Pointwise Exponential Stability of State Consensus with Intermittent Communication

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***Abstract*— In this paper, we propose a solution to the problem of achieving global consensus of the states of scalar integrator systems over a directed graph when the network connecting the agents is available only at isolated (and possibly aperiodic) time instances. We propose decentralized consensus protocols that, using such intermittent information obtained at communication times, globally and asymptotically drives the values of their states to an agreement value, with stability and robustness to perturbations on the dynamics, the information exchanged over the network, and the communication times. Using stability analysis tools for hybrid systems, we recast the consensus problem as a set stabilization problem and leverage Lyapunov stability tools for the analysis of the networked system, both in the nominal and perturbed cases. When communication between the agents occurs synchronously, we show that the set of points characterizing consensus is globally exponentially stable, and, under some mild additional conditions, is partially pointwise globally exponentially stable. On the other hand, when communication occurs asynchronously, we show global asymptotic stability of consensus, for which we exploit well-posedness of the hybrid system modeling the network and an hybrid invariance principle. Results certifying robustness of the proposed consensus protocols, to a wide class of perturbations, are presented. Numerical examples illustrate the main results.**

I. Introduction

A. Motivation, Related Work, and Challenges

A common problem in distributed coordinated control of multi-agent systems is information agreement – or, equivalently, consensus – among all agents. This problem consists of multiple agents connected over a network sharing their local state with their neighbors to converge to a common value. The set of points to which their states shall converge is given by

$$x_1 = x_2 = \cdots = x_N. \tag{1}$$

where $x_i \in \mathbb{R}$ is the state of agent i. In this work, we consider a network of $N > 1$ agents with scalar integrator dynamics given by

$$\dot{x}_i = u_i \in \mathbb{R}, \qquad i \in \mathcal{V} := \{1, 2, \ldots, N\}. \tag{2}$$

Some of the main challenges in designing robust control protocols for consensus in real-world systems come from the unavoidable intermittency of information and asynchronous communication in the network. The question driving this work is how to design a protocol to achieve global consensus, robustly and with stable behavior, when information shared among the neighbors is available only intermittently without a priori information or local knowledge of initial conditions.

Consensus algorithms have been thoroughly employed in many scientific and engineering applications due to the prevalence of low-cost microcontrollers, sensors, and networks. The idea of consensus is rooted in the field of computer science, wherein agreement algorithms for multiagent systems were used for distributed computation; see [1]. From computer science, consensus of multiple agents entered the rigor of control theory which lead to continuous-time and discrete-time consensus algorithms [2]–[4]. However, these algorithms require that all agents pass information continuously at each time instant, or discretely and synchronously. In fact, one can find in the literature that when each agent can communicate *continuously* to their neighbors, the distributed control law given by[1]

$$u_i = -\gamma \sum_{k \in \mathcal{N}(i)} (x_i - x_k), \qquad \gamma > 0,$$

drives each agent, under certain network connectivity assumptions, in (2) to the average of the initial conditions of the agents; see, e.g., [2]. A key assumption for such desired convergence property is that the information between the agents is available continuously, namely, for all ordinary time t. In this paper, we are interested in the realistic case when communication between agents may occur intermittently – more specifically, at (not necessarily periodic) isolated time instances.

To reduce the communication times necessary to reach consensus, the combination of continuous-time and impulsive updates in terms of event-driven communication has been proposed in the literature. Recently, the topic of consensus of continuous-time systems with discrete update times is gaining some traction due to the efforts in cyber-physical systems and hybrid systems [5], [6]. In [7], the authors study a case of consensus (called therein as synchronization) where agents have nonlinear continuous-time dynamics with continuous coupling and impulsive perturbations. In [8], the authors investigate a self-triggered approach and establish robustness of consensus for the case of information delay and clock errors for quantized dynamic control inputs. In [9], the authors use Lyapunov-like analysis to derive sufficient conditions for the synchronization of continuously coupled nonlinear systems

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[1] $\mathcal{N}(i)$ is defined as the set of neighbors for agent i; see Section II-B for more information.

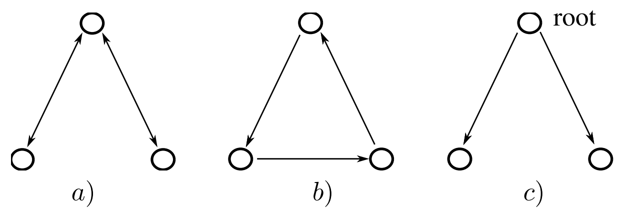

Fig. 1. a) An undirected connected graph; b) a directed strongly connected and weight balanced graph; c) a directed spanning tree with the root vertex labeled.

with impulsive resets on the difference between neighboring agents. In [10], the authors present a sampled-data consensus problem with both periodic and non-uniform update times for synchronous updates. In [11], investigates the maximum allowable transmission times between updates to yield asymptotic consensus. Similar to impulsive systems, consensus in systems where feedback controllers are designed as state-triggered discrete events appeared in [12]–[15]. In [12], a distributed event-triggered control strategy was developed to drive the outputs of the agents in a network to consensus. An observer-based event policy was developed in [13] for a network of linear time-invariant systems where communication events occur when the distance between the local state and its estimate is larger than a threshold. In [15], an event-based protocol was developed for continuous-time systems to achieve consensus using a Lyapunov based analysis. In [16], a controller to achieve consensus for multi-agent systems is proposed for the case when each agent transmits information to their neighbors continuously. In [17], an algorithm for leader-follower consensus under the presence of delays is designed by appropriately choosing the sampling period and the coupling strength. For a similar leader-follower consensus problem, the work in [18] proposes the design of a consensus algorithm under saturation constraints and with intermittent information exchanged over a network modeled by a directed graph – a similar problem but with limited information, for which an observer-based algorithm is devised, is addressed in [19]. In [20], [21], a heterogeneous timer synchronization algorithm was developed for interconnected systems. In [22], a case study of task allocation algorithms using a consensus-based approach over a dynamic graph. A setting in which the control input is only available over periodic time windows for consensus of multiple agents operating over a network is considered in [23]. The protocols designed in this paper extend our previous work in [24] to achieve consensus using measurements and communicated information, do not rely on a leader-follower architecture and, instead, are distributed, and allow for communication events to occur aperiodically.

For dynamic and control systems, the study of asymptotic stability is continued to be an important concept and is often a baseline requirement for control system development. More specifically, asymptotic stability is the notion that solutions to the dynamical system are convergent and Lyapunov stable to a point or a set of points, i.e., when a solution is initialized close (in some notion of distance) to this set of points it stays close. Pointwise asymptotic stability for a dynamical system is a property of a set of points that requires each point to be Lyapunov stable and that every solution to the system be convergent and have a limit in the set, [25]. This notion for both continuous-time and discrete-time systems are considered in [25]. In [26], pointwise asymptotic stability is considered for the case of hybrid systems, namely, for when the dynamics of a system may have both continuous-time and discrete-time dynamics.

B. Contributions and Organization

To the best of our knowledge, besides our preliminary results in [27], protocols that guarantee consensus with pointwise stability and robustness, when the information is only available at intermittent time instances are not available in the literature. Namely, many protocol design methods for consensus make assumptions on the times when communication occurs, [14], [28]. For instance, a common assumption is that communication occurs at time instances $t_k \in \mathbb{R}_{\geq 0}$ where $k \in \mathbb{N}$ indexes the sequential event times such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_N$$

where $\lim_{k\to\infty} t_k = \infty$ implying that t_0 $(= 0)$ is contained in the sequence of times; see e.g., [11], [29]–[32]. In this paper, we only require that the time instances are upper and lower bounded by positive constants, more over successive communication event times may occur at any time instance within this interval of time. We also do not assume that communication occurs at the initial time instant $t_0 = 0$. More specifically, we allow for some time to elapse before an initial communication to occur after the initial time. This assumption is more realistic in deployed systems due to the time it may take between initializing the consensus algorithm and establishing communication between nodes.

In this paper, we design a hybrid first-order-hold state-feedback protocol for consensus that undergoes an instantaneous change in its state when new information is available and evolves continuously between such events. Due to the combination of continuous state variables and the impulsive communication events, along with the continuous and discrete consensus protocol, we model the networked system as a closed-loop hybrid system as in [33], [34]. For the resulting hybrid closed-loop system, we recast the consensus problem as a set stability problem and apply tools for the study of Lyapunov stability to show the global stability properties of the consensus set. More specifically, the main contributions in this paper are as follows:

- Synchronous communication: when the connected agents receive neighboring information at the same event times,
  1) The consensus set in (1), under certain conditions, is shown to be globally exponentially stable through a Lyapunov-based stability analysis;
  2) We characterize the point to which solutions converge given the initial conditions of the hybrid closed-loop system and initial communication time;
  3) The consensus set is shown to be partially pointwise globally exponentially stable[2] (defined in Section III-G) with respect to the states of the agents and the (to be defined) hybrid controller states;
  4) The hybrid system is shown to be input-to-state stable with respect to the consensus set and relative to communication noise.

[2]Partial pointwise global exponential stability is defined for a given closed set that enjoys the typical global exponential stability property, but with the addition that each point in the set is also Lyapunov stable; more information on pointwise asymptotic stability can be found in [35].

- Asynchronous communication (the agents receive information from their neighbors at different event times).
  1) The consensus set in (1) is shown to be globally asymptotically stable through the application of an invariance principle;
  2) The consensus set is shown to be robust to a wide class of perturbations through a class-$\mathcal{KL}$ equivalence argument.

The remainder of this article is outlined as follows. Section II lists the notations used and some preliminaries on graph theory. In Section III, we consider the case of synchronous communications between agents and present the main results for partial pointwise asymptotic stability. We then consider the case for asynchronous communications in Section IV. In Section V, we give the robustness results.

Compared to [27], this article includes detailed proofs – [27] does not include any proofs –, Theorem 3.8 provides an explicit bound on solutions, Proposition 3.11 provides a new sufficient condition for consensus, and Section V thoroughly studies robustness of the algorithm – only measurement noise is considered in [27].

II. Notation and Preliminaries on Graph Theory

A. Notation

The set of natural numbers is denoted as $\mathbb{N}$, i.e., $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. The set $\mathbb{B} \subset \mathbb{R}^n$ is the closed unit circle centered at the origin, i.e., $\mathbb{B} := \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$. Given a square matrix A, the set $\text{eig}(A)$ collects the eigenvalues of A. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and the notation $[u^\top \ v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t \geq 0} |m(t)|$. Given a vector $x \in \mathbb{R}^n$ and a set $\mathcal{A} \subset \mathbb{R}^n$, the distance from x to the set $\mathcal{A}$ is given by $|x|_\mathcal{A} = \inf_{y \in \mathcal{A}} |x - y|$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \ \lambda \in \text{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \lambda \in \text{eig}(P)\}$. Given matrices A and B with appropriate dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$. Given $N \in \mathbb{N}$, $I_N \in \mathbb{R}^{N \times N}$ defines the N dimensional identity matrix, the vector $\mathbf{1}_N$ is a column vector of N ones, and the vector $\mathbf{0}_N$ is a column vector of N zeros. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, also written $\alpha \in \mathcal{K}$, if α is zero at zero, continuous, and strictly increasing; it is said to belong to class-$\mathcal{K}_\infty$, also written $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and is unbounded; α is positive definite, also written $\alpha \in \mathcal{PD}$, if $\alpha(s) > 0$ for all $s > 0$ and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$.

B. Preliminaries on Graph Theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \dots, N\}$ and the edges are pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two different nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can send information to agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} \in \mathbb{R}^{N \times N}$ with elements $g_{ik} \in \{0, 1\}$, where $g_{ik} = 1$ if $(i, k) \in \mathcal{E}$, and $g_{ik} = 0$ otherwise. The in-degree and out-degree of agent i are defined by $d^{in}(i) = \sum_{k=1}^N g_{ki}$ and $d^{out}(i) = \sum_{k=1}^N g_{ik}$. The largest (smallest) in-degree in the digraph is given by $\overline{d} = \max_{i \in \mathcal{V}} d^{in}(i)$ (respectively, $\underline{d} = \min_{i \in \mathcal{V}} d^{in}(i)$). The in-degree matrix $\mathcal{D}$ is the diagonal matrix with entries $\mathcal{D}_{ii} = d^{in}(i)$ for all $i \in \mathcal{V}$. The Laplacian matrix of the digraph Γ, denoted by $\mathcal{L}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{G}$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (k, i) \in \mathcal{E}\}$.

A digraph is said to be *weight balanced* if, at each node $i \in \mathcal{V}$, the out-degree and in-degree are equal; i.e., for each $i \in \mathcal{V}$, $d^{out}(i) = d^{in}(i)$; *complete* if every pair of vertices is connected by a unique edge, namely, $g_{ik} = 1$ for each $i, k \in \mathcal{V}$, $i \neq k$; and *strongly connected* if and only if any two nodes of the digraph can be connected via a path that traverses the directed edges of the digraph. A *spanning tree* is a directed graph whose vertices have exactly one parent except for one which is called a the root vertex. We say that a graph contains a spanning tree (or has a spanning tree) if the vertices $\mathcal{V}$ and a subset of edges in $\mathcal{E}$ can form a spanning tree. A graph is *undirected* if and only if $(i, k) \in \mathcal{E}$ and $(k, i) \in \mathcal{E}$. See Figure 1 for examples of network structures.

For a digraph, the Laplacian matrix with eigenvalues $\{\lambda_i\}_{i=1}^N$ is such that $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$. If the digraph contains a spanning tree, then zero is a simple eigenvalue of $\mathcal{L}$ and all other eigenvalues have positive real parts [36]. Moreover, there exists an orthogonal matrix U such that $U^\top A U = \begin{bmatrix} 0 & 0 \\ 0 & \star \end{bmatrix}$, where $\star$ represents any nonsingular matrix with an appropriate dimension and $\mathcal{L}$ has a zero eigenvalue with eigenvector $\mathbf{1}_N \in \mathbb{R}^N$ [37]. Given $\Pi = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$ it follows that $\Pi\mathcal{L} = \mathcal{L}$, furthermore, if the digraph is weight balanced $\Pi\mathcal{L} = \mathcal{L}\Pi = \mathcal{L}$. See [38] for more information on algebraic graph theory.

III. Pointwise Asymptotic Stability through Synchronous Communication

A. Problem Description

As mentioned in Section I-A, the goal of this paper is to design protocols that achieve global consensus for agents in (2) with intermittent communication. Namely, we want each agent with state x_i and dynamics in (2) to converge asymptotically to

$$x_1 = x_2 = \cdots = x_N,$$

which characterizes the consensus set. Such a property is typically referred to as static consensus and defined explicitly in the forthcoming Section III-B; see [36], [39].

In this paper, we consider each agent to be able to access the state information of their neighbors at time instances $t \in \{t_s\}_{s=1}^\infty$, where $s \in \mathbb{N} \setminus \{0\}$ is the communication event time index. Given the scalars $T_2 \geq T_1 > 0$, the sequence of times $\{t_s\}_{s=1}^\infty$ must satisfy

$$\begin{aligned} &T_1 \leq t_{s+1} - t_s \leq T_2 \qquad \forall\ s \in \{1, 2, \dots\} \\ &t_1 \leq T_2 \end{aligned} \tag{3}$$

where the positive scalars T_1 and T_2 define the lower and upper bounds on the minimum and maximum time allowed to elapse between consecutive communication instances, respectively. Note that different from much of the literature on intermittent communication and control, this particular formulation does not assume an initial communication event at $t_0 = 0$; see e.g., [10], [11], [29]–[32] for more information. Allowing such communications at the initial time implicitly implies that the controller states in each agent can be initialized to appropriate values. The formulation in (3) allows any (bounded) amount of continuous time (within $[0, T_2]$) before an initial communication event occurs. In the asynchronous communication case, this captures the real-world situation of agents receiving information at different times and rates than others. The sequence of times $\{t_s\}_{s=1}^{\infty}$ is not assumed to be known to any agent a priori. Due to the nonperiodic arrival of information and impulsive dynamics, classical analysis tools (for continuous-time or discrete-time systems) do not apply to the design of the proposed controller. This motivates us to design the proposed controller by recasting the interconnected systems, the impulsive network, and the proposed control protocol within a hybrid system framework; specifically, the one given in [33], [40].

The remainder of this section is dedicated to the modeling and stability analysis of the interconnected networked system with synchronous communication. In Section III-B we introduce the particular model technique we will leverage as well as determine a set characterizing consensus. In Section III-C, we use the connectivity properties of the graph to change the coordinates of the state to prepare for the analysis. In Section III-E, we give sufficient conditions for global exponential stability of the consensus set. Section III-F determines the point to which solutions converge. In Section III-G, we provide the main results of this paper in the form of sufficient conditions for partial pointwise exponential stability. Lastly, we provide some numerical simulations showcasing the results.

B. Hybrid Modeling and Consensus Protocol

The dynamics of the closed-loop system inherently contains both continuous-time – generated by the differential equations modeling the agents' dynamics in (2) – and discrete-time – governed by the impulsive communication times in (3) – dynamics which lead to a hybrid system model. Moreover, the protocols designed in this paper to achieve consensus also give rise to hybrid dynamics due to including a state variable that is updated both continuously and discretely. In this way, the state of the resulting hybrid closed-loop system is allowed to evolve continuously between update times and, at such time instances satisfying (3), the agents pass their internal states to their neighbors and update their controller state discretely.

In this work, we will leverage the hybrid systems framework defined in [33], [41] for modeling and Lyapunov-based analysis. A hybrid system $\mathcal{H}$ with state $\xi \in \mathbb{R}^n$ is defined by four objects (C, f, D, G). The single-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is the flow map and defines the continuous dynamics; the set $C \subset \mathbb{R}^n$ is the flow set and defines where the continuous dynamics are applied; the set-valued mapping $G : \mathbb{R}^n \quad \mathbb{R}^n$ is the jump map and defines the discrete changes in ξ; and, the set $D \subset \mathbb{R}^n$ is the jump set and defines where the discrete changes in ξ can occur. Then, this system can be written in compact form as

$$\mathcal{H}: \quad \left\{ \begin{array}{ll} \dot{\xi} \;\; = f(\xi) & \quad \xi \in C \\ \xi^+ \in G(\xi) & \quad \xi \in D. \end{array} \right. \qquad (4)$$

Solutions to the general hybrid system $\mathcal{H}$ in (4) are allowed to evolve both continuously and discretely. As such, a solution ϕ to the hybrid system $\mathcal{H}$ in (4) is parametrized by two independent variables, $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\bigcup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when ϕ jumps and j counts the number of jumps. The set $\mathcal{S}_{\mathcal{H}}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_{\mathcal{H}}(\xi_0)$ contains all maximal solutions to $\mathcal{H}$ from ξ_0. A solution to $\mathcal{H}$ is called maximal if it cannot be extended; i.e., if it is not a truncated version of another solution. A complete solution is a solution that has an unbounded domain. A Zeno solution is a complete solution with bounded domain in the t direction.

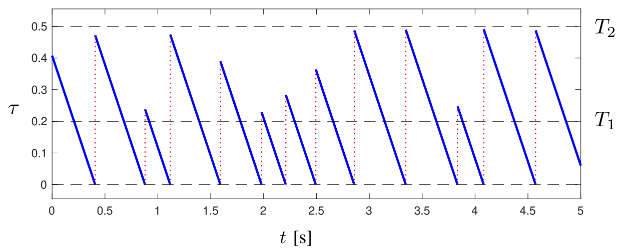

Fig. 2. A sample numerical solution to the decreasing nonperiodic hybrid timer in (5).

Definition 3.1: A hybrid system $\mathcal{H}$ in (4) is said to satisfy the *hybrid basic conditions* if

a) the sets C and D are closed;
b) the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous;
c) the set-valued mapping $G : \mathbb{R}^n \quad \mathbb{R}^n$ is outer semi-continuous and locally bounded relative to D, and $D \subset \operatorname{dom} G$.

Interestingly, as shown in [33], satisfying the hybrid basic conditions implies that hybrid systems are well-posed and, with asymptotic stability of a compact set, automatically gives robustness to small enough perturbations.

A convenient example of a hybrid system is given by a *decreasing nonperiodic hybrid timer*. More specifically, this hybrid timer consists of a continuously decreasing timer state $\tau \in [0, T_2]$ that, upon reaching zero, is reset to a point in the interval $[T_1, T_2]$. Inspired by [42], we can capture the dynamics of such a timer via

$$\begin{array}{ll} \dot{\tau} \;\; = -1 & \qquad \tau \in [0, T_2] \\ \tau^+ \in [T_1, T_2] & \qquad \tau = 0 \end{array} \qquad (5)$$

It follows that the timer formulation in (5) is capable of capturing any sequence of communication times given by $\{t_s\}_{s=1}^{\infty}$ satisfying (3). Figure 2 provides a sample numerical solution of (5) with $T_1 = 0.2$s and $T_2 = 0.5$s. In the remainder of this paper, we will leverage this hybrid system model for the timer to trigger the communication event times between each agent in the network.

Recalling the informal definition of consensus in Section III-A and using the definition of solutions to hybrid systems, we formally define static consensus next.

Definition 3.2 (static consensus): Given the agents in (2) over a digraph Γ, a control protocol u_i is said to globally solve the consensus problem if every resulting maximal solution $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ with $u = (u_1, u_2, \ldots, u_N)$ is complete and satisfies

$$\lim_{t+j\to\infty} |\phi_i(t,j) - \phi_k(t,j)| = 0$$

for each $i,k \in \mathcal{V}$, $i \neq k$.

To achieve this property under intermittent information, we propose the following distributed hybrid consensus protocol, which assigns the input of the i-th agent u_i based on the communicated value of the states of the neighboring agents obtained at the isolated communication events generated by τ reaching zero.

Protocol 3.3: Given the parameter $T_2 > 0$ of the network, the i-th hybrid controller has state η_i with the following dynamics:

$$\begin{aligned} u_i &= \eta_i \\ \dot{\eta}_i &= -h\eta_i & \tau \in [0, T_2] \\ \eta_i^+ &= -\gamma \sum_{k\in\mathcal{N}(i)} (x_i - x_k) & \tau = 0 \end{aligned} \tag{6}$$

where $h \geq 0$ and $\gamma > 0$ are the controller parameters to be designed.

Remark 3.4: The parameter h affects the change of the state η_i of each agent during flows, namely, between update times, making it another useful design parameter. If $h > 0$ (or $h < 0$), then the state η_i exponentially decreases (or increases, respectively) in between updates times. If $h = 0$, then the state η_i remains constant in between such updates, leading to a zero-order hold controller. Allowing h to be nonzero provides some control on the rate of consensus, for example, to not overshoot consensus. The parameter γ scales the so-called (local) consensus error for the i-th agent, which is given by the sum of the difference between the neighboring states and the state of the i-th agent.

We denote $\mathcal{H}$ as the closed-loop hybrid system resulting from the interconnection of the agents dynamics in (2), the algorithm in Protocol 3.3, and the hybrid model of the aperiodic decreasing timer in (5) to trigger the communication events at any instants satisfying (3). The state of $\mathcal{H}$ is given by $\xi = (x, \eta, \tau) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T_2] =: \mathcal{X}$, where $x = (x_1, x_2, \ldots, x_N)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ comprise the system states and controller states of all agents, respectively. We have that $\mathcal{H}$ is given by

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} \eta \\ -h\eta \\ -1 \end{bmatrix} =: f(\xi) & \xi \in C := \mathcal{X}, \\ \xi^+ &\in \begin{bmatrix} x \\ -\gamma\mathcal{L}x \\ [T_1, T_2] \end{bmatrix} =: G(\xi) & \xi \in D := \mathbb{R}^N \times \mathbb{R}^N \times \{0\}. \end{aligned} \tag{7}$$

Using the consensus notion in Definition 3.2, the states of the agents have to converge to $x_i = x_k$ for each $i, k \in \mathcal{V}$. Since the state η_i in Protocol 3.3 is updated to the (local) consensus error at communication events, when consensus occurs the state η must converge to zero. Therefore, the goal is to render the set

$$\mathcal{A} := \{\xi = (x, \eta, \tau) \in \mathcal{X} : x_i - x_k = 0, \eta_i = 0 \ \forall i, k \in \mathcal{V}\} \tag{8}$$

globally exponentially stable and, under extra conditions, partially pointwise exponentially stable with respect to (x, η) for the hybrid system $\mathcal{H}$ in (7), which is notion that is defined in the forthcoming Section III-G. In the next section, we consider a change of coordinates exploiting the natural structure of the network that facilitates analysis and design of the protocol.

C. Change of Coordinates

We establish the main results of this work through a change of coordinates for (7) that leverages key properties of the graph structure. More precisely, let Γ contain a directed spanning tree. Using the properties for graphs summarized in Section II, the Laplacian $\mathcal{L}$ associated with Γ is positive semi-definite and, as such, there exists a nonsingular matrix that satisfies $U = [u_1, U_1]$ such that $U^\top \mathcal{L} U = \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathcal{L}} \end{bmatrix}$, where $u_1 = \alpha \mathbf{1}_N$, $\alpha \in \mathbb{R}$, and $U_1 \in \mathbb{R}^{N\times(N-1)}$. The matrix $\bar{\mathcal{L}}$ is a diagonal matrix that has diagonal elements given by the positive eigenvalues of $\mathcal{L}$, namely, $\lambda_i > 0$ for each $i \in \{2, 3, \ldots, N\}$. The change of coordinates $\bar{x} = U^\top x$ and $\bar{\eta} = U^\top \eta$ applied to $\mathcal{H}$ in (7) leads to

$$\begin{aligned} \dot{\bar{x}} &= U^\top \dot{x} = U^\top \eta = \bar{\eta} \\ \dot{\bar{\eta}} &= U^\top \dot{\eta} = -hU^\top \eta = -h\bar{\eta}. \end{aligned} \tag{9}$$

At jumps and using the definition of $\mathcal{H}$ in (7), $\bar{x}$ and $\bar{\eta}$ are updated via

$$\begin{aligned} \bar{x}^+ &= U^\top x^+ = U^\top x = \bar{x} \\ \bar{\eta}^+ &= -\gamma U^\top \mathcal{L} x = -\gamma U^\top \mathcal{L} U \bar{x} = -\gamma \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathcal{L}} \end{bmatrix} \bar{x} \end{aligned}$$

due to the properties of U in Section II.

The resulting hybrid system in the new coordinates is denoted as $\widetilde{\mathcal{H}}$. Its state χ is defined by collecting the scalar states $\bar{x}_1$ and $\bar{\eta}_1$ into $\bar{z}_1 = (\bar{x}_1, \bar{\eta}_1)$ and stacking the remaining states of $\bar{x}$ and $\bar{\eta}$ into $\bar{z}_2 = (\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_N, \bar{\eta}_2, \bar{\eta}_3, \ldots, \bar{\eta}_N)$. Then, the state χ is given by $\chi = (\bar{z}_1, \bar{z}_2, \tau) \in \mathcal{X}$. The new coordinates lead to the hybrid system $\widetilde{\mathcal{H}}$ with the following data:

$$\begin{aligned} \tilde{f}(\chi) &:= \begin{bmatrix} A_{f1}\bar{z}_1 \\ A_{f2}\bar{z}_2 \\ -1 \end{bmatrix} & \forall \chi \in \widetilde{C} := \mathcal{X} \\ \widetilde{G}(\chi) &:= \begin{bmatrix} A_{g1}\bar{z}_1 \\ A_{g2}\bar{z}_2 \\ [T_1, T_2] \end{bmatrix} & \forall \chi \in \widetilde{D} := \{\chi \in \mathcal{X} : \tau = 0\} \end{aligned} \tag{10}$$

where

$$\begin{aligned} A_{f1} &= \begin{bmatrix} 0 & 1 \\ 0 & -h \end{bmatrix}, & A_{g1} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ A_{f2} &= \begin{bmatrix} 0 & I_{N-1} \\ 0 & -hI_{N-1} \end{bmatrix}, & A_{g2} &= \begin{bmatrix} I_{N-1} & 0 \\ -\gamma\bar{\mathcal{L}} & 0 \end{bmatrix} \end{aligned} \tag{11}$$

$h \geq 0$ and $\gamma > 0$. Moreover, in the new coordinates and from the definition of $\mathcal{A}$ in (8), the set to stabilize for the hybrid system $\widetilde{\mathcal{H}}$ is given as

$$\widetilde{\mathcal{A}} := \{\chi = (\bar{z}_1, \bar{z}_2, \tau) \in \mathcal{X} : \bar{z}_1 = (x^*, 0), x^* \in \mathbb{R}, \bar{z}_2 = 0\}. \tag{12}$$

The set $\widetilde{\mathcal{A}}$ is derived directly from the change of coordinates $\bar{x}$ and $\bar{\eta}$ and the properties of Laplacian matrices discussed in Section II; see the proof of the forthcoming Lemma 3.13 in Section III-E. Note that the first component of $\bar{z}_1$ in $\widetilde{\mathcal{A}}$ is free. In the next section, we present basic equivalency properties between $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ and their solutions.

D. Basic properties of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$

In this section, we provide some basic properties of solutions for both $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ as well as show that these hybrid systems satisfy the hybrid basic conditions introduced in Definition 3.1.

Lemma 3.5: Suppose the positive scalars T_1 and T_2 satisfy $T_1 \leq T_2$. The hybrid systems $\mathcal{H}$ in (7) *and $\widetilde{\mathcal{H}}$ in* (10) *satisfy the hybrid basic conditions.*

Proof: The jump and flow sets of both systems are closed by definition. The flow maps $\widetilde{f}$ and f are continuous. The jump map $\widetilde{G}$ is outer semicontinuous via [33, Lemma 5.10] since its graph, which is given by $\{(x, y) : x \in \widetilde{D}, y \in \widetilde{G}(x)\}$ is closed due to the interval $[T_1, T_2]$ being closed. Furthermore, $\widetilde{G}(\chi)$ is bounded and nonempty for each $\chi \in \widetilde{D}$. Similar arguments can be used to show that G is outer semicontinuous. ■

With the definition of solutions to hybrid systems in Section III-B, we have the following result.

Lemma 3.6: Let $0 < T_1 \leq T_2$ be given. Every maximal solution ϕ to the hybrid system $\mathcal{H}$ in (7) *satisfies the following:*

1) *ϕ is complete, i.e., $\operatorname{dom}\phi$ is unbounded.*
2) *for each $(t, j) \in \operatorname{dom}\phi$, $(j-1)T_1 \leq t \leq (j+1)T_2$ for all $j \geq 1$.*

The same properties hold for every maximal solution to $\widetilde{\mathcal{H}}$ in (10).

Proof: This proof leverages the results in [33, Proposition 6.10] to establish completeness of maximal solutions to $\mathcal{H}$. Due to spatial constraints, we will only show that these properties hold for $\mathcal{H}$; however, similar steps establish these properties for $\widetilde{\mathcal{H}}$. Given the hybrid system $\mathcal{H}$ with $0 < T_1 \leq T_2$, we first show completeness of maximal solutions. Note that for any $\xi \in C \setminus D$, we have that the tangent cone[3] satisfies $T_C(\xi) \cap f(\xi) \neq \emptyset$ since $T_C(\xi) = \mathbb{R}^{2n+1}$ for all $\xi \in C \setminus D$, which implies that every solution from $\xi \in C \setminus D$ is nontrivial; namely, every maximal solution from $\xi \in C \setminus D$ has a domain with at least two points [33]. Moreover, for each $\xi \in C \cap D$, solutions from ξ cannot be extended via flow. Due to the fact that the flow map is linear, finite escape time during flows is impossible. Lastly, it is straightforward to check that $G(D) \subset C \cup D$. Then, since the hybrid system $\mathcal{H}$ satisfies the hybrid basic conditions, every maximal solution to $\mathcal{H}$ is complete by [33, Proposition 6.10].

Next, we show item 2). Note that the jumps are triggered when $\xi \in D$, implying that jumps are triggered only by the timer state τ and jump times of solutions satisfy (3). For a solution ϕ with domain $\operatorname{dom}\phi$, let the jump times be given by $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_j$. From (3), we have that $0 \leq t_1 \leq T_2$, by definition. Then, it follows that $T_1 \leq t_2 \leq 2T_2$, $2T_1 \leq t_3 \leq 3T_2$, $3T_1 \leq t_4 \leq 4T_2$, which can be generalized to $(j-1)T_1 \leq t_j \leq jT_2$. Therefore, since $t \in [t_j, t_{j+1}]$ for each $(t, j) \operatorname{dom}\phi$, we obtain item 2). ■

E. Global Exponential Stability Results

In this section, we study the exponential stability of the set $\widetilde{\mathcal{A}}$ for the hybrid system $\widetilde{\mathcal{H}}$ and reveal some of its consequences. We also show the equivalence between uniform global exponential stability (UGES) for $\widetilde{\mathcal{H}}$ in (10) and UGES for $\mathcal{H}$ in (7). The notion of exponential stability used is given as follows; see [40, Definition 3.11].

[3] The tangent cone to a set $S \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^N$ is the set of all vectors $w \in \mathbb{R}^N$ for which there exist $x_i \in S$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \searrow 0$, and $w = \lim_{i \to \infty} \frac{x_i - x}{\tau_i}$; see [33, Definition 5.12] for more information.

Definition 3.7: (uniform global exponential stability) Let a hybrid system $\mathcal{H}$ with the state in $\mathbb{R}^n$ be given and $\mathcal{A} \subset \mathbb{R}^n$ be nonempty and closed. The set $\mathcal{A}$ is said to be *uniformly globally exponentially stable* (UGES) for $\mathcal{H}$ if there exist $\kappa, \alpha > 0$ such that every maximal solution ϕ to $\mathcal{H}$ is complete and satisfies

$$|\phi(t, j)|_{\mathcal{A}} \leq \kappa \exp(-\alpha(t + j))|\phi(0, 0)|_{\mathcal{A}} \tag{13}$$

for all $(t, j) \in \operatorname{dom}\phi$.

Inspired by [42] and using Lemma 3.13, we have the following stability result for $\widetilde{\mathcal{H}}$.

Theorem 3.8: Let T_1 and T_2 be two positive scalars satisfying $T_1 \leq T_2$. Let the digraph Γ contain a directed spanning tree. The set $\widetilde{\mathcal{A}}$ in (12) *is uniformly globally exponentially stable for the hybrid system $\widetilde{\mathcal{H}}$ with data in* (10) *if scalars $\gamma > 0$ and $h \geq 0$ are selected such that there exists $P = P^\top > 0$ satisfying*

$$A_{g2}^\top \exp(A_{f2}^\top \nu) P \exp(A_{f2}\nu) A_{g2} - P < 0 \tag{14}$$

for all $\nu \in [T_1, T_2]$, where A_{g2} and A_{f2} are defined in (11). *Furthermore, every maximal solution ϕ to $\widetilde{\mathcal{H}}$ satisfies*

$$|\phi(t, j)|_{\widetilde{\mathcal{A}}} \leq \exp\left(\frac{R}{2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}} \exp\left(-\frac{\alpha}{2}(t + j)\right) |\phi(0, 0)|_{\widetilde{\mathcal{A}}} \tag{15}$$

for all $(t, j) \in \operatorname{dom}\phi$ where $\alpha \in \left(0, \frac{|\lambda_d|}{1+T_2}\right]$, $R \in \left[\frac{T_2|\lambda_d|}{1+T_2}, \infty\right)$, $\lambda_d = \ln\left(1 - \frac{\beta}{\alpha_2}\right)$, $\beta = \min\{\kappa, 1\}$ and

$$\alpha_1 = \min_{s \in [0, T_2]} \left\{\underline{\lambda}\left(\exp(A_{f1}^\top s) \exp(A_{f1}s)\right), \underline{\lambda}\left(\exp(A_{f2}^\top s) P \exp(A_{f2}s)\right)\right\}$$

$$\alpha_2 = \max_{s \in [0, T_2]} \left\{\overline{\lambda}\left(\exp(A_{f1}^\top s) \exp(A_{f1}s)\right), \overline{\lambda}\left(\exp(A_{f2}^\top s) P \exp(A_{f2}s)\right)\right\}$$

$$\kappa \in \left(0, -\min_{\nu \in [T_1, T_2]} \underline{\lambda}(A_{g2}^\top \exp(A_{f2}^\top \nu) P \exp(A_{f2}\nu) A_{g2} - P)\right]. \tag{16}$$

Proof: First, note that from Lemma 3.6 we have that all maximal solutions to $\widetilde{\mathcal{H}}$ are complete. Next, consider the Lyapunov function candidate

$$V(\chi) = V_1(\chi) + V_2(\chi) \tag{17}$$

to establish uniform global exponential stability of $\widetilde{\mathcal{A}}$ for $\widetilde{\mathcal{H}}$, where, for each $\chi \in \mathcal{X}$, we have

$$V_1(\chi) = \exp(-2h\tau)\bar{\eta}_1^2$$
$$V_2(\chi) = \bar{z}_2^\top \exp(A_{f2}^\top \tau) P \exp(A_{f2}\tau)\bar{z}_2$$

with $P = P^\top > 0$. Due to the definition of $\widetilde{\mathcal{A}}$, the state $\bar{x}_1$ and τ are free in the domain of the state space, i.e., $\bar{x}_1 \in \mathbb{R}$ and $\tau \in [0, T_2]$. It follows that for each $\chi \in \widetilde{\mathcal{A}}$, $V(\chi) = 0$ and, for each $\chi \in (\widetilde{C} \cup \widetilde{D}) \setminus \widetilde{\mathcal{A}}$, $V(\chi) > 0$. Furthermore, V satisfies

$$\alpha_1|\chi|_{\widetilde{\mathcal{A}}}^2 \leq V(\chi) \leq \alpha_2|\chi|_{\widetilde{\mathcal{A}}}^2 \tag{18}$$

for all $\chi \in \mathcal{X}$, where α_1 and α_2 are defined in (16). Then, for each $\chi \in \widetilde{C}$, we have that

$$\begin{aligned}\langle \nabla V(\chi), \widetilde{f}(\chi)\rangle &= -2h\exp(-2h\tau)\bar{\eta}_1^2 + 2h\exp(-2h\tau)\bar{\eta}_1^2 \\ &\quad + 2\bar{z}_2^\top \exp(A_{f2}^\top \tau) P \exp(A_{f2}\tau) A_{f2}\bar{z}_2 \\ &\quad - 2\bar{z}_2^\top \exp(A_{f2}^\top \tau) P \exp(A_{f2}\tau) A_{f2}\bar{z}_2 = 0\end{aligned}$$

by using the property of commutability for exponential matrices. At jumps, $\tau = 0$ and, after the jump, τ is updated to a scalar ν in the interval $[T_1, T_2]$. Recall from Section III-C that the change in coordinates above (9) requires the graph to have a directed spanning tree. This property allows for the partition of the states $\bar{z}_1$ and $\bar{z}_2$ with the diagonal Laplacian used during jumps. Moreover, also at jumps, the states $\bar{z}_1$ and $\bar{z}_2$ are updated to $A_{g1}\bar{z}_1$ and $A_{g2}\bar{z}_2$, respectively. Due to the form of the matrix A_{g1}, the change in V_1 at such points satisfies $V_1(g) - V_1(\chi) = -\bar{\eta}_1^2 \leq 0$ for each $\chi \in \widetilde{D}, g \in \widetilde{G}(\chi)$, implying that $V(g) - V(\chi) \leq V_2(g) - V_2(\chi)$. It follows that, for each $\chi \in \widetilde{D}$, and each $g \in \widetilde{G}(\chi)$, we have that the change in V_2 is given by

$$\begin{aligned} V_2(g) - V_2(\chi) &\leq \bar{z}_2^\top A_{g2}^\top \exp(A_{f2}^\top \nu) P \exp(A_{f2}\nu) A_{g2}\bar{z}_2 \\ &\qquad - \bar{z}_2^\top P \bar{z}_2 \\ &\leq \bar{z}_2^\top (A_{g2}^\top \exp(A_{f2}^\top \nu) P \exp(A_{f2}\nu) A_{g2} - P)\bar{z}_2 \end{aligned}$$

where ν is the third component of g. Due to the continuity of (14) with respect to ν, there exists a sufficiently small κ in (16) such that

$$V(g) - V(\chi) \leq -\bar{\eta}_1^2 - \kappa|\bar{z}_2|^2 \leq -\min\{\kappa, 1\}|\chi|^2_{\widetilde{\mathcal{A}}},$$

We employ the bounds in (18) to arrive to $V(g) - V(\chi) \leq -\frac{\beta}{\alpha_2}V(\chi)$. Let $\lambda_d = \ln(1 - \beta/\alpha_2)$, which is negative since β can be chosen to be arbitrarily small and positive. Then, $V(g) \leq \exp(\lambda_d)V(\chi)$ for each $\chi \in \widetilde{D}$ and each $g \in \widetilde{G}(\chi)$.

It remains to show that the distance of solutions to the set $\widetilde{\mathcal{A}}$ is bounded above by an exponentially decreasing function with respect to hybrid time and the initial conditions in the form of (13). With some abuse of notation, consider a maximal solution $\phi = (\phi_1, \phi_2, \phi_\tau)$ to $\widetilde{\mathcal{H}}$ where ϕ_1 and ϕ_2 correspond to states $\bar{z}_1$ and $\bar{z}_2$, respectively. Since $\langle \nabla V(\chi), \widetilde{f}(\chi)\rangle = 0$ between jumps, direct integration of $(t, j) \mapsto V(\phi(t, j))$ over $\operatorname{dom}\phi$ leads to $V(\phi(t, j)) \leq \exp(\lambda_d j)V(\phi(0, 0))$ for each $(t, j) \in \operatorname{dom}\phi$. Pick $\alpha \in \left(0, \frac{|\lambda_d|}{1+T_2}\right]$ and $R \in \left[\frac{T_2|\lambda_d|}{1+T_2}, \infty\right)$. In light of Lemma 3.6, straightforward computations lead to $\lambda_d j \leq R - \alpha(t + j)$ for all $(t, j) \in \operatorname{dom}\phi$. More specifically, we have that from (18),

$$\begin{aligned} \alpha_1|\phi(t, j)|^2_{\widetilde{\mathcal{A}}} &\leq V(\phi(t, j)) \leq \exp(\lambda_d j)V(\phi(0, 0)) \\ &\leq \exp(R - \alpha(t + j))V(\phi(0, 0)) \\ &\leq \alpha_2 \exp(R)\exp(-\alpha(t + j))|\phi(0, 0)|^2_{\widetilde{\mathcal{A}}} \end{aligned}$$

which leads to (15). Hence, $\widetilde{\mathcal{A}}$ is UGES for $\widetilde{\mathcal{H}}$ in (10). ■

Note that the graphical topology is implicitly considered within the condition (14). Theorem 3.8 considers the case of a network topology having a directed spanning tree. The next result is for the case with a completely connected graph.

Corollary 3.9: Let T_1 and T_2 be two positive scalars satisfying $T_1 \leq T_2$ and the digraph Γ be completely connected. Suppose that scalars $h \geq 0$ and $\gamma > 0$ are chosen such that there exists $P = P^\top > 0$ satisfying

$$\tilde{A}_g^\top \exp(\tilde{A}_f^\top \nu) P \exp(\tilde{A}_f \nu)\tilde{A}_g - P < 0 \tag{19}$$

for all $\nu \in [T_1, T_2]$ where

$$\tilde{A}_f = \begin{bmatrix} 0 & 1 \\ 0 & -h \end{bmatrix}, \qquad \tilde{A}_g = \begin{bmatrix} 1 & 0 \\ -\gamma N & 0 \end{bmatrix},$$

and N is the number of agents in the network. Then, the set $\widetilde{\mathcal{A}}$ is uniformly globally exponentially stable for $\widetilde{\mathcal{H}}$.

Proof: Using the change of coordinates in (9), the result follows from the proof of Theorem 3.8, where, for a completely connected network $\bar{\mathcal{L}} = NI_{N-1}$. More specifically, this results in the hybrid system $\mathcal{H}$ in (10) with matrices in (11). Moreover, the elements on the diagonal of $\bar{\mathcal{L}}$ are identical and the block elements in (11) are all diagonal matrices, which leads to the subsystem dynamics

$$\begin{aligned} \dot{\bar{x}}_i &= \bar{\eta}_i, \\ \dot{\bar{\eta}}_i &= -h\bar{\eta}_i \end{aligned}$$

during flows, while during jumps we have

$$\begin{aligned} \bar{x}_i^+ &= \bar{x}_i, \\ \bar{\eta}_i^+ &= -\gamma N x_i \end{aligned}$$

for each $i \in \{2, 3, \ldots, N\}$. Note that each i-th system is independent and identical. Then, following the steps in the proof of Theorem 3.8 leads to condition (14). Due to the structure of each i-th system, it follows that (14) is reduced to (19). ■

Remark 3.10: Condition (14) may be difficult to satisfy numerically as it is not convex in γ, h and P. Moreover, it needs to be satisfied for infinitely many values of $\nu \in [T_1, T_2]$. In [43], the authors use a polytopic embedding strategy to reduce the problem into a linear matrix inequality problem in which one needs to find finitely many matrices X_i such that, for each $\nu \in [T_1, T_2]$, the exponential matrix $\exp(A_{f2}\nu)$ is contained in the convex hull of the X_i matrices. That methodology can be leveraged to find parameters that satisfy (14).

Condition (14) has a form that is similar to the discrete Lyapunov equation $A^\top PA - P < 0$. Using this observation, the following result gives a sufficient condition to satisfy (14).

Proposition 3.11: Given $0 < T_1 \leq T_2$ and a digraph Γ containing a directed spanning tree, if there exists $P = P^\top > 0$ such that (14) holds then there exist $\gamma > 0$ and $h \geq 0$ satisfying

$$\left|1 + \frac{\lambda_N \gamma(\exp(-hT_2) - 1)}{h}\right| < 1$$

where λ_N is the largest eigenvalue of $\mathcal{L}$.

Proof: Since P is positive definite and satisfies (14), the spectral radius of the matrix $\exp(A_{f2}\nu)A_{g2}$ can be found as follows[4]. Due to the form of A_{f2} and A_{g2}, it follows that[5]

$$\exp(A_{f2}\nu)A_{g2} = \begin{bmatrix} I + \frac{\gamma(\exp(-h\nu)-1)}{h}\bar{\mathcal{L}} & 0 \\ -\frac{\gamma\exp(-h\nu)}{h}\bar{\mathcal{L}} & 0 \end{bmatrix} =: \widetilde{A}.$$

Note that since Γ is strongly connected, the eigenvalues λ_i of the Laplacian are such that $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$ and the diagonal matrix $\bar{\mathcal{L}} = \operatorname{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N)$. Since the $(1, 1)$ block matrix of $\widetilde{A}$ is diagonal, it follows that the eigenvalues of $\widetilde{A}$ are equal to the diagonal elements of the (1,1) block, i.e., $1 + \frac{\gamma(\exp(-h\nu)-1)}{h}\lambda_i$ for each $i \in \{2, \ldots, N\}$ and zero with a multiplicity of $N - 1$. The spectral radius is given by the maximum value of the eigenvalue of $1 + \frac{\gamma(\exp(-h\nu)-1)}{h}\lambda_i$, namely, with $\nu \in [T_1, T_2]$,

$$\begin{aligned} \max_{i\in\{2,3,\ldots,N\}, \nu\in[T_1,T_2]} &\left|1 + \frac{\lambda_i\gamma(\exp(-h\nu) - 1)}{h}\right| \\ &\leq \left|1 + \frac{\lambda_N\gamma(\exp(-hT_2) - 1)}{h}\right| \end{aligned}$$

[4]The spectral radius of a matrix M is the maximum absolute values of the eigenvalues of M.

[5]Note that the matrix exponential of a lower block diagonal matrix $X = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ is given by $\exp(Xt) = \begin{bmatrix} \exp(At) & 0 \\ \int_0^t \exp(t-\tau)AB\exp(C\tau)d\tau & \exp(Ct) \end{bmatrix}$.

which, by assumption is less than one, where λ_N is the largest eigenvalue of $\mathcal{L}$. ■

Remark 3.12: Under the assumptions in Proposition 3.11, which require the digraph to contain a directed spanning tree, a similar analysis to the proof therein can be done to determine the sufficient conditions for a sample-and-hold consensus controller, i.e., $h = 0$. It can be found that there exists a positive definite and symmetric matrix P that satisfies (14) if $\gamma \in (0, 2/(\lambda_N T_2))$.

Next, we show that uniform global exponential stability of $\widetilde{\mathcal{A}}$ for $\widetilde{\mathcal{H}}$ implies that $\mathcal{A}$ is uniformly globally exponentially stable for $\mathcal{H}$.

Lemma 3.13: Let T_1 and T_2 be two positive scalars satisfying $T_1 \le T_2$ and a digraph Γ containing a directed spanning tree be given. The set $\mathcal{A}$ is uniformly globally exponentially stable for $\mathcal{H}$ if and only if the set $\widetilde{\mathcal{A}}$ is uniformly globally exponentially stable for $\widetilde{\mathcal{H}}$.

Proof: Given a solution $\widetilde{\phi}$ to $\widetilde{\mathcal{H}}$, we build a solution $\bar{\phi}$ using the associated coordinates in (9). Denote $\Upsilon = \operatorname{diag}(U, U, 1)$, which is also orthogonal since its block diagonal elements are orthogonal. Due to the fact that $x = U\bar{x}$ and $\eta = U\bar{\eta}$, following from the change of coordinates (since U is orthogonal), we have that $\phi = (U\phi_{\bar{x}}, U\phi_{\bar{\eta}}, \phi_\tau)$ is a solution to $\mathcal{H}$. Following [21, Lemma 5.5], we have that $|\chi|_{\widetilde{\mathcal{A}}} \le |\Upsilon^{-1}||\xi|_{\mathcal{A}}$ and $|\xi|_{\mathcal{A}} \le |\Upsilon||\chi|_{\widetilde{\mathcal{A}}}$ for each χ and ξ such that $\xi = \Upsilon\chi$; however, with Υ being an orthogonal matrix, it follows that $|\chi|_{\widetilde{\mathcal{A}}} = |\xi|_{\mathcal{A}}$.

To show sufficiency, since $\widetilde{\mathcal{A}}$ is UGES for $\widetilde{\mathcal{H}}$, each solution $\widetilde{\phi}$ to $\widetilde{\mathcal{H}}$ satisfies

$$|\widetilde{\phi}(t,j)|_{\widetilde{\mathcal{A}}} \le \kappa \exp(-\alpha(t+j))|\widetilde{\phi}(0,0)|_{\widetilde{\mathcal{A}}} \tag{20}$$

for each $(t,j) \in \operatorname{dom}\widetilde{\phi}$, where $\kappa, \alpha > 0$. From (20), we have that

$$\begin{aligned}|\phi(t,j)|_{\mathcal{A}} = |\widetilde{\phi}(t,j)|_{\widetilde{\mathcal{A}}} &\le \kappa \exp(-\alpha(t+j))|\widetilde{\phi}(0,0)|_{\widetilde{\mathcal{A}}} \\ &\le \kappa \exp(-\alpha(t+j))|\phi(0,0)|_{\mathcal{A}}\end{aligned}$$

which leads to $\mathcal{A}$ being UGES for $\mathcal{H}$. To show the necessity of the claim, let $\mathcal{A}$ be UGES for $\mathcal{H}$, which, for some $\alpha, \kappa > 0$, leads to each solution ϕ satisfying $|\phi(t,j)|_{\mathcal{A}} \le \kappa\exp(-\alpha(t+j))|\phi(0,0)|_{\mathcal{A}}$ for each $(t,j) \in \operatorname{dom}\phi$. We have that

$$\begin{aligned}|\widetilde{\phi}(t,j)|_{\widetilde{\mathcal{A}}} = |\phi(t,j)|_{\mathcal{A}} &\le \kappa \exp(-\alpha(t+j))|\phi(0,0)|_{\mathcal{A}} \\ &\le \kappa \exp(-\alpha(t+j))|\widetilde{\phi}(0,0)|_{\widetilde{\mathcal{A}}}\end{aligned}$$

which leads to $\widetilde{\mathcal{A}}$ being UGES for $\widetilde{\mathcal{H}}$. ■

F. Consensus Convergence Point

The following result characterizes the point to which maximal solutions $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ to $\mathcal{H}$ converge, where $\phi_x = (\phi_{x_1}, \phi_{x_2}, \dots, \phi_{x_N})$ and $\phi_\eta = (\phi_{\eta_1}, \phi_{\eta_2}, \dots, \phi_{\eta_N})$.

Proposition 3.14: Let T_1 and T_2 be two positive scalars satisfying $T_1 \le T_2$ and the digraph Γ be weight balanced and contain a directed spanning tree. If γ and h are chosen such that $\mathcal{A}$ is UGES for $\mathcal{H}$, then every solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ to $\mathcal{H}$ satisfies $\lim_{t+j\to\infty} \phi_\eta(t,j) = 0$ and, for each $i \in \mathcal{V}$,

$$\lim_{t+j\to\infty} \phi_{x_i}(t,j) = \frac{1}{N}\sum_{i=1}^{N}(\phi_{x_i}(0,0) + \phi_{\eta_i}(0,0)\phi_\tau(0,0)) \tag{21}$$

when $h = 0$, and when $h > 0$ it follows that

$$\begin{aligned}\lim_{t+j\to\infty}\phi_{x_i}(t,j) = \frac{1}{N}\sum_{i=1}^{N}\Bigg(&\phi_{x_i}(0,0) \\ &+ \frac{\exp(-h\phi_\tau(0,0)) - 1}{h}\phi_{\eta_i}(0,0)\Bigg).\end{aligned}$$

Proof: Let $\delta_x = \frac{1}{N}\sum_{i\in\mathcal{V}} x_i$ and $\delta_\eta = \frac{1}{N}\sum_{i\in\mathcal{V}}\eta_i$ be the average of the system states and controller states, respectively. The dynamics of $(\delta_x, \delta_\eta, \tau)$ is given by

$$\left\{\begin{array}{ll}\dot{\delta}_x &= \delta_\eta \\ \dot{\delta}_\eta &= -h\delta_\eta \\ \dot{\tau} &= -1\end{array}\right. \qquad \tau \in [0, T_2]. \tag{22}$$

Since the digraph is weight balanced, we have $d^{in}(i) = d^{out}(i)$ for each $i \in \mathcal{V}$. Note that δ_η can be rewritten as $\delta_\eta = \frac{1}{N}\mathbf{1}_N^\top\eta$. At jumps, when $\tau = 0$, we have

$$\delta_x^+ = \delta_x, \quad \delta_\eta^+ = \frac{1}{N}\mathbf{1}_N^\top\eta = \frac{-\gamma}{N}\mathbf{1}_N^\top\mathcal{L}x = 0$$

since $\mathbf{1}_N^\top\mathcal{L} = 0^\top$ for weight balanced graphs [44, Theorem 1.37]. Now, consider a solution $\phi = (\phi_x, \phi_\eta, \phi_\tau) \in \mathcal{S}_{\mathcal{H}}(\phi_x(0,0), \phi_\eta(0,0), \phi_\tau(0,0))$. First, consider the case when $h = 0$. By direct integration of (22) and some abuse of notation, we have

$$\begin{aligned}\delta_x(t,0) &= \delta_x(0,0) + \delta_\eta(0,0)t \\ &= \frac{1}{N}\sum_{i\in\mathcal{V}}(\phi_{x_i}(0,0) + \phi_{\eta_i}(0,0)\phi_\tau(0,0))\end{aligned} \tag{23}$$

for each $t \in [0, t_1]$, where $t_1 = \phi_\tau(0,0)$. When $t = t_1$, a jump occurs, and after the jump, $\delta_\eta(t_1, 1) = 0$. From the dynamics in (22), both δ_x and δ_η are constant due to $\dot{\delta}_x(t,j) = 0$ and $\dot{\delta}_\eta(t,j) = 0$ for all $(t,j) \in \operatorname{dom}\phi$ such that $j \ge 1$. Furthermore, due to Lemma 3.13 and Theorem 3.8, we know that $\mathcal{A}$ is UGES for $\mathcal{H}$. Since δ_x is constant for all $j \ge 1$ and solutions to $\mathcal{H}$ converge to $\mathcal{A}$ exponentially, then we have that every solution ϕ_{x_i} must converge to the average in (23), which leads to (21) for each $i \in \mathcal{V}$. Furthermore, at points on $(x, \eta, \tau) \in \mathcal{A}$, $\eta = \mathbf{0}_N$, which leads to $\lim_{t+j\to\infty}\phi_\eta(t,j) = \mathbf{0}_N$. For the case when $h > 0$, we have that direct integration of the continuous dynamics of (x_i, η_i) leads to $\phi_{x_i}(t,0) = \phi_{x_i}(0,0) + \frac{1}{h}(\exp(ht) - 1)\phi_{\eta_i}(0,0)$ for each $t \in [0, t_1]$. Then, by following an approach similar to the one above, one can establish that (23) holds for each $i \in \mathcal{V}$. ■

Remark 3.15: When the initial condition of the timer is set to zero, namely, $\phi_\tau(0,0) = 0$, Proposition 3.14 indicates that maximal solutions to $\mathcal{H}$ converge to the average of the initial conditions of the state $\phi_{x_i}(0,0)$, $i \in \mathcal{V}$.

G. Partial pointwise exponential stability for $\mathcal{H}$

Using the previous results, we now provide sufficient conditions for the set $\mathcal{A}$ to be partially pointwise globally exponentially stable with respect to (x, η) for $\mathcal{H}$ – namely, in addition to $\mathcal{A}$ being uniformly globally exponentially stable as a set, each point in $\mathcal{A}$ is also stable. The notion of partial pointwise exponential stability is given as follows; see [35] and [45] for more details.

Definition 3.16: (partial pointwise global exponential stability) Consider a hybrid system $\mathcal{H}$ with state $\xi = (p, q) \in \mathbb{R}^n$. The closed set $\mathcal{A} \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$, where $r \in \mathbb{N}$ and $0 < r \le n$ is *partially pointwise globally exponentially stable with respect to the state component $p \in \mathbb{R}^r$ for $\mathcal{H}$* if

1) every maximal solution ϕ to $\mathcal{H}$ is complete and has a limit belonging to $\mathcal{A}$;
2) $\mathcal{A}$ is uniformly globally exponentially stable for $\mathcal{H}$; and
3) for each $p^* \in \mathbb{R}^r$ such that there exists $q \in \mathbb{R}^{n-r}$ satisfying $(p^*, q) \in \mathcal{A}$, it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $\phi = (\phi_p, \phi_q)$ to $\mathcal{H}$ with $\phi_p(0,0) \in p^* + \delta\mathbb{B}$ satisfies $|\phi_p(t,j) - p^*| \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$.

The motivation behind the partial pointwise notion in Definition 3.16 is due to potential state components that may not converge to a set, for example, timers and boolean logic states (or modes). While not exercised in this article, if $r = n$, then the notion in Definition 3.16 reduces to pointwise globally exponentially stable as in [35].

In the next result, we exploit Theorem 3.8 to establish that the (diagonal-like) set $\mathcal{A}$ is partially pointwise globally exponentially stable with respect to (x, η) for $\mathcal{H}$

Theorem 3.17: Let T_1 and T_2 be two positive scalars satisfying $T_1 \leq T_2$ and the digraph Γ be weight balanced and contain a directed spanning tree. Suppose that scalars $h \geq 0$ and $\gamma > 0$ are chosen such that there exists $P = P^\top > 0$ satisfying (14) *in Theorem 3.8 holds. Then, the set $\mathcal{A}$ is partially pointwise globally exponentially stable with respect to (x, η) for $\mathcal{H}$.*

Proof: Items 1 and 2 in Definition 3.16 are satisfied via Lemma 3.6 and Theorem 3.8, respectively. It remains to show item 3. Pick $x^* \in \mathbb{R}$. Denote $\tilde{x} = x - x^*\mathbf{1}_N$, $\tilde{\chi} = (\tilde{x}, \eta, \tau)$, and define $\mathcal{H}^*$ as

$$\dot{\tilde{\chi}} = \begin{bmatrix} \eta \\ -h\eta \\ -1 \end{bmatrix} =: f^*(\tilde{\chi}) \qquad \tilde{\chi} \in C,$$

$$\tilde{\chi}^+ \in \begin{bmatrix} \tilde{x} \\ -\gamma\mathcal{L}\tilde{x} \\ [T_1, T_2] \end{bmatrix} =: G^*(\tilde{\chi}) \qquad \tilde{\chi} \in D.$$

where C and D are defined in (7). In these coordinates, the set to stabilize is $\mathcal{A}^* = \{0_N\} \times \{0_N\} \times [T_1, T_2]$. Consider the function $\tilde{\chi} \mapsto V(\tilde{\chi}) = \mu^\top \exp(A_{f2}^\top \tau) P \exp(A_{f2}\tau)\mu$, where $\mu = (\tilde{x}, \eta)$, $P = P^\top > 0$, and A_{f2} in (11). Note that

$$\alpha_1 |\tilde{\chi}|^2_{\mathcal{A}} \leq V(\tilde{\chi}) \leq \alpha_2 |\tilde{\chi}|^2_{\mathcal{A}} \qquad \forall \tilde{\chi} \in C \cup D$$

where $\alpha_1 = \min_{\tau\in[0,T_2]} \underline{\lambda}(\exp(A_{f2}^\top\tau) P \exp(A_{f2}\tau))$ and $\alpha_2 = \max_{\tau\in[0,T_2]} \overline{\lambda}(\exp(A_{f2}^\top\tau) P \exp(A_{f2}\tau))$. During flows, since $\dot{\mu} = A_{f2}\mu$, we have

$$\langle \nabla V(\tilde{\chi}), f^*(\tilde{\chi}) \rangle = 0$$

for all $\tilde{\chi} \in C$. Furthermore, for each $\tilde{\chi} \in D$, and each $g \in G^*(\tilde{\chi})$, since μ also satisfies $\dot{\mu} = A_{f2}\mu$ during flows and $\mu^+ = A_{g2}\mu$ at jumps, where A_{f2} and A_{g2} are given in (11), we have

$$V(g) - V(\tilde{\chi}) \leq 0,$$

namely, $V(g) \leq V(\tilde{\chi})$. Then, since $\mathcal{A}^*$ is compact, [40, Theorem 3.19, item 1] implies that $\mathcal{A}^*$ is stable. ∎

IV. Asymptotic Stability of Synchronization with Asynchronous Intermittent Information

A. Problem Statement

In a more realistic setting, agents may have access to their neighbors' information at asynchronous time instances. In this section, we consider the case where for each $i \in \mathcal{V}$, the i-th agent receives information from its neighbors at times in the sequence $\{t_s^i\}_{s=1}^\infty$ satisfying

$$\begin{aligned} & T_1^i \leq t_{s+1}^i - t_s^i \leq T_2^i \qquad \forall\, s \in \{1, 2, \ldots\} \\ & t_1^i \leq T_2^i \end{aligned} \tag{24}$$

where the positive scalars T_1^i and T_2^i define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive information updates for agent i. Note that, for each $i \in \mathcal{V}$, the bounds T_1^i and T_2^i are assumed to be known, but they might be independently determined by each agent and not necessarily the same among agents.

Similar to the case when all agents communicate synchronously, we assign a timer to trigger local communication events at times $\{t_s^i\}_{s=1}^\infty$ satisfying (24). We attach to each agent a timer state $\tau_i \in [T_1^i, T_2^i]$, which evolves with dynamics in (25), i.e., for each $i \in \mathcal{V}$, we have the following hybrid system modeling the communication times:

$$\begin{aligned} \dot{\tau}_i &= -1 & \tau_i &\in [0, T_2^i] \\ \tau_i^+ &\in [T_1^i, T_2^i] & \tau_i &= 0 \end{aligned} \tag{25}$$

Due to the set-valued jump map of the timer states, these dynamics are capable of modeling any sequence of local communication events at times $\{t_s^i\}_{s=1}^\infty$ satisfying (24).

Remark 4.1: From the communication time law in (24) (or in (3)), it is possible that such times are stochastically driven where the sequence $\{t_s^i\}_{s=1}^\infty$ can be generated by evaluating a random variable. For example, a random variable Y_i with a uniform distribution can be employed, in which case Y_i takes values in $[T_1^i, T_2^i]$ and $t_{s+1}^i - t_s^i = Y_i$ for each agent index $i \in \mathcal{V}$ and integer $s > 1$.

B. Hybrid Modeling and Consensus Protocol

Similar to Protocol 3.3, we consider a dynamic hybrid consensus protocol with state η_i that is assigned to the input u_i of the agent, for each $i \in \mathcal{V}$. For agent i and when τ_i reaches zero, η_i is updated based on information provided by its neighbors, as explained next.

Protocol 4.2: For each $i \in \mathcal{V}$, given the parameter $T_2^i > 0$, the i-th hybrid controller has state η_i with the following dynamics:

$$\begin{aligned} u_i &= \eta_i \\ \dot{\eta}_i &= -h\eta_i & \tau_i &\in [0, T_2^i] \\ \eta_i^+ &= -\gamma \sum_{k\in\mathcal{N}(i)} (x_i - x_k) & \tau_i &= 0 \end{aligned}$$

where $h \geq 0$ and $\gamma > 0$ are the controller parameters to be designed.

Following [42], [46], and following (but slightly abusing) the notation in Section III, we consider the following change of coordinates:

$$\begin{aligned} \bar{x}_i &= x_i - \frac{1}{N}\sum_{k=1}^{N} x_k \\ \theta_i &= -\gamma \sum_{k\in\mathcal{N}(i)} (x_i - x_k) - \eta_i \end{aligned} \tag{26}$$

for each $i \in \mathcal{V}$. We have that

$$\theta = -\gamma\mathcal{L}\bar{x} - \eta \tag{27}$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$, $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$, $\tau = (\tau_1, \tau_2 \ldots, \tau_N)$, and $\mathcal{L}$ is the Laplacian matrix given by the directed graph Γ of the network. Using the change in coordinates in (26), the continuous dynamics of

$\bar{x}_i$ are given by

$$\dot{\bar{x}}_i = -\gamma \sum_{k\in\mathcal{N}(i)} (\bar{x}_i - \bar{x}_k) - \theta_i + \sum_{k=1}^{N} \theta_k + \frac{\gamma}{N} \sum_{k=1}^{N} \sum_{r\in\mathcal{N}(i)} (x_k - x_r)$$

leading to[6]

$$\dot{\bar{x}} = -\gamma \mathcal{L}\bar{x} - \Pi\theta \tag{28}$$

where $\Pi = I - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$. By taking the derivative of (27) and applying (28), the dynamics of the auxiliary state θ are given by $\dot{\theta} = (\gamma^2\mathcal{L}\mathcal{L} - \gamma h\mathcal{L})\bar{x} + (-hI + \gamma\mathcal{L}\Pi)\theta$.

Through the change of coordinates in (26), we define a hybrid system $\mathcal{H}_\theta = (C_\theta, f_\theta, D_\theta, G_\theta)$ as the collection of all agents with dynamics in (2) and the control dynamics in Protocol 4.2 with associated timers. Let the state of $\mathcal{H}_\theta$ be given by $\xi_\theta = (z_\theta, \tau) \in \mathcal{X}_\theta$, $z_\theta = (\bar{x}, \theta)$, where $\mathcal{X}_\theta := \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{T}$ and $\mathcal{T} = [0, T_2^1] \times [0, T_2^2] \times \cdots \times [0, T_2^N]$. The data of $\mathcal{H}_\theta$ is given by

$$\begin{aligned} f_\theta(\xi_\theta) &:= (A_\theta z_\theta, -\mathbf{1}_N) && \forall \xi_\theta \in C_\theta := \mathcal{X}_\theta \\ G_\theta(\xi_\theta) &:= \{G_i(\xi_\theta) : \xi_\theta \in D_i, i \in \mathcal{V}\} && \forall \xi_\theta \in D := \cup_{i\in\mathcal{V}} D_i \end{aligned} \tag{29}$$

where $D_i = \{\xi_\theta \in \mathcal{X}_\theta : \tau_i = 0\}$ and the i-th jump map is given by

$$G_i(\xi_\theta) := \begin{bmatrix} \bar{x} \\ (\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_N) \\ (\tau_1, \ldots, \tau_{i-1}, [T_1^i, T_2^i], \tau_{i+1}, \ldots, \tau_N) \end{bmatrix}. \tag{30}$$

Note that G_i only updates the i-th component of θ and τ. The matrix A_θ for the continuous dynamics in (29) is given by

$$A_\theta = \begin{bmatrix} -\gamma\mathcal{L} & -\Pi \\ \gamma^2\mathcal{L}^2 - \gamma h\mathcal{L} & \gamma\mathcal{L}\Pi - hI \end{bmatrix}$$

where $\mathcal{L}$ is the Laplacian matrix, and γ and h are to be designed.

The following result is immediate and its proof follows along the lines of the proof of Lemma 3.5.

Lemma 4.3: Suppose the positive scalars T_1^i and T_2^i satisfy $T_1^i \le T_2^i$ for each $i \in \mathcal{V}$. The hybrid system $\mathcal{H}_\theta = (C_\theta, f_\theta, D_\theta, G_\theta)$ in (29) *satisfies the hybrid basic conditions.*

Due to the construction of (29), we have that maximal solutions are complete, non-Zeno, and have a uniformly bounded (from below) amount of flow time between consecutive jumps, much in the same way that Lemma 3.6 characterizes properties of solutions to $\mathcal{H}_\theta$.

Lemma 4.4: *[47, Lemma 3.5] For each $i \in \mathcal{V}$, let $0 < T_1^i \le T_2^i$ be given. Every maximal solution ϕ to $\mathcal{H}_\theta$ satisfies the following:*

1) *ϕ is complete;*
2) *ϕ is not Zeno;*
3) *for each $(t, j) \in \operatorname{dom}\phi$ it follows that $\left(\frac{j}{N} - 1\right)\widetilde{T}_1 \le t \le \frac{j}{N}\widetilde{T}_2$ where $\widetilde{T}_1 = \min_{i\in\mathcal{V}} T_1^i$ and $\widetilde{T}_2 = \max_{i\in\mathcal{V}} T_2^i$.*

The objective of each agent in the hybrid system $\mathcal{H}_\theta$ is to drive the states x_i to consensus, i.e., to asymptotically drive the difference between the agents' state x_i to zero. Note that the definition of η and θ implies that these states converge to zero as the error converges to consensus. Therefore, in the $(\bar{x}, \theta, \tau)$ coordinates, the set of interest is given by

$$\mathcal{A}_\theta := \{0_N\} \times \{0_N\} \times \mathcal{T} \tag{31}$$

for the hybrid system $\mathcal{H}_\theta$ defined in (29).

[6]Due to the definition of $d^{in}(i) = \sum_{k=1}^{N} g_{ki}$, we can expand the summation terms in $\sum_{k=1}^{N}\sum_{r\in\mathcal{N}(i)}(x_k - x_r)$ to obtain $\sum_{k=1}^{N}\sum_{r\in\mathcal{N}(i)}(x_k - x_r) = 0$.

C. Global Asymptotic Stability

In this section, we establish sufficient conditions to guarantee global asymptotic stability of the consensus set $\mathcal{A}_\theta$ in (31) under asynchronous communication. We consider the following notion of global asymptotic stability [33].

Definition 4.5: (global asymptotic stability) Let a hybrid system $\mathcal{H}$ with state in $\mathbb{R}^n$ be given. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be

- *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_\mathcal{A} \le \delta$ satisfies $|\phi(t,j)|_\mathcal{A} \le \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;
- *globally attractive* for $\mathcal{H}$ if every maximal solution ϕ to $\mathcal{H}$ is complete and $\lim_{t+j\to\infty} |\phi(t,j)|_\mathcal{A} = 0$;
- *globally asymptotically stable* for $\mathcal{H}$ if it is both stable and globally attractive.

Under the change of coordinates in (26), we will leverage the quadratic-like function

$$V_\theta(\xi_\theta) = \bar{x}^\top P\bar{x} + \theta^\top Q(\tau)\theta \tag{32}$$

where P is symmetric positive definite and $Q(\tau)$ is diagonal and positive definite for all $\tau \in \mathcal{T}$. Through this choice of V, regardless of which timer τ_i triggers a jump, this function satisfies the property that $V_\theta(\xi_\theta^+) - V_\theta(\xi_\theta)$ is upper bounded by a nonpositive function of θ_i for all $\xi_\theta \in D_\theta$. Such a property is possible due to the convenient choice of the auxiliary state θ_i at jumps, which, when applied to the control law in Protocol 4.2, steers θ_i to zero. Under the conditions of the following result, we have that during flows, namely, for each $\xi_\theta \in C_\theta$, the change in V_θ is upper bounded by a nonpositive function of $z_\theta = (\bar{x}, \theta)$. These properties and the fact that $\mathcal{H}_\theta$ satisfies the hybrid basic conditions (Lemma 4.3) are exploited in the following result through an application of the invariance principle for hybrid systems involving a nonincreasing function [40, Theorem 3.23].

Theorem 4.6: Let T_1^i and T_2^i be two positive scalars such that $T_1^i \le T_2^i$ for each $i \in \mathcal{V}$ and a digraph Γ contain a directed spanning tree and be weight balanced. If the scalars $\gamma < 0$ and $h \in \mathbb{R}$ are selected such that there exist $\sigma > 0$, a positive definite symmetric matrix $P \in \mathbb{R}^{N\times N}$, and a positive definite diagonal matrix function $\nu \mapsto Q(\nu) \in \mathbb{R}^{N\times N}$ such that

$$\mathcal{M}(\nu) \le 0 \tag{33}$$

for each $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathcal{T}$, where

$$\mathcal{M}(\nu) := \begin{bmatrix} -\gamma \operatorname{He}(P, \mathcal{L}) & -P\Pi + (\gamma^2\mathcal{L}^2 - \gamma h\mathcal{L})Q(\nu) \\ \star & -\sigma Q(\nu) + \operatorname{He}(Q(\nu), (\gamma\mathcal{L}\Pi - hI)) \end{bmatrix} \tag{34}$$

and $Q(\nu) = \operatorname{diag}(q_1 \exp(\sigma\nu_1), q_2 \exp(\sigma\nu_2), \ldots, q_N \exp(\sigma\nu_N))$, then $\mathcal{A}_\theta$ in (31) *is globally asymptotically stable for $\mathcal{H}_\theta$.*

Proof: Let $V_\theta : \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{T} \to \mathbb{R}_{\ge 0}$ be given by (32). We have that

$$\alpha_1 |\xi_\theta|^2_\mathcal{A} \le V_\theta(\xi_\theta) \le \alpha_2 |\xi_\theta|^2_\mathcal{A} \tag{35}$$

for all $\xi_\theta \in C_\theta$, where $\alpha_1 = \min\{\underline{\lambda}(P), \underline{\lambda}(Q)\}$ and $\alpha_2 = \max_{\nu\in\mathcal{T}}\{\overline{\lambda}(P), \overline{\lambda}(Q(\nu))\}$. For each $\xi_\theta \in C_\theta$, we have that

$$\langle \nabla V_\theta(\xi_\theta), f_\theta(\xi_\theta) \rangle = z_\theta^\top \mathcal{M}(\tau) z_\theta$$

where $\mathcal{M}(\tau)$ is given in (34). Therefore, by the fact that $\mathcal{M}(\tau) \le 0$, we have that $\langle \nabla V_\theta(\xi_\theta), f_\theta(\xi_\theta) \rangle \le 0$ for each $\xi_\theta \in C_\theta$. Define $u_c(\xi_\theta) = z_\theta^\top \mathcal{M}(\tau) z_\theta$ for each $\xi_\theta \in C_\theta$ and $u_C(\xi_\theta) = -\infty$ otherwise. Then, the zero-level set of u_C is given by $u_C^{-1}(0) = \{\xi_\theta \in \mathcal{X}_\theta : z_\theta^\top \mathcal{M}(\tau) z_\theta = 0\}$ Note that $\mathcal{M}(\tau)$ can be decomposed as $\mathcal{M}(\tau) = \Omega^\top R(\tau) + R(\tau)\Omega$,

where
$$\Omega = \begin{bmatrix} -\gamma\mathcal{L} & -\Pi \\ \gamma^2\mathcal{L}^2 - \gamma h\mathcal{L} & \gamma\mathcal{L}\Pi - hI - \frac{\sigma}{2}I \end{bmatrix}, R(\tau) = \begin{bmatrix} P & 0 \\ 0 & Q(\tau) \end{bmatrix}.$$

Note that $R(\tau)$ is positive definite and therefore invertible for each $\tau \in \mathcal{T}$. Furthermore, note that $z_\theta^\top \mathcal{M}(\tau) z_\theta = 0$ when $z_\theta^\top R(\tau)\Omega z_\theta = 0$ which implies that either $z_\theta = 0$ or $\Omega z_\theta = 0$ – namely, z_θ is zero or in the null space of Ω. Since the digraph contains a directed spanning tree, the Laplacian matrix $\mathcal{L}$ is such that $\mathcal{L}\mathbf{1}_N = 0$ corresponds to a singular zero eigenvalue for $\mathcal{L}$.

The nullspace of Ω is the set of points $(a, b) \in \mathbb{R}^N \times \mathbb{R}^N$ such that
$$\begin{bmatrix} -\gamma\mathcal{L} & -\Pi \\ \gamma^2\mathcal{L}^2 - \gamma h\mathcal{L} & \gamma\mathcal{L}\Pi - hI - \frac{\sigma}{2}I \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{36}$$
From (36), we have that $-\gamma\mathcal{L}a = \Pi b$ and $(\gamma^2\mathcal{L}^2 - \gamma h\mathcal{L})a + (\gamma\mathcal{L}\Pi - hI - \frac{\sigma}{2})b = 0$. By substituting the former equation into the latter equation and recalling the definition of Π, we have that
$$-\left(\frac{h}{N}\mathbf{1}_N\mathbf{1}_N^\top + \frac{\sigma}{2}I\right)b = 0.$$
Due to $\left(\frac{h}{N}\mathbf{1}_N\mathbf{1}_N^\top + \frac{\sigma}{2}I\right)$ being full rank for all $\sigma, N > 0$, $h \geq 0$, it follows that $b = 0$. From (36), we have that $\gamma\mathcal{L}x = 0$. Due to the fact that the digraph is weight balanced and contains a directed spanning tree, there exists a singular eigenvalue at zero (i.e., $\lambda_1 = 0$) which corresponds to eigenvalue $\mathbf{1}_N$, then $\gamma\mathcal{L}x = 0$ when $x_i = x_j$ for all $i, j \in \mathcal{V}$. Due to the definition of nullspace, it follows that $u_C^{-1}(0) = \mathcal{A}_\theta$. Therefore, for each $\xi_\theta \in \mathcal{A}_\theta$, $u_C(\xi_\theta) = 0$ and, in light of (33), $u_C(\xi_\theta) < 0$ for each point $\xi_\theta \in C_\theta \setminus \mathcal{A}_\theta$.

Now, we analyze the change in V_θ at jumps. Namely, for each $\xi_\theta = (x, \theta, \tau) \in D_\theta$ and for each $g_\theta \in G_\theta(\xi_\theta)$, there exists $i \in \mathcal{V}$ such that $\tau_i = 0$. From the definition in (30), if there are multiple timers that reach zero simultaneously, then the jump map is set-valued and the state is sequentially updated by the G_i map that corresponds to the i-th expired timer. Therefore, without loss of generality, we consider the case of a single τ_i reaching zero, noting that if multiple timers simultaneously reach zero, there would be multiple successive jumps. Recalling that $Q(\nu) = \mathrm{diag}(q_1 \exp(\sigma\nu_1), q_2 \exp(\sigma\nu_2), \ldots q_N \exp(\sigma\nu_N))$, we denote the k-th diagonal element of $Q(\nu)$ as $\tilde{q}_k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Then, for each $\xi_\theta \in D_\theta$ such that $\tau_i = 0$ and each $g_\theta \in G_\theta(\xi_\theta)$, we have that
$$V_\theta(g_\theta) - V_\theta(\xi_\theta) = \sum_{k=1, k\neq i}^{N} \tilde{q}_k(\tau_k)\theta_k^2 - \sum_{k=1}^{N} \tilde{q}_k(\tau_k)\theta_k^2$$
$$= -\tilde{q}_i(\tau_i)\theta_i^2 \leq 0.$$
Define $u_D(\xi_\theta) := -\tilde{q}_i(\tau_i)\theta_i^2$ for each $\xi_\theta \in D_\theta$ and $u_D(\xi_\theta) = -\infty$ otherwise. The zero-level set of u_D is given by $u_D^{-1}(0) = \{\xi_\theta \in \mathcal{X}_\theta : \theta_i = 0, \tau_i = 0, i \in \mathcal{V}\}$ and note that $G_\theta(u_D^{-1}(0)) = \{\xi_\theta \in \mathcal{X}_a : \theta_i = 0, \tau_i \in [T_1, T_2], i \in \mathcal{V}\}$.

From Lemma 4.4 we have that all maximal solutions to $\mathcal{H}_\theta$ are complete and non-Zeno. Moreover, from Lemma 4.3, $\mathcal{H}_\theta$ satisfies the hybrid basic conditions. In light of (35) and noting that, for any $r > 0$, the largest weakly invariant set contained in $V_\theta^{-1}(r) \cap \mathcal{X}_\theta \cap \left(\overline{u_C^{-1}(0)} \cup (u_D^{-1}(0) \cup G(u_D^{-1}(0)))\right)$ is empty. Then, by [48, Theorem 8.8] and Lemma 4.3, it follows that the set $\mathcal{A}_\theta$ is globally asymptotically stable for $\mathcal{H}_\theta$. ■

The Lyapunov analysis in the proof of Theorem 4.6 establishes global asymptotic stability of $\mathcal{A}_\theta$. Note that this analysis does not imply that each point in the consensus set is Lyapunov stable; hence, as a difference to the synchronous case, it does not show that the consensus set is pointwisely asymptotically stable for the case of asynchronous communication.

Note that condition (33) must be satisfied for an infinite number of points in the compact set $\mathcal{T}$. We can relax (33) by noting that the interval $[0, T_2^i] \subset [0, \widetilde{T}_2]$, where $\widetilde{T}_2 = \max_{i\in\mathcal{V}} T_2^i$ and checking (33) on the boundary points of $[0, \widetilde{T}_2]$, as stated in the following result. Due to space constraints, the proof is omitted here, but it follows similar steps as in the proof of [47, Propositions 3.8 and 3.9].

Proposition 4.7: Given the conditions of Theorem 4.6, it follows that (33) *holds if* $\mathcal{M}(\mathbf{0}_N) \leq 0$ *and* $\mathcal{M}(\widetilde{T}_2\mathbf{1}_N) \leq 0$ *where* $\widetilde{T}_2 = \max_{i\in\mathcal{V}} T_2^i$.

D. Numerical Examples

In this section, we consider two examples illustrating the consensus results.

Example 4.8: Consider five agents with dynamics as in (2) over the following graph with adjacency matrix
$$\mathcal{G} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}. \tag{37}$$
Let $T_1 = 0.5$ and $T_2 = 1.5$. It can be shown that the parameters
$$P \approx \begin{bmatrix} 7:38 & 0 & 0 & 0 & 0:12 & 0 & 0 & 0 \\ 0 & 6:58 & 0 & 0 & 0 & 0:03 & 0 & 0 \\ 0 & 0 & 8:59 & 0 & 0 & 0 & :40 & 0 \\ 0 & 0 & 0 & 9:05 & 0 & 0 & 0 & 1:18 \\ 0:12 & 0 & 0 & 0 & 5:83 & 0 & 0 & 0 \\ 0 & 0:03 & 0 & 0 & 0 & 5:83 & 0 & 0 \\ 0 & 0 & 0:40 & 0 & 0 & 0 & 5:20 & 0 \\ 0 & 0 & 0 & 1:77 & 0 & 0 & 0 & 3:80 \end{bmatrix},$$
$h = 0.3$, and $\gamma = 0.3$ satisfy condition (14). Figure 3 shows the x_i components $i \in \{1, 2, 3, 4, 5\}$ of a solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ from initial conditions given by $\phi_x(0, 0) = (-3, -5, 3.5, -3.5, 2.6)$, $\phi_\eta(0, 0) = (0.84, 1.77, 0.65, 1.7, -4)$, and $\phi_\tau(0, 0) = 1$ as well as the function V_2 below Theorem 3.8 evaluated along ϕ projected onto the ordinary time domain. In Section V-C, we consider the case of perturbations on the agents' dynamics and communications. △

Example 4.9: Consider ten agents with dynamics as in (2) over a random graph given by Figure 5. Let $T_1^i = 0.2$ and $T_2^i = 0.4$ for each $i \in \mathcal{V}$ be given. Then, using parameters $\gamma = 0.3$, $h = 0.1$, and $\sigma = 50$, we find the matrices $P = 1 * 10^7 I$ and $Q = 0.26I$ satisfy condition (33) in Theorem 4.6. Figure 4 shows a solution ϕ to the hybrid system with agent dynamics (2) using Protocol 4.2, and asynchronous timers τ_i to trigger communications. The solution is from random initial conditions as follows $\phi_{x_i}(0, 0) \sim U([-5, 5])$, $\phi_{\eta_i}(0, 0) \sim U([-5, 5])$ and $\phi_\tau(0, 0) \sim U([0, 0.4])$ where $U(\cdot)$ is the uniform distribution over the argument interval. Furthermore, as indicated by Figure 4, the bottom plot shows the Lyapunov function V_θ given by (32) over the solution ϕ. In particular, it is worth noting that V_θ is non-increasing over continuous flow time t. △

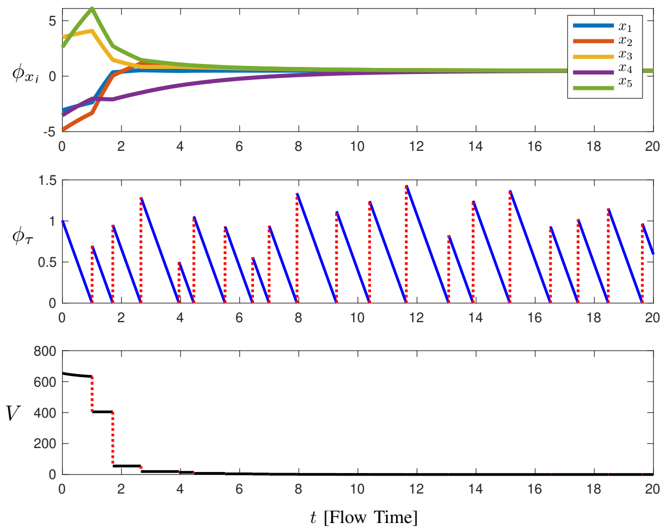

Fig. 3. (top) The x and components of a solution $= (x; ;)$ to H with G in (37) using Protocol 3.3 which satisfies Theorem 3.8. (bottom) Note that since $V_2( )$ for deceases to zero with respect to flow time, it indicates that the solution reaches consensus.

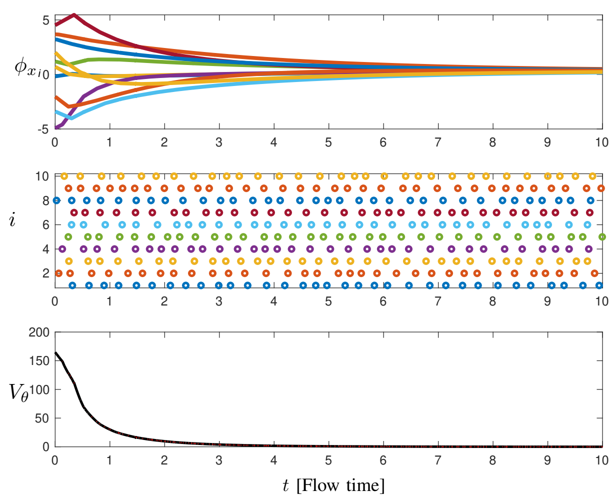

Fig. 4. The x_i components of a solution $= (x; ;)$ to a hybrid system with with agent dynamics (2) where asynchronous communication is present. The asynchronous communication is shown on the middle chart where an 'o' marker indicates a communication of agent i, more specifically, $_i(t; j) = 0$ for each $(t; j) \in \operatorname{dom}$. The Lyapunov function V in (32) is plotted over time.

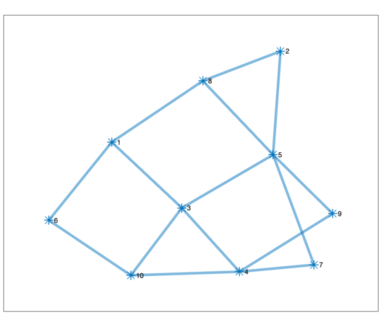

Fig. 5. Network configuration for Example 4.8.

V. Robustness of Consensus with Intermittent Communication

Note that the results in the previous sections are for idealized models with no noise, perturbations, or unmodeled dynamics. However, in real-world settings, systems are affected by such uncertainty. This section expands the results in Sections III and IV as follows. Section V-A considers nominal robustness of consensus for $\mathcal{H}_\theta$ in (29) leveraging the properties of the system – namely, well-posedness. In Section V-B, we consider the hybrid system in (7) and provide sufficient conditions for input-to-state stability of the consensus set $\mathcal{A}$ in (8) with respect to communication noise.

A. Nominal Robustness to General Perturbations

In this section, we consider perturbations in the agent models and communication networks. Specifically, we consider the agent dynamics given by

$$\dot{x}_i = u_i + b_i \tag{38}$$

where $|b_i| \leq b_i^*$ is a (possibly state-dependent) perturbation; for example, actuator/input noise or bias can be considered. The information available to the agents, either measured or communicated, may also be affected by some noise. We consider communication noise c_k^c from agent k, leading to

$$\tilde{x}_k^c = x_k + c_k^c, \tag{39}$$

and noise in the information measured at agent i, denoted c_i^k, is given by

$$\tilde{x}_i^m = x_i + c_i^m. \tag{40}$$

Furthermore, the timers triggering communication events at each node may be perturbed due to uncertainty in the communication network. We model such effects by the perturbed timer system given by

$$\begin{aligned} \dot{\tau}_i &= -1 + \iota_i & \tau_i &\in [0, T_2^i + \vartheta_2^i] \\ \tau_i^+ &\in [T_1^i + \vartheta_1^i, T_2^i + \vartheta_2^i] & \tau_i &= 0, \end{aligned}$$

where $\iota_i < 1$ is a constant modeling a possible skew on the timer dynamics for τ_i, and $\vartheta_i = (\vartheta_1^i, \vartheta_2^i)$ is a constant that satisfies $0 < T_1^i + \vartheta_1^i \leq T_2^i + \vartheta_2^i$ modeling the perturbations on the known nominal values of the parameters T_1^i and T_2^i.

Following Protocol 4.2, the perturbed versions of the dynamics of the proposed control algorithm are

$$\dot{\eta}_i = -h\eta_i,$$

when $\tau_i \in [0, T_2^i + \vartheta_2^i]$. At every event time given by $\tau_i = 0$, the η_i state is updated to

$$\eta_i^+ = -\gamma \sum_{k \in \mathcal{N}(i)} (\tilde{x}_i^m - \tilde{x}_k^c) = -\gamma \sum_{k \in \mathcal{N}(i)} (x_i - x_k) - \tilde{c}_i,$$

where $\tilde{c}_i = \gamma \sum_{k \in \mathcal{N}(i)} (c_i^m - c_k^c)$, and $\tilde{x}_i^m$ and $\tilde{x}_i^c$ are given in (39) and (40), respectively. Then, following the definition of θ_i in (26), we define a perturbation of $\mathcal{H}_\theta$, denoted $\widetilde{\mathcal{H}}_\theta$ with data $(\widetilde{C}_\theta, \widetilde{f}_\theta, \widetilde{D}_\theta, \widetilde{G}_\theta)$ given as follows. For each $\xi_\theta \in \widetilde{C}_\theta := \mathbb{R}^N \times \mathbb{R}^N \times \widetilde{\mathcal{T}} =: \widetilde{\mathcal{X}}_\theta$, $\widetilde{\mathcal{H}}_\theta$ flows are governed according to the perturbed flow map

$$\widetilde{f}_\theta(\xi_\theta) = f_\theta(\xi_\theta) + \begin{bmatrix} b \\ \gamma b \\ \iota \end{bmatrix}$$

where $b = (b_1, b_2, \ldots, b_N)$, $\iota = (\iota_1, \iota_2, \ldots, \iota_N)$ and $\widetilde{\mathcal{T}} := [0, T_2^1 + \vartheta_2^1] \times [0, T_2^2 + \vartheta_2^2] \times \cdots \times [0, T_2^N + \vartheta_2^N]$. Since jumps occur when $\tau_i = 0$, the jump set for the perturbed hybrid system is $\widetilde{D}_\theta = D_\theta$ as given above (29), and the perturbed jump maps is

$$\widetilde{G}_\theta(\xi_\theta) := \{\widetilde{G}_i(\xi_\theta) : \xi_\theta \in \widetilde{D}_i, i \in \mathcal{V}\},$$

where $\widetilde{D}_i = D_i$ defined after (29) and

$$\widetilde{G}_i(z) := \begin{bmatrix} \bar{x} \\ (\theta_1, \ldots, \theta_{i-1}, -\tilde{c}_i, \theta_{i+1}, \ldots, \theta_N) \\ (\tau_1, \ldots, \tau_{i-1}, [T_1^i + \vartheta_1^i, T_2^i + \vartheta_2^i], \tau_{i+1}, \ldots, \tau_N) \end{bmatrix}.$$

In light of $\mathcal{H}_\theta$ satisfying the hybrid basic conditions in Definition 3.1 and $\mathcal{A}_\theta$ in (31) being compact, we have the following result.

Theorem 5.1: Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Suppose the conditions in Theorem 4.6 for the unperturbed hybrid system $\mathcal{H}_\theta$ with data in (29)*. Then, there exists $\beta \in \mathcal{KL}$ such that, for every compact set $K \subset \widetilde{\mathcal{X}}_\theta$ and $\varepsilon > 0$, there exists $\rho^\star > 0$ such that if*

$$\max\{|b|, |c|, |\iota|\} \leq \rho^\star$$

then, every $\phi \in \mathcal{S}_{\mathcal{H}}\ (K)$ satisfies

$$|\phi(t,j)|_{\mathcal{A}}\ \leq \beta(|\phi(0,0)|_{\mathcal{A}}\ , t+j) + \varepsilon$$

for all $(t,j) \in \operatorname{dom}\phi$, i.e., $\mathcal{A}_\theta$ is semiglobally practically robustly $\mathcal{KL}$ asymptotically stable for $\mathcal{H}_\theta$.

Proof: Given any continuous function $\rho : \widetilde{\mathcal{X}}_\theta \to \mathbb{R}_{\geq 0}$, the ρ-perturbation of $\mathcal{H}_\theta = (C_\theta, f_\theta, D_\theta, G_\theta)$, denoted $\mathcal{H}_{\theta,\rho}$, is given by

$$\begin{cases} \xi \in \widetilde{C}_\rho & \dot{\xi} \in F_\rho(\xi) \\ \xi \in \widetilde{D}_\rho & \xi^+ \in G_\rho(\xi) \end{cases}$$

where[7]

$$\widetilde{C}_\rho = f \quad 2\ \widetilde{X}_\theta : (\quad + \quad (\quad)\mathbb{B}) \setminus C_\theta \not\in 0g$$

$$F_\rho(\ ) = \overline{\mathrm{con}}\, f_\theta((\quad + \quad (\quad)\mathbb{B}) \setminus C_\theta) + \quad (\quad)\mathbb{B} \quad 8 \quad 2\ C_\theta \setminus D_\theta$$

$$\widetilde{D}_\rho = f \quad 2\ \widetilde{X}_\theta : (\quad + \quad (\quad)\mathbb{B}) \setminus D_\theta \not\in 0g$$

$$G_\rho(\ ) = fv\ 2\ \widetilde{X}_\theta : v\ 2\ g + \quad (g)\mathbb{B}; g\ 2\ G_\theta(\quad + \quad (\quad)) \setminus D_\theta g \quad 8 \quad 2\ C_\theta \setminus D_\theta$$

Note that by Theorem 4.6, the set $\mathcal{A}_\theta$ is globally asymptotically stable for $\mathcal{H}_\theta$. Since ρ is continuous and $\mathcal{H}_\theta$ satisfies the hybrid basic conditions, by [33, Theorem 6.8], $\mathcal{H}_{\theta,\rho}$ is nominally well-posed and, moreover, by [33, Proposition 6.28] is well-posed. Then, [33, Theorem 7.20] implies that $\mathcal{A}_\theta$ is semiglobally practically robustly $\mathcal{KL}$ pre-asymptotically stable for $\mathcal{H}_{\theta,\rho}$. Namely, for every compact set $K \subset \mathbb{R}^N \times \mathbb{R}^N \times \widetilde{\mathcal{T}}$ and for every $\varepsilon > 0$, there exists $\widetilde{\rho} \in (0,1)$ such that every maximal solution ϕ to $\mathcal{H}_{\theta,\widetilde{\rho}\rho}$ from K that satisfies

$$|\phi(t,j)|_{\mathcal{A}}\ \leq \beta(|\phi(0,0)|_{\mathcal{A}}\ , t+j) + \varepsilon$$

for all $(t,j) \in \operatorname{dom}\phi$. Lastly, ρ is such that there exists $\rho^* > 0$ such that $\rho^* \leq \rho\widetilde{\rho}$ which concludes the proof. ∎

Remark 5.2: The perturbations considered above for $\mathcal{H}_\theta$ in (29) can be extended to the single timer case in (7). Naturally, the hybrid system in (7) can be affected by unmodeled dynamics, perturbations in the communication times, and additive communication noise as well. The set $\mathcal{A}$ in (8) is globally exponentially stable for $\mathcal{H}$ per Theorem 3.8 and Lemma 3.13 and satisfies the hybrid basic conditions. In this case, $\mathcal{A}$ is not compact, therefore, we will have to consider the results as being over a compact subset S of $\mathcal{A}$, note that this is not restrictive as S can be arbitrarily large. Under these assumptions, similar steps can be followed as the proof of Theorem 5.1 to show that $S \subset \mathcal{A}$ is semiglobally practically robustly $\mathcal{KL}$ asymptotically stable for the associated perturbed hybrid system $\mathcal{H}_\rho$.

[7]Given a set $X \in \mathbb{R}^n$, the notation $\overline{\mathrm{con}}X$ is closed convex hull of X, more specifically, the smallest closed convex set containing X.

B. Robustness to Perturbations on Communication Noise

In this section, we consider the case of potentially large communication and measurement noise for the synchronous network case in Section III. More specifically, we consider the hybrid system $\mathcal{H} = (C, f, D, G)$ given by (7) where the agents are perturbed according to (39)–(40), where $\tilde{x}_k$ is the perturbed information communicated from agent k to agent i and $\tilde{x}_i$ is the perturbed measured information of agent i. In the following results, we consider the case when $c_i^m = c_i^c = c_i^*$. In such a case, the controller from Protocol 3.3 becomes

$$\dot{\eta}_i = -h\eta_i \qquad \tau \in [0, T_2]$$

$$\eta_i^+ = -\gamma \sum_{k \in \mathcal{N}(i)} (\tilde{x}_i - \tilde{x}_k) \qquad \tau = 0$$

which, different than (6), leads to an update law with communication noise given by

$$\eta_i^+ = -\gamma \sum_{k \in \mathcal{N}(i)} (x_i - x_k) - \gamma \sum_{k \in \mathcal{N}(i)} (c_i^* - c_k^*).$$

We show that the hybrid system $\mathcal{H}$ in (7) has $\mathcal{A}$ input-to-state stable (ISS) with respect to noises c_i^m and c_i^c, which is defined as follows.

Definition 5.3 (Input-to-state stability [49]): A hybrid system $\mathcal{H}$ with input m is input-to-state stable with respect to $\mathcal{A}$ if there exist $\beta \in \mathcal{KL}$ and $\kappa \in \mathcal{K}$ such that each solution pair (ϕ, m) to $\mathcal{H}$ satisfies[8,9]

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\{\beta(|\phi(0,0)|_{\mathcal{A}}, t+j), \kappa(|m|_\infty)\}$$

for each $(t,j) \in \operatorname{dom}\phi$.

Let $c^* = (c_1^*, c_2^*, \ldots, c_N^*)$. The controller state $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ is given by

$$\eta^+ = -\gamma\mathcal{L}x - \gamma\mathcal{L}c^*$$

where $\mathcal{L}$ is the Laplacian. Recall that the set from stabilize for $\mathcal{H}$ is (8). Moreover, recall that from the proof of Lemma 3.13, the distance from solutions to $\mathcal{H}$ to $\mathcal{A}$ is equivalent to the distance from solutions to $\widetilde{\mathcal{H}}$ to $\widetilde{\mathcal{A}}$. Using the change of coordinates involving the matrix U, as in Section III-C, namely,

$$\bar{x} = U^{-1}x \qquad \bar{\eta} = U^{-1}\eta \qquad \bar{c}^* = U^{-1}c^*$$

it follows that, at jumps, the update of the new state $\bar{\eta}$ is given by

$$\bar{\eta}^+ = (0, -\gamma\bar{\mathcal{L}}\bar{x} - \gamma\bar{\mathcal{L}}\bar{c}^*).$$

Note that the first component of $\bar{\eta}^+$ does not depend on the communication noise due to the change of coordinates. Defining the perturbed error hybrid system as $\mathcal{H}_c$ and with states $\chi = (\bar{z}_1, \bar{z}_2, \tau) \in \mathcal{X} := \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{T}$, $\bar{z}_1 = (\bar{x}_1, \bar{\eta}_1)$, and $\bar{z}_2 = (\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_N, \bar{\eta}_2, \bar{\eta}_3, \ldots, \bar{\eta}_N)$, the data of $\mathcal{H}_c$ is

[8]Given a hybrid arc x and a hybrid input u, (x, u) is a solution pair to a hybrid system $\mathcal{H} = (C, f, D, G)$ if $(x(0,0), u(0,0) \in \overline{C} \cup D$, $\operatorname{dom} x = \operatorname{dom} u$, for every $j \in \mathbb{N}$ such that I_j has nonempty interior $\dot{x}(t,j) \in f(x(t,j), u(t,j))$ for almost all $t \in I_j$, $(x(t,j), u(t,j)) \in C$ for all $t \in [\inf I_j, \sup I_j)$, and for every $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x = \operatorname{dom} u$ we have that $x(t, j+1) \in G((x(t,j), u(t,j)), x(t,j), u(t,j))) \in D$.

[9]The $\mathcal{L}\ 1$ norm n of $(t,j) \mapsto u(t,j)$ is given by $|u|\ 1 := \max\ \operatorname{ess\,sup}_{(t^0;j^0)2\,\mathrm{dom}\,un\ (u);t^0+j^0\ t+j} |u(t^0, j^0)|$, o $\sup_{(t^0;j^0)2\ (u);t^0+j^0\ t+j} |u(t^0, j^0)|$ where $\Upsilon(u) = \{(t,j) \in \operatorname{dom} u : (t, j+1) \in \operatorname{dom} u\}$; see [49, Definition 2.1] for details.

given by

$$\begin{aligned} f_c(\chi) &= (A_{f1}\bar{z}_1, A_{f2}\bar{z}_2, -1) \qquad \forall \chi \in C_c := \mathcal{X} \\ G_c(\chi, \bar{c}^*) &= (A_{g1}\bar{z}_1, A_{g2}\bar{z}_2 - B_g\bar{c}^*, -1) \qquad (41) \\ &\qquad \forall \chi \in D_c := \{\chi \in \mathcal{X} : \tau = 0\} \end{aligned}$$

where A_{f1}, A_{f2}, A_{g1}, and A_{g2} are given in (11) and $B_g = (0, \gamma\bar{\mathcal{L}})$. Then, using the change of coordinates as in Section III-C we can show that global exponential stability of $\widetilde{\mathcal{A}}$ in (12) for $\widetilde{\mathcal{H}}$ in (10) is robust to communication noise in the sense of input-to-state stability.

Theorem 5.4: Let $0 < T_1 \le T_2$ be given. Suppose the digraph $\Gamma = (\mathcal{E}, \mathcal{V}, \mathcal{G})$ contains a directed spanning tree. If the scalars $\gamma > 0$ and $h \ge 0$ are selected so that there exists $P = P^\top > 0$ satisfying (14) *holds for all $\nu \in [T_1, T_2]$, then the hybrid system $\widetilde{\mathcal{H}}$ with input $\bar{m}$ is ISS with respect to $\mathcal{A}$ as in* (12).

Proof: Consider the same Lyapunov function from the proof of Theorem 3.8, i.e., $V(\chi) = V_1(\chi) + V_2(\chi)$ as in (17). Note that for $\mathcal{H}_c$, the communication noise does not appear on the flow map. Therefore, the analysis on flows applies from the proof of Theorem 3.8, namely, for each $\chi \in C_c$

$$\langle \nabla V(\chi), f_c(\chi)\rangle = 0. \qquad (42)$$

When $\tau = 0$, a jump occurs mapping τ to some point $\nu \in [T_1, T_2]$. The state $\bar{z}_2$ is updated by $G_c(\chi, \bar{c}^*)$ in (41) to $A_{g2}\bar{z}_2 - B_g\bar{c}^*$. At jumps, we have that, for each $\chi \in D_c$ and $g \in G_c(\chi, \bar{c}^*)$, the change in V is given by

$$\begin{aligned} V(g) - V(\chi) = {} & \bar{z}_2^\top A_{g2}^\top \exp(A_{f2}^\top\nu) P \exp(A_{f2}\nu) A_{g2}\bar{z}_2 \\ & - 2\bar{c}^* B_g^\top \exp(A_{f2}^\top\nu) P \exp(A_{f2}\nu) A_{g2}\bar{z}_2 \\ & + (\bar{c}^*)^\top B_g^\top \exp(A_{f2}^\top\nu) P \exp(A_{f2}\nu) B_g\bar{c}^* \\ & - \bar{z}_2^\top P \bar{z}_2 \end{aligned} \qquad (43)$$

Following the steps in the proof of Theorem 3.8, from (14), there exists a scalar $\beta > 0$ such that

$$\bar{z}_2^\top (A_{g2}^\top \exp(A_{f2}^\top\nu) P \exp(A_{f2}\nu) A_{g2} - P)\bar{z}_2 < -\beta \bar{z}_2^\top \bar{z}_2$$

which also implies that $|\exp(A_{f2}\nu)A_{g2}| < 1$. Furthermore, the second term of (43) can be decomposed using Young's Inequality[10]. Let $a = \bar{z}_2$, $b^\top = (\bar{c}^*)^\top B_g^\top \exp(A_{f2}^\top\nu) P \exp(A_{f2}\nu) \bar{A}_{g2}$, and $c = \frac{\beta}{2}$. Then (43) becomes

$$V(g) - V(\chi) \le -\frac{\beta}{2}\bar{z}_2^\top \bar{z}_2 + \gamma^2\kappa|\bar{c}^*|^2 \qquad (44)$$

where $\kappa = \lambda_N^2 |P|(1 + \frac{2}{\beta}|P|) \max_{\nu \in [T_1, T_2]} |\exp(A_{f2}\nu)|^2$, $\gamma > 0$, and λ_N is the maximum eigenvalue of $\bar{\mathcal{L}}$. Then, from (42) and (44), we have that

$$V(g) \le \exp(\lambda_d) V(\chi) + \gamma^2\kappa|\bar{c}^*|^2 \qquad (45)$$

for all $\chi \in D_c, g \in G_c(\chi, \bar{c}^*)$, where $\lambda_d = \ln(1 - \beta/\alpha_2)$ and α_2 is defined in (16). Therefore, given any maximal solution pair $(\phi, \bar{c}^*)$ to $\mathcal{H}_c$ in (41), we have that during flows $V(\phi(t,0)) = V(\phi(0,0))$ for all $t \in [0, t_1]$ and with jumps of the solution given by (45), V over any maximal solution is given by

$$\begin{aligned} V(\phi(t,j)) \le {} & \exp(\lambda_d j) V(\phi(0,0)) \\ & + \kappa\gamma^2 \sum_{i=0}^{j-1} \exp(\lambda_d(j-1-i)) |c^*(t_{i+1}, i+1)|^2 \end{aligned}$$

for all $(t,j) \in \operatorname{dom}\phi$, with $j \ge 1$.

Following the proof of Theorem 3.8, we have $V(\phi(t,j)) \le e^{\lambda_d j} V(\phi(0,0)) + \frac{\kappa e^{-\lambda_d}\gamma^2}{e^{-\lambda_d}-1} |\bar{c}^*|^2_{(t,j)}$ for each $(t,j) \in \operatorname{dom}\phi$ such that $j \ge 1$. Recall from the proof of Lemma 3.13 that the set $\mathcal{A}$ for $\mathcal{H}$ and the set $\mathcal{A}$ for $\mathcal{H}$ satisfies $|\xi|_{\mathcal{A}} = |\chi|_{\mathfrak{A}}$. By following similar arguments as in Theorem 3.8, we have

$$\begin{aligned} |\phi(t,j)|^2_{\mathfrak{A}} &\le \frac{\alpha_2}{\alpha_1} e^{\lambda_d j} |\phi(0,0)|^2_{\mathfrak{A}} + \frac{\kappa e^{-\lambda_d}\gamma^2}{(e^{-\lambda_d}-1)\alpha_1} |\bar{c}^*|^2_{(t,j)} \\ &\le \frac{\alpha_2}{\alpha_1} e^{-\alpha(t+j)} e^{R} |\phi(0,0)|^2_{\mathfrak{A}} + \frac{\kappa e^{-\lambda_d}\gamma^2}{(e^{-\lambda_d}-1)\alpha_1} |\bar{c}^*|^2_{(t,j)} \end{aligned}$$

where $\alpha \in \left(0, \frac{\lambda_d}{1+T_2}\right]$ and $R = \left[\frac{T_2\lambda_d}{1+T_2}, \infty\right)$, which leads to

$$\begin{aligned} |\phi(t,j)|_{\mathfrak{A}} = \max\Bigg\{ & \sqrt{2\frac{\alpha_2}{\alpha_1}} e^{-\frac{(t+j)}{2}} e^{\frac{R}{2}} |\phi(0,0)|_{\mathfrak{A}}, \\ & \sqrt{2\frac{\kappa e^{-\lambda_d}}{(e^{-\lambda_d}-1)\alpha_1}} \gamma |\bar{c}^*|_{(t,j)} \Bigg\}. \end{aligned}$$

which concludes the proof. ∎

The following remark leverages Lemma 3.14 to provide a discussion on the perturbation on initial conditions.

Remark 5.5: From Lemma 3.5, $\mathcal{H}$ satisfies the hybrid basic conditions implying that the hybrid systems are nominally well-posed [33, Definition 6.2]. Due to the fact that all maximal solutions to $\mathcal{H}$ are complete, [33, Proposition 6.14] implies that solutions with perturbed initial conditions stay close to the unperturbed solutions. Consider any compact set $K \subset \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{T}$. Then, for every $\tau' > 0$, $\varepsilon' \ge 0$ there exists $\sigma > 0$ such that, for every maximal solution $\phi_\sigma \in \mathcal{S}_{\mathcal{H}}(K + \sigma\mathbb{B})$, there exists a solution ϕ to $\mathcal{H}$ with $\phi(0,0) \in K$ such that ϕ_σ and ϕ are (τ', ε')-close.[11] Recall the existence of a limit point for $\mathcal{H}$ as given by Lemma 3.14. Then, we can show that given a perturbation on the initial conditions results in a perturbation on the limit point. To show this, consider an unperturbed initial condition $\phi(0,0)$ for $\phi_\sigma(0,0) \in \phi(0,0) + \sigma$ where $\sigma \in [-\sigma'_x, \sigma'_x]^N \times [-\sigma'_\eta, \sigma'_\eta]^N \times [-\sigma'_\tau, \sigma'_\tau]$ to $\mathcal{H}$ with data in (10) where $\phi_\sigma = (\phi_{\sigma_x}, \phi_{\sigma}, \phi_{\sigma})$, $\phi = (\phi_x, \phi_\eta, \eta_\tau)$. With a slight abuse of notation, denote the initial condition of a solution $\phi(0,0)$ as ϕ^0. From the proof of Proposition 3.14, it follows that when $h = 0$ the limit point ϕ_{x_i} of the perturbed solutions are given by

$$\begin{aligned} \lim_{t+j\to\infty} \phi_{x_i}(t,j) = {} & \frac{1}{N}\sum_{i=1}^{N} \left(\phi^0_{x_i} + \phi^0_\tau \phi^0_{\eta_i}\right) \\ & + \frac{1}{N}\sum_{i=1}^{N} \left(\phi^0_{\sigma_{x_i}} + \phi^0_\tau \phi^0_{\sigma\eta_i} + \phi^0_{\sigma}\left(\phi^0_{\eta_i} + \phi^0_{\sigma_i}\right)\right), \end{aligned}$$

and when $h \neq 0$ it follows that the limit point of the perturbed solution is given by

$$\begin{aligned} \lim_{t+j\to\infty} \phi_{x_i}(t,j) = {} & \frac{1}{N}\sum_{i=1}^{N} \Bigg(\phi^0_{x_i} + \phi^0_{\sigma_{x_i}} \\ & + \frac{\exp(-h(\phi^0_\tau + \phi^0_{\sigma})) - 1}{h} (\phi^0_{\eta_i} + \phi^0_{\sigma_i})\Bigg). \end{aligned}$$

C. Numerical Examples

Example 5.6: In this example, we consider the case of parameters in Example 4.8 to exercise the results on perturbations. In particular, we consider the case of perturbations

[10] Young's Inequality is defined as $2a^\top b \le c a^\top a + \frac{1}{c} b^\top b$ for $c > 0$.

[11] Given $\tau^0, \varepsilon^0 > 0$, two hybrid arcs ϕ_1 and ϕ_2 are (τ^0, ε^0)-close if the following is satisfied: for all $(t,j) \in \operatorname{dom}\phi_1$ with $t + j \le \tau^0$ there exists s such that $(s,j) \in \operatorname{dom}\phi_2$, $|t-s| < \varepsilon^0$, and $|\phi_1(t,j) - \phi_2(s,j)| < \varepsilon^0$; for all $(t,j) \in \operatorname{dom}\phi_2$ with $t + j \le \tau^0$ there exists s such that $(s,j) \in \operatorname{dom}\phi_1$, $|t-s| < \varepsilon^0$, and $|\phi_2(t,j) - \phi_1(s,j)| < \varepsilon^0$.

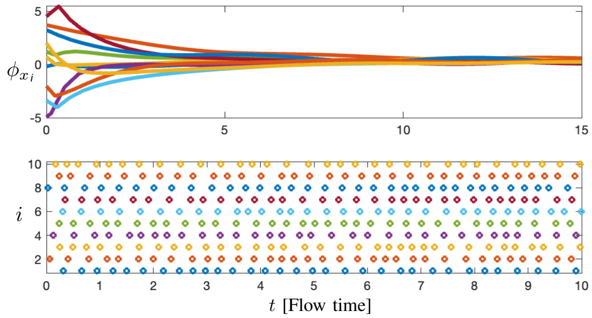

Fig. 6. Robustness of consensus to communication noise for asynchronous communications. The $\boldsymbol{x_i}$ components of a solution = ($_x$; ;) to a hybrid system with agent dynamics (2) where asynchronous communication is present. The asynchronous communication is shown on the bottom where an 'o' marker indicates a communication of agent $\boldsymbol{i}$, more specifically, $_i(t;j) = 0$ for each $(t;j)$ 2 dom .

on the dynamics given by (38), communication and measurement perturbations given by (40) and (39). In particular, let $b_i \sim U([-0.25, 0.25])$ and $c_i^s \sim U([-0.25, 0.25])$ for $s \in \{c, m\}$. Figure 6 illustrates Theorem 5.1 showing that there exists $\varepsilon > 0$ such that the consensus set $\mathcal{A}_\theta$ in (31) is semiglobally practically robustly $\mathcal{KL}$ asymptotically stable for $\mathcal{H}_\theta$ in (29). In particular, it is worth noting that the solutions never converge to consensus explicitly due to the perturbations, however, they converge to a region nearby consensus. △

Example 5.7: In this example, we will showcase the robustness results in Section V-B, which considers input-to-state stability of the consensus set for synchronous communication. In particular, we consider using the parameters and network in Example 4 with perturbations on the communication given by c^*. Let $c_i^* \sim U([-1, 1])$ where $U(\cdot)$ is the uniform random distribution over the argument interval. Figure 7 shows the solution from the initial conditions considered in Example 4. In this figure, we showcase the agents' states, the timer state, and the Lyapunov function evaluated over the numerical solution. Note that due to the communication noise inducing perturbations on the η^+ term on jumps, the Lyapunov function does not explicitly converge to the origin, however, the Lyapunov function reduces significantly after the first several jumps and converges to a region around the consensus set $\mathcal{A}$ in (12). △

VI. Conclusion

In this work, we design hybrid consensus protocols to achieve consensus under intermittent communication events. Using a hybrid systems framework, we define the communication events between the systems using a hybrid decreasing timer. Recasting consensus as a set stability problem, we leverage the network topology and employed a Lyapunov-based approach to certify that the consensus set is partially pointwise globally exponentially stable. For the case of asynchronous communication events, we show that, under certain conditions, the consensus set is globally asymptotically stable which is, in fact, robust to a rich class of perturbations.

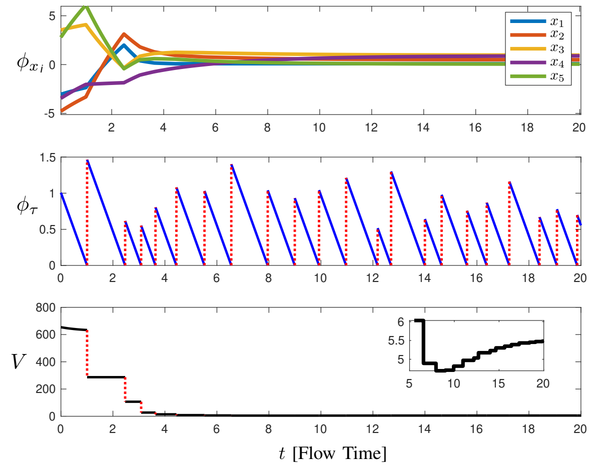

Fig. 7. Input-to-state stability of consensus with communication noise. The $\boldsymbol{x_i}$ and components of a solution = ($_x$; ;) to a hybrid system with agent dynamics (2) with synchronous communications. The bottom figure shows the Lyapunov function $\boldsymbol{V}$ in (17) evaluated over the solution.

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