 < b/a$, which implies that $f_{\mathrm{A}}^\top(x)f_{\mathrm{A}}(x) = (-ax + b)^2 \geqslant (b - ab_1)^2 > 0$. Then, by Remark 5.10, $M_{\mathrm{A}} \cap C_{\mathrm{A}}$ contains no equilibrium set for the flow dynamics $\dot{x} = f_{\mathrm{A}}(x)\ \ x \in M_{\mathrm{A}} \cap C_{\mathrm{A}}$, where $(M_{\mathrm{A}} \cap C_{\mathrm{A}})^\circ = (0, b_1)$ is nonempty. By the definitions of f_{A} and g_{A}, the set M_{A} is forward invariant and each $\phi \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}(M_{\mathrm{A}} \cap C_{\mathrm{A}})$ is unique and complete with $\operatorname{dom}\phi$ unbounded in the t direction. Next, from the data of $\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}$, each solution $\phi \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}(M_{\mathrm{A}} \cap C_{\mathrm{A}})$ to $\dot{x} = f_{\mathrm{A}}(x)\ \ x \in M_{\mathrm{A}} \cap C_{\mathrm{A}}$ is not complete and ends at a point in $M_{\mathrm{A}} \cap C_{\mathrm{A}}$. Therefore, for each maximal solution ϕ from $\xi \in [b_2, b_1]$ given by, for each $(t, j) \in \mathbb{R}_{\geqslant 0} \times \mathbb{N}$,

$$\phi(t,j) = \begin{cases} (\xi - \frac{b}{a})e^{-at} + \frac{b}{a} & t \in [0, t_1], j = 0 \\ (b_2 - \frac{b}{a})e^{-a(t - t_1')} + \frac{b}{a} & t \in [t_1', t_1] + jT^*, j \in \mathbb{N}\backslash\{0\} \end{cases}$$

where $t_1' = (j-1)T^* + t_1$, $t_1 = \frac{1}{a}\ln\frac{a\xi - b}{ab_1 - b}$, and $T^* = \frac{1}{a}\ln\frac{ab_2 - b}{ab_1 - b}$, by Theorem 5.9, $\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}$ has a nonempty ω-limit set $\Omega(\phi) := \{x \in [0, b_1] : x = \phi(t, 1), t \in [t_1, t_1 + T^*]\}$. Finally, the ZLAS property of each $\phi \in \mathcal{S}_{\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}}$ has been verified in Example 5.3. In addition, from the construction of $\Omega(\phi)$ and the condition $b_1 > b_2 > 0$, $\Omega(\phi) \cap (M_{\mathrm{A}} \cap C_{\mathrm{A}})^\circ = [b_2, b_1)$ is nonempty. Therefore, by Theorem 5.9, $\Omega(\phi)$ is the hybrid limit cycle for $\mathcal{H}_{\mathrm{A}}|_{M_{\mathrm{A}}}$ with period $T^* = \frac{1}{a}\ln\frac{ab_2 - b}{ab_1 - b}$ and one jump per period.

[15] If this conclusion holds for $j_m = 1$, the remaining proofs will show that $\mathcal{H}|_M$ has a hybrid limit cycle with one jump in each period.

[16] Since $p \in \Omega(\phi) \cap (M \cap C)^\circ$ and $\Phi_{\bar{t}}(p + \sigma\mathbb{B}) \subset (M \cap C)^\circ$, there always exists (t_m, j_m) such that $\phi_3(\tau(t_m), j_m) \in (p + \sigma\mathbb{B})\backslash D$. In fact, if that were not the case, for each $(t_m, j_m) \in \operatorname{dom}\phi_1$ satisfying $t_m \geqslant 0$, $j_m \geqslant 1$ and $t_m + j_m \geqslant T$, we would have $\phi_3(\tau(t_m), j_m) \in D$. Since $|\phi_1(t_m, j_m) - p| \leqslant \sigma/2$ and $|\phi_1(t_m, j_m) - \phi_3(\tau(t_m), j_m)| \leqslant \sigma/2$, and since σ is arbitrary, $\lim_{m\to\infty}\phi_3(\tau(t_m), j_m) = p \in D$, which contradicts with the fact $p \in \Omega(\phi) \cap (M \cap C)^\circ$.

[17] The operator $\circ$ defines a function composition, i.e., $H^{-1} \circ \Gamma \circ H(x) = H^{-1}(\Gamma(H(\xi)))$ for all $\xi \in (-1,1)^n$.

In light of Example 5.11, one may wonder if incremental graphical asymptotic stability would serve as a necessary condition for the existence of a hybrid limit cycle. Unfortunately, the fact that incremental graphical asymptotic stability is a property for all solutions starting in a neighborhood makes it difficult to allow for the existence of a hybrid limit cycle. The following result establishes a sufficient condition for the nonexistence of hybrid limit cycles for systems that are δLAS.

Theorem 5.12: *For a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1, consider the hybrid system $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ and assume that each solution $\phi \in \mathcal{S}_{\mathcal{H}|_M}(M \cap C)$ is complete with $\operatorname{dom}\phi$ unbounded in the t direction. If the hybrid system $\mathcal{H}|_M$ is δLAS, then $\mathcal{H}|_M$ has no hybrid limit cycles for $\mathcal{H}|_M$ with period given by some $T^* > 0$.*

A proof can be found in [39, Theorem 5.17].

VI. Sufficient Conditions for Asymptotic Stability of Hybrid Limit Cycles

A. Notions

Following the stability notion introduced in [16, Definition 3.6], we employ the following notion for stability of hybrid limit cycles.

Definition 6.1: Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact hybrid limit cycle $\mathcal{O}$. Then, the hybrid limit cycle $\mathcal{O}$ is said to be

- *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{O}} \leqslant \delta$ satisfies $|\phi(t,j)|_{\mathcal{O}} \leqslant \varepsilon$ for each $(t,j) \in \operatorname{dom}\phi$;
- *globally attractive* for $\mathcal{H}$ if every maximal solution ϕ to $\mathcal{H}$ from $\bar{C} \cup D$ is complete and satisfies $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *globally asymptotically stable* for $\mathcal{H}$ if it is both stable and globally attractive;
- *locally attractive* for $\mathcal{H}$ if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ starting from $|\phi(0,0)|_{\mathcal{O}} \leqslant \mu$ is complete and satisfies $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *locally asymptotically stable* for $\mathcal{H}$ if it is both stable and locally attractive.

Given $M \subset \mathbb{R}^n$ and $\mathcal{H} = (C, f, D, g)$, for $x \in M \cap (C \cup D)$, define the "distance" function $d : M \cap (C \cup D) \to \mathbb{R}_{\geqslant 0}$ as

$$d(x) := \sup_{t \in [0, T_I(x)],\ (t,j) \in \operatorname{dom}\phi,\ \phi \in \mathcal{S}_{\mathcal{H}|_M}(x)} |\phi(t,j)|_{\mathcal{O}}$$

when $0 \leqslant T_I(x) < \infty$, and

$$d(x) := \sup_{(t,j) \in \operatorname{dom}\phi,\ \phi \in \mathcal{S}_{\mathcal{H}|_M}(x)} |\phi(t,j)|_{\mathcal{O}}$$

if $T_I(x) = \infty$, where T_I is the time-to-impact function defined in (5). Note that d vanishes on $\mathcal{O}$. Then, following [5, Lemma 4], the following property of the function d can be established.

Lemma 6.2: *Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that every maximal solution to $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ is complete and $\mathcal{H}|_M$ has a flow periodic solution ϕ^* with period $T^* > 0$ that defines a hybrid limit cycle $\mathcal{O} \subset M \cap C$. Then, the function $d : M \cap C \to \mathbb{R}_{\geqslant 0}$ is well-defined and continuous on $\mathcal{O}$.*

A proof can be found in [39, Lemma 6.2].

B. Asymptotic Stability Properties of $\mathcal{O}$

To establish conditions for asymptotic stability of a hybrid limit cycle, let us introduce a Poincaré map for hybrid systems. Referred to as the *hybrid Poincaré map*, given a maximal solution ϕ to $\mathcal{H}|_M$, we denote it as $P : M \cap D \to M \cap D$ and define it as[18]

$$P(x) := \begin{Bmatrix} \phi(T_I(g(x)), j) : \phi \in \mathcal{S}_{\mathcal{H}|_M}(g(x)), \\ (T_I(g(x)), j) \in \operatorname{dom}\phi \end{Bmatrix} \ \forall x \in M \cap D, \tag{19}$$

where T_I is the time-to-impact function defined in (5).

The importance of the hybrid Poincaré map in (19) is that it allows one to determine the stability of hybrid limit cycles. Before revealing the stability properties of a hybrid limit cycle, we introduce the following stability notions for the hybrid Poincaré map P in (19). Let P^k denote k compositions of the hybrid Poincaré map P with itself; namely, $P^k(x) = \underbrace{P \circ P \cdots \circ P}_{k}(x)$.

Definition 6.3: A fixed point x^* of a hybrid Poincaré map $P : M \cap D \to M \cap D$ defined in (19) is said to be

- *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in M \cap D$, $|x - x^*| \leqslant \delta$ implies $|P^k(x) - x^*| \leqslant \epsilon$ for all $k \in \mathbb{N}$;
- *globally attractive* if for each $x \in M \cap D$, $\lim_{k\to\infty} P^k(x) = x^*$;
- *globally asymptotically stable* if it is both stable and globally attractive;
- *locally attractive* if there exists $\mu > 0$ such that for each $x \in M \cap D$, $|x - x^*| \leqslant \mu$ implies $\lim_{k\to\infty} P^k(x) = x^*$;
- *locally asymptotically stable* if it is both stable and locally attractive.

A relationship between stability of fixed points of hybrid Poincaré maps and stability of the corresponding hybrid limit cycles is established next.

Theorem 6.4: *Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that every maximal solution to $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ is complete and $\mathcal{H}|_M$ has a flow periodic solution ϕ^* with period $T^* > 0$ that defines a hybrid limit cycle $\mathcal{O} \subset M \cap C$. Then, the following statements hold:*

1) *$x^* \in M \cap D$ is a stable fixed point of the hybrid Poincaré map P in (19) if and only if the hybrid limit cycle $\mathcal{O}$ of $\mathcal{H}|_M$ generated by a flow periodic solution ϕ^* with period T^* from $\phi^*(0,0) = x^*$ is stable for $\mathcal{H}|_M$,*
2) *$x^* \in M \cap D$ is a globally asymptotically stable fixed point of the hybrid Poincaré map P if and only if $\mathcal{H}|_M$ has a unique hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ^* with period T^* from $\phi^*(0,0) = x^*$*

[18] The hybrid Poincaré map P in (19) is different from the Poincaré map $\Gamma : \Sigma \to \Sigma$ in (18). The map P in (19) maps $M \cap D$ to $M \cap D$ within one jump, while the map Γ in (18) maps a closed set $\Sigma \subset (M \cap C)^\circ$ to Σ and allows for multiple jumps.

that is globally asymptotically stable for $\mathcal{H}|_M$ with basin of attraction containing every point in[19] $M \cap C$.

Proof: We first prove the sufficiency of item 1). By Assumption 4.1, every maximal solution to $\mathcal{H}|_M$ is unique via [16, Proposition 2.11]. Consider the hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution to $\mathcal{H}|_M$ from x^* with $x^* \in M \cap D$. Since $\mathcal{O}$ is stable for $\mathcal{H}|_M$, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution ϕ to $\mathcal{H}|_M$, $|\phi(0,0)|_{\mathcal{O}} \leqslant \delta$ implies $|\phi(t,j)|_{\mathcal{O}} \leqslant \varepsilon$ for each $(t,j) \in \operatorname{dom}\phi$. Since ϕ is complete and $P^k(x^*) = \phi(T_I(g(x^*)), j)$ for some j, in particular, we have that $|P^k(x^*)|_{\mathcal{O}} \leqslant \varepsilon$ for each $k \in \mathbb{N}$. Therefore, $x^* \in M \cap D$ is a stable fixed point of the hybrid Poincaré map P.

Next, we prove the necessity of item 1) as in the proof of [5, Theorem 1]. Suppose that $x^* \in M \cap D$ is a stable fixed point of P. Then, $P(x^*) = x^*$ due to the continuity of P in (19) and, for any $\bar{\epsilon} > 0$, there exists $\bar{\delta} > 0$ such that

$$\tilde{x} \in (x^* + \bar{\delta}\mathbb{B}) \cap (M \cap D)$$

implies $P^k(\tilde{x}) \in (x^* + \bar{\epsilon}\mathbb{B}) \cap (M \cap D)$ for all $k \in \mathbb{N}$. Moreover, by assumption, every maximal solution ϕ to $\mathcal{H}|_M$ from $\tilde{x} \in (x^* + \bar{\delta}\mathbb{B}) \cap (M \cap D)$ is complete. Since solutions are guaranteed to exist from $M \cap D$, there exists a complete solution ϕ from every such point $\tilde{x}$. Furthermore, the distance between ϕ and the hybrid limit cycle $\mathcal{O}$ satisfies[20]

$$\sup_{(t,j)\in \operatorname{dom}\phi} |\phi(t,j)|_{\mathcal{O}} \leqslant \sup_{x \in (x^* + \bar{\delta}\mathbb{B}) \cap (M \cap D)} d \circ g(x).$$

By Lemma 6.2, d is continuous at x^*. Since $\mathcal{O}$ is transversal to $M \cap D$, $\mathcal{O} \cap (M \cap D)$ is a singleton, $g(x^*) \in \mathcal{O}$, and g is continuous, we have that $d \circ g$ is continuous at x^*. Moreover, since $d \circ g(x^*) = 0$, it follows by continuity that given any $\epsilon > 0$, we can pick $\bar{\epsilon}$ and $\bar{\delta}$ such that $0 < \bar{\epsilon} < \epsilon$ and

$$\sup_{x \in (x^* + \bar{\delta}\mathbb{B}) \cap (M \cap D)} d \circ g(x) \leqslant \epsilon.$$

Therefore, an open neighborhood of $\mathcal{O}$ given by $\mathcal{V} := \{x \in \mathbb{R}^n : d(x) \in [0, \bar{\epsilon})\}$ is such that any solution ϕ to $\mathcal{H}|_M$ from $\phi(0,0) \in \mathcal{V}$ satisfies $|\phi(t,j)|_{\mathcal{O}} \leqslant \epsilon$ for each $(t,j) \in \operatorname{dom}\phi$. Thus, the necessity of item 1) follows immediately.

The stability part of item 2) follows similarly. Sufficiency of the global attractivity part in item 2) is proved as follows. Suppose the hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution to $\mathcal{H}|_M$ from x^* is globally attractive for $\mathcal{H}|_M$ with basin of attraction containing every point in $M \cap C$. Then, given $\epsilon > 0$, for any solution ϕ to $\mathcal{H}|_M$, there exists $\bar{T} > 0$ such that $|\phi(t,j)|_{\mathcal{O}} \leqslant \epsilon$ for each $(t,j) \in \operatorname{dom}\phi$ with $t + j \geqslant \bar{T}$. Note that ϕ is precompact since ϕ is complete and the set M is compact by Assumption 4.1. Therefore, via [46, Lemma 2.7], $\operatorname{dom}\phi$ is unbounded in the t-direction as Assumption 4.1 prevents solutions from being Zeno. It follows that $|P^k(x^*)|_{\mathcal{O}} \leqslant \epsilon$ for sufficiently large k. Therefore, x^* is a globally attractive fixed point of P.

Finally, we prove the necessity of the global attractivity property in item 2). Assume that $x^* \in M \cap D$ is a globally attractive fixed point. Then, for any $\bar{\epsilon} > 0$, there exists $\bar{\delta} > 0$ such that, for all $k \in \mathbb{N}$, $\tilde{x} \in (x^* + \bar{\delta}\mathbb{B}) \cap (M \cap D)$ implies $\lim_{k\to\infty} P^k(\tilde{x}) = x^*$. Moreover, following from Definition 6.3, it is implied that a maximal solution ϕ to $\mathcal{H}|_M$ from x^* is complete. Then, by continuity of d and g,

$$\lim_{k\to\infty} d \circ g(P^k(\tilde{x})) = d \circ g(x^*) = 0,$$

from which it follows that

$$\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} \leqslant d \circ g(x^*) = 0.$$

The proof is complete. □

Remark 6.5: In [5], sufficient and necessary conditions for stability properties of periodic orbits in impulsive systems are established using properties of the fixed points of the corresponding Poincaré maps. Compared to [5], Theorem 6.4 enables the use of the Lyapunov stability tools in [16] to certify asymptotic stability of a fixed point without even computing the hybrid Poincaré map.

At times, one might be interested only on local asymptotic stability of the fixed point of the hybrid Poincaré map. Such case is handled by the following result.

Corollary 6.6: *Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that every maximal solution to $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ is complete and $\mathcal{H}|_M$ has a flow periodic solution ϕ^* with period $T^* > 0$ that defines a hybrid limit cycle $\mathcal{O} \subset M \cap C$. Then, $x^* \in M \cap D$ is a locally asymptotically stable fixed point of the hybrid Poincaré map P if and only if $\mathcal{H}|_M$ has a unique hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ^* with period T^* from $\phi^*(0,0) = x^*$ that is locally asymptotically stable for $\mathcal{H}|_M$.*

The proof can be found in [39, Corollary 6.6].

Remark 6.7: In [5] and [25], the authors extend the Poincaré method to analyze the stability properties of periodic orbits in nonlinear systems with impulsive effects. In particular, the solutions to the systems considered therein are right-continuous over (not necessarily closed) intervals of flow. In particular, the models therein (as well as those in [18]) require $C \cap D = \emptyset$, which prevents the application of the robustness results in [16] due to the fact that the hybrid basic conditions would not hold. On the other hand, our results allow us to establish robustness properties of hybrid limit cycles as shown in Section VII.

Remark 6.8: In [18], within a contraction framework, conditions guaranteeing local orbital stability of limit cycles for a class of hybrid systems are provided, where, as a difference to the notion used here, orbital stability is solely defined as an attractivity (or convergence) property. Note that the case of limit cycles with multiple jumps for hybrid systems is not explicitly analyzed in [18], while the results here are applicable to the situation where a hybrid limit cycle may contain multiple jumps within a period; see our preliminary results in [33].

Example 6.9: Consider the hybrid congestion control sys-

[19] A "global" property for $\mathcal{H}|_M$ implies a "global" property of the original system $\mathcal{H}$ only when M contains C. For tools to establish the asymptotic stability property, see [16].

[20] Given two functions $d : M \cap C \to \mathbb{R}_{\geqslant 0}$ and $g : M \cap D \to M \cap D$, the operator $\circ$ defines a function composition, i.e., $d \circ g(x) = d(g(x))$ for all $x \in M \cap D$.

tem in Example 4.6. A solution ϕ^* to $\mathcal{H}_{\mathrm{TCP}}|_{M_{\mathrm{TCP}}} = (M_{\mathrm{TCP}} \cap C_{\mathrm{TCP}}, f_{\mathrm{TCP}}, M_{\mathrm{TCP}} \cap D_{\mathrm{TCP}}, g_{\mathrm{TCP}})$ from $\phi^*(0,0) = (q_{\max}, 2Bm/(1+m)) \in M_{\mathrm{TCP}} \cap C_{\mathrm{TCP}}$ is a flow periodic solution with $T^* = 2B(1-m)/(a+ma)$, which defines a hybrid limit cycle $\mathcal{O} \subset M_{\mathrm{TCP}} \cap C_{\mathrm{TCP}}$. We verify the sufficient condition 2) in Theorem 6.4 as follows. Due to the specific form of the flow map of $\mathcal{H}_{\mathrm{TCP}}$, the Jacobian of the hybrid Poincaré map has an explicit analytic form. The flow solution ϕ^f to the flow dynamics $\dot{x} = f_{\mathrm{TCP}}(x)$ from ξ is given by

$$\phi^f(t,\xi) = \begin{bmatrix} \xi_1 + (\xi_2 - B)t + \frac{at^2}{2} \\ \xi_2 + at \end{bmatrix}. \tag{20}$$

From the definition of the hybrid Poincaré map and the solution of the flow dynamics from $x = (q_{\max}, r)$ with $x \in D_{\mathrm{TCP}}$, it follows that

$$P_{\mathrm{TCP}}(x) = \begin{bmatrix} q_{\max} + (mr - B)\hat{T} + \frac{a\hat{T}^2}{2} \\ mr + a\hat{T} \end{bmatrix},$$

where $\hat{T} = 2(B - mr)/a$, which leads to

$$P_{\mathrm{TCP}}(x) = (q_{\max}, 2B - mr). \tag{21}$$

Then, the eigenvalues of the Jacobian of the hybrid Poincaré map P_{TCP} are computed as $\lambda_1 = 0$ and $\lambda_2 = -m$, which, since $m \in (0,1)$, are inside the unit circle. According to Theorem 6.4, the hybrid limit cycle $\mathcal{O}$ of the hybrid system $\mathcal{H}_{\mathrm{TCP}}|_{M_{\mathrm{TCP}}}$ is asymptotically stable with basin of attraction containing every point in $M_{\mathrm{TCP}} \cap D_{\mathrm{TCP}}$.

VII. Robustness of Asymptotically Stable Hybrid Limit Cycles

A. Robustness to General Perturbations

First, we present results guaranteeing robustness to generic perturbations of asymptotically stable hybrid limit cycles. More precisely, we consider the perturbed continuous dynamics of the hybrid system $\mathcal{H}|_M = (M \cap C, f, M \cap D, g)$ given by $\dot{x} = f(x+d_1)+d_2 \quad x+d_3 \in M \cap C$, where d_1 corresponds to state noise (e.g., measurement noise), d_2 captures unmodeled dynamics or additive perturbations, and d_3 captures generic disturbances on the state when checking if the state belongs to the constraint. Similarly, we consider the perturbed discrete dynamics $x^+ = g(x+d_1)+d_2 \quad x+d_4 \in M \cap D$, where d_4 captures generic disturbances on the state when checking if the state belongs to the constraint $M \cap D$. The hybrid system $\mathcal{H}|_M$ with such perturbations results in the perturbed hybrid system

$$\tilde{\mathcal{H}}|_M \begin{cases} \dot{x} &= f(x+d_1)+d_2 \quad x+d_3 \in M \cap C, \\ x^+ &= g(x+d_1)+d_2 \quad x+d_4 \in M \cap D. \end{cases} \tag{22}$$

The perturbations d_i $(i = 1,2,3,4)$ might be state or hybrid time dependent, but are assumed to have Euclidean norm bounded by $\bar{M}_i \geqslant 0$ $(i = 1,2,3,4)$, and to be admissible, namely, $\operatorname{dom} d_i$ $(i = 1,2,3,4)$ is a hybrid time domain and the function $t \mapsto d_i(t,j)$ is measurable on $\operatorname{dom} d_i \cap (\mathbb{R}_{\geqslant 0} \times \{j\})$ for each $j \in \mathbb{N}$.

Theorem 7.1: *Consider a hybrid system $\mathcal{H} = (C,f,D,g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. If $\mathcal{O}$ is a locally asymptotically stable hybrid limit cycle for $\mathcal{H}|_M$ with basin of attraction $\mathcal{B}_{\mathcal{O}}$, then $\mathcal{O}$ is $\mathcal{KL}$ asymptotically stable[21] on the basin of attraction $\mathcal{B}_{\mathcal{O}}$ of the set $\mathcal{O}$.*

Proof: First, it is proved in Lemma 4.4 that $\mathcal{O}$ is a compact set. Second, note that for a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1, $\mathcal{H}|_M$ is well-posed [16, Definition 6.29]. Then, it is also nominally well-posed. Therefore, according to [16, Theorem 7.12], $\mathcal{B}_{\mathcal{O}}$ is open and $\mathcal{O}$ is $\mathcal{KL}$ asymptotically stable on $\mathcal{B}_{\mathcal{O}}$. □

The following result establishes that the stability of $\mathcal{O}$ for $\mathcal{H}|_M$ is robust to the class of perturbations defined above.

Theorem 7.2: *Consider a hybrid system $\mathcal{H} = (C,f,D,g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. If $\mathcal{O}$ is a locally asymptotically stable compact set for $\mathcal{H}|_M$ with basin of attraction $\mathcal{B}_{\mathcal{O}}$, then for every proper indicator ω of $\mathcal{O}$ on $\mathcal{B}_{\mathcal{O}}$ there exists $\tilde{\beta} \in \mathcal{KL}$ such that for every $\varepsilon > 0$ and every compact set $K \subset \mathcal{B}_{\mathcal{O}}$, there exist $\bar{M}_i > 0$, $i \in \{1,2,3,4\}$, such that for any admissible perturbations d_i, $i \in \{1,2,3,4\}$, with Euclidean norm bounded by $\bar{M}_i$, respectively, every solution $\tilde{\phi}$ to $\tilde{\mathcal{H}}|_M$ with $\tilde{\phi}(0,0) \in K$ satisfies*

$$\omega(\tilde{\phi}(t,j)) \leqslant \tilde{\beta}(\omega(\tilde{\phi}(0,0)), t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom} \tilde{\phi}.$$

Proof: Following [16, Section 6.4], we introduce the following perturbed hybrid system $\mathcal{H}|_M^\rho$ with constant $\rho > 0$:

$$\mathcal{H}|_M^\rho \begin{cases} \dot{x} \in F_\rho(x) & x \in C_\rho, \\ x^+ \in G_\rho(x) & x \in D_\rho, \end{cases} \tag{23}$$

where

$$\begin{aligned} C_\rho &:= \{x \in \mathbb{R}^n : (x+\rho\mathbb{B}) \cap (M \cap C) \neq \emptyset\}, \\ F_\rho(x) &:= \overline{\mathrm{co}} f((x+\rho\mathbb{B}) \cap (M \cap C)) + \rho\mathbb{B} \quad \forall x \in \mathbb{R}^n, \\ D_\rho &:= \{x \in \mathbb{R}^n : (x+\rho\mathbb{B}) \cap (M \cap D) \neq \emptyset\}, \\ G_\rho(x) &:= \{v \in \mathbb{R}^n : v \in \eta + \rho\mathbb{B}, \eta \in g((x+\rho\mathbb{B}) \cap (M \cap D))\} \ \forall x \in \mathbb{R}^n. \end{aligned}$$

Then, every solution to $\tilde{\mathcal{H}}|_M$ with admissible perturbations d_i having Euclidean norm bounded by $\bar{M}_i$, $i \in \{1,2,3,4\}$, respectively, is a solution to the hybrid system $\mathcal{H}|_M^\rho$ with $\rho \geqslant \max\{\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4\}$, which corresponds to an outer perturbation of $\mathcal{H}|_M$ and satisfies the convergence property [24, Assumption 3.25]. Then, the claim follows by [24, Theorem 3.26] and the fact that every solution to $\tilde{\mathcal{H}}|_M$ is a solution to (23). In fact, using [24, Theorem 3.26], for every proper indicator ω of $\mathcal{O}$ on $\mathcal{B}_{\mathcal{O}}$ there exists $\tilde{\beta} \in \mathcal{KL}$ such that for each compact set $K \subset \mathcal{B}_{\mathcal{O}}$ and each $\varepsilon > 0$, there exists $\rho^* > 0$ such that for each $\rho \in (0, \rho^*]$, every solution ϕ_ρ to (23) from K satisfies $\omega(\phi_\rho(t,j)) \leqslant \tilde{\beta}(\omega(\phi_\rho(0,0)), t+j) + \varepsilon$ for each $(t,j) \in \operatorname{dom} \phi_\rho$. The proof concludes using the relationship between the solutions to $\tilde{\mathcal{H}}|_M$ and (23), and picking $\bar{M}_i$, such that $\max\{\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4\} \in (0, \rho^*]$. □

Remark 7.3: Robustness results of stability of compact sets for general hybrid systems are available in [16]. Since $\mathcal{O}$ is an asymptotically stable compact set for $\mathcal{H}|_M$, Theorem 7.2 is novel in the context of the literature of Poincaré maps. In particular, if one was to write the systems in [5] and [25] within the framework of [16], then one would not be able to apply the results on robustness for hybrid systems in [16] since the hybrid basic conditions would not be satisfied and the hybrid limit cycle may not be given by a compact set. Furthermore, through an application of [16, Lemma 7.19], it can be shown that the hybrid limit cycle is robustly $\mathcal{KL}$

[21] See [16, Definition 7.10] for a definition of $\mathcal{KL}$ asymptotic stability.

asymptotically stable on $\mathcal{B}_{\mathcal{O}}$.

Remark 7.4: Recently, the authors in [34], [47] present static or dynamic decentralized (event-based) controllers for robust stabilization of hybrid periodic orbits against possible disturbances and established results on H_2/H_∞ optimal decentralized event-based control design. In contrast to our work, they use input-to-state stability for robust stability properties of hybrid periodic orbits with respect to disturbance inputs in the discrete dynamics. Note that the results in [34], [47] consider possible disturbances only on the discrete dynamics and are only suitable for nonlinear impulsive systems that have jumps on switching surfaces. On the other hand, in this paper, we establish conditions for robustness of hybrid limit cycles that allow disturbances in the continuous/discrete dynamics and are applicable for hybrid dynamical systems with nonempty intersection between the flow set and the jump set.

Remark 7.5: Very recently, the authors in [48] propose a reachability-based approach to compute regions-of-attraction for hybrid limit cycles in a class of hybrid systems with a switching surface and bounded disturbance. Note that the approach in [48] deals with bounded disturbance only on the continuous dynamics and is only suitable for hybrid systems that have jumps on switching surfaces.

B. Robustness to Inflations of C and D

We consider the following specific parametric perturbation on h, in both the flow and jump sets, with $\epsilon > 0$ denoting the parameter: the perturbed flow set is an inflation of the original flow set while the condition $h(x) = 0$ in the jump set is replaced by $h(x) \in [-\epsilon, \epsilon]$. The resulting system is denoted as $\mathcal{H}|^\epsilon_M$ and is given by

$$\mathcal{H}|^\epsilon_M \begin{cases} \dot{x} &= f(x) \qquad x \in C_\epsilon \cap M, \\ x^+ &= g(x) \qquad x \in D_\epsilon \cap M, \end{cases} \tag{24}$$

where the flow set and the jump set are replaced by $C_\epsilon = \{x \in \mathbb{R}^n : h(x) \geqslant -\epsilon\}$ and $D_\epsilon = \{x \in \mathbb{R}^n : h(x) \in [-\epsilon, \epsilon], L_f h(x) \leqslant 0\}$, respectively, while the flow map and jump map are the same as for $\mathcal{H}|_M$. We have the following result, whose proof follows from the proof of Theorem 7.2.

Theorem 7.6: *Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. If $\mathcal{O}$ is a locally asymptotically stable compact set for $\mathcal{H}|_M$ with basin of attraction $\mathcal{B}_{\mathcal{O}}$, then there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for every $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}_{\mathcal{O}}$, there exists $\bar{\epsilon} > 0$ such that for each $\epsilon \in (0, \bar{\epsilon}]$ every solution ϕ to $\mathcal{H}|^\epsilon_M$ in (24) with $\phi(0,0) \in K$ satisfies*

$$|\phi(t,j)|_{\mathcal{O}} \leqslant \tilde{\beta}(|\phi(0,0)|_{\mathcal{O}}, t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom}\phi. \tag{25}$$

Theorem 7.6 implies that the asymptotic stability property of the hybrid limit cycle $\mathcal{O}$ is robust to a parametric perturbation on h. Note that the $\mathcal{KL}$ bound (25) is obtained when the parametrically perturbed system $\mathcal{H}|^\epsilon_M$ in (24) should also exhibit a hybrid limit cycle. At times, a relationship between the maximum value $\bar{\epsilon}$ of the perturbation and the factor ε in the semiglobal and practical $\mathcal{KL}$ bound in (25) can be established numerically. Next, Theorem 7.6 and this relationship are illustrated in the TCP congestion control example.

Example 7.7: Let us revisit the hybrid congestion control system (4) in Section II, where, now, the flow set C_{TCP} and the jump set D_{TCP} are replaced by $C^\epsilon_{\text{TCP}} = \{x \in \mathbb{R}^2 : q_{\max} - q \geqslant -\epsilon\}$, $D^\epsilon_{\text{TCP}} = \{x \in \mathbb{R}^2 : q_{\max} - q \in [-\epsilon, \epsilon], r \geqslant B\}$, respectively. To validate Theorem 7.6, multiple simulations are performed to show a relationship between $\bar{\epsilon}$, the maximal value of the perturbation parameter ϵ, and ε, the desired level of closeness to the hybrid limit cycle $\mathcal{O}$. Given the compact set $K = [0.68, 0.72] \times [0.58, 0.64]$ and different desired level $\varepsilon \in \{0.01, 0.02, 0.03, 0.04\}$ of closeness to the hybrid limit cycle, it indicates that the relationship between $\bar{\epsilon}$ and ε can be approximated as $\bar{\epsilon} \approx 2.8\varepsilon$.

C. Robustness to Computation Error of Hybrid Poincaré Map

The hybrid Poincaré map defined in (19) indicates the evolution of a trajectory of a hybrid system from a point on the jump set $M \cap D$ to another point in the same set $M \cap D$. As stated in Theorem 6.4 and Corollary 6.6, stability of hybrid limit cycles can be verified by checking the eigenvalues of the Jacobian of the hybrid Poincaré map at its fixed point. However, errors in the computation of the hybrid Poincaré map may influence the statements made about asymptotic stability. Typically, the hybrid Poincaré map is computed numerically by discretizing the flows, using integration schemes such as Euler, Runge-Kutta, and multi-step methods [49], which unavoidably lead to an approximation of Poincaré maps.

Following the ideas in [49] about perturbations introduced by computations, the discrete-time system associated with the (exact) hybrid Poincaré map P in (19) is given by[22]

$$\mathcal{H}_P : x^+ = P(x) \quad x \in M \cap D, \tag{26}$$

which we treat as a hybrid system without flows. As argued above, due to unavoidable errors in computations and computer implementations, only approximations of the map P and of the sets M and D are available. In particular, given a point $x \in M \cap D$, the value of the step size, denoted $s > 0$, used in the computation of P at a point x affects the precision of the resulting approximation, which, in turn, may prevent the solution to (26) to remain in $M \cap D$ and be complete. Due to this, we denote by P_s the results of computing P, and by M_s and D_s the approximations of M and D, respectively. With some abuse of notation, the discrete-time system associated with P_s, M_s, and D_s is defined as

$$\mathcal{H}_{P_s} : x^+ = P_s(x) \quad x \in M_s \cap D_s. \tag{27}$$

The approximations of P_s, M_s, and D_s are assumed to satisfy the following properties.

Assumption 7.8: Given $M \subset \mathbb{R}^n$ and $\mathcal{H} = (C, f, D, g)$, the function $P_s : \mathbb{R}^n \to \mathbb{R}^n$ parameterized by $s > 0$ is such that, for some continuous function $\varrho : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, there exists $s^* > 0$ such that, for all $x \in M \cap D$,

$$P_s(x) \in P_\varrho(x) \qquad \forall s \in (0, s^*] \tag{28}$$

where $P_\varrho(x) := \{v \in \mathbb{R}^n : v \in g + \varrho(g)\mathbb{B},\ g \in P(x + \varrho(x)\mathbb{B})\}$ and the set $M_s \cap D_s$ satisfies, for any positive sequence $\{s_i\}_{i=1}^\infty$

[22] By some abuse of notation, though it is not hybrid, we label as $\mathcal{H}_P$ the discrete-time system in (26) associated to the Poincaré map P and we use $j \in \mathbb{N}$ as time instead of $(0, j)$ for $\mathcal{H}_P$ afterwards.

such that $s_i \searrow 0$,

$$\limsup_{i\to\infty} M_{s_i} \cap D_{s_i} \subset M \cap D. \tag{29}$$

Remark 7.9: The property in (28) is a consistency condition on the integration scheme used to compute the flows involved in (19). For instance, when the forward Euler method is used to approximate those flows, the numerical values of ϕ are generated using the scheme $x + sf(x)$, which, under Lipschitzness of f and boundedness of solutions (and its derivatives) to $\dot{x} = f(x)$, is convergent of order 1; in particular, the error between P_s and P is $O(s)$, which implies that (28) holds for some function ϱ. Vaguely, the property in (29) holds when a distance between $M_s \cap D_s$ and $M \cap D$ approaches zero as the step size vanishes, which is an expected property as precision improves with a decreasing step size. Condition (29) is satisfied when, for small enough $s > 0$, $M_s \cap D_s$ is contained in an outer perturbation of $M \cap D$. Very often, the jump set $M \cap D$ can be implemented accurately in the computation of the hybrid Poincaré map, i.e., it may be possible to take $M_s = M$ and $D_s = D$.

The following closeness result between solutions to $\mathcal{H}_P$ and $\mathcal{H}_{P_s}$ holds.

Theorem 7.10: *(closeness between solutions and approximations on compact domains) Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Assume the computed Poincaré map P_s approximating P and the sets M_s and D_s approximating M and D, respectively, satisfy Assumption 7.8. Then, for every compact set $K \subset M \cap D$, every $\varepsilon > 0$, and every simulation horizon $J \in \mathbb{N}$, there exists $s^* > 0$ with the following property: there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, for each $s \in (0, s^*]$ and any solution $\phi_{P_s} \in \mathcal{S}_{\mathcal{H}_{P_s}}(K + \delta\mathbb{B})$ there exists a solution $\phi_P \in \mathcal{S}_{\mathcal{H}_P}(K)$ with $\operatorname{dom}\phi_P \subset \mathbb{N}$ such that ϕ_{P_s} and ϕ_P are (J, ε)-close.* [23]

A proof can be found in [39, Theorem 7.12].

Inspired by [49, Theorem 5.3], the following stability result shows that when Assumption 7.8 holds, asymptotic stability of the fixed point of P (assumed to be unique) is preserved under the computation of P.

Theorem 7.11: *(stability preservation under computation error of P) Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Assume that the computed Poincaré map P_s approximating P and the sets M_s and D_s approximating M and D, respectively, satisfy Assumption 7.8, and that x^* is a unique globally asymptotically stable fixed point of P. Then, x^* is a unique semiglobally practically asymptotically stable fixed point of P_s with basin of attraction containing every point in $M \cap D$, i.e., there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for every $\varepsilon > 0$, each compact set $K \subset M_s \cap D_s$, and every simulation horizon $J \in \mathbb{N}$, there exists $s^* > 0$ such that, for each $s \in (0, s^*]$, every solution $\phi_{P_s} \in \mathcal{S}_{\mathcal{H}_{P_s}}(K)$ to $\mathcal{H}_{P_s}$ satisfies for each $j \in \operatorname{dom}\phi_{P_s}$*

$$|\phi_{P_s}(j) - x^*| \leqslant \tilde{\beta}(|\phi_{P_s}(0) - x^*|, j) + \varepsilon.$$

[23] See [49, Definition 3.2] for a definition of (T, J, ε)-close to quantify the distance between hybrid arcs (and solutions). Here, it is just the hybrid case but with $t = 0$.

Proof: Since the hybrid system $\mathcal{H}_P$ without flows satisfies the hybrid basic conditions A1) and (A2) in Section III-A and x^* is a unique globally asymptotically stable fixed point of P, by [49, Theorem 3.1], there exists $\tilde{\beta} \in \mathcal{KL}$ such that each solution $\phi_P \in \mathcal{S}_{\mathcal{H}_P}(M \cap D)$ to $\mathcal{H}_P$ satisfies

$$|\phi_P(j) - x^*| \leqslant \tilde{\beta}(|\phi_P(0) - x^*|, j) \quad \forall j \in \operatorname{dom}\phi_P.$$

Given $\delta > 0$, let $s \in (0, \delta]$. Given a compact set $K \subset M_s \cap D_s$ and a simulation horizon $J \in \mathbb{N}$, by the assumptions, [49, Lemma 5.1] implies that for a state dependent perturbation determined by the constant δ and a continuous function $\varrho : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, the outer perturbation $\mathcal{H}_{P_\delta}$ of $\mathcal{H}_P$ given by

$$\mathcal{H}_{P_\delta} : x^+ \in P_\delta(x) \quad x \in D_\delta, \tag{30}$$

where $P_\delta(x) := \big\{v \in \mathbb{R}^n : v \in g + \delta\varrho(g)\mathbb{B},\ g \in P(x + \delta\varrho(x)\mathbb{B})\big\}$, $D_\delta := \{x \in \mathbb{R}^n : x + \delta\varrho(x)\mathbb{B} \cap (M \cap D) \neq \emptyset\}$, satisfies the convergence property in [49, Definition 3.3]. Then, using K above, [49, Theorem 3.5] implies that for each $\varepsilon > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, every solution $\phi_{P_\delta} \in \mathcal{S}_{\mathcal{H}_{P_\delta}}(K + \delta\mathbb{B})$ to $\mathcal{H}_{P_\delta}$ satisfies for each $j \in \operatorname{dom}\phi_{P_\delta}$

$$|\phi_{P_\delta}(j) - x^*| \leqslant \tilde{\beta}(|\phi_{P_\delta}(0) - x^*|, j) + \varepsilon.$$

By Assumption 7.8, the properties of solutions to $\mathcal{H}_{P_\delta}$ also hold for solutions ϕ_{P_s}. The result follows by this preservation and the $\mathcal{KL}$ bound of solutions to $\mathcal{H}_P$. $\square$

Note that the property in Theorem 7.11 holds for small enough step size s. The step size bound s^* decreases with the desired level of closeness to x^*, which is given by ε. The next result shows that the computed Poincaré map P_s has a semiglobally asymptotically stable (semi-GAS) compact set $\mathcal{A}_s$ with basin of attraction containing every point in $M \cap D$ that reduces to a singleton $\{x^*\}$ as s approaches zero.

Theorem 7.12: *(continuity of asymptotically stable fixed points) Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Assume that x^* is a unique globally asymptotically stable fixed point of the hybrid Poincaré map P and the computed Poincaré map P_s approximating P and the sets M_s and D_s approximating M and D, respectively, satisfy Assumption 7.8. Then, there exists $s^* > 0$ such that for each $s \in (0, s^*]$, the computed Poincaré map P_s has a semi-GAS compact set $\mathcal{A}_s$ with basin of attraction containing every point in $M \cap D$ satisfying $\lim_{s\searrow 0} \mathcal{A}_s = x^*$.*

Proof: Let K be any compact set such that for some $\varepsilon > 0$, $x^* + 2\varepsilon\mathbb{B} \subset K \subset \mathbb{R}^n$. Using K as above and an arbitrary simulation horizon $J \in \mathbb{N}$, consider the perturbed system $\mathcal{H}_{P_\delta}$ in (30) and define $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$ with

$$\tilde{P}_\delta(x) = \begin{cases} P_\delta(x) \cup \{x^*\} & x \in D_\delta \\ \{x^*\} & x \in \mathbb{R}^n \backslash D_\delta \end{cases}$$

and $\tilde{D}_\delta = \mathbb{R}^n$. Using K and ε as above, [49, Theorem 3.5] implies that for each $\varepsilon > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, every solution $\phi_{\tilde{P}_\delta} \in \mathcal{S}_{\tilde{\mathcal{H}}_{\tilde{P}_\delta}}(K)$ to $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$ satisfies for each $j \in \operatorname{dom}\phi_{\tilde{P}_\delta}$

$$|\phi_{\tilde{P}_\delta}(j) - x^*| \leqslant \tilde{\beta}(|\phi_{\tilde{P}_\delta}(0) - x^*|, j) + \varepsilon. \tag{31}$$

For a simulation horizon $J \in \mathbb{N}$, let $\mathrm{Reach}_{J,\tilde{\mathcal{H}}_{\tilde{P}_\delta}}(x^* + 2\varepsilon\mathbb{B})$ be the reachable set of $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$ from $x^* + 2\varepsilon\mathbb{B}$ up to J, i.e.,

$$\mathrm{Reach}_{J,\tilde{\mathcal{H}}_{\tilde{P}_\delta}}(x^*+2\varepsilon\mathbb{B}) := \{\phi_{\tilde{P}_\delta}(j) : \phi_{\tilde{P}_\delta} \text{ is a solution to } \tilde{\mathcal{H}}_{\tilde{P}_\delta}, \\ \phi_{\tilde{P}_\delta}(0) \in x^* + 2\varepsilon\mathbb{B}, j \in \operatorname{dom}\phi, j \leqslant J\}.$$

Now, following a similar step as in the proof of [49, Theorem 5.4], let

$$B_\varepsilon := \overline{\mathrm{Reach}_{\infty,\tilde{\mathcal{H}}_{\tilde{P}_\delta}}(x^* + 2\varepsilon\mathbb{B})}.$$

By (31), B_ε is bounded. Moreover, since B_ε is closed by definition, it follows that it is compact. Next, we show that it is forward invariant. Consider a solution $\phi_{\tilde{P}_\delta} \in \mathcal{S}_{\tilde{\mathcal{H}}_{\tilde{P}_\delta}}(B_\varepsilon)$ to $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$. Assume that there exists $j' \in \operatorname{dom}\phi_{\tilde{P}_\delta}$ for which $\phi_{\tilde{P}_\delta}(j') \notin B_\varepsilon$. By definition of B_ε, since $\phi_{\tilde{P}_\delta}(0) \in B_\varepsilon$, the solution $\phi_{\tilde{P}_\delta}$ belongs to B_ε for each $j \in \operatorname{dom}\phi_{\tilde{P}_\delta}$. This is a contradiction. Next, we show that solutions to $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$ starting from K converge to B_ε uniformly. (31) implies that for the given K and ε, there exists $N > 0$ such that for every solution $\phi_{\tilde{P}_\delta} \in \mathcal{S}_{\tilde{\mathcal{H}}_{\tilde{P}_\delta}}(K)$ to $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$ and for each $j \in \operatorname{dom}\phi_{\tilde{P}_\delta}$, $j \geqslant N$: $|\phi_{\tilde{P}_\delta}(j) - x^*| \leqslant 2\varepsilon$. Then, since B_ε is compact, forward invariant, and uniformly attractive from K, by [24, Theorem 3.26], B_ε is a semi-GAS set for $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$. By the construction of $\tilde{\mathcal{H}}_{\tilde{P}_\delta}$ and Assumption 7.8, semiglobal asymptotic stability of B_ε for $\mathcal{H}_{P_s}$ with basin of attraction containing every point in $M \cap D$ follows. Finally, note that $B_0 = \{x^*\}$ and that as $\varepsilon \to 0$, $\lim_{\varepsilon \searrow 0} B_\varepsilon = x^*$. By (31), $\varepsilon \searrow 0$ implies $\delta \searrow 0$. Moreover, from the proof of Theorem 7.10, we have $s \searrow 0$ as $\delta \searrow 0$. It follows that $s \searrow 0$ as $\varepsilon \searrow 0$. Therefore, the result follows by $\mathcal{A}_s = B_\varepsilon$. □

VIII. Conclusion

Notions and tools for the analysis of existence and stability of hybrid limit cycles in hybrid dynamical systems were proposed. Necessary conditions were established for the existence of hybrid limit cycles. The Zhukovskii stability notion for hybrid systems was introduced and a relationship between Zhukovskii stability and the incremental graphical stability was presented. A sufficient condition relying on Zhukovskii stability of the hybrid system was established for the existence of hybrid limit cycles. Sufficient and necessary conditions for the stability of hybrid limit cycles were presented. Moreover, comparing to previous results in the literature, we established conditions for robustness of hybrid limit cycles with respect to small perturbations and to computation error of the hybrid Poincaré map, which is a very challenging problem in systems with impulsive effects. Examples were included to aid the reading and illustrate the concepts and the methodology of applying the new results. Future work includes exercising the presented conditions on systems of higher dimension and more intricate dynamics, and hybrid control design for asymptotic stabilization of limit cycles as well as their robust implementation.

References

[1] M. Adimy, F. Crauste, A. Halanay, M. Neamţu, and D. Opriş, “Stability of limit cycles in a pluripotent stem cell dynamics model”, *Chaos, Solitons & Fractals,* vol. 27, no. 4, pp. 1091-1107, 2006.

[2] S. Rodrigues, J. Gonçalves, and J. R. Terry, “Existence and stability of limit cycles in a macroscopic neuronal population model”, *Physica D: Nonlinear Phenomena,* vol. 233, no. 1, pp. 39-65, 2007.

[3] E. González-Olivares, H. Meneses-Alcay, B. González-Yañez, J. Mena-Lorca, A. Rojas-Palma, and R. Ramos-Jiliberto, “Multiple stability and uniqueness of the limit cycle in a Gause-type predator-prey model considering the Allee effect on prey”, *Nonlinear Analysis: Real World Applications,* vol. 12, no. 6, pp. 2931-2942, 2011.

[4] A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of Oscillators,* International Series of Monographs in Physics, Vol. 4, Pergamon Press, 1966.

[5] J. W. Grizzle, G. Abba, and F. Plestan, “Asymptotically stable walking for biped robots: analysis via systems with impulse effects”, *IEEE Transactions on Automatic Control,* vol. 46, no. 1, pp. 51-64, 2001.

[6] Q. Shu and R. G. Sanfelice, “Dynamical properties of a two-gene network with hysteresis”, *Information and Computation,* vol. 236, pp. 102-121, 2014.

[7] V. I. Slyn’ko, “Constructing a Poincaré map for a holonomic mechanical system with two degrees of freedom subject to impacts”, *Internat. Appl. Mech.*, vol. 44, no. 5, pp. 575-581, 2008.

[8] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko and N. V. Skripnik, *Differential Equations with Impulse Effects: Multivalued Right-hand Sides with Discontinuities* De Gruyter, 2011.

[9] V. Lakshmikantham, D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations,* volume 6. World scientific, 1989.

[10] A. Samoilenko and N. Perestyuk, *Impulsive Differential Equations,* World Scientific, 1995.

[11] M. U. Akhmetov and N. A. Perestyuk, “Periodic and almost periodic solutions of strongly nonlinear impulse systems”, *Prikl. Mat. Mekh.*, vol. 56, no. 6, pp. 926-934, 1992.

[12] A. M. Samoilenko and N. A. Perestyuk, “N. N. Bogoljubov’s second theorem for systems of differential equations with impulse action”, *Differencialnye Uravnenija,* vol. 10, pp. 2001-2009, 2083, 1974.

[13] A. M. Samoilenko and N. A. Perestyuk, “The method of averaging in systems with an impulsive action”, *Ukrain. Mat. Ž.*, vol. 26, pp. 411-418, 431, 1974.

[14] M. Akhmet and M. Turan, “Bifurcation of discontinuous limit cycles of the Van der Pol equation”, *Mathematics and Computers in Simulation,* vol. 95, pp. 39-54, 2014.

[15] Y. M. Perestyuk, “Discontinuous oscillations in one impulsive system,” *Journal of Mathematical Sciences,* vol. 194, pp. 404-413, 2013.

[16] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness,* Princeton University Press, 2012.

[17] T. V. Gorbachuk and M. O. Perestyuk, “On the existence of discontinuous limit cycles for one system of differential equations with pulse influence”, *Ukr Math J,* vol. 49, pp. 1265-1275, 1997.

[18] J. Z. Tang and I. R. Manchester, “Transverse contraction criteria for stability of nonlinear hybrid limit cycles”, *Proceedings of the 53rd IEEE Conference on Decision and Control,* December 15-17, 2014, Los Angeles, CA, USA, pp. 31-36.

[19] A. Barreiro, A. Baños, S. Dormidoc, and J. A. González-Prieto, “Reset control systems with reset band: Well-posedness, limit cycles and stability analysis”, *Systems & Control Letters,* vol. 63, no. 1, pp. 1-11, 2014.

[20] X. Lou, Y. Li, and R. G. Sanfelice, “Existence of hybrid limit cycles and Zhukovskii stability in hybrid systems”, *2017 American Control Conference,* May 24-26, 2017, Seattle, USA, pp. 1187-1192.

[21] W. Clark, A. Bloch, and L. Colombo, “A Poincaré-Bendixson theorem for hybrid systems”, *Mathematical Control & Related Fields,* vol. 10, no. 1, pp. 27-45, 2020.

[22] W. Clark and A. Bloch, “A Poincaré-Bendixson theorem for hybrid dynamical systems on directed graphs”, *Mathematics of Control, Signals, and Systems,* vol. 32, no. 1, pp. 1-18, 2020.

[23] J. Goodman and L. Colombo, “On the existence and uniqueness of Poincaré maps for systems with impulse effects”, *IEEE Transactions on Automatic Control,* vol. 65, no. 4, pp. 1815-1821, 2020.

[24] R. G. Sanfelice, *Hybrid Feedback Control,* Princeton University Press, 2021.

[25] S. Nersesov, V. Chellaboina, and W. Haddad, “A generalization of Poincaré theorem to hybrid and impulsive dynamical systems”, *Proceedings of the 2002 American Control Conference,* Anchorage, AK May 8-10, 2002, pp. 1240-1245.

[26] J. M. Gonçalves, *Constructive Global Analysis of Hybrid Systems*. Ph.D. dissertation, Massachusetts Institute of Technology, September 2000, pp. 44-48.

[27] M. Benmiloud, A. Benalia, M. Djemai, and M. Defoort, “On the local stabilization of hybrid limit cycles in switched affine systems”, *IEEE Transactions on Automatic Control,* vol. 64, no. 2, pp. 841-846, 2019.

[28] B. Morris and J. W. Grizzle, “Hybrid invariant manifolds in systems with impulse effects with application to periodic locomotion in bipedal robots”, *IEEE Transactions on Automatic Control,* vol. 54, no. 8, pp. 1751-1764, 2009.

[29] H. Wang, H. Zhang, Z. Wang, and Q. Chen, “Finite-time stabilization of periodic orbits for under-actuated biped walking with hybrid zero dynamics”, *Communications in Nonlinear Science and Numerical Simulation,* vol. 80, article no. 104949, 2020.

[30] K. A. Hamed and A. D. Ames, “Nonholonomic hybrid zero dynamics for the stabilization of periodic orbits: application to underactuated robotic walking”, *IEEE Transactions on Control Systems Technology,* vol. 28, no. 6, pp. 2689-2696, 2020.

[31] W. Znegui, H. Gritli, and S. Belghith, “Stabilization of the passive walking dynamics of the compass-gait biped robot by developing the analytical expression of the controlled Poincaré map,” *Nonlinear Dynamics,* vol. 101, pp. 1061-1091, 2020.

[32] X. Lou, Y. Li, and R. G. Sanfelice, “Results on stability and robustness of hybrid limit cycles for a class of hybrid systems”, *Proceedings of the 54th IEEE Conference on Decision and Control,* December 15-18, 2015, Osaka, Japan, pp. 2235-2240.

[33] X. Lou, Y. Li, and R. G. Sanfelice, “Robust stability of hybrid limit cycles with multiple jumps in hybrid dynamical systems,” *IEEE Transactions on Automatic Control,* vol. 63, no. 4, pp. 1220-1226, 2018.

[34] K. A. Hamed and R. G. Gregg, “Decentralized feedback controllers for robust stabilization of periodic orbits of hybrid systems: application to bipedal walking”, *IEEE Transactions on Control Systems Technology,* vol. 25, no. 4, pp. 1153-1167, 2017.

[35] C. M. Ding, “The limit sets of uniformly asymptotically Zhukovskij stable orbits”, *Computers and Mathematics with Applications,* vol. 47, no. 6, pp. 859-862, 2004.

[36] X. S. Yang, “Remarks on three types of asymptotic stability”, *Systems & Control Letters,* vol. 42, pp. 299-302, 2001.

[37] G. A. Leonov, “Generalization of the Andronov-Vitt theorem”, *Regular and Chaotic Dynamics,* vol. 11, no. 2, pp. 281-289, 2006.

[38] Y. Li and R. G. Sanfelice, “On necessary and sufficient conditions for incremental stability of hybrid systems using the graphical distance between solutions”, *Proceedings of the 54th IEEE Conference on Decision and Control,* December 15-18, 2015, Osaka, Japan, pp. 5575-5580.

[39] X. Lou, Y. Li, and R. G. Sanfelice, “Notions, stability, existence, and robustness of limit cycles in hybrid dynamical systems”, arXiv: 2208.02785

[40] J. P. Hespanha, S. Bohacek, K. Obraczka, and J. Lee, “Hybrid modeling of TCP congestion control”, *Hybrid Systems: Computation and Control, LNCS (M.D. Di Benedetto, A. Sangiovanni-Vincentelli (Eds.)),* New York: Springer-Verlag, 2001, pp. 291-304.

[41] J. Chai and R. G. Sanfelice, “Forward invariance of sets for hybrid dynamical systems (Part I)”, *IEEE Transactions on Automatic Control,* vol. 64, no. 6, pp. 2426-2441, 2019.

[42] V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations,* Princeton University Press, pp. 333, 1966.

[43] M. Maghenem and R. G. Sanfelice, “Lipschitzness of minimal-time functions in constrained continuous-time systems with applications to reachability analysis”, *Proceedings of the 2020 American Control Conference,* July 1-3, 2020, Denver, pp. 937-942.

[44] A. Ghaffari, M. Tomizuka, and R. A. Soltan, “The stability of limit cycles in nonlinear systems”, *Nonlinear Dynamics,* vol. 56, pp. 269-275, 2009.

[45] J. Palis and W. de Melo, *Geometric Theory of Dynamical Systems*, New York: Springer, pp. 40-41, 1982.

[46] R. G. Sanfelice, R. Goebel, and A. R. Teel, “Invariance principles for hybrid systems with connections to detectability and asymptotic stability”, *IEEE Transactions on Automatic Control,* vol. 52, no. 12, pp. 2282-2297, 2007.

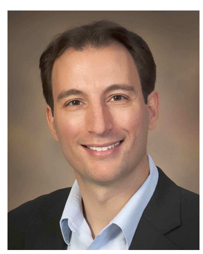
[47] K. A. Hamed and R. G. Gregg, “Decentralized event-based controllers for robust stabilization of hybrid periodic orbits: application to underactuated 3-D bipedal walking”, *IEEE Transactions on Automatic Control,* vol. 64, no. 6, pp. 2266-2281, 2019.

[48] J. J. Choi, A. Agrawal, K. Sreenath, C. J. Tomlin, and S. Bansal, “Computation of regions of attraction for hybrid limit cycles using reachability: an application to walking robots”, *IEEE Robotics and Automation Letters,* vol. 7, no. 2, pp. 4504-4511, 2022.

[49] R. G. Sanfelice and A. R. Teel, “Dynamical properties of hybrid systems simulators”, *Automatica,* vol. 46, no. 2, pp. 239-248, 2010.

Xuyang Lou received the Ph.D. degree in control theory and control engineering from the Jiangnan University, Wuxi, China, in 2009. In 2010, he was a Postdoctoral Fellow in the Department of Electrical Engineering, Katholieke Universiteit Leuven, Belgium. In 2014, he visited the Computer Engineering Department, University of California, Santa Cruz, USA. He is currently a Full Professor with the Jiangnan University, China. His research interests include hybrid systems, data-driven discovery and flexible robotics.

Yuchun Li received the B.S. and M.S. degree in Mechanical Engineering from Zhejiang University, Hangzhou, China, in 2007 and 2010, respectively. In 2016, he received the Ph.D. degree in Computer Engineering from the University of California, Santa Cruz, CA, USA. He is currently working at Google as a Senior Software Engineer. His research interests include modeling, stability, observer design, control and robustness analysis of hybrid/network systems.



Ricardo G. Sanfelice received the B.S. degree in Electronics Engineering from the Universidad de Mar del Plata, Buenos Aires, Argentina, in 2001, and the M.S. and Ph.D. degrees in Electrical and Computer Engineering from the University of California, Santa Barbara, CA, USA, in 2004 and 2007, respectively. In 2007 and 2008, he held postdoctoral positions at the Laboratory for Information and Decision Systems at the Massachusetts Institute of Technology and at the Centre Automatique et Systèmes at the École de Mines de Paris. In 2009, he joined the faculty of the Department of Aerospace and Mechanical Engineering at the University of Arizona, Tucson, AZ, USA, where he was an Assistant Professor. In 2014, he joined the University of California, Santa Cruz, CA, USA, where he is currently Professor in the Department of Electrical and Computer Engineering. Prof. Sanfelice is the recipient of the 2013 SIAM Control and Systems Theory Prize, the National Science Foundation CAREER award, the Air Force Young Investigator Research Award, the 2010 IEEE Control Systems Magazine Outstanding Paper Award, and the 2020 Test-of-Time Award from the Hybrid Systems: Computation and Control Conference. He is Associate Editor for Automatica and a Fellow of the IEEE. His research interests are in modeling, stability, robust control, observer design, and simulation of nonlinear and hybrid systems with applications to power systems, aerospace, and biology.