Multi-channel hybrid time domains and clustering protocols for large-scale interconnections

Andrew R. Teel, Rafal Goebel, and Ricardo G. Sanfelice

***Abstract*— Multi-channel hybrid time domains and clustering protocols are introduced. These concepts are shown to be well-suited for the modeling and description of interconnections of hybrid dynamical systems. Compared to a hybrid time domain, a multi-channel hybrid time domain incorporates multiple jump counters, one for each subsystem in an interconnection. The interconnection's clustering protocol then determines how the different counters are coordinated. A distributed, hybrid average consensus algorithm is used to illustrate these concepts.**

I. Introduction

Hybrid dynamical systems combine continuous-time dynamics, resulting in flows, with discrete-time dynamics, resulting in jumps. For example, see [1], [2]. Here, we are interested in the interconnection of hybrid systems. The literature contains several results on hybrid systems interconnections, many of them related to stability analysis. See, for example, [3]- [15]. Results that use category theory to address open hybrid systems and their compositions have appeared recently; see [16], [17] and the references therein.

When interconnecting open hybrid systems, special attention should be given to how jumps of the different subsystems interact [18]. There are many possibilities in this regard. On one extreme, the subsystems could be required to jump in unison; on another extreme, the interconnection may insist on only one subsystem jumping at a time. Various other intermediate "protocols" can be considered that prescribe this interaction. In this paper, we introduce clustering protocols that are used to impose structure on the interaction of jumps in interconnected hybrid systems. The utility of the clustering protocol is that it captures, in a single vector, the structure of the jump set and jump map of the interconnection.

A solution of a hybrid system is usually defined on a "hybrid time domain" that keeps track of the duration of flows as well as the number of jumps that have occurred in leading the state to its current value; see [1], with precursors in [19], [20], [21], [22].

To promote modularity in the design of complex, interconnected hybrid systems, we introduce multi-channel hybrid time domains for open hybrid systems, where each system in the interconnection is assigned its own channel and each channel corresponds to a standard hybrid time domain. Using multi-channel domains permits describing the behaviors of the subsystems independent of the clustering protocol that will be imposed eventually. Subsequently, the interconnection is formulated by specifying interconnection constraints that prescribe data flow as well as a clustering protocol that prescribes jump interactions.

Since a multi-channel hybrid time domain is one for which each channel corresponds to a standard hybrid time domain, a multi-channel hybrid time domain involves multiple jump counters. Time domains with multiple jump counters have been used before in other hybrid systems contexts, mostly when attempting to address the passage of ordinary time in systems, including interconnections of subsystems, that exhibit Zeno solutions, i.e., solutions with an infinite number of jumps in a finite amount of ordinary time. See [23] and [24, Chapter 4], and [25, Section 5.2] for example.

The paper is organized as follows: we discuss hybrid systems with inputs and solutions defined on single-channel hybrid time domains in Section II. Then multi-channel hybrid time domains, arcs, and solutions are introduced in Section III. Section IV presents the concept of clustering protocols, including examples. The link between interconnected models and clustering protocols is established in Section V. Section VI contains a discussion of interconnection constraints to specify data flow. Lastly, in Section VII, we illustrate the paper's main ideas using a distributed, hybrid algorithm for average consensus.

While not discussed here, extensions that allow for applying multiple clustering protocols to a given interconnection, e.g., time-varying clustering protocols, are straightforward.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}_{\geq 0}$ the nonnegative real numbers, $\mathbb{N}$ is the natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Given $N \in \mathbb{N}$, $\mathbb{N}_{\leq N} := \{1, 2, \dots, N\}$ and $\mathbb{N}_{\geq N} := \{N, N+1, \dots\}$. $\mathbf{e}_i$ denotes the standard i-th basis vector in $\mathbb{R}^n$ and $\mathbf{1}_n$ is the vector in $\mathbb{R}^n$ with every coordinate equal to one.

II. Hybrid dynamical systems and (single channel) solutions

This paper addresses hybrid dynamical systems with inputs of the form

$$(x, u) \in C \qquad \dot{x} \in F(x, u) \tag{1a}$$
$$(x, u) \in D \qquad x^+ \in G(x, u, u^+) \tag{1b}$$

Andrew R. Teel is with the Electrical and Computer Engineering Department, University of California, Santa Barbara, CA 93106-9560. teel@ucsb.edu. Research supported in part by AFOSR grant no. FA9550-21-1-0452. Rafal K. Goebel is with the Department of Mathematics & Statistics, Loyola University Chicago, Chicago, IL 60660. rgoebel1@luc.edu. Ricardo G. Sanfelice is with the Electrical and Computer Engineering Department, University of California, Santa Cruz, CA 95064. ricardo@ucsc.edu. Research supported in part by NSF Grants no. CNS-2039054 and CNS-2111688, by AFOSR Grants nos. FA9550-19-1-0169, FA9550-20-1-0238, FA9550-23-1-0145, and FA9550-23-1-0313, by AFRL Grant nos. FA8651-22-1-0017 and FA8651-23-1-0004, by ARO Grant no. W911NF-20-1-0253, and by DoD Grant no. W911NF-23-1-0158.

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $C \subset \mathbb{R}^{n+m}$ is the flow set, $D \subset \mathbb{R}^{n+m}$ is the jump set, $F : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^n$ is the flow map, and $G : \mathbb{R}^{n+2m} \rightrightarrows \mathbb{R}^n$ is the jump map. The symbol $\dot{x}$ represents the velocity of the state x and x^+ represents the value of the state after a jump. The value of the input after a jump is denoted u^+.

In the existing literature, solutions of (1) are often defined on hybrid time domains defined as follows. A *compact hybrid time domain* is a set of the form

$$E := \bigcup_{i=0}^{J-1} \Big([t_i, t_{i+1}] \times \{i\} \Big) \tag{2}$$

where $J \in \mathbb{N}$, and $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. A *hybrid time domain* is the union of a nondecreasing sequence of compact hybrid time domains $E_1 \subset E_2 \subset E_3 \subset \ldots$.

A mapping $x : \operatorname{dom} x \to \mathbb{R}^n$ is called a *hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and, for each $\ell \in \mathbb{N}_0$, $t \mapsto x(t,\ell)$ is locally absolutely continuous; it is called a *generalized hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and, for each $\ell \in \mathbb{N}_0$, $t \mapsto x(t,\ell)$ is continuous.

A pair of mappings $x : \operatorname{dom} x \to \mathbb{R}^n$, $u : \operatorname{dom} u \to \mathbb{R}^m$ constitute a solution of (1) if

1) x is a hybrid arc,
2) u is a generalized hybrid arc,
3) $\operatorname{dom} x = \operatorname{dom} u$,
4) if $(t_1,\ell),(t_2,\ell) \in \operatorname{dom} x$ and $t_1 < t_2$ then

$$\begin{aligned} &(x(t,\ell),u(t,\ell)) \in C && \forall t \in (t_1,t_2)\,, \\ &\dot{x}(t,\ell) \in F(x(t,\ell),u(t,\ell)) && \text{for a.a. } t \in [t_1,t_2]\,, \end{aligned}$$

5) if $(t,\ell_1),(t,\ell_2) \in \operatorname{dom} x$ and $\ell_2 - \ell_1 = 1$ then

$$\begin{aligned} &(x(t,\ell_1),u(t,\ell_1)) \in D \quad \& \\ &\qquad x(t,\ell_2) \in G(x(t,\ell_1),u(t,\ell_1),u(t,\ell_2)). \end{aligned}$$

III. Multi-channel Hybrid Time Domains, Arcs, and Solutions

A. Multi-channel domains

For hybrid systems with inputs, like in (1), that are to be modular and available for insertion into complex, large-scale interconnections, it is important to develop a solution notion that anticipates the existence of jumps due to other components in the system. We pursue an approach that defines solutions on time domains with multiple jump counters. We consider doing so in a manner such that, when using any one of these counters, we obtain a hybrid time domain. In particular, allowing for $N \in \mathbb{N}_{\geq 2}$ jump sources, we consider a time domain E that is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}^N_{\geq 0}$ having the property that, for each $i \in \mathbb{N}_{\leq N}$, the set

$$E_i := \bigcup_{(t,\ell) \in E} (t, \mathbf{e}_i^T \ell) \tag{3}$$

is a hybrid time domain. We call E an *N-channel time domain*. Let Π_c be the projection in $\mathbb{R}^n$, with n determined by the context, onto the first coordinate. In particular, given an N-channel time domain E,

$$\Pi_c(E) = \left\{ t \in \mathbb{R}_{\geq 0} : (t,j) \in E \text{ for some } j \in \mathbb{N}_0^N \right\}. \tag{4}$$

An N-channel time domain can be constructed from N (single-channel) hybrid time domains as follows:

Proposition 1: Let $N \in \mathbb{N}_{\geq 2}$ and, for each $i \in \mathbb{N}_{\leq N}$, suppose that E_i is a hybrid time domain. If

$$\Pi_c(E_i) = \Pi_c(E_j) \qquad \forall i,j \in \mathbb{N}_{\leq N} \tag{5}$$

then the set

$$E := \left\{ (t,\ell) \in \mathbb{R}_{\geq 0} \times \mathbb{N}^N_{\geq 0} : (t,\ell_i) \in E_i \ \forall i \in \mathbb{N}_{\leq N} \right\} \tag{6}$$

is an N-channel time domain and $\Pi_c(E) = \Pi_c(E_i)$ for all $i \in \mathbb{N}_{\leq N}$.

B. Arcs

We call a mapping $x : \operatorname{dom} x \to \mathbb{R}^n$ an *ith (out of N) channel hybrid arc* if

1) $\operatorname{dom} x$ is an N-channel time domain,
2) for each $\ell \in \mathbb{N}_0^N$, $t \mapsto x(t,\ell)$ is locally absolutely continuous, and
3) if (t,ℓ_1) and (t,ℓ_2) belong to $\operatorname{dom} x$ and satisfy $\mathbf{e}_i^T(\ell_2 - \ell_1) = 0$ then $x(t,\ell_1) = x(t,\ell_2)$.

We call the mapping an *ith (out of N) channel generalized hybrid arc* when the local absolute continuity in the second condition above is relaxed to continuity. By virtue of the third condition above, an ith channel hybrid arc can make a jump only when the ith counter changes; that is, it ignores changes in every counter except the ith one.

C. Solutions

Having defined multi-channel time domains, and single-channel hybrid arcs on such domains, we turn to defining systems whose solutions are defined on multi-channel time domains. Given $N \in \mathbb{N}$ and $i \in \mathbb{N}_{\leq N}$, we will say that a pair of mappings $x : \operatorname{dom} x \to \mathbb{R}^n$, $u : \operatorname{dom}(u) \to \mathbb{R}^m$ constitute an *ith (out of N) channel solution* to (1) if

1) x is an ith (out of N) channel hybrid arc,
2) u is an ith (out of N) channel generalized hybrid arc,
3) $\operatorname{dom} x = \operatorname{dom} u$
4) if $(t_1,\ell),(t_2,\ell) \in \operatorname{dom} x$ and $t_1 < t_2$ then

$$\begin{aligned} &(x(t,\ell),u(t,\ell)) \in C && \forall t \in (t_1,t_2)\,, \\ &\dot{x}(t,\ell) \in F(x(t,\ell),u(t,\ell)) && \text{for a.a. } t \in [t_1,t_2]\,, \end{aligned}$$

5) if $(t,\ell_1),(t,\ell_2) \in \operatorname{dom} x$ and $\mathbf{e}_i^T(\ell_2 - \ell_1) = 1$ then

$$\begin{aligned} &(x(t,\ell_1),u(t,\ell_1)) \in D \ \& \\ &\qquad x(t,\ell_2) \in G(x(t,\ell_1),u(t,\ell_1),u(t,\ell_2)). \end{aligned}$$

This solution concept is very similar to the single-channel solution concept given in Section II with the feature that changes in ℓ that do not change $\mathbf{e}_i^T \ell$ are ignored.

IV. Clustering Protocols for Multi-channel Hybrid Time Domains

There are many ways in which the jump sets and jump maps of a network of hybrid systems can be coordinated to end up with a composite jump set and jump map for the interconnection. For example, one protocol may insist that jumps happen sequentially, i.e., at different hybrid times; another protocol may insist that all subsystems jump simultaneously, i.e., at the same hybrid time. Other protocols may be in between, insisting that some subsystems jump together while others do not.

One way to anticipate and capture this variety mathematically is through a protocol parameter that can be applied to a multi-channel time domain. The intention of the protocol parameter is to cluster together some of the channels, corresponding to a group of hybrid systems, while not affecting the rest of the channels. The result is a new multi-channel time domain with fewer channels. Suppose we want to reduce the number of channels in a time domain from $N \in \mathbb{N}_{\geq 2}$ to $M \in \mathbb{N}_{\leq N-1}$ by collapsing $N-(M-1)$ channels into a single channel. We associate this operation with interconnecting $N-(M-1)$ hybrid systems according to the desired protocol while leaving the remaining channels open for other systems. Let the index set $\mathcal{I} \subset \mathbb{N}_{\leq N}$ denote the set of channels to be combined. Let $\lambda \in [0,1]^N$ be such that $\lambda_i > 0$ if and only if $i \in \mathcal{I}$. We say that λ is a *valid clustering protocol* for the indices $\mathcal{I}$ and the N-channel time domain E if

$$E_\lambda := \bigcup_{(t,\ell)\in E} (t, \lambda^T \ell) \tag{7}$$

is a hybrid time domain.

Example 1: We consider some possible protocol parameters for clustering N channels into a single channel, i.e., $\mathcal{I} = \mathbb{N}_{\leq N}$.

1) $\underline{\lambda = \mathbf{1}_N}$. This clustering protocol requires the various jump counters in E to increment one at a time. In particular, if $(t,\ell_a),(t,\ell_b) \in E$ and $\mathbf{1}_N^T(\ell_b-\ell_a) = n$ with $n \in \mathbb{N}$ then there exist n distinct values $(t,\ell_i) \in E$, $i \in \mathbb{N}_{\leq n}$ such that $\ell_1 = \ell_a$, $\ell_n = \ell_b$, and $\mathbf{1}_N^T(\ell_{i+1}-\ell_i) = 1$ for all $i \in \mathbb{N}_{\leq n-1}$.
2) $\underline{\lambda = \frac{1}{N}\mathbf{1}_N}$. This clustering protocol requires all N jump counters in E to increment simultaneously. In particular, if $(t,\ell_a),(t,\ell_b) \in E$ and $\mathbf{1}_N^T(\ell_b-\ell_a) = nN$ with $n \in \mathbb{N}$ then $(t,\ell_a + i\mathbf{1}_N) \in E$ for all $i \in \mathbb{N}_{\leq n-1}$ and $\ell_b = \ell_a + n\mathbf{1}_N$.
3) $\underline{\lambda = \frac{1}{M}\mathbf{1}_N,\ M \in \{2,\ldots,N-1\}}$. This clustering protocol requires any M jump counters in E to increment simultaneously. At different jump times, the set of M jump counters that increment can be different. In particular, $(t,\ell_a),(t,\ell_b) \in E$ and $\mathbf{1}_N^T(\ell_b-\ell_a) = nM$ with $n \in \mathbb{N}$ then there exist n distinct values $(t,\ell_i) \in E$, $i \in \mathbb{N}_{\leq n}$ such that $\ell_1 = \ell_a$, $\ell_n = \ell_b$, and $\mathbf{1}_N^T(\ell_{i+1}-\ell_i) = M$ for all $i \in \mathbb{N}_{\leq n-1}$.
4) $\underline{\lambda = \left(\frac{1}{L_1}\mathbf{1}_M^T, \frac{1}{L_2}\mathbf{1}_{N-M}^T\right)^T \text{ with } L_1 \in \mathbb{N}_{\leq M},\ L_2 \in \mathbb{N}_{\leq N-M},\ L_1 \neq L_2}$. This clustering protocol requires any L_1 jump counters in E, among the first M, to increment simultaneously but separately from the final $N-M$ jump counters, among which L_2 are required to increment together.
5) $N_1, N_2 \in \mathbb{N}$, $N := N_1 + N_2$ and $\lambda \in (0,1)^N$ is such that its first N_1 elements form a set with distinct subset sums (as in [26]) that are all less than 0.5 and then each of the remaining elements of λ, λ_i with $i \in \{N_1+1,\ldots,N_1+N_2\}$, satisfy $\lambda_i = 1 - s_i$ where s_i denotes the sum of the indices in $\mathbb{N}_{\leq N_1}$ of λ with which the i component of the interconnection must jump. We denote this set as $\mathcal{I}_i \subset \mathbb{N}_{\leq N_1}$. This protocol requires the ith jump counter, $i \in \{N_1+1,\ldots,N_1+N_2\}$ to increment simultaneously with all of the jump counters with indices in $\mathcal{I}_i$. ■

The next assumption will be used in upcoming results.

Assumption 1: The following conditions hold:

1) $N \in \mathbb{N}_{\geq 2}$,
2) $M \in \mathbb{N}_{\leq N-1}$ and $L := N - M + 1$,
3) the indices $\{i_1, i_2, \ldots, i_L\}$ are distinct and $\mathcal{I} := \{i_1, i_2, \ldots, i_L\} \subset \mathbb{N}_{\leq N}$,
4) E is an N-channel time domain,
5) λ is a valid clustering protocol for $\mathcal{I}$ and E, and
6) $\Lambda \in \mathbb{R}^{N\times M}$ is a full column-rank matrix with one column equal to λ and the remaining columns being the basis vectors in $\mathbb{R}^N$ corresponding to the indices not in $\mathcal{I}$. Let j_λ represent the column of Λ that equals λ.

Under Assumption 1, an M-channel domain that clusters the indices in $\mathcal{I}$ can be constructed from the N-channel domain E by using the matrix Λ.

Proposition 2: If Assumption 1 holds then

$$E_{N\to M,\Lambda} := \bigcup_{(t,\ell)\in E} (t, \Lambda^T \ell) \tag{8}$$

is an M-channel time domain.

V. Models from Protocols

In this section, we indicate how the model of a large-scale interconnection of hybrid systems is determined by a given valid clustering protocol. Throughout the section, (C_i, F_i, D_i, G_i) are hybrid systems with state/input pairs $(x_i, u_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{m_i}$, $i \in \mathcal{N}$, and we impose Assumption 1. The systems to be interconnected are (C_i, F_i, D_i, G_i), $i \in \mathcal{I}$, and, presumably, they have appropriately understood channel solutions on the N-channel domain E.

For the interconnection (C, F, D, G), the state/input pair is $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ where $n := \sum_{i\in\mathcal{I}} n_i$, $m := \sum_{i\in\mathcal{I}} m_i$, and

$$x := (x_{i_1}, \ldots, x_{i_L}), \quad u := (u_{i_1}, \ldots, u_{i_L}). \tag{9}$$

The construction of the flow set C and the flow map F is straightforward:

$$C := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m : (x_{i_k}, u_{i_k}) \in C_{i_k}\} \tag{10a}$$

$$F(x,u) := \{f = (f_{i_1}, \ldots, f_{i_L}) : f_{i_k} \in F_{i_k}(x_{i_k}, u_{i_k})\}. \tag{10b}$$

This way, the flow conditions on (x_i, u_i) for (C_i, F_i, D_i, G_i), $i \in \mathcal{I}$, are equivalent to the flow condition on (x, u) for $(C, F, \cdot, \cdot)$.

The construction of the jump set D and jump map G is more intricate and explicitly involves the protocol parameter. Let p be the number of distinct index sets $\mathcal{J} \subset \mathcal{I}$ such that $\sum_{j \in \mathcal{J}} \lambda_j = 1$. Denote these sets $\{\mathcal{J}_j\}_{j=1}^{p}$. By construction, the increase by 1 of the j_λ-th counter in $E_{N \to M, \Lambda}$ corresponds to increases by 1 of all counters in E indexed by one of the sets $\mathcal{J}_j$. Accordingly, to each $\mathcal{J}_j$ associate a set

$$\widetilde{D}_j := \Big\{(x,u) \in \mathbb{R}^{n+m} : (x_i, u_i) \in D_i \quad \forall i \in \mathcal{J}_j\Big\} \tag{11}$$

and define

$$D := \bigcup_{j=1}^{p} \widetilde{D}_j. \tag{12}$$

This way, the protocol enables a jump each time the state/input pair is in at least one of the sets $\widetilde{D}_j$. Further, to each set $\widetilde{D}_j$, we associate a jump map $\widetilde{G}_j(x, u, w)$. This mapping is empty outside of $\widetilde{D}_j$. Inside of $\widetilde{D}_j$, it resets the state components whose indices belong to $\mathcal{J}_j$ while leaving the other state components unchanged. That is, for all $w \in \mathbb{R}^m$,

$$\widetilde{G}_j(x,u,w) := \emptyset \qquad \forall (x,u) \notin \widetilde{D}_j \tag{13a}$$

$$\widetilde{G}_j(x,u,w) := \Bigg\{ g = \big(g_{i_1}, \ldots, g_{i_{N-(M-1)}}\big) : \tag{13b}$$

$$\begin{cases} g_{i_k} \in G_{i_k}(x_{i_k}, u_{i_k}, w_{i_k}) & i_k \in \mathcal{J}_j \\ g_{i_k} = x_{i_k} & \text{otherwise} \end{cases} \Bigg\} \quad \forall (x,u) \in \widetilde{D}_j.$$

The overall jump map allows any of these transitions when the state/input pair belongs to the jump set D. Hence

$$G(x,u,w) := \bigcup_{j=1}^{p} \widetilde{G}_j(x,u,w). \tag{14}$$

This way, the jump conditions on (x, u) are equivalent to the jump conditions on (x_i, u_i) with i in one of the sets $\mathcal{J}_j$. Consequently, a jump of a presumed j_λ-th (out of N) multi-channel solution to (C, F, D, G) on $E_{N \to M, \Lambda}$ corresponds to jumps in presumed i-th (out of N) multi-channel solutions (x_i, u_i) to (C_i, F_i, D_i, G_i). This, and the construction of $E_{N \to M, \Lambda}$, ensure that:

Proposition 3: The following are equivalent:

(a) The pair (x_i, u_i) is an i-th (out of N) multi-channel solution to (C_i, F_i, D_i, G_i) on E for every $i \in \mathcal{I}$.

(b) $(x_{i_1}, \ldots, x_{i_L}, u_{i_1}, \ldots, u_{i_L})$ is a j_λth (out of M) multi-channel solution to (C, F, D, G) defined in (10)-(14) on $E_{N \to M, \Lambda}$.

We are often interested in models that satisfy the following conditions.

Definition 1: The data (C, F, D, G) is said to satisfy the *hybrid basic conditions* if

1) C and D are closed;
2) F and G are outer semicontinuous and locally bounded;
3a) $F(x, u)$ is nonempty and convex for all $(x, u) \in C$,
3b) $G(x, u, w)$ is nonempty for all $(x, u, w) \in D \times \mathbb{R}^m$. ■

We can add that, also by construction:

Lemma 1: If (C_i, F_i, D_i, G_i) satisfy the hybrid basic conditions for each $i \in \mathcal{I}$ then (C, F, D, G) defined in (10)-(14) satisfies the hybrid basic conditions.

VI. Interconnection Constraints and Data Flow

In addition to specifying a clustering protocol to describe how the jumps of an interconnection of hybrid systems interact, we also specify interconnection constraints that specify how data flows in the interconnection. We may be interested in data flow on a subset of the channels, perhaps because systems have not been been assigned to all of the channels. Suppose we are interested in specifying data flow on $L \in \mathbb{N}_{\geq 2}$ different channels, corresponding to the indices $\{i_1, \ldots, i_L\}$ with state and input pairs $(x_{i_j}, u_{i_j}) \in \mathbb{R}^{n_{i_j}} \times \mathbb{R}^{m_{i_j}}$ for $j \in \mathbb{N}_{\leq L}$. Define

$$n := \sum_{j=1}^{L} n_{i_j}, \quad m := \sum_{j=1}^{L} m_{i_j} \tag{15}$$

and

$$x := (x_{i_1}, \ldots, x_{i_L}) \in \mathbb{R}^n, u := (u_{i_1}, \ldots, u_{i_L}) \in \mathbb{R}^m. \tag{16}$$

We then choose to specify data flow through a closed constraint

$$(x, u) \in H \subset \mathbb{R}^{n+m}, \tag{17}$$

i.e., H is a closed set.

Example 2: Suppose each system is given an output y_{i_j} that is a continuous function of (x_{i_j}, u_{i_j}) and the inputs u of the various systems are related to outputs $y := (y_{i_1}, \ldots, y_{i_L})$ via a linear equation $y = \Gamma u$, where Γ is a matrix of appropriate dimension. This equation can be written as $y(x, u) = \Gamma u$ and then H is the relation

$$H := \{(x, u) : y(x, u) - \Gamma u = 0\} \tag{18}$$

which is closed since $y(\cdot, \cdot)$ is assumed to be continuous. Under the conditions of the global implicit function theorem, e.g., y depends only on x and Γ is invertible, the constraint H resolves to a continuous equation for u in terms of x. ■

The interconnection constraint H adds an additional constraint to the flow and jump sets of the hybrid system. In the previous section, the construction of (C, F, D, G) corresponding to a given protocol can be extended to a construction that includes an interconnection constraint by restricting (x, u) to the set H. The following observation is straightforward.

Proposition 4: The hybrid system obtained by including an interconnection constraint H and then specifying a clustering protocol λ is the same as the hybrid system obtained by specifying the clustering protocol λ and then including the interconnection constraint H, i.e., the operations commute. If each hybrid system in the interconnection satisfies the hybrid basic conditions and H is a closed relation then the hybrid system resulting from the interconnection and clustering satisfies the hybrid basic conditions.

VII. A Distributed Hybrid Average Consensus Algorithm with Locally Coordinated Jumps

In this section, we augment a continuous-time average consensus algorithm with locally coordinated jumps that do not alter convergence to average consensus while adding some flexibility to the algorithm.

A. Incidence matrix background

We consider a hybrid average consensus algorithm with locally coordinated jumps using the framework developed in this paper. We consider the setting of $\widehat{N} \in \mathbb{N}_{\geq 2}$ agents communicating over a connected, undirected graph. Let $L \in \mathbb{N}$ denote the number of edges in the graph. For $i \in \mathbb{N}_{\leq \widehat{N}}$, let L_i denote the number of nodes connected to agent i and let $\mathcal{N}_i$ denote the set of L_i agents connected to agent i, i.e., the "neighbors" of agent i. Since the graph is undirected, we have that $\sum_{i=1}^{\widehat{N}} L_i = 2L$. To define the graph's incidence matrix $\mathcal{B} \in \mathbb{R}^{\widehat{N} \times 2L}$, let $\Sigma_0 := 0$ and $\Sigma_i := \sum_{j=1}^{i} L_j$ for $i \in \mathbb{N}_{\leq \widehat{N}}$. Then, for each $i \in \mathbb{N}_{\leq \widehat{N}}$ and each $j \in \{\Sigma_{i-1}+1, \ldots, \Sigma_i\}$, associate a unique agent $\kappa_j \in \mathcal{N}_i$ and let $\mathcal{B}^T(j,i) = 1$ and $\mathcal{B}^T(j,\kappa_j) = -1$; all other entries of $\mathcal{B}$ are equal to zero. The connectedness assumption guarantees that $\mathcal{B}\mathcal{B}^T + \mathbf{1}_{\widehat{N}}\mathbf{1}_{\widehat{N}}^T$ is positive definite. In addition, $\mathcal{B}^T \mathbf{1}_{\widehat{N}} = 0$.

B. Problem formulation and algorithm basics

The goal of the hybrid average consensus algorithm is for each agent to asymptotically determine the average of the components of a constant vector $\xi^* \in \mathbb{R}^{\widehat{N}}$, denoted $\text{avg}(\xi^*)$, using communication compatible with the incidence matrix.

Each agent is given a state $z_i \in \mathbb{R}^{L_i}$ with dynamics of the form

$$u_{i,C} \in [0,1] \qquad \dot{z}_i = u_{i,F} \tag{19a}$$

$$u_{i,D} \in \{1\} \qquad z_i^+ = g_i(u_{i,G}, u_{i,G}^+) \tag{19b}$$

and the composite state for this part of the system is $z := (z_1^T, \ldots, z_{\widehat{N}}^T)^T \in \mathbb{R}^{2L}$.

Let $K \in \mathbb{R}^{2L \times 2L}$ be a diagonal, positive definite matrix. Following [27], the interconnection constraints for the network of agents will be chosen to induce the flow dynamics

$$\dot{z} = -K\mathcal{B}^T\zeta, \qquad \zeta = \mathcal{B}z + \xi^* \tag{20}$$

where ζ_i is node i's estimate of $\text{avg}(\xi^*)$. A Lyapunov function that establishes average consensus for (20) is

$$V(z,\xi^*) := \left|\zeta - \text{avg}(\xi^*) \cdot \mathbf{1}_{\widehat{N}}\right|_2^2. \tag{21}$$

In particular, it can be shown (see [27]) that

$$\begin{aligned}&\langle \nabla V(z,\xi^*), -K\mathcal{B}^T(\mathcal{B}z+\xi^*)\rangle \\ &\leq -\lambda_{min}(K)\lambda_{min}\left(\mathcal{B}\mathcal{B}^T + \mathbf{1}_{\widehat{N}}\mathbf{1}_{\widehat{N}}^T\right)V(z,\xi^*).\end{aligned} \tag{22}$$

Therefore, average consensus is achieved without introducing any jumps, as long as the graph is connected. However, there may be interest in introducing jumps to further influence the progress toward average consensus or to provide a discrete-time communication patch to a network that loses its connectivity property due to a loss of continuous-time communication over a link or set of links.

In this direction, in addition to the dynamics of the agents, the network has an array of $\rho \in \mathbb{N}$ processing modules, where each module collects data locally from a relatively small group of connected agents. For the jth processing module, $j \in \mathbb{N}_{\leq \rho}$, we let $\mathcal{I}_j \subset \mathbb{N}_{\leq \widehat{N}}$ denote the indices of the agents influenced by the jth processing agent. Without loss of generality, we assume that each such set has cardinality $r_j \geq 2$, that the number of edges that connect agents in $\mathcal{I}_j$, denoted p_j, is positive, and that each agent in $\mathcal{I}_j$ is connected to some other agent in $\mathcal{I}_j$. For each $j \in \mathbb{N}_{\leq \rho}$, the jth processing unit has hybrid dynamics of the form

$$(\alpha_j, \tau_j) \in \mathbb{R}^{2p_j} \times [0,1] \qquad \begin{bmatrix} \dot{\alpha}_j \\ \dot{\tau}_j \end{bmatrix} \in \begin{bmatrix} 0 \\ [\underline{\omega}_j, \overline{\omega}_j] \end{bmatrix} \tag{23a}$$

$$(\alpha_j, \tau_j) \in \mathbb{R}^{2p_j} \times \{1\} \qquad \begin{bmatrix} \alpha_j^+ \\ \tau_j^+ \end{bmatrix} = \begin{bmatrix} u_j \\ 0 \end{bmatrix} \tag{23b}$$

where $0 \leq \underline{\omega}_j < \overline{\omega}_j$.

The jumps of α_j, which will determine jumps of z_i for all $i \in \mathcal{I}_j$, due to the clustering protocol described later, will be designed to not increase (and often decrease) the Lyapunov function candidate, i.e.,

$$\begin{aligned}V(z^+,\xi^*) &= \left|\zeta^+ - \text{avg}(\xi^*) \cdot \mathbf{1}_{\widehat{N}}\right|_2^2 \\ &\leq \left|\zeta - \text{avg}(\xi^*) \cdot \mathbf{1}_{\widehat{N}}\right|_2^2 = V(z,\xi^*).\end{aligned} \tag{24}$$

The processing modules need to be able to make such a choice for ζ^+ without knowing $\text{avg}(\xi^*)$.

The jth processing module will update the sub-components of z_i, $i \in \mathcal{I}_j$, that correspond to the edges that connect agents in the set $\mathcal{I}_j$. Given the definition of ζ in (20), we see that, among all of the entries of ζ, these sub-components appear only in the entries ζ_i that satisfy $i \in \mathcal{I}_j$. We use $\widehat{\zeta}_j \in \mathbb{R}^{r_j}$ for the components of ζ whose indices belong to $\mathcal{I}_j$, $\widehat{z}_j \in \mathbb{R}^{2p_j}$ for the z variables that correspond to edges that connect agents in $\mathcal{I}_j$, $\widehat{\mathcal{B}}_j$ to be the rows and columns of $\mathcal{B}$ that relate $\widehat{z}_j$ to $\widehat{\zeta}_j$, and $\widehat{v}_j$ to be such that

$$\widehat{\zeta}_j = \widehat{\mathcal{B}}_j \widehat{z}_j + \widehat{v}_j. \tag{25}$$

We then pick u_j in (23b) to satisfy

$$u_j = -\widehat{\mathcal{B}}_j^{\dagger} \widehat{v}_j \tag{26}$$

where $\widehat{\mathcal{B}}_j^{\dagger}$ denotes the Moore-Penrose pseudo-inverse of $\widehat{\mathcal{B}}_j$. We now explain why this choice has the desired effect when $\alpha_j^+ = u_j = \widehat{z}_j^+$.

Since $\mathbf{1}_{\widehat{N}}^T \mathcal{B} = 0$, any change to z cannot change the average of ζ. Then, since changes to $\widehat{z}_j$ change only $\widehat{\zeta}_j$ and not the other components of ζ, any changes to $\widehat{z}_j$ cannot change the average of $\widehat{\zeta}_j$. Next, we note that

$$\begin{aligned}&\left|\zeta^+ - \text{avg}(\xi^*) \cdot \mathbf{1}_{\widehat{N}}\right|_2^2 - \left|\zeta - \text{avg}(\xi^*) \cdot \mathbf{1}_{\widehat{N}}\right|_2^2 \\ &= \left|\widehat{\zeta}_j^+ - \text{avg}(\xi^*) \cdot \mathbf{1}_{r_j}\right|_2^2 - \left|\widehat{\zeta}_j - \text{avg}(\xi^*) \cdot \mathbf{1}_{r_j}\right|_2^2.\end{aligned} \tag{27}$$

Now suppose that the processing module achieves

$$\left|\widehat{\zeta}_j^+ - \mathrm{avg}(\widehat{\zeta}_j)\cdot \mathbf{1}_{r_j}\right|_2^2 \le \left|\widehat{\zeta}_j - \mathrm{avg}(\widehat{\zeta}_j)\cdot \mathbf{1}_{r_j}\right|_2^2. \tag{28}$$

Then note that

$$\begin{aligned}
&\left|\widehat{\zeta}_j^+ - \mathrm{avg}(\xi^*)\cdot \mathbf{1}_{r_j}\right|_2^2 \\
&= \left|\widehat{\zeta}_j^+ - \mathrm{avg}(\widehat{\zeta}_j)\cdot \mathbf{1}_{r_j} + \mathrm{avg}(\widehat{\zeta}_j)\cdot \mathbf{1}_{r_j} - \mathrm{avg}(\xi^*)\cdot \mathbf{1}_{r}\right|_2^2 \\
&= \left|\widehat{\zeta}_j^+ - \mathrm{avg}(\widehat{\zeta}_j)\cdot \mathbf{1}_{r_j}\right|^2 + r_j\left(\mathrm{avg}(\widehat{\zeta}_j) - \mathrm{avg}(\xi^*)\right)^2 \\
&\le \left|\widehat{\zeta}_j - \mathrm{avg}(\widehat{\zeta}_j)\cdot \mathbf{1}_{r_j}\right|^2 + r_j\left(\mathrm{avg}(\widehat{\zeta}_j) - \mathrm{avg}(\xi^*)\right)^2 \\
&= \left|\widehat{\zeta}_j - \mathrm{avg}(\xi^*)\cdot \mathbf{1}_{r_j}\right|_2^2
\end{aligned}$$

thereby making the quantity in (27) not positive, thereby achieving (24).

Motivated by the desire to satisfy (28) and using (25), we pick $u_j = \widehat{z}_j^+ = \alpha_j^+$ to be the minimum norm solution from the set of values u_j that minimize

$$\left|\widehat{\mathcal{B}}_j u_j + \widehat{v}_j - \mathrm{avg}(\widehat{\zeta}_j)\cdot \mathbf{1}_{r_j}\right|_2^2$$

which is (26) since it can be verified that $\widehat{\mathcal{B}}_j^T \mathbf{1}_{r_j} = 0$ and since, in general, $A^\dagger = (A^T A)^\dagger A^T$; see [28, Theorem 3.8.1].

C. Clustering protocol

The overall system involves the interconnection of $\widehat{N} + \rho$ hybrid systems where ρ is the number of processing modules that are used. The clustering protocol imposed is that given in Example 1 item 5) with $N_1 := \widehat{N}$, $N_2 := \rho$ and, for $i \in \left\{\widehat{N}+1, \ldots, \widehat{N}+\rho\right\}$, $\mathcal{I}_i \subset \mathbb{N}_{\le \widehat{N}}$ is the set of agents that the ith processing module influences.

D. Interconnection constraints and data flow

The interconnection constraints are such that, in the model (23), u_j is given as in (26) and, in the models (19):

1) the inputs $u_{i,F}$ in (19a) are such that the flow dynamics for the partial composite state z are given by (20);
2) $u_{i,C} = u_{i,D} = \max_{\{j: i\in\mathcal{I}_j\}} \tau_j$ or, if there is no such j then $u_{i,C} = u_{i,D} = 0$.
3) the inputs $u_{i,G}$ in (19b) are such that z_i updates via the update z_{ij}^+, the latter being the update for z_i that corresponds to the update α_j^+ in the state of the processing module j described above (when $i \in \mathcal{I}_j$). The appropriate module can be determined by recognizing which (if any) module with index j such that $i \in \mathcal{I}_j$ satisfies $\tau_j^+ - \tau_j \le -0.5$. There will be, at most, just one such index j, due to the clustering protocol specification of the previous subsection. We note that z_i^+ depends on both α^+ and τ^+, thereby motivating our use of u^+ in the model (1b) and $u_{i,G}^+$ in the model (19b).

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