On the Feasibility and Continuity of Feedback Controllers Defined by Multiple Control Barrier Functions

Axton Isaly, Masoumeh Ghanbarpour, Ricardo G. Sanfelice, Warren E. Dixon

***Abstract*—Control barrier functions are a popular method for encoding safety specifications for dynamical systems. In this paper, a notion of control barrier function is defined that permits vector-valued barrier functions and flow constraints involving both the state and the control input. Control barrier functions induce constraints on the control input that, when satisfied, guarantee the forward invariance of a safe set of states. The constraints are enforced using a pointwise-optimal feedback controller. Sufficient conditions for the continuity of the controller are given. The existence of a control barrier function is defined to be equivalent to the feasibility of the optimal feedback controller. Polynomial optimization problems based on sums of squares are formulated that can be used to certify that a given function is a control barrier function. An example of the control barrier function design procedure is presented illustrating the process of formulation, synthesis, and verification.**

I. Introduction

The use of control barrier functions (CBF) to synthesize feedback controllers that render sets of states forward invariant has recently gained significant interest because of the tight relationship between forward invariance and safety. The CBF literature has demonstrated that CBFs are a practical method for enforcing complex safety specifications defined by multiple, sometimes conflicting requirements such as obstacle avoidance, shifting goal locations, dynamic constraints, and control input limitations [1]–[4]. In practice, these specifications are often described using multiple CBFs (e.g., [3] and [1, Sec. V]), whereas the majority of theoretical results are developed for scalar barrier functions. While it is possible to combine multiple barrier functions into a scalar one using max and min operations, as in works like [3] and [5], the resulting functions are generally nonsmooth, leading to discontinuous controllers. A framework for studying forward invariance with multiple barrier functions was developed in [6] in the context of uncontrolled systems. For controlled systems, the conditions therein can be interpreted as constraints on the control input that can be enforced using optimization-based controllers; see [7, Ch. 11]. The constraints define a set of safety-ensuring control inputs. Enforcing multiple input constraints defined by multiple continuously differentiable CBF candidates is a promising way to obtain control laws that are continuous functions of the state. This paper aims to augment the existing body of practical work for CBFs by developing a framework for solving problems with multiple CBFs that is cohesive throughout the process of problem formulation, controller synthesis, and feasibility verification.

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A. Feasibility

Traditionally, a CBF is defined to guarantee that a safety-ensuring controller exists, meaning that all objectives in the safety specification can be met simultaneously. However, tools for verifying that a given function is a CBF are not fully developed. While analytical conditions exist to determine whether a scalar-valued function is a CBF (cf. [8], [9, Prop. 1]), the problem is significantly more challenging in the presence of multiple CBFs. In general, a CBF candidate defines a set of constraints in the decision variable (control input) that vary with an external parameter (the state of the dynamical system), and it must be verified that control inputs satisfying the constraints exist for all states in a given set. This verification should be done during the design phase so that controllers are certified as feasible before deployment. The authors of [10] leverage a tool for checking that multiple constraints have at least one feasible solution at a particular point in the state space, but it is not clear how to verify this property on a given (uncountable) set of states. One method to ensure feasibility is by adding slack variables or similar relaxations to the optimization problem at the cost of losing safety guarantees [2], [11]. The authors in [11] use slack variables to ensure feasibility only on the interior of the safe set, while still enforcing conditions on the boundary of the safe set that guarantee forward invariance. However, the slack variable method does not constrain the control input at points in the interior of the safe set, which permits trajectories to approach the boundary of the safe set with high velocity. Aggressive control action must then be used to prevent the trajectory from exiting the safe set. A more gradual transition to an invariance-ensuring control can be designed by removing slack variables and adding a user-prescribed performance function that constrains the control input at points in the interior of the set, although the feasibility problem becomes more challenging.

To address the feasibility problem, we use sum of squares (SoS) programming, which requires the more restrictive assumptions that the constraints defining the feasible set are polynomials and affine in the control input. SoS programming can be used to verify that given polynomials are nonnegative on a subset of their domain [12], [13]. Our technique verifies feasibility on level-sets of a given function (typically the CBF candidate), which is useful for safe synthesis and computationally simpler than techniques that search simultaneously for a controller and CBF (or control Lyapunov function) as in [14]–[16]. The SoS problems in these works depend on conditions that are bilinear in the decision variables, leading to complex iterative procedures. In [15], an iterative procedure was developed to search for a scalar CBF defining a safe set that was in the complement of given unsafe regions. The safe set was rendered forward invariant by a feasible controller. Iterative techniques are valuable when a CBF candidate is unavailable, whereas our approach is targeted toward verification that a given candidate is a CBF. Our approach has broad applications for verifying that an optimization-based controller is feasible before deployment.

B. Continuity

Optimization-based control laws are a natural way to implement the control input constraints defined by CBFs. For many common classes of dynamical systems, the input constraints are convex or affine in the control input, leading to optimization problems that are convex or quadratic programs. As mentioned in Section I-A, the control laws are parametric optimization problems that vary with the state of the dynamical system. Continuous control laws simplify analysis and, in many cases, provide robustness properties to the closed-loop system. The vast majority of CBF literature restricts attention to parametric optimization problems with one or two constraints, for which results certifying local Lipschitz continuity are available [17], [18]. Other works have arbitrary numbers of constraints but do not study the continuity of the control laws [5], [11]. The authors of [19, Thm. 3] provide a continuity result for a quadratic program with an arbitrary number of constraints, but the result requires information about the set of active constraints at the minimizing value. Since the active set changes with the state, it can be difficult to make global conclusions about the active set.

Berge's maximum theorem is an effective method for analyzing the continuity of parameterized optimization problems with arbitrary numbers of constraints. Berge's maximum theorem does not require information about the active constraint set, but it does require the feasible set to be compact. It is widely accepted in the parametric optimization literature that compactness of the feasible set can be replaced with uniform compactness in the parameter space of the level sets of the cost function [20], [21]. Interestingly, uniform compactness holds for the majority of convex optimization problems. Using this observation, we are able to obtain general results certifying the continuity of optimization-based control laws.

C. Contributions

In Section II, we define a notion of vector-valued CBF for continuous-time differential inclusions with constraints on the state and control input. Our construction carefully considers the case where multiple CBF candidates define the safe set. Differential inclusions are useful for robust control applications as they can model uncertainty in the dynamics, and the constraints capture state-dependent input constraints as a special case. In Section III, we show that forward (pre-)invariance of the safe set defined by a CBF is guaranteed using control inputs from the safety-ensuring set. We also provide conditions for when the safe set is asymptotically stable. Our primary notion of CBF allows for continuous control laws, whereas the majority of literature imposes the stronger condition of local Lipschitz continuity, which is more challenging to verify for optimization-based control laws. In Section IV, we provide sufficient conditions under which the CBF-induced pointwise optimal control law is continuous. These conditions generalize available results by allowing broader classes of cost functions and not requiring the feasible set of control inputs to be compact. In Section V, we develop SoS optimization tools that can be used to verify that a CBF candidate is a CBF. Methods for applying the SoS technique, even when the actual dynamics of the system are not polynomial, are discussed. An example is presented in Section VI illustrating the process of problem formulation, feasibility verification, and control synthesis for a system with uncertain, non-polynomial dynamics.

Relative to our preliminary work in [22], we include a second notion of CBF called a tangent-cone CBF (t-CBF) that uses alternative conditions for forward invariance which are comparable to notions of CBF in the literature based on Nagumo's theorem. The notion of t-CBF helps solve a complication in [22] where control inputs are required to be selected based on the tangent cone to the safe set. However, the notion of t-CBF requires control laws that are locally Lipschitz whereas our standard notion of CBF only requires continuous control laws. We include a simple example to aid in understanding our results for forward invariance. Our result for asymptotic stability is an addition over [22], and, to the best of our knowledge, is the first asymptotic stability result for multiple CBFs in the literature. We improve our results for continuity of optimization-based control laws by removing a redundant assumption from [22, Lem. 2]. This paper includes proofs that were excluded from [22] due to space limitations. The detailed example in Section VI is also a new addition.

Forward invariance with multiple CBFs is studied in [11] using a tangent-cone-based approach. However, the results require the safe set to be compact, and require the feasible set of control inputs to be compact. Additionally, the definition of a CBF in [11] is problematic when multiple CBFs are present as it does not require the existence of control inputs that simultaneously satisfy all of the CBF-induced constraints. Notably, [11] allows for control laws that are only Lebesgue measurable by taking an alternative analytical approach focusing on the differential inclusion defined by all the possible safety-ensuring control inputs.

It should be noted that the method for feasibility verification in Section V is complimentary to works that seek persistent feasibility such as [23] and [24]. The latter methods add new CBF candidates to the problem that are designed to modify the safe set by removing states where the input constraints are infeasible. The methods in this paper could be used to determine whether the newly defined CBF candidate is a CBF, while the methods from [23] and [24] would be useful if it cannot be verified that a given candidate is a CBF.

D. Preliminaries

For vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $|x|$ denotes the Euclidean norm, $(x, y) \triangleq \left[x^T, y^T\right]^T$, and $|x|_A \triangleq \inf_{z \in A} |x - z|$ denotes the distance of x from the set $A \subset \mathbb{R}^n$. The shorthand $[d] \triangleq \{1, 2, \ldots, d\}$ is used, and $\mathbb{B}^n$ denotes the n-dimensional unit ball. Given a function $B : \mathbb{R}^n \to \mathbb{R}^d$, the components are indexed as $B(x) \triangleq (B_1(x), B_2(x), \ldots, B_d(x))$ and the inequality $B(x) \leq 0$ means that $B_i(x) \leq 0$ for all $i \in [d]$. For a set $A \subset \mathbb{R}^n$, the notation ∂A denotes its boundary, $\overline{A}$ its closure, $\mathrm{Int}(A)$ its interior, and $U(A)$ denotes an open neighborhood around A. A set $C \subset A$ is relatively closed in A if $C = A \cap \overline{C}$.

Given a set $X \subset \mathbb{R}^n$, a set-valued mapping $M : X \rightrightarrows \mathbb{R}^m$ associates every point $x \in X$ with a set $M(x) \subset \mathbb{R}^m$. The mapping M is locally bounded if, for every $x \in X$, there exists a neighborhood $U_X(x) \triangleq U(x) \cap X$ such that $M(U_X(x))$ is bounded, M is outer semicontinuous if $\mathrm{Graph}_X(M) \triangleq \{(x, u) \in X \times \mathbb{R}^m : u \in M(x)\}$ is relatively closed in $X \times \mathbb{R}^m$, and M is lower semicontinuous if, for any open set $\mathcal{G} \subset \mathbb{R}^m$, the inverse image $M^{-1}(\mathcal{G}) \triangleq \{x \in X : M(x) \cap \mathcal{G} \neq \emptyset\}$ is open.

II. Control Barrier Functions

Consider a constrained control differential inclusion (F, C_u) with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ modeled by

$$\dot{x} \in F(x, u) \qquad (x, u) \in C_u \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the set-valued flow map, and $C_u \subset \mathbb{R}^n \times \mathbb{R}^m$ is the flow set. The set-valued nature of the differential inclusion in (1) can model uncertainty by allowing trajectories to move in a variety of directions for a given state and control input (x, u). Differential inclusions are useful for robust control design because ensuring safety requires every possible trajectory to remain safe. This work therefore generalizes work on robust CBFs such as [25]. To facilitate the subsequent development, let

$$\Pi(C_u) \triangleq \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \ s.t. \ (x, u) \in C_u\} \tag{2}$$

denote the set of all states for which flowing is allowed, and for each $x \in \Pi(C_u)$ let

$$\Psi(x) \triangleq \{u \in \mathbb{R}^m : (x, u) \in C_u\} \tag{3}$$

denote the set of admissible control inputs at each state.

CBFs are defined to guarantee the existence of control inputs that ensure forward invariance (i.e., safety) of a given set of states $\mathcal{S} \subset \Pi(C_u)$. Compared to works such as [2], we use a notion of CBF that accommodates safe sets defined by multiple scalar functions. Defining a CBF in this case requires special care because there are multiple constraints on the control input that must be satisfied simultaneously. For notational convenience, as in [6], we use vector-valued functions $B : \mathbb{R}^n \to \mathbb{R}^d$ to represent multiple CBFs. Our development is based on the work for closed-loop hybrid systems in [6] and for hybrid systems with inputs in [7], which we adapt for the open-loop continuous-time dynamics in (1).

Definition 1. A vector-valued function $B : \mathbb{R}^n \to \mathbb{R}^d$ is called a CBF candidate defining the safe set $\mathcal{S} \subset \Pi(C_u)$ if

$$\mathcal{S} = \{x \in \Pi(C_u) : B(x) \leq 0\}.$$

Also define $\mathcal{S}_i \triangleq \{x \in \mathbb{R}^n : B_i(x) \leq 0\}$ for every $i \in [d]$.

We restrict our attention to continuously differentiable CBF candidates because of advantages they offer towards synthesizing continuous controllers. Given a continuously differentiable CBF candidate, define a function $\Gamma : C_u \to \mathbb{R}^d$ such that the i-th component is

$$\Gamma_i(x, u) \triangleq \sup_{f \in F(x,u)} \langle \nabla B_i(x), f \rangle, \quad \forall (x, u) \in C_u. \tag{4}$$

The value of $\Gamma_i(x, u)$ represents the worst-case growth of $B_i(x)$ for any possible direction of flow in the set-valued map $F(x, u)$ defining the control system in (1). When $F(x, u)$ is nonempty and bounded, the supremum in (4) is finite. Thus, the following mild assumption is imposed to ensure that Γ is well-defined. Allowing for an unbounded flow map would lead to solutions that flow arbitrarily fast, which is not physically meaningful.

Assumption 1. The set $F(x, u)$ is nonempty and bounded for every $(x, u) \in C_u$.

We also introduce the primary design parameter in the form of a performance function γ, which is used to define a set of control inputs that constrain the worst-case growth function Γ according to conditions derived from [6] that guarantee forward invariance of the safe set $\mathcal{S}$. We impose the following assumption. Figure 1 illustrates a safe set defined by multiple CBFs showing the regions on which γ_i is constrained.

Assumption 2. The function $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$ is such that, for each $i \in [d]$, $\gamma_i(x) \geq 0$ for all $x \in (U(M_i) \backslash \mathcal{S}_i) \cap \Pi(C_u)$, where $M_i \triangleq \{x \in \partial\mathcal{S} : B_i(x) = 0\}$.

Definition 2. Let (F, C_u) satisfy Assumption 1. A continuously differentiable CBF candidate $B : \mathbb{R}^n \to \mathbb{R}^d$ defining the set $\mathcal{S} \subset \Pi(C_u)$ is a CBF for (F, C_u) and $\mathcal{S}$ on a set $\mathcal{O} \subset \Pi(C_u)$ with respect to a function $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$

if there exists a neighborhood of the boundary of $\mathcal{S}$ such that $U(\partial\mathcal{S}) \cap \Pi(C_u) \subset \mathcal{O}$, γ satisfies Assumption 2, and the set

$$K_c(x) \triangleq \{u \in \Psi(x) : \Gamma(x, u) \leq -\gamma(x)\} \tag{5}$$

is nonempty for every $x \in \mathcal{O}$.

The mapping K_c defines, at each state $x \in \mathcal{O}$, a set of safety-ensuring control inputs. We will subsequently show that control inputs selected from K_c (i.e., mappings $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ for which $\kappa(x) \in K_c(x)$ for every $x \in \mathcal{O}$) ensure the forward (pre-)invariance of the safe set $\mathcal{S}$.

Remark 1. The performance function γ is a relaxation of the conditions in works like [2], which require that $\gamma_i(x) = \alpha(B_i(x))$ where α is an extended class $\mathcal{K}$ function (i.e., strictly increasing with $\alpha(0) = 0$). Removing the dependency of the performance function on the CBF candidate provides some additional design flexibility. Additionally, extended class $\mathcal{K}$ functions are required to be strictly positive outside of the safe set $\mathcal{S}$. Section III shows that being strictly positive outside the safe set is stronger than required for forward invariance (Theorem 2) but does lead to asymptotic stability of the safe set (Theorem 3). Whenever the set $\mathcal{D} \subset \mathbb{R}^n$ in [2, Defn. 5] contains a neighborhood of the safe set, any function that is a zeroing CBF according to [2, Defn. 5] is also a CBF on $\mathcal{D}$ for some function γ according to Definition 2.

Remark 2. In Definition 2, the performance function γ is not required to be continuous. The least conservative selection of γ that satisfies Assumption 2 is $\gamma_i(x) \triangleq 0$ for $x \in (U(M_i) \setminus \mathcal{S}_i) \cap \Pi(C_u)$ and $\gamma_i(x) \triangleq -\infty$ otherwise, in which case $K_c(x) = \Psi(x)$ outside of $(U(M_i) \setminus \mathcal{S}_i) \cap \Pi(C_u)$, where $\mathcal{S}_i$ is introduced in Definition 1. However, discontinuous choices of γ will lead to the mapping K_c having poor regularity. As we find in Section IV, using a continuous γ facilitates the systematic design of continuous, safety-ensuring control laws.

A. Tangent-Cone Conditions

Assumption 2 imposes conditions on the function γ that must hold on a region outside the set $\mathcal{S}$. In contrast to more common notions of CBF based on Nagumo's theorem (cf. [2]), the conditions in Definition 2 apply to a more general class of systems and are valid even if the gradients of the component CBF candidates are degenerate (i.e., $\nabla B_i(x) = 0$ for some $x \in M_i$). It is useful to have conditions that, like Nagumo's theorem, restrict γ only on the boundary of $\mathcal{S}$, which we provide in this section based on [6, Thm. 2]. The following assumption is known as a transversality condition, and reduces to the assumption that $\nabla B(x) \neq 0$ for all $x \in \partial\mathcal{S}_e \cap \Pi(C_u)$ when B is scalar, where $\mathcal{S}_e \triangleq \{x \in \mathbb{R}^n : B(x) \leq 0\}$.

Assumption 3. For every $x \in \partial\mathcal{S}_e \cap \Pi(C_u)$, there exists $v \in \mathbb{R}^n$ such that $\langle \nabla B_i(x), v\rangle < 0$ for every $i \in [d]$ such that $B_i(x) = 0$.

We also impose stronger assumptions on the regularity of the flow map F. The assumption below is more restrictive than the assumption used in [6, Thm. 2], and is used here to simplify the development.

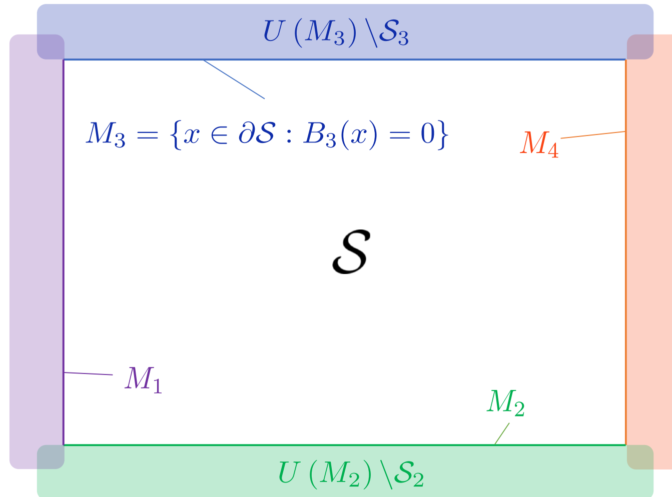

Figure 1. Illustration of the safe set for Example 1 showing the regions where the performance function γ is constrained according to Assumptions 2 and 5. When multiple CBFs are present, the function γ_i need only be constrained on the region $U(M_i) \setminus \mathcal{S}_i$, which is a region outside the safe set nearby where B_i defines the boundary of the safe set $\mathcal{S}$. Assumption 5 constrains γ_i only on M_i under stricter assumptions.

Assumption 4. The set-valued mapping $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally Lipschitz on $(U(\partial\mathcal{S}) \times \mathbb{R}^m) \cap C_u \triangleq A$ in the sense that, for every compact set $K \subset A$, there exists a constant $L > 0$ such that, for all $z_1, z_2 \in K$,

$$F(z_1) \subset F(z_2) + L\,|z_1 - z_2|\,\mathbb{B}^n.$$

In this case, we relax Assumption 2 as follows, and use it to define an alternative notion of CBF.

Assumption 5. The function $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$ is such that, for each $i \in [d]$, $\gamma_i(x) \geq 0$ for all $x \in M_i$.

Definition 3. Let the data (F, C_u) of (1) and the continuously differentiable CBF candidate $B : \mathbb{R}^n \to \mathbb{R}^d$ defining the set $\mathcal{S}$ satisfy Assumptions 1 and 3. The candidate B is a tangent-cone CBF (t-CBF) for (F, C_u) and $\mathcal{S}$ on a set $\mathcal{O} \subset \Pi(C_u)$ with respect to a function $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$ if $U(\partial\mathcal{S}) \cap \Pi(C_u) \subset \mathcal{O}$, γ satisfies Assumption 5, and $K_c(x)$ in (5) is nonempty for every $x \in \mathcal{O}$.

The key property of t-CBFs is that control inputs selected from K_c ensure that vectors in the closed-loop dynamics lie in the tangent cone to the safe set [26, Def. 5.12]. The following straightforward corollary of Lemma 3 in [6] shows this result. The tangent cone of a set $\mathcal{S} \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is defined as $T_{\mathcal{S}}(x) \triangleq \{v \in \mathbb{R}^n : \liminf_{h\to 0^+} |x + hv|_{\mathcal{S}}/h = 0\}$.

Lemma 1. *Suppose B is a t-CBF for (F, C_u) and $\mathcal{S} = \{x \in \Pi(C_u) : B(x) \leq 0\}$ on a set $\mathcal{O} \subset \Pi(C_u)$ with respect to a function $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$. For a given set $M \subset \mathbb{R}^n$, suppose that $\mathcal{S} = \{x \in M : B(x) \leq 0\}$ and let $x \in \mathcal{S} \cap$ Int(M). If $u \in K_c(x)$, then $f \in T_{\mathcal{S}}(x)$ for every $f \in F(x, u)$.*

Proof: If $x \in$ Int $(\mathcal{S})$, then by the definition of the tangent cone, $T_{\mathcal{S}}(x) = \mathbb{R}^n$. Thus, for $u \in \Psi(x)$, $F(x, u)$ is nonempty so the claim that $f \in T_{\mathcal{S}}(x)$ for every $f \in F(x, u)$ is trivial.

Assume $x \in \partial\mathcal{S}$ and $u \in K_c(x)$. Using Assumption 3, Lemma 3 in [6] shows that $T_S(x) = \{v \in \mathbb{R}^n : \langle \nabla B_i(x), v\rangle \leq 0,\ \forall i \in I_x\}$, where $I_x = \{i \in [d] : B_i(x) = 0\} = \{i \in [d] : x \in M_i\}$. For each $i \in I_x$, Assumption 5 implies that $\Gamma_i(x,u) \leq 0$. Using the definition of Γ_i, it follows that if $f \in F(x,u)$, $\langle \nabla B_i(x), f\rangle \leq 0$, which completes the proof. ■

The following example demonstrates some of the notions introduced above. It will be used throughout the next section to illustrate other key points.

Example 1. Consider the dynamical system in (1) defined by $F(x,u) = u$ and $C_u = \{(x,u) \in \mathbb{R}^2 \times \mathbb{R}^2 : \max\{|x_1|, |x_2|\} \leq \bar{x}\}$, where $\bar{x} > 0$. The flow set represents hard constraints on the system that, by the definition of solutions, cannot be violated. For example, the set $\Pi(C_u) = \{x \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} \leq \bar{x}\}$ may represent the walls of a square room centered at the origin in $\mathbb{R}^2$. In practice, it is desirable to prevent agents from closely approaching the boundary of $\Pi(C_u)$, so that the safe set for this example may be $\mathcal{S} \triangleq \{x \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} \leq \bar{x}_s\}$ for $0 < \bar{x}_s < \bar{x}$. The safe set could be defined by the CBF candidate $B : \mathbb{R}^2 \to \mathbb{R}^4$ with components $B_1(x) = -x_1 - \bar{x}_s$, $B_2(x) = -x_2 - \bar{x}_s$, $B_3(x) = x_2 - \bar{x}_s$, and $B_4(x) = x_1 - \bar{x}_s$ (see Figure 1). Note that it is difficult to define $\mathcal{S}$ using a single continuously differentiable function.

We have $\Gamma(x,u) = (-u_1, -u_2, u_2, u_1)$ for every $(x,u) \in C_u$. It is common to choose $\gamma_i \triangleq B_i$ for each $i \in [4]$, which always satisfies both Assumptions 2 and 5 since $B_i(x) > 0$ if $x \notin \mathcal{S}_i$, and, clearly, $B_i(x) = 0$ for $x \in M_i = \{x \in \partial\mathcal{S} : B_i(x) = 0\}$. With this choice of γ, it can be found that $K_c(x) = \{u \in \mathbb{R}^2 : |u_j + x_j| \leq \bar{x}_s,\ j \in \{1,2\}\}$ for all $x \in \Pi(C_u)$. The set $K_c(x)$ is clearly nonempty for every $x \in \Pi(C_u)$, so that B is a CBF for (F, C_u) and $\mathcal{S}$ on $\Pi(C_u)$ with respect to γ. Note that B is also a t-CBF. In particular, the vector $v = -x$ satisfies Assumption 3 for every $x \in \partial\mathcal{S}_e \cap \Pi(C_u)$.

III. Forward (pre-)Invariance Using Selections of K_c

A. Forward pre-Invariance

We next relate the notion of CBF in Definition 2 to forward pre-invariance of the safe set $\mathcal{S} = \{x \in \Pi(C_u) : B(x) \leq 0\}$. Consider a closed-loop system (F_{cl}, C) defined by (F, C_u) in (1) and a control law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ as

$$\dot{x} \in F_{cl}(x) \qquad x \in \overline{C} \tag{6}$$

where $C \triangleq \{x \in \mathbb{R}^n : (x, \kappa(x)) \in C_u\} = \{x \in \mathbb{R}^n : \kappa(x) \in \Psi(x)\}$, $F_{cl}(x) \triangleq F(x, \kappa(x))$ if $x \in \overline{C}$, and $F_{cl}(x) \triangleq \emptyset$ if $x \notin \overline{C}$. A solution to (F_{cl}, C) starting from $\phi_0 \in C$ is a locally absolutely continuous function $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ such that $\phi(0) = \phi_0$, $\phi(t) \in C$ for all $t \in \text{Int}(\operatorname{dom}\phi)$, and $\dot{\phi}(t) \in F_{cl}(\phi(t))$ for almost all $t \in \operatorname{dom}\phi$, where $\operatorname{dom}\phi \subset [0,\infty)$ is an interval containing zero. A solution is said to be complete if $\operatorname{dom}\phi$ is unbounded, and it is maximal if there is no solution ϕ' such that $\phi(t) = \phi'(t)$ for all $t \in \operatorname{dom}\phi$ with $\operatorname{dom}\phi$ a proper subset of $\operatorname{dom}\phi'$. The following notions of forward invariance are adapted from [6] for the case of constrained differential inclusions.

Definition 4. A set $\mathcal{S} \subset C$ is forward pre-invariant for (F_{cl}, C) if, for each $\phi_0 \in \mathcal{S}$ and each maximal solution ϕ starting from ϕ_0, $\phi(t) \in \mathcal{S}$ for all $t \in \operatorname{dom}\phi$. The set $\mathcal{S}$ is forward invariant for (F_{cl}, C) if it is forward pre-invariant and, for each $\phi_0 \in \mathcal{S}$, every maximal solution ϕ starting from ϕ_0 is complete.

Note that the flow set C is always forward pre-invariant for (6) but not necessarily forward invariant. The following assumption and lemma relate regularity conditions imposed on the control system (F, C_u) in (1) to common regularity conditions for the closed-loop system that will be used in the next two theorems. When the dynamics are single-valued, outer semicontinuity of the dynamics is equivalent to continuity.

Assumption 6. Given the data (F, C_u) of (1), the following hold:
A) The flow map $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semicontinuous, and has nonempty and convex values on C_u.
B) The flow set C_u is a closed subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Lemma 2. *Given the data (F, C_u) defining the control system in (1), suppose $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ and $C = \{x \in \mathbb{R}^n : (x, \kappa(x)) \in C_u\}$ are such that κ is continuous on $\overline{C}$. If Assumption 6A) holds, then $F_{cl} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semicontinuous, and has nonempty and convex images on C. If Assumption 6B) holds, then C is a closed subset of $\mathbb{R}^n$. Moreover, if $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally Lipschitz on $A \subset C \times \mathbb{R}^m$ and κ is locally Lipschitz on $\Pi(A)$, then F_{cl} is locally Lipschitz on $\Pi(A)$.*

Proof: See Appendix A. ■

Remark 3. Relative to our preliminary work in [22], we make a correction in Lemma 2 by assuming that the controller κ is continuous on the closure of C. This is to prevent an issue involving the fact that C may not be closed even if C_u is closed. The correction is also reflected in the subsequent Theorems 1 and 2 by the additional assumption that the closed-loop control law is continuous on a closed set. In practice, the modification is minor as the tools for verifying feasibility and continuity of closed-loop controllers in Section V already work with closed sets.

The following result provides conditions under which continuous controllers selected from the mapping K_c in (5) render the set $\mathcal{S}$ forward pre-invariant for the closed-loop dynamics in (6). In Section IV we provide a strategy for designing continuous safety-ensuring controllers using optimization.

Theorem 1. ***(Forward pre-Invariance)*** *Let Assumption 6A) hold for the control system in (1) with data (F, C_u) and suppose $B : \mathbb{R}^n \to \mathbb{R}^d$ is either a CBF or a t-CBF for (F, C_u) and $\mathcal{S} \subset \Pi(C_u)$ on $\mathcal{O} \subset \Pi(C_u)$ with respect to $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$. Let the control law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\overline{\mathcal{O}}$ with $\kappa(x) \in K_c(x)$ for all $x \in \mathcal{O}$ and $\kappa(x) \in \Psi(x)$ for all $x \in \mathcal{S}$, where K_c is defined in (5) and Ψ is defined in (3). When B is exclusively a t-CBF, assume* *additionally that Assumptions 4 and 6B) hold, and κ is locally Lipschitz on $\mathcal{O}$. If $\mathcal{S} = \{x \in \Pi(C_u) : B(x) \leq 0\}$ is closed*[1] *in $\mathbb{R}^n$, then $\mathcal{S}$ is forward pre-invariant for the closed-loop system (F_{cl}, C) defined in (6) by (F, C_u) and κ.*

Proof: Given a neighborhood of $\partial\mathcal{S}$ such that $\overline{U(\partial\mathcal{S})} \cap \Pi(C_u) \subset \mathcal{O}$, which exists by Definition 2, we consider the restriction of the control system (F, C_u) to flow only on the set $\tilde{C}_u \triangleq (\overline{U(\partial\mathcal{S})} \times \mathbb{R}^m) \cap C_u$. Using the definition of $\Pi(\cdot)$, we have $\Pi(\tilde{C}_u) = \overline{U(\partial\mathcal{S})} \cap \Pi(C_u)$. Because B is either a CBF or t-CBF, $K_c(x) \neq \emptyset$ for all $x \in \Pi(\tilde{C}_u) \subset \mathcal{O}$. The closed-loop system $(F_{cl}, \tilde{C})$ is defined by $(F, \tilde{C}_u)$ and κ according to (1) with $\tilde{C} \triangleq \{x \in \mathbb{R}^n : (x, \kappa(x)) \in \tilde{C}_u\}$. Since $\kappa(x) \in K_c(x) \subset \Psi(x)$ for all $x \in \Pi(\tilde{C}_u)$, it follows that $\tilde{C} = \Pi(\tilde{C}_u)$. The function B is a barrier function candidate defining $\mathcal{S} \cap \tilde{C}$ [6, Defn. 3]. Using equivalences established above, it follows that $\mathcal{S} \cap \tilde{C} = \mathcal{S} \cap \overline{U(\partial\mathcal{S})} \cap \Pi(C_u) = \mathcal{S} \cap \overline{U(\partial\mathcal{S})}$, which is closed by the assumption that $\mathcal{S}$ is closed. Since κ is continuous on[2] $\mathrm{cl}(\tilde{C}) \subset \overline{\mathcal{O}}$, Lemma 2 shows that F_{cl} meets the basic assumptions in Section 2.3 of [6].

For each $i \in [d]$ and $x \in \tilde{C}$, $\langle \nabla B_i(x), f \rangle \leq -\gamma_i(x)$ for all $f \in F(x, \kappa(x))$ because $\kappa(x) \in K_c(x)$. First assume that B is a CBF. Since γ satisfies Assumption 2, we conclude that $\langle \nabla B_i(x), f \rangle \leq 0$ for all $x \in (U(M_i) \backslash \mathcal{S}_i) \cap \tilde{C}$ and $f \in F_{cl}(x)$. Since $\mathcal{S} \cap \tilde{C}$ is closed, we apply Theorem 1 in [6] to conclude that $\mathcal{S} \cap \tilde{C}$ is forward pre-invariant for $(F_{cl}, \tilde{C})$.

If B is a t-CBF, we apply Theorem 2 in [6]. Assumption 5 implies that $\langle \nabla B_i(x), f \rangle \leq 0$ for all $x \in M_i \cap \tilde{C}$ and $f \in F_{cl}(x)$, where $\tilde{C} = \overline{U(\partial\mathcal{S})} \cap \Pi(C_u)$. Since F_{cl} is locally Lipschitz on $U(\partial\mathcal{S}) \cap \tilde{C}$ via Assumption 4 and Lemma 2, condition (20) in [6] holds (see [6, Rem. 9]). Under Assumption 6B), Lemma 2 shows that $\tilde{C}$ is closed. Thus, by definition of F_{cl}, $F_{cl}(x) = \emptyset$ for $x \notin \tilde{C}$, so that condition (21) in [6] holds vacuously. Using Assumption 3, we apply [6, Thm. 2] to conclude that $\mathcal{S} \cap \tilde{C}$ is forward pre-invariant for $(F_{cl}, \tilde{C})$.

Forward pre-invariance of $\mathcal{S}$ for the unrestricted closed-loop system (F_{cl}, C) follows from the definition of forward pre-invariance, since solutions to (F_{cl}, C) starting from $\mathcal{S}$ cannot exit $\mathcal{S}$ without passing through $\tilde{C}$. Such solutions remain in $\mathcal{S}$ by forward pre-invariance of $\mathcal{S} \cap \tilde{C}$ for the restricted dynamics. ∎

When the performance function γ satisfies stronger conditions than those imposed in Assumption 2, selections of K_c, designed to enforce all of the barrier function-induced constraints, not only render $\mathcal{S}$ forward pre-invariant, but also some larger sets defined by a subset of the barrier functions. This situation is different from redefining K_c by removing some of the constraints. The result is motivated by the observation that a common selection for γ is $\gamma_i \triangleq B_i$, where by definition $B_i(x) > 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{S}_i$.

Corollary 1. *Under the assumptions of Theorem 1, assume additionally that $\mathcal{O} = \Pi(C_u)$ and $\gamma_i(x) \geq 0$ for all $x \in \mathcal{O} \backslash \mathcal{S}_i$, for each $i \in [d]$. For any index set $\mathcal{I} \subset \{1, 2, \ldots, d\}$, if the set $\mathcal{S}_\mathcal{I} \triangleq \{x \in \Pi(C_u) : B_i(x) \leq 0, \forall i \in \mathcal{I}\}$ is closed in $\mathbb{R}^n$, then $\mathcal{S}_\mathcal{I}$ is forward pre-invariant for the closed-loop system (F_{cl}, C) defined in (6) by (F, C_u) and κ.*

[1]Since B is assumed to be continuous, a sufficient condition for $\mathcal{S}$ to be closed is that $\Pi(C_u)$ is closed.

[2]In some places, we use $\mathrm{cl}(A)$ instead of $\overline{A}$ to denote the closure for aesthetic reasons.

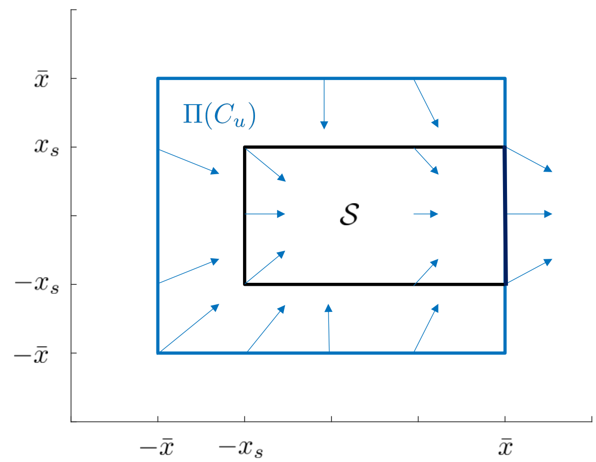

Figure 2. Modification of Example 1 where the CBF candidate $B_4(x) = x_1 - \bar{x}_s$ is removed. The vector field represents the closed-loop dynamics under a control law that renders the safe set forward pre-invariant. Trajectories starting in the right half-plane will terminate on the boundary of the flow set.

The corollary follows by applying Theorem 1 to the CBF candidate $B_\mathcal{I} : \mathbb{R}^n \to \mathbb{R}^{|\mathcal{I}|}$ defined by only the components of B in $\mathcal{I}$.

Example 2. For some applications, forward pre-invariance may not be a strong enough property. In Example 1, consider a situation where the safe set is defined by only three of the CBF candidates (see Figure 2). In this case, the set of safety-ensuring controls is given by $K_c(x) = \{u \in \mathbb{R}^2 : |u_2 + x_2| \leq \bar{x}_s,\ u_1 + x_1 \geq -\bar{x}_s\}$. An example of a continuous selection of K_c is $\kappa(x) \triangleq -x$ if $x_1 \leq 0$ and $\kappa(x) \triangleq (x_1, -x_2)$ if $x_1 > 0$. Theorem 1 shows that $\mathcal{S}$ is forward pre-invariant for the closed-loop dynamics $\dot{x} = \kappa(x)$. Figure 2 displays the closed-loop dynamics.

Forward pre-invariance implies that trajectories do not exit the safe set, but may terminate on the boundary of the safe set due to being unable to continue flowing inside the flow set. In this example, termination of flow may correspond to the agent crashing into the wall of the room. The issue occurs because portions of the boundary of $\mathcal{S} = \{x \in \Pi(C_u) : B(x) \leq 0\}$ are not defined by the CBF, but rather are defined by $\Pi(C_u)$. In this example, we should ensure that the set $\mathcal{S}$ is forward invariant.

B. Forward Invariance

The forward pre-invariance notion in Definition 4 does not guarantee that maximal solutions to the closed-loop system are complete. In addition to terminating on the boundary of the flow set as illustrated in Example 2, solutions may escape in finite time inside of $\mathcal{S}$. To select control inputs that prevent solutions from terminating on the boundary of the flow set, following from [27, Eqn. 31] we define the map

$$\Theta_{\mathcal{S}}(x) \triangleq \begin{cases} \{u \in \Psi(x) : F(x,u) \cap T_{\mathcal{S}}(x) \neq \emptyset\} \\ \qquad\qquad\qquad\qquad \text{if } x \in \partial\Pi(C_u) \cap \mathcal{S}, \\ \Psi(x) \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{7}$$

Relative to the assumptions of Theorem 1, it is notable that in the next result we assume that the controller is continuous on a set that contains the entire safe set. Doing so provides regularity of the closed-loop system with which we can establish completeness of maximal solutions.

Theorem 2. ***(Forward Invariance)*** *Let Assumption 6A) hold for the control system in (1) with data (F, C_u) and suppose $B : \mathbb{R}^n \to \mathbb{R}^d$ is either a CBF or a t-CBF for (F, C_u) and $\mathcal{S} \subset \Pi(C_u)$ on $\mathcal{O}$ with respect to γ when $K_c(x)$ in (5) is replaced with $K_c(x) \cap \Theta_{\mathcal{S}}(x)$, meaning that $K_c(x) \cap \Theta_S(x)$ is nonempty for every $x \in \mathcal{O}$. Suppose $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\overline{\mathcal{O} \cup \mathcal{S}}$ with $\kappa(x) \in K_c(x) \cap \Theta_{\mathcal{S}}(x)$ for all $x \in \mathcal{O}$ and $\kappa(x) \in \Psi(x)$ for all $x \in \mathcal{S}$. When B is exclusively a t-CBF, assume additionally that Assumptions 4 and 6B) hold, and κ is locally Lipschitz on $\mathcal{O}$. If $\mathcal{S}$ is closed and one of the following conditions hold:*

2.1) $\mathcal{S}$ is compact,

2.2) F_{cl} is bounded on $\mathcal{S}$, or

2.3) F_{cl} has linear growth on $\mathcal{S}$, namely, there exists $c > 0$ such that, for all $x \in \mathcal{S}$, $sup_{v \in F_{cl}(x)} |v| \leq c(|x| + 1)$,

then $\mathcal{S}$ is forward invariant for the closed-loop system (F_{cl}, C) defined in (6) by (F, C_u) and κ.

Proof: Forward pre-invariance of $\mathcal{S}$ for the closed-loop dynamics follows from Theorem 1. It remains to show that maximal solutions to the closed-loop system starting from $\mathcal{S}$ are complete. Using continuity of κ on $\mathcal{S}$ and Lemma 2, the map F_{cl} is outer semicontinuous and locally bounded on $\mathcal{S}$ with nonempty, convex values. Since $\kappa(x) \in \Psi(x)$ for all $x \in \mathcal{S}$ and $C = \{x \in \mathbb{R}^n : \kappa(x) \in \Psi(x)\}$, we have $\Pi(C_u) \cap \mathcal{S} = C \cap \mathcal{S}$. Because $\mathcal{S} \subset C$, we have $\partial C \cap \mathcal{S} \subset \partial\mathcal{S}$. From the definition of a CBF (or t-CBF), the set $\mathcal{O}$ contains $\partial\mathcal{S}$, from which we conclude that $\kappa(x) \in \Theta_{\mathcal{S}}(x)$ for all $x \in \partial C \cap \mathcal{S}$. Thus, Proposition 3 in [6] implies that a nontrivial flow exists from every point in $\partial C \cap \mathcal{S}$, and thus $\mathcal{S}$ is forward invariant for (F_{cl}, C) if maximal solutions cannot escape in finite time inside the set $\mathcal{S}$. Finite-time escape is eliminated by assuming Condition 1), 2), or 3) (see [28, Thm. 10.1.4] and the subsequent discussion, treating $\mathcal{S}$ as the viability domain). ∎

Due to the tangent cone condition defining the mapping $\Theta_{\mathcal{S}}$, it will generally be difficult to obtain an analytical form of (7) that can be used to make a selection from the mapping $K_c \cap \Theta_{\mathcal{S}}$ as required by Theorem 2. Note based on (7) that the complication is only present if the safe set intersects the boundary of the flow set ($\partial\Pi(C_u) \cap \mathcal{S} \neq \emptyset$). The notion of t-CBF offers a solution to the problem when a CBF candidate is available to define the portion of the boundary of $\mathcal{S}$ that intersects $\partial\Pi(C_u)$. Proposition 1 can be combined with Theorem 2 to remove the complication of selecting inputs from $K_c \cap \Theta_{\mathcal{S}}$.

Proposition 1. ***(Forward Invariance with t-CBFs)*** *Suppose $B : \mathbb{R}^n \to \mathbb{R}^d$ is a t-CBF for (F, C_u) and $\mathcal{S} \subset \Pi(C_u)$ on $\mathcal{O}$ with respect to γ. If $\mathcal{S} = \{x \in \mathbb{R}^n : B(x) \leq 0\}$, then $K_c(x) = K_c(x) \cap \Theta_{\mathcal{S}}(x)$ for every $x \in \Pi(C_u)$.*

Proof: Pick $x \in \Pi(C_u)$. The claim is trivial if $x \notin \partial\Pi(C_u) \cap \mathcal{S}$ since $K_c(x) \subset \Psi(x)$, so assume that $x \in \partial\Pi(C_u) \cap \mathcal{S}$. For $u \in K_c(x)$, applying Lemma 1 with $M = \mathbb{R}^n$, we find that $F(x,u) \cap T_{\mathcal{S}}(x) \neq \emptyset$. Thus, $u \in \Theta_{\mathcal{S}}(x)$, which completes the proof. ∎

Example 3. The problem in Example 2 can be remedied using the tools developed in this section. One solution is to add the additional CBF candidate $B_4(x) = x_1 - \bar{x}_s$, in which case $\partial\Pi(C_u) \cap \mathcal{S} = \emptyset$ and $K_c \cap \Theta_{\mathcal{S}} = K_c$ so that Theorem 2 can be applied because the original candidate B is both a CBF and a t-CBF. If one wishes to describe the flow set exactly, then it is possible to choose $B_4(x) = x_1 - \bar{x}$. In this case, $\partial\Pi(C_u) \cap \mathcal{S} \neq \emptyset$, but $\mathcal{S} = \{x \in \mathbb{R}^2 : B(x) \leq 0\}$. Thus, Proposition 1 and Theorem 2 imply that any selection of K_c renders the compact set $\mathcal{S}$ forward invariant.

C. Asymptotic Stability

Comparable notions of CBF in the literature, such as the zeroing CBFs of [2], are defined so that the safe set is not just forward invariant, but rather asymptotically stable. Asymptotic stability implies forward invariance and ensures that complete solutions starting outside the safe set converge (in distance) to the set. The following result provides conditions for asymptotic stability using multiple CBFs. For brevity, we make the simplifying assumption that the safe set is compact. The result is stated first for pre-asymptotic stability which, like forward pre-invariance, enjoys the properties of asymptotic stability except that maximal solutions need not be complete [7, Def. 3.1]. To obtain asymptotic stability, we must ensure that solutions do not terminate on the boundary of the flow set by selecting vectors according to the mapping $\Theta_{\Pi(C_u)}$, which is defined as in (7) with $\mathcal{S}$ replaced by $\Pi(C_u)$. Control inputs in $\Theta_{\Pi(C_u)}$ could be included in K_c by using a t-CBF to represent the flow set.

Theorem 3. ***(Asymptotic Stability)*** *Let Assumptions 6A) and 6B) hold for the control system in (1) with data (F, C_u) and suppose $B : \mathbb{R}^n \to \mathbb{R}^d$ is a CBF for (F, C_u) and $\mathcal{S} \subset \Pi(C_u)$ on $\mathcal{O} \subset \Pi(C_u)$ with respect to γ. Suppose that $\mathcal{S}$ is compact and, for every $i \in [d]$, $\gamma_i(x) > 0$ for all $x \in \mathcal{O} \backslash \mathcal{S}_i$, where $\mathcal{S}_i = \{x \in \mathbb{R}^n : B_i(x) \leq 0\}$. Suppose $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\overline{\mathcal{O} \cup \mathcal{S}}$ with $\kappa(x) \in K_c(x)$ for all $x \in \mathcal{O}$ and $\kappa(x) \in \Psi(x)$ for every $x \in \mathcal{S}$. Then $\mathcal{S}$ is pre-asymptotically stable for the closed-loop system (F_{cl}, C) defined in (6) by (F, C_u) and κ. If additionally $\kappa(x) \in \Theta_{\Pi(C_u)}(x)$ for all $x \in \mathcal{O}$, then $\mathcal{S}$ is asymptotically stable for the closed-loop system.*

Proof: Using continuity of κ along with Assumptions 6A) and 6B), Lemma 2 shows that the closed-loop system meets the basic conditions in [7, Def. 2.18]. Consider the nonsmooth Lyapunov function candidate $V_{max} \triangleq \max\{0, B_1, \ldots, B_d\}$ and the function $\gamma_{max} \triangleq \max_{i \in [d]} \gamma_i$. The function V_{max} is locally Lipschitz [29, Prop. 7], nonnegative, and $V_{max}(x) > 0$

for all $x \in \Pi(C_u) \setminus \mathcal{S}$. To prove that $\mathcal{S}$ is pre-asymptotically stable, we apply case 2a) of Theorem 3.19 in [7], for which it remains only to show that

$$\dot{V}_{max}(x) \triangleq \max_{\zeta \in \partial V_{max}(x)} \max_{f \in F(x,\kappa(x))} \langle \zeta, f \rangle < 0$$

for all $x \in \mathcal{O} \setminus \mathcal{S}$, where ∂V_{max} denotes the generalized gradient of V_{max} [29, Eqn. 37]. Let $I_{max}(x) \triangleq \{i \in \{0, 1, \ldots, d\} : B_i(x) = V_{max}(x)\}$, where we use the convention $B_0 \triangleq 0$. From Proposition 7 in [29] and using the fact that each B_i is continuously differentiable, $\partial V_{max}(x) \subset \mathrm{co} \cup_{i \in I_{max}(x)} \{\nabla B_i(x)\}$, where $\mathrm{co}\{\cdot\}$ denotes the convex hull. Note that based on convention, $0 \in \cup_{i \in I_{max}(x)} \{\nabla B_i(x)\}$ when $V_{max}(x) = 0$. Since each $\zeta \in \partial V_{max}(x)$ is a convex combination of the gradients of each B_i, we have

$$\begin{aligned} \dot{V}_{max}(x) &\leq \max_{\lambda} \max_{f \in F(x,\kappa(x))} \sum_{i \in I_{max}(x)} \lambda_i \langle \nabla B_i(x), f \rangle \\ &= \max_{\lambda} \sum_{i \in I_{max}(x)} \lambda_i \Gamma_i(x, \kappa(x)) \end{aligned}$$

where $\lambda_i \geq 0$ and $\sum_{i \in I_{max}(x)} \lambda_i = 1$. Thus, $\dot{V}_{max}(x) \leq \max_{i \in I_{max}(x)} \Gamma_i(x, \kappa(x))$, where $\Gamma_0 = 0$. Since $\kappa(x) \in K_c(x) = \{u : \Gamma(x, u) \leq -\gamma(x)\}$, $\dot{V}_{max}(x) \leq \max_{i \in I_{max}(x)} -\gamma_i(x)$. For any $x \in \mathcal{O} \setminus \mathcal{S}$ and $i \in I_{max}(x)$, $V_{max}(x) = B_i(x) > 0$. Because $B_i(x) > 0$, we have that $x \in \mathcal{O} \setminus \mathcal{S}_i$ and hence $\gamma_i(x) > 0$. We conclude that $\dot{V}_{max}(x) < 0$ for every $x \in \mathcal{O} \setminus \mathcal{S}$, from which [7, Thm. 3.19] shows that $\mathcal{S}$ is pre-asymptotically stable for the closed-loop dynamics.

Since $\mathcal{S}$ is pre-asymptotically stable, showing that $\mathcal{S}$ is asymptotically stable requires showing that there exists $\delta > 0$ such that every maximal solution to the closed-loop system starting from $\mathcal{S}_\delta \triangleq \{x \in \Pi(C_u) : |x|_{\mathcal{S}} \leq \delta\}$ is complete. From pre-asymptotic stability of $\mathcal{S}$ and the fact that $\mathcal{O}$ contains $U(\partial \mathcal{S}) \cap \Pi(C_u)$ (via the definition of a CBF), given any $\epsilon > 0$ such that $\mathcal{S}_\epsilon \subset \mathcal{O} \cup \mathcal{S}$, there exists $0 < \delta < \epsilon$ for which any trajectory of the closed-loop dynamics starting in $\mathcal{S}_\delta$ does not exit $\mathcal{S}_\epsilon$. In particular, ϵ can always be selected so there is a neighborhood of $\mathcal{S}_\epsilon$ for which $U(\mathcal{S}_\epsilon) \cap \Pi(C_u) \subset \mathcal{O} \cup \mathcal{S}$. Since $\kappa(x) \in \Psi(x)$ for all $x \in \mathcal{O} \cup \mathcal{S}$ and $C = \{x \in \mathbb{R}^n : \kappa(x) \in \Psi(x)\}$, we have $\Pi(C_u) \cap (\mathcal{O} \cup \mathcal{S}) = C \cap (\mathcal{O} \cup \mathcal{S})$. Because $\mathcal{S}_\epsilon \subset C$, it follows that $\partial C \cap \mathcal{S}_\epsilon \subset \partial \mathcal{S}_\epsilon$. The set $\mathcal{O}$ contains $\partial \mathcal{S}_\epsilon$, from which it follows that for any $x \in \mathcal{S}_\epsilon \cap \partial C$, there is a neighborhood U of x for which $\kappa(x) \in \Theta_{\Pi(C_u)}(x)$ for all $x \in U \cap C$. We conclude using [7, Prop 2.34] that a nontrivial flow exists from every point in $\mathcal{S}_\epsilon$. Combined with the fact that $\mathcal{S}_\epsilon$ is compact, this implies that every maximal solution starting from $\mathcal{S}_\delta$ is complete, establishing the asymptotic stability of $\mathcal{S}$ for the closed-loop dynamics. ∎

Remark 4. Under the assumptions of Theorem 3, suppose K_c is nonempty on $\mathcal{L}_B(\beta) = \{x \in \Pi(C_u) : B(x) \leq \beta\}$ for some $\beta > 0$. Then $\mathcal{L}_B(\beta)$ is forward pre-invariant for the closed-loop system, and pre-asymptotic stability of $\mathcal{S}$ then implies that any complete solution starting from $\mathcal{L}_B(\beta)$ converges to $\mathcal{S}$. This observation motivates verifying that K_c is nonempty on level sets of B, which will be explored in Section V. The asymptotic stability of $\mathcal{S}$ follows whenever $\mathcal{L}_B(\beta)$ is forward invariant for the closed-loop dynamics via Theorem 2.

IV. Design of Optimal Safety-Ensuring Feedback

In this section, we study the continuity (as a function of the state) of a class of optimization-based control laws that can be used to make selections from mappings such as the set of safety ensuring control inputs K_c defined in (5). The results here allow for a general class of cost functions and an arbitrary number of constraints. Given a CBF, Theorems 1 and 2 show that continuous selections of the mapping K_c render the safe set forward (pre-)invariant. When working with t-CBFs, Theorems 1 and 2 require selections of K_c that are locally Lipschitz. Unfortunately, as has been noted in works like [19], studying the local Lipschitz continuity of the broad class of optimization-based control laws considered here is more challenging, and is beyond the scope of this work. A result for the local Lipschitz continuity of a more restrictive class of quadratic programs with a maximum of two constraints is available in [2, Thm. 3].

To obtain an implementable form for the controller, we impose the following condition on the set-valued map Ψ of admissible controls. We emphasize that the set Ψ represents arbitrary, state-dependent constraints on the control input.

Assumption 7. There exists a function $\psi : \Pi(C_u) \times \mathbb{R}^m \to \mathbb{R}^k$ such that $\Psi(x) = \{u \in \mathbb{R}^m : \psi(x, u) \leq 0\}$ for all $x \in \Pi(C_u)$.

Assumption 7 is commonly used when input constraints are present [10], [11]. If B is a CBF for (F, C_u) and $\mathcal{S}$ on $\mathcal{O}$ with respect to γ, define the controller $\kappa^* : \mathcal{O} \to \mathbb{R}^m$ as[3]

$$\begin{aligned} \kappa^*(x) \triangleq &\arg\min_{u \in \mathbb{R}^m} Q(x, u) \\ &s.t.\ \Gamma(x, u) \leq -\gamma(x), \\ &\quad\ \psi(x, u) \leq 0, \end{aligned} \tag{8}$$

where $Q : \mathcal{O} \times \mathbb{R}^m \to \mathbb{R}$ is a cost function and Γ is defined in (4). Because K_c in (5) is the feasible set for (8), it is clear that κ^* is a selection of K_c; we write (8) equivalently as $\kappa^*(x) = \arg\min_{u \in K_c(x)} Q(x, u)$. When K_c is nonempty on $\mathcal{O}$ as required in the definition of a CBF in Definition 2, the optimization in (8) is feasible.

Remark 5. The optimization in (8) is generally a nonlinear program. While solvers for nonlinear programs exist, they can be computationally expensive leading to practical difficulties. However, (8) reduces to a quadratic program if the cost function Q is quadratic and the constraints are affine in the control input. The functions Γ_i in (4) are affine when the dynamics are affine in the control input [2], [30]. In the setting of differential inclusions, the dynamics should have the form $F(x) + g(x) u$, where F is set-valued and g is single-valued. Results such as [1] and the references therein have demonstrated that a significant number of relevant control

[3]For κ^* to be well-defined, the function Γ should be extended to points $(x, u) \in \Pi(C_u) \times \mathbb{R}^m$ where $u \notin \Psi(x)$. This extension can be done arbitrarily since such points are infeasible.

problems feature affine constraints and that quadratic programs can be viably computed online in real-time applications.

Although $\kappa^*(x)$ is feasible at $x \in \mathcal{O}$ if $K_c(x) \neq \emptyset$, it is not necessarily continuous. We provide a result for continuity of κ^* in Lemma 2 of our preliminary work [22], where it is assumed that the cost function Q in (8) is level-bounded in u, locally uniformly in x. This property is equivalent to the local boundedness of the mapping $x \mapsto \ell_Q(x, \lambda) \triangleq \{u \in \mathbb{R}^m : Q(x,u) \leq \lambda\}$ for every $\lambda \in \mathbb{R}$, where local boundedness is the same as the notion of local compactness in [31] and is closely related to the notion of uniform compactness in [32]. Local boundedness holds whenever Q is continuous and, for each fixed x, the mapping $u \mapsto Q(x,u)$ is convex and inf-compact (cf. [31, Lem. 5.7]). We thus can obtain the following simplified form of [22, Lem. 2]. In comparison to the more commonly-used Berge's maximum theorem, Lemma 3 does not require the feasible set to be compact.

Lemma 3. *Let X be a metric space and $\mathcal{U}$ a finite-dimensional normed space. Suppose $K : X \rightrightarrows \mathcal{U}$ is lower and outer semicontinuous with nonempty, convex values, and the function $Q : X \times \mathcal{U} \to \mathbb{R}$ is continuous and, for each $x \in X$, $u \mapsto Q(x,u)$ is strictly convex and inf-compact[4] on $K(x)$. Then $\kappa^* : X \to \mathcal{U}$ defined as $\kappa^*(x) \triangleq \arg\min_{u \in K(x)} Q(x,u)$ is single-valued and continuous.*

Proof: Our proof is based on [32], from which we note that the notion of closed mappings is equivalent to outer semicontinuity [32, Thm. 2], and a mapping is open if and only if it is lower semicontinuous [32, Cor. 1.1]. Strict convexity and inf-compactness of $u \mapsto Q(x,u)$ ensure that the set of minimizers $\kappa^*(x)$ contains a single unique element for every $x \in \mathcal{O}$. Thus, Corollary 9.1 in [32] shows that $\kappa^*(x)$ is uniformly compact near x. Corollary 8.1 in [32] then shows that $\kappa^*(x)$ is continuous at every $x \in \mathcal{O}$. ■

Remark 6. Lemma 3 is more general than results that have previously appeared in the controls literature. The authors in [33] provide a continuity result that leverages the generalization of Berge's maximum theorem in [21, Thm. 1.2]. However, the assumptions imposed on the cost function are restrictive, and in fact Lemma 3 shows that condition (O2) in [33, Thm. 3] is redundant if the cost function is also inf-compact. The authors in [34] study the continuity of optimization-based control laws by leveraging [31, Lem. 5.7], which shows uniform compactness of the level sets of the cost function for convex optimization problems. Relative to the results in [31], Lemma 3 applies on a more general class of spaces and allows the convexity of the cost function to be restricted to the feasible set $K(x)$.

Next, we establish the continuity of the controller in (8). We impose the following assumptions on the constraints, which lead to the continuity properties of the mapping K_c required by Lemma 3.

Assumption 8. For each $i \in [d]$ and each $j \in [k]$,

A) For each $x \in \mathcal{O}$, the functions $u \mapsto \Gamma_i(x,u)$ and $u \mapsto \psi_j(x,u)$ are convex on $K_c(x)$.

B) The functions $(x,u) \mapsto \Gamma_i(x,u) + \gamma_i(x)$ and $(x,u) \mapsto \psi_j(x,u)$ are continuous on $C_u \cap (\mathcal{O} \times \mathbb{R}^m)$ and $\mathcal{O} \times \mathbb{R}^m$, respectively.

Theorem 4. ***(Continuity of κ^*)*** *Let $C_u \subset \mathbb{R}^n \times \mathbb{R}^m$, $\mathcal{O} \subset \Pi(C_u)$, $\Gamma : C_u \to \mathbb{R}^d$, and $\gamma : \Pi(C_u) \to \mathbb{R}^d$ be given. Suppose Assumptions 7 and 8 hold, the cost function $Q : \mathcal{O} \times \mathbb{R}^m \to \mathbb{R}$ is continuous and, for each $x \in \mathcal{O}$, $u \mapsto Q(x,u)$ is strictly convex and inf-compact on $K_c(x)$, and the set*

$$K_c^\circ(x) \triangleq \left\{ \begin{array}{c} u \in \mathbb{R}^m : \Gamma(x,u) < -\gamma(x) \\ \psi(x,u) < 0 \end{array} \right\} \tag{9}$$

is nonempty for every $x \in \mathcal{O}$. Then $\kappa^* : \mathcal{O} \to \mathbb{R}^m$ defined in (8) is continuous.

Proof: For every $x \in \mathcal{O}$, the functions defining K_c are assumed to be continuous and convex on $\{x\} \times K_c(x)$. Since $K_c^\circ(x)$ is nonempty, Theorem 12 in [32] shows that $K_c : \mathcal{O} \rightrightarrows \mathbb{R}^m$ is lower semicontinuous. We recall that $K_c(x) = \{u \in \Psi(x) : \Gamma(x,u) \leq -\gamma(x)\}$. From the definition of Ψ, the graph of K_c is equivalent to $\text{Graph}_{\mathcal{O}}(K_c) = \big\{(x,u) \in C_u \cap (\mathcal{O} \times \mathbb{R}^m) : \Gamma(x,u) \leq -\gamma(x)\big\}$, which is relatively closed in $C_u \cap (\mathcal{O} \times \mathbb{R}^m)$ by continuity of $\Gamma + \gamma$. Using Assumption 7, $C_u \cap (\mathcal{O} \times \mathbb{R}^m) = \{(x,u) \in \mathcal{O} \times \mathbb{R}^m : \psi(x,u) \leq 0\}$, which is relatively closed in $\mathcal{O} \times \mathbb{R}^m$ by continuity of ψ. Thus, $\text{Graph}_{\mathcal{O}}(K_c)$ is also relatively closed in $\mathcal{O} \times \mathbb{R}^m$, i.e., the mapping K_c is outer semicontinuous. Moreover, $K_c(x)$ is convex for every $x \in \mathcal{O}$ since it is a sublevel set of convex functions. Thus, the assumptions of Lemma 3 are satisfied and κ^* is continuous. ■

Remark 7. By invoking Proposition 2.9 of [35], the functions Γ_i in (4) are continuous when the flow map $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semicontinuous, and lower semicontinuous. When the needed regularity is not present in the dynamics, one can replace Γ with a continuous upper bound $\bar{\Gamma} : C_u \to \mathbb{R}^d$ such that $\bar{\Gamma}_i(x,u) \geq \Gamma_i(x,u)$ for every $(x,u) \in C_u \cap (\mathcal{O} \times \mathbb{R}^m)$ and $i \in [d]$. It follows that $K_c^S(x) \triangleq \{u \in \Psi(x) : \bar{\Gamma}(x,u) \leq -\gamma(x)\} \subset K_c(x)$ for all $x \in \Pi(C_u) \cap \mathcal{O}$, so that redefining κ^* to be a selection of the subset mapping K_c^S still leads to a selection of K_c. Similar replacements can be made for the functions γ and ψ.

Remark 8. A common practice in the CBF literature is to use slack variables [11] or adaptive slack parameters [36] to improve the feasibility of the safety-ensuring control law. For example, the performance function γ in (8) could be selected as $\gamma_i(x, \delta_i) \triangleq \delta_i \min\{B_i(x), 0\}$, where $\delta_i \geq 0$ is a decision variable in the optimization problem. The feasible set for this optimization problem can be modeled in a simpler way by setting $\gamma_i(x) = -\infty$ if $B_i(x) < 0$ (cf. Remark 2). Moreover, the continuity of an optimization-based controller featuring slack variables can be analyzed directly using Theorem 4 by including δ_i as a state variable. However, slack variables can lead to abrupt control action as discussed in Section I-A.

V. Feasibility Verification with Sum of Squares

A challenging aspect of verifying that a given CBF candidate is a CBF (or a t-CBF) is determining whether the set

[4] A function $f : X \to \mathbb{R}$ is inf-compact if for every $\lambda \in \mathbb{R}$, the sublevel set $\mathcal{L}_f(\lambda) \triangleq \{x \in X : f(x) \leq \lambda\}$ is compact.

$K_c(x)$ is nonempty for every $x \in \mathcal{O}$. Since K_c is the feasible set for the control law κ^* in (8), checking if a function is a CBF is the same as checking if the optimization defining κ^* is feasible. Moreover, certifying that K_c° in (9) has nonempty values guarantees continuity of κ^* under the assumptions of Theorem 4. In this section, we develop sum of squares (SoS) polynomial optimization tools for certifying that K_c and K_c° have nonempty values under more restrictive assumptions on the constraints defining the mappings. Namely, we assume that the constraints are polynomials and affine in the control input. However, the tools can be used in the case of non-polynomial constraints to obtain conservative estimates of the feasible region by replacing the constraints with polynomial upper bounds. This procedure is similar to Remark 7 except the polynomial upper bounds are used only for verification and we do not need to redefine κ^*.

There are well-characterized computational limitations of SoS programming [37]. Although we have made efforts to develop SoS programs that are less complex than previous results, such programs scale poorly with the number of state variables. Yet, there are many examples of control problems where SoS techniques have proven useful [14]–[16].

Let $\mathcal{P}[x]$ be the set of all polynomials in the variables $x \in \mathbb{R}^n$. The set of SoS polynomials is $\Sigma[x] \triangleq \{p \in \mathcal{P}[x] : p = \sum_{i=1}^N f_i^2,\ f_1, \ldots, f_N \in \mathcal{P}[x]\}$, where $p \in \Sigma[x]$ implies that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. We will also use $\mathcal{P}^{m_1 \times m_2}[x]$ to denote the set of matrix-valued functions $p : \mathbb{R}^n \to \mathbb{R}^{m_1 \times m_2}$ with polynomial entries.

SoS programming involves a series of relaxations of originally NP-hard polynomial optimization problems that lead to tractable semidefinite programs [12]. The class of problems that can be solved involve optimizing the coefficients of polynomials $p_i \in \mathcal{P}[x]$ subject to constraints of the form $a_0 + \sum_{i=1}^N p_i a_i \in \Sigma[x]$, where $a_0, a_1, \ldots, a_N \in \mathcal{P}[x]$ are given, constant coefficient polynomials (see [13], SoS Program 2). The aforementioned constraint is linear in the coefficients of the polynomials p_i.

To describe how SoS optimization can be used to certify whether a given function is a CBF, first consider a global feasibility problem. Let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping defined by a system of inequalities as

$$K(x) \triangleq \{u \in \mathbb{R}^m : A(x)\, u + b(x) \leq 0\}, \qquad (10)$$

where $A : \mathbb{R}^n \to \mathbb{R}^{n_c \times m}$ and $b : \mathbb{R}^n \to \mathbb{R}^{n_c}$ are polynomial, i.e., $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$. The assumption that the constraints are affine is needed to obtain a proper SoS program, as discussed above. Given constraints of the form in (10), the following SoS program will certify that $K(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$.

Problem 1. (Global Feasibility) Given $x \in \mathbb{R}^n$ and polynomials $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$, find a constant $\epsilon \geq 0$ and a polynomial $u \in \mathcal{P}^m[x]$ such that, for all $i \in [n_c]$,

$$-A_{i*}(x)\, u(x) - b_i(x) - \epsilon \in \Sigma[x],$$

where $A_{i*}(x)$ denotes that i-th row of $A(x)$. The parameter ϵ could either be a fixed value or a decision variable. If $\epsilon > 0$, then $K^\circ(x) \triangleq \{u \in \mathbb{R}^m : A(x)\, u + b(x) < 0\}$ is nonempty.

Although the polynomial controller u found in Problem 1 is a selection of K (i.e., $u(x) \in K(x)$), it is not an optimal selection like κ^* in Section IV. Thus, we use u only for feasibility verification purposes, while κ^* is used to define a closed-loop system for control purposes. To apply the techniques in this section to K_c in Section II, we will need to assume the existence of a polynomial and affine upper bound of the functions defining K_c.

Assumption 9. Given $\Gamma : C_u \to \mathbb{R}^d$, $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$, and $\psi : \Pi(C_u) \times \mathbb{R}^m \to \mathbb{R}^k$, let $n_c \triangleq d + k$ and assume there exists $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$ such that $A(x)\, u + b(x) \geq (\Gamma(x, u) + \gamma(x), \psi(x, u))$ for all $(x, u) \in C_u$.

For many practical controls problems, especially those involving constraints on the magnitude of the control input, one will likely not find (or need) a CBF that exists on the entire state space. More often, feasibility verification can be restricted to a particular operating region. Thus, a method is needed to verify that $K_c(x)$ in (5) is nonempty on a subset of $\mathbb{R}^n$. A natural way to define the operating region is with sublevel sets of a CBF candidate B defining $\mathcal{S} \subset \Pi(C_u)$, which is especially useful when convergence to the safe set is desired (see Remark 4). By certifying that $K_c(x)$ is nonempty on a set $\mathcal{L}_B(\beta) \triangleq \{x \in \mathbb{R}^n : B(x) \leq \beta\}$, with $\beta > 0$, we certify that B is a CBF on $\mathcal{L}_B(\beta)$, and that the controller κ^* in (8) exists on the entire safe set $\mathcal{S} \subset \mathcal{L}_B(\beta)$. Since working with B in this context requires assuming that B is polynomial, we subsequently consider a generic polynomial $\tilde{B} \in \mathcal{P}^{n_b}[x]$.

While being a SoS polynomial is a global property, there exist hierarchies of relaxations that have close relationships to the set of polynomials that are nonnegative only on a particular subset of $\mathbb{R}^n$ [12]. The relaxation that will be most useful for the feasibility verification problem is the following, based on Putinar's Positivstellensatz [38].

Lemma 4. *Let $\tilde{B} \in \mathcal{P}^{n_b}[x]$ and define $\mathcal{L}_{\tilde{B}}(\beta) \triangleq \{x \in \mathbb{R}^n : \tilde{B}(x) \leq \beta\}$ for some $\beta \in \mathbb{R}$. A function $p \in \mathcal{P}[x]$ is nonnegative on $\mathcal{L}_{\tilde{B}}(\beta)$ if there exist $s_0, s_1, \ldots, s_{n_b} \in \Sigma[x]$ such that, for all $x \in \mathbb{R}^n$,*

$$p(x) \geq s_0(x) + \sum_{j=1}^{n_b} s_j(x) \left(\beta - \tilde{B}_j(x)\right). \qquad (11)$$

Proof: The result follows from the facts that $s_j(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\beta - \tilde{B}_j(x) \geq 0$ for all $x \in \mathcal{L}_{\tilde{B}}(\beta)$. ∎

Putinar's Positivstellensatz shows that every polynomial that is strictly positive on $\mathcal{L}_{\tilde{B}}(\beta)$ can be decomposed in the form on the right-hand side of (11) under the assumption that the functions defining $\mathcal{L}_{\tilde{B}}(\beta)$ have an Archimedean property [12, Thm 3.20]. While results guaranteeing the existence of SoS decompositions when the Archimedean property is not present have been applied to controls problems in, e.g., [14], these methods scale poorly with the number of components in $\tilde{B}$. Additionally, the multiplicative monoid in [14] is known to lead to multiplicative combinations of decision variables that require developing complex iterative procedures, thereby adding conservativeness to the problem.

Recalling the definition of the mapping K in (10), the following program certifies that the set $K(x)$ is nonempty for all $x \in \mathcal{L}_{\tilde{B}}(\beta) = \{x \in \mathbb{R}^n : \tilde{B}(x) \leq \beta\}$.

Problem 2. (Feasibility on Level Sets) Given $x \in \mathbb{R}^n$, $A \in \mathcal{P}^{n_c \times m}[x]$, $b \in \mathcal{P}^{n_c}[x]$, $\tilde{B} \in \mathcal{P}^{n_b}[x]$, and $\beta \in \mathbb{R}$, find polynomials $u \in \mathcal{P}^m[x]$, $s_0, s_1, \ldots, s_{n_b} \in \Sigma[x]$, and a constant $\epsilon \geq 0$ such that, for all $i \in [n_c]$,

$$-A_{i*}(x)\,u(x) - b_i(x) - \epsilon \\ - s_0(x) - \sum_{j=1}^{n_b} s_j(x)\left(\beta - \tilde{B}_j(x)\right) \in \Sigma[x]. \quad (12)$$

The main result of this section follows. It states that a CBF candidate B can be certified as a CBF on a set $\mathcal{L}_{\tilde{B}}(\beta) \supset U(\partial\mathcal{S}) \cap \Pi(C_u)$ by finding a feasible solution to Problem 2. An identical result can be given for t-CBFs provided the additional assumptions in Section II-A hold. Unfortunately, the inability to find a feasible solution to Problem 2 does not mean that no such feasible solution exists. One major cause for conservativeness is that the degree of the involved polynomials must be restricted in practice, and feasible solutions may exist for higher degree polynomials.

Theorem 5. ***(Verification of CBF)*** *Consider the dynamical system (F, C_u) in (1) and a set $\mathcal{S} \subset \Pi(C_u)$. Suppose Assumption 7 holds for a function ψ, $B : \mathbb{R}^n \to \mathbb{R}^d$ is a CBF candidate defining $\mathcal{S}$, and $\gamma : \Pi(C_u) \to [-\infty, \infty]^d$ satisfies Assumption 2. Given Γ defined in (4), let Assumption 9 hold for some $A \in \mathcal{P}^{n_c \times m}[x]$ and $b \in \mathcal{P}^{n_c}[x]$. If Problem 2 has a solution for some $\tilde{B}$ and β for which there exists a neighborhood of the boundary of $\mathcal{S}$ such that $U(\partial\mathcal{S}) \cap \Pi(C_u) \subset \mathcal{L}_{\tilde{B}}(\beta)$, then $K_c(x)$ in (5) is nonempty for all $x \in \mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ and B is a CBF for (F, C_u) and $\mathcal{S}$ on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ with respect to γ. Moreover, if the solution to Problem 2 is such that $\epsilon > 0$, then $K_c^\circ(x)$ in (9) is nonempty for all $x \in \mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$.*

Proof: Using Definition 2 and the assumptions of the theorem, we need only show that K_c in (5) is nonempty on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ to prove that B is a CBF. Problem 2 and Lemma 4 tell us that there exists $u \in \mathcal{P}^m[x]$ such that $A(x)\,u(x) + b(x) \leq -\epsilon$ for all $x \in \mathcal{L}_{\tilde{B}}(\beta)$. From Assumption 9, $(\Gamma(x, u(x)) + \gamma(x), \psi(x, u(x))) \leq -\epsilon$ for all $x \in \mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$. It follows by definition that K_c is nonempty on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ and, if $\epsilon > 0$, so is K_c°. ■

Remark 9. When $\mathcal{L}_{\tilde{B}}(\beta) \setminus \Pi(C_u) \neq \emptyset$, the procedure developed above will be conservative because it is unnecessary to consider points outside $\Pi(C_u)$. If we assume that $\Pi(C_u)$ can be described by a polynomial $\pi \in \mathcal{P}^{n_p}[x]$ as $\Pi(C_u) = \{x \in \mathbb{R}^n : \pi(x) \leq 0\}$, then these constraints can be included in Problem 2 to reduce conservativeness.

VI. Example: Non-Polynomial Dynamics with Unknown Parameters and Input Constraints

In this section, we apply the tools developed in this paper to certify the feasibility of a safety-ensuring control law in the presence of uncertain dynamics and input constraints. A particular challenge in this example is that the functions Γ_i defined by (4) are not polynomial. Due to this, we develop polynomial upper bounds of the functions that are used for feasibility verification purposes, and show that these upper bounds are not overly conservative. Consider the second-order system

$$\dot{x} = \underbrace{\begin{bmatrix} x_1^2 & \sin(x_2) & 0 & 0 \\ 0 & x_2\cos(x_1) & |x_1| & x_1 x_2 \end{bmatrix}}_{Y(x)} \theta + \underbrace{\begin{bmatrix} 1 & 3 \\ 1/2 & -1 \end{bmatrix}}_{g} u, \quad (13)$$

with $u \in \mathbb{R}^2$, $\theta \in \mathbb{R}^4$ a vector of unknown parameters, and $C_u \triangleq \{(x, u) \in \mathbb{R}^4 : |u_i| \leq u_{max},\ \forall i \in \{1, 2\}\}$ for some constant $u_{max} > 0$. We assume that the unknown parameters are bounded such that $\theta \in \Theta \triangleq \{\theta \in \mathbb{R}^4 : |\theta_i| \leq \bar{\theta}_i,\ \forall i \in [4]\}$, with each $\bar{\theta}_i > 0$. In [9], an adaptive control scheme was developed for systems of the class in (13) using a CBF-based control law. It was shown that safety is guaranteed if the pointwise optimal controller is feasible for the worst-case values of the parameters. Given the assumption that θ is bounded, the needed analysis can be performed by treating (13) as a differential inclusion with θ taking values in the set Θ, given as in (1) with $F(x, u) \triangleq \{Y(x)\theta + gu : \theta \in \Theta\}$ and C_u given above.

Consider a CBF candidate

$$B(x) = \begin{bmatrix} x_1 + x_2 - c \\ -x_1 + x_2 - c \\ \frac{1}{c}x_1^2 - x_2 - c \end{bmatrix}, \quad (14)$$

defining the set $\mathcal{S} \triangleq \{x \in \Pi(C_u) : B(x) \leq 0\}$, with $c = 5$ and $\Pi(C_u) = \mathbb{R}^2$. The feasibility condition used here is a slight refinement of the one in [9], which was based on the 2-norm. Considering that the set Θ is compact, the function Γ in (4) has components

$$\Gamma_i(x, u) = \max_{\theta \in \Theta}\left\{\nabla B_i^T(x)\,Y(x)\,\theta\right\} + \nabla B_i^T(x)\,gu. \quad (15)$$

Note that Γ is linear in the control input u. Because of the expression for Y, the functions Γ_i are not polynomial. Using the upper bounds $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$, it can be shown that for $i \in \{1, 2\}$,

$$\Gamma_i(x, u) \leq x_1^2\bar{\theta}_1 + (1 + |x_2|)\,\bar{\theta}_2 + |x_1|\,\bar{\theta}_3 \\ + |x_1 x_2|\,\bar{\theta}_4 + \nabla B_i^T(x)\,gu,$$

and

$$\Gamma_3(x, u) \leq \frac{2}{c}\left|x_1^3\right|\bar{\theta}_1 + \left(\frac{2}{c}|x_1| + |x_2|\right)\bar{\theta}_2 + |x_1|\,\bar{\theta}_3 \\ + |x_1 x_2|\,\bar{\theta}_4 + \nabla B_3^T(x)\,gu.$$

The upper bounds above take the form $A(x)\,u + b(x)$, where the function b contains monomials composed with the absolute value function. Although the absolute value function is not polynomial, constraints of this form can be replaced with a set of polynomial constraints (see Appendix B for a description of the procedure). The upper bounds of Γ are used only for verification with SoS, while the actual value of Γ in (15) can be used to compute the controller κ^*.

The feasibility verification was implemented in MATLAB (2019b). The SOS optimization problems were formulated using the SPOT toolbox (see [39]). The SoS optimization was solved on a workstation running Windows 11 with a 2.2 GHz Intel Core i7-1360P processor and 32 GB of RAM. The program took approximately 0.2 seconds to solve.

The function γ is selected as $\gamma_i(x) = K_b B_i(x)$ with $K_b = 13$, which satisfies Assumption 2 since $B_i(x) > 0$ for all $x \in \mathbb{R}^2 \backslash \mathcal{S}_i$, where $\mathcal{S}_i = \{x \in \mathbb{R}^n : B_i(x) \leq 0\}$. The maximum value of the control input is selected as $u_{max} = 40$ and the upper bound of the unknown parameters is $\bar{\theta} = (1, 2, 2, 1)$. As required by Assumption 7, the set Ψ, in this case not dependent on x, is the zero-sublevel set of the functions $\psi_i(u) = |u_i| - u_{max}$ for $i \in \{1, 2\}$, where the absolute value function can be replaced by polynomials as described above. With these choices, the mapping K_c is defined in terms of Γ, γ, and ψ, and the constraints can be implemented in a SoS program like Problem 2. Since B is polynomial, we defined $\tilde{B} \triangleq B$ and performed a search on the set $\mathcal{L}_B(\beta) = \{x \in \mathbb{R}^2 : B(x) \leq \beta\}$. The problem is feasible with $\beta = 1.05$. Since $\mathcal{L}_B(\beta)$ contains a neighborhood of $\mathcal{S}$, Theorem 5 shows that B is a CBF for (F, C_u) and $\mathcal{S}$ on $\mathcal{L}_\beta(B)$ with respect to γ. In fact, the problem is feasible with $\epsilon = 0.1$, which implies that $K_c^\circ(x) \neq \emptyset$ for all $x \in \mathcal{L}_B(\beta)$.

Theorem 4 shows that the controller $\kappa^*(x) \triangleq \arg\min_{u \in K_c(x)} |u - u_{nom}(x)|$ is continuous on $\mathcal{L}_B(\beta)$ when $u_{nom} : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous. In the simulation in Figure 3, $u_{nom}(x) \triangleq g^{-1}(Rx - Y(x)\theta)$, where $R = [0.15, 1.3; -1, 0.1]$ is a 2×2 matrix. Without modification for safety, the controller u_{nom} would make the system unstable. Since $\partial\Pi(C_u) \cap \mathcal{S} = \emptyset$, we have $K_c(x) = K_c(x) \cap \Theta_{\mathcal{S}}(x)$ for all $x \in \mathbb{R}^n$. Since $\mathcal{S}$ is compact and $\kappa^*(x) \in K_c(x) \subset \Psi$ for all $x \in \mathcal{L}_B(\beta)$, we apply Theorem 2 to conclude that $\mathcal{S}$ is forward invariant for the closed-loop dynamics. In fact, $\partial\Pi(C_u) = \partial\mathbb{R}^2 = \emptyset$ and, for each $i \in [d]$, $\gamma_i(x) > 0$ for all $x \in \mathbb{R}^2 \backslash \mathcal{S}_i$, so that Theorem 3 and Remark 4 show that $\mathcal{S}$ is asymptotically stable from $\mathcal{L}_B(\beta)$.

Remark 10. For the feasible set $K(x) = \{u \in \mathbb{R}^m : A(x)u + b(x) \leq 0\}$, one can compute the so-called width of the feasible set $w^* : \mathbb{R}^n \to \mathbb{R}$ (see [10]) using

$$w^*(x) = \max_{(u,w) \in \mathbb{R}^{m+1}} w \qquad (16)$$
$$s.t.\ A_{i*}(x)u + b_i(x) + w \leq 0 \quad \forall i \in [d].$$

The problem in (16) is always feasible, and if $w^*(x) \geq 0$, then $K(x) \neq \emptyset$. Computing $w^*(x)$ at various points in the state space provides a useful visual approximation of the feasible set for problems of low dimension. However, in contrast to the approach in Section V, the computational approach cannot verify feasibility on the entire safe set, since this would involve computing $w^*(x)$ at an uncountable number of points.

Figure 3 shows an approximation of the feasible region for Example 1, i.e., the set on which $K_c(x)$ is nonempty. The feasible region was approximated by solving (16) with the constraints $(\Gamma(x, u) + \gamma(x), \psi(u))$. The actual value of the function Γ was used when solving (16) instead of the polynomial upper bound. As can be seen, the set $\mathcal{L}_B(\beta)$ with $\beta = 1.05$ could not be expanded significantly without

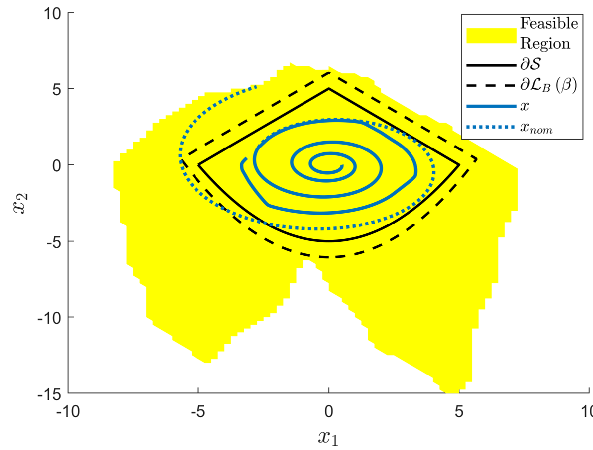

Figure 3. The feasible region for Example 1 was approximated by solving (16) at various points in the state space. A SoS program like Problem 2 certified that B in (14) is a CBF on the region inside the dotted line representing the boundary of $\mathcal{L}_B(\beta) = \{x \in \mathbb{R}^n : B(x) \leq \beta\}$. Note that the problem would not be feasible if β were increased slightly. The trajectory x is generated by a controller that modifies an unstable nominal controller subject to the safety constraints. The nominal controller without safety constraints produced the trajectory x_{nom}.

including points outside the feasible region, showing that the SoS program with polynomial upper bounds was not overly conservative compared to solving (16) with the actual value of Γ. The feasible region is larger than the area captured by $\mathcal{L}_B(\beta)$ and could be characterized more fully using the level sets of an alternative function. However, this characterization would not provide a significant benefit because forward invariance is only guaranteed on level sets of B. Thus, trajectories starting at other points in the feasible region may flow out of the feasible region.

In Figure 3, the trajectory produced by the safe controller remains within the set $\mathcal{S}$ although the nominal trajectory lies outside the set. The safe trajectory deviates from the nominal trajectory only when approaching the boundary of the set $\mathcal{S}$. The trajectory is not allowed to approach closely to the boundary of $\mathcal{S}$ because the controller is compensating for the worst-case value of the unknown parameters. In this way, the controller is robust to parameter uncertainty. The interested reader is referred to the adaptive technique in [9], which was developed to reduce uncertainty through estimation and allow trajectories to more closely approach the safe set boundary.

VII. Conclusion

This paper defined a notion of vector-valued CBF that is amenable to problems where the mapping of safety-ensuring control inputs is defined by multiple constraints. Selections of the safety-ensuring map render the safe set of states forward (pre-)invariant under mild conditions. Tools for certifying the continuity and feasibility of optimal selections from the map were developed.

Appendix

A. Proof of Lemma 2

Proof: If $x \in C$ then $(x, \kappa(x)) \in C_u$, which, when Assumption 6A) holds, implies that $F(x, \kappa(x)) = F_{cl}(x)$ is nonempty and convex on C. To show that F_{cl} is outer semicontinuous, note that $\text{Graph}(F_{cl}) = \{(x, y) \in \overline{C} \times \mathbb{R}^n : y \in F(x, \kappa(x))\}$. Let (x, y) be a limit point of $\text{Graph}(F_{cl})$ and let $\{x_n, y_n\}_{n \in \mathbb{N}}$ be a sequence from $\text{Graph}(F_{cl})$ converging to (x, y). Each element of the sequence is such that $(x_n, \kappa(x_n), y_n) \in \text{Graph}(F) = \{(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n : y \in F(x, u)\}$. Since $\lim_{n \to \infty} x_n = x \in \overline{C}$ and κ is continuous on $\overline{C}$, $\lim_{n \to \infty} \kappa(x_n) = \kappa(x)$. Thus, $\{x_n, \kappa(x_n), y_n\}_{n \in \mathbb{N}}$ converges, and since $\text{Graph}(F)$ is closed by outer semicontinuity of F, it converges in $\text{Graph}(F)$. It follows that $(x, y) \in \text{Graph}(F_{cl})$, which shows that $\text{Graph}(F_{cl})$ is closed and F_{cl} is outer semicontinuous. To show local boundedness of F_{cl}, fix $x \in \mathbb{R}^n$. If $x \notin \overline{C}$, then there exists $U(x)$ such that $F_{cl}(U(x)) = \emptyset$. Thus, assume that $x \in \overline{C}$. Since F is locally bounded, there exists a neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}^m$ of $(x, \kappa(x))$ such that $F(U)$ is bounded. Since κ is continuous on $\overline{C}$, there exists a neighborhood $U'(x)$ such that $U_C \times \kappa(U_C) \subset U$ with $U_C \triangleq U'(x) \cap \overline{C}$. Thus, $F_{cl}(U_C) = F(U_C, \kappa(U_C))$ is bounded and so is $F_{cl}(U'(x))$ using the emptiness of F_{cl} outside $\overline{C}$.

Suppose that Assumption 6B) holds, i.e., C_u is closed. Let $x \in \mathbb{R}^n$ be a limit point of $C = \{x \in \mathbb{R}^n : (x, \kappa(x)) \in C_u\}$. Then there is a sequence $\{x_n\}_{n \in \mathbb{N}}$ from C converging to x. Each element of the sequence (since it lies in C) is such that $(x_n, \kappa(x_n)) \in C_u$. Since $\lim_{n \to \infty} x_n = x \in \overline{C}$ and κ is continuous on $\overline{C}$, $\lim_{n \to \infty} \kappa(x_n) = \kappa(x)$. Since C_u is closed, $(x, \kappa(x)) \in C_u$, which shows that $x \in C$. Thus, C is closed.

Let F be locally Lipschitz on $A \subset C \times \mathbb{R}^m$. Let $K \subset \Pi(A)$ be compact. Since κ is continuous on $\Pi(A)$, $K \times \kappa(K)$ is also compact. Thus, for any $z_1, z_2 \in K$, the fact that F is locally Lipschitz on A implies that there exists $L > 0$ for which

$$F(z_1, \kappa(z_1)) \subset F(z_2, \kappa(z_2)) + L\left|(z_1, \kappa(z_1)) - (z_2, \kappa(z_2))\right| \mathbb{B}^n.$$

Because $F_{cl}(z) = F(z, \kappa(z))$, to show that F_{cl} is locally Lipschitz it suffices to show that the scalar quantity $L\left|(z_1, \kappa(z_1)) - (z_2, \kappa(z_2))\right|$ can be upper bounded linearly in terms of $|z_1 - z_2|$ on K. Since κ is locally Lipschitz on $\Pi(A)$, there exists $L_\kappa > 0$ such that $|\kappa(z_1) - \kappa(z_2)| \leq L_\kappa |z_1 - z_2|$ for every $z_1, z_2 \in K$. Thus, for any $z_1, z_2 \in K$, $L\left|(z_1, \kappa(z_1)) - (z_2, \kappa(z_2))\right| \leq L|z_1 - z_2| + L|\kappa(z_1) - \kappa(z_2)| \leq L(1 + L_\kappa)|z_1 - z_2|$, which completes the proof. ■

B. Polynomial Replacement of Absolute Value

As seen in Section VI, constraints in controls applications often take the form $A(x)u + b(x)$, where $b : \mathbb{R}^n \to \mathbb{R}$ contains some monomial expressions composed with the absolute value function. It has been demonstrated in [41] that upper bounds of this form are useful for practical applications involving Euler-Lagrange dynamics. Assume that b is defined by monomials with integer exponents as

$$b(x) = \sum_{i=1}^{n_c} c_i \left| x_1^{\alpha_i} x_2^{\beta_i} \cdots x_n^{\gamma_i} \right|,$$

where $c_i > 0$. For every $i \in [n_c]$, let $\mathcal{I}_i \subset \mathbb{N}$ denote the components of x with odd exponents. For example, the monomial $x_1^3 x_3^4 x_5$ has index set $\mathcal{I}_i = \{1, 5\}$. Let $E \triangleq \{(e_1, e_2, \ldots, e_n) : e_i \in \{-1, 1\}\}$ denote the set of vectors pointing into each orthant of $\mathbb{R}^n$. Note that the cardinality of E is 2^n. For every $e^k \in E$, we define the polynomial

$$\mathcal{P}_k(x) \triangleq \sum_{i=1}^{n_c} \left[\left(c_i x_1^{\alpha_i} x_2^{\beta_i} \cdots x_n^{\gamma_i} \right) \cdot \prod_{j \in \mathcal{I}_i} e_j^k \right].$$

In each orthant, one of the polynomials $\mathcal{P}_k(x)$ is dominant and corresponds to $b(x)$. From this observation, we obtain the following proposition.

Proposition 2. *For any $x \in \mathbb{R}^n$, $b(x) = \max_{k \in [2^n]} \mathcal{P}_k(x)$. Thus, for any $u \in \mathbb{R}^m$, $A(x)u + b(x) \leq 0$ if and only if $A(x)u + \mathcal{P}_k(x) \leq 0$ for every $k \in [2^n]$.*

The proposition shows that all of the 2^n polynomial constraints $A(x)u + \mathcal{P}_k(x) \leq 0$ can be used to replace the constraint $A(x)u + b(x) \leq 0$ in a SOS program. As an example, consider the polynomial $b(x) = |x_1 x_2| + |x_2^3|$. The set E is $E = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$ and the index sets for the monomials are $\mathcal{I}_1 = \{1, 2\}$ and $\mathcal{I}_2 = \{2\}$. Then the polynomial replacements are $\mathcal{P}_1(x) = x_1 x_2 + x_2^3$, $\mathcal{P}_2(x) = -x_1 x_2 + x_2^3$, $\mathcal{P}_3(x) = x_1 x_2 - x_2^3$, and $\mathcal{P}_4(x) = -x_1 x_2 - x_2^3$.

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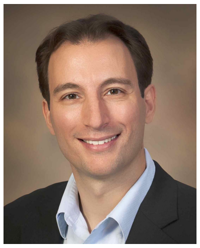

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