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Hybrid Control, Morse Theory and Ivan Kupka.

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Summary. We begin with memories of Ivan Kupka. In the body of the paper we use Morse theory to construct a hybrid feedback law that robustly and globally asymptotically stabilizes the system to any desired point of any compact connected manifold. The method is a straightforward generalization of an example of performing this trick on the circle, found in a textbook by the second author. The logic variable part of "hybrid" is a single bit indicating whether or not to switch, with hysteresis, between two smooth vector fields on the manifold. One vector field is minus the gradient of a Morse function, to be constructed, whose global minimum is the desired point. The other vector field represents a steady breeze blowing by all the unstable equilibria of the gradient flow and pointing roughly parallel to their unstable manifolds. In order to motivate the use of hybrid control, we discuss how one might formulate the ideas of robustness, measurement, and measurement error to feedback systems on manifolds.

6.1 Ivan Kupka

6.1.1 Montgomery

Ivan Kupka and I became friends through mathematics. He remained close to my heart ever after our initial meetings.

We met through subRiemannian geometry and its interactions with control theory. Mike Enos[3], Ivan Kupka had been at a conference together. I had recently uncovered the phenomenon of topologically stable strictly abnormal geodesics in rank 2 subRiemannian geometries (see [14]). Enos explained my basic example to Ivan in the back row during a boring talk.

Ivan became intrigued and wrote several papers around the phenomenon and a survey of subRiemannian geometry. See [2], [11], and [1].

As a result of that introduction, Ivan and I had several visits. I particularly remember walking through Île Saint-Louis with Ivan in a downpour in early Spring. We ducked under the eaves of a cafe. He was grumpy about the prices and poshness of the island. He told me about growing up there when the central streets of the island were a slum. He grew up poor. I began to get a deep appreciation for the French education system whose notion of equality allowed a slum kid like Ivan to rise to the top. Later, Ivan took me on a tour of Versailles, not far from his home outside of Paris and afterwards we had a simple delicious dinner at his house with his wife.

[3] Enos was a retired gymnast who had switched into mathematical control theory, wanting to do "falling cat" type optimal control problems with the dream of designing new gymnastic moves.

As a young man, I had dropped out of society and lived in a tree house and made a living on rivers teaching people to kayak. Ivan had joined some version of the French merchant marine (the legends are various) and somehow ended up in Brazil where he reconnected with mathematics, and got his PhD under Peixoto. I felt like we were some type of alter egos - alternative selves. I loved his mathematical taste. He engaged in all areas of mathematics. I did not always understand him. His love and skill in genericity arguments and singularity theory as exemplified by the Kupka-Smale theorem (see [16] and references therein) combined in wonderful surprising ways with his deep appreciation and skill in hard down-to-earth explicit computations involving special functions. He had a particular love of elliptic functions which shined through in his work with Bonnard et al. I feel blessed for our friendship and the time we had together.

6.1.2 Sanfelice

I never met Ivan Kupka in person, however, I gained a deep appreciation for his work on observers, also known a state estimators, for the purpose of reconstructing the full state of a dynamical system from measurements of a (likely nonlinear and noninvertible) function on state space. I became aware of this work during my short stint at the Ecole de Mines de Paris in Fall 2008, working with Laurent Praly on high gain observers using adaptive gains.

Laurent introduced me to Ivan's book on observers coauthored with J-P. Gauthier [7]. This book presents, in a deep and concise manner, a general theory for analysis and design of observers. It gives a much detailed presentation of their general approach then their seminal 1994 SIAM Journal on Control and Optimization article "Observability and observers for nonlinear systems," which, currently, has more than 600 citations.

Being shortly after I finished my PhD (in 2007), I was really thirsty for knowledge on state estimation, as my PhD focused mostly on control theory for the solution of state feedback problems. Infused with Laurent's courage (and great espresso), I carefully read the formulation, results, and proofs in Ivan's book. His work is mathematically deep and rigorous, arguably, among the most impactful ones on the topic. The generality of the mathematical development is also unique – a particular feature of it is that, unlike much of the work in the literature, his results do not assume that (maximal) solutions exist for all time. Many of our recent articles on observers for dynamical systems propose solutions that are inspired from the constructions in his book. It is evident that his work has made a long lasting impact on the field.

6.1.3 Acknowledgement

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6.2 Introduction and Setup

Many control problems are difficult to solve due to topological obstructions intrinsic to the system being controlled. Such obstructions emerge in most autonomous vehicles problems. We focus here on the problem of stabilizing a system on a manifold to a single fixed point using feedback. If the point is stable for a vector field then its basin of attraction is contractible. The flow itself yields the contraction to the the stable equilibrium point! But compact boundaryless manifolds are not contractible. It follows that finding a global Lipshitz feedback law for a smooth system on such a manifold is impossible. See [3] and [12] for more concerning topological obstructions to Lipshitz feedback stabilization.

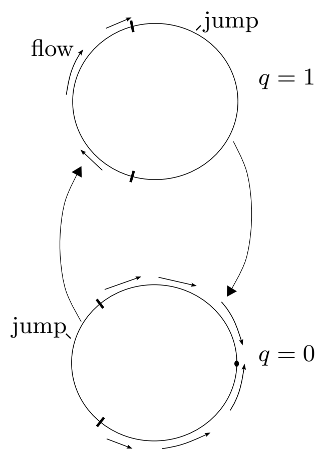


Fig. 6.1: Turning a gradient flow on the circle into a hybrid system with a single global attractor. The main trick is that the flow set for $q = 1$, the subset of the circle where you see the arrows, contains the jump set for $q = 0$.

Consider the problem of achieving *robust* global asymptotic stability of a desired point for the attitude of a planar rigid body. The goal is to render the desired point stable – trajectories starting nearby the point stay nearby – and globally attractive – every trajectory limits to the desired point as time approaches positive infinity – and, perhaps most importantly, to achieve these goals with robustness to perturbations such as noise in the measurements telling us our current attitude. The state space of the planar rigid body is the group $SO(2)$ of rotations of the plane, a group which is diffeomorphic to the circle $\S^1$ in the standard way:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO(2)$$

being parameterized by the single angle θ. The circle is not contractible so we cannot design a smooth feedback system driving us to our goal, the identity, which corresponds to $\theta = 0$. Nevertheless, let us try. Introduce the control system

$$\dot{\theta} = u. \tag{6.1}$$

The feedback law

$$u = -\sin(\theta)$$

yields the negative gradient flow $\dot{\theta} = -\sin(\theta)$ for the function $\phi(\theta) = -\cos(\theta)$. It has the origin $\theta = 0$ as stable equilibrium. The basin of attraction of $\theta = 0$ is all of the circle minus the single point $\theta = \pi$ antipodal to $\theta = 0$. The point $\theta = \pi$ is an unstable equilibrium so our feedback law leaves it fixed where it is. We have not achieved global asymptotic stability. Almost - we missed by a hair: one point, $\theta = \pi$ just sits there forever, all the others limit to $\theta = 0$. We have failed to achieve global asymptotic stability of $\theta = 0$, in line with the basic fact from topology that the circle is not contractible. See also [4].

We can regain global asymptotic stability by using a discontinuous feedback control law. Any feedback law which interpolates in a convex manner between $u = -\text{sgn}(\pi - \theta)$ near $\theta = \pi$ and $u = -\sin(\theta)$ near $\theta = 0$ will do the trick. (Here $\text{sgn}(x)$ is the sign function, so that $\text{sgn}(x) = +1, x > 0$ and $\text{sgn}(x) = -1, x \ \leq 0$.)

However, introducing this discontinuity to our feedback law destroys robustness to measurement error. Suppose that m represents measurement error in the angle θ. Near $\theta = \pi$, the actual recieved feedack by the system would then be $u = -\text{sgn}(\pi - \theta + m)$. An arbitrarily small oscillatory measurement noise m can render the previously unstable equilbrium point $\theta = \pi$ into a stable equilibrium! [4]

The notion of robustness and measurement error are central to this paper. Hermes, in [9] brought the importance of measurement error its potentially devastating effects when feedback laws are discontinuous, and its beautiful connections to the Fillipov lemma to the attention of the control community. In Section 6.3.3 below we touch on his paper and define robustness to measurement noise so as to make sense on manifolds.

We can achieve global robust asymptotic stability by moving into the world of hybrid systems where we mix analog and digital. See [5] and [17]. Introduce a logic variable, or simply, a single bit $q \in \{0, 1\}$ which we carry around with us and monitor as we travel about the circle. The bit acts as a state-dependent switch to between two vector fields, say[5] $-\mathrm{d}\theta$ for $q = 1$ and $-\sin(\theta)\mathrm{d}\theta$ for $q = 0$, switching depending both on where we are on the circle and what the current state of the bit is. See figure 6.1. This is a basic example in the subject of hybrid feedback controllers. See p. 21, Section 1.2.1 of [17]

The point of this note is to show how we can use Morse theory to generalize the circle example so as to work on any compact manifold M. We "hybridize" M in the same way as we did the circle, by introducing a single bit $q \in \{0, 1\}$. The hybrid feedback law allows us to carry on with two interpenetrating vector fields which we can switch between depending on where we are and the value of our bit and in this way achieve a global robust asymptotic feedback stablizer on M. See theorem 9.27 at the end of the next-to-last section of this article.

6.2.1 Setup, Goal, and Strategy

Let M be a compact connected manifold and $m_0 \in M$ be our target. Our goal is to design a control system having a robust global feedback law with m_0 as its global attractor. As described above, for

[4] The basic phenomenon of stabilizing an unstable fixed point by imposing small amplitude high frequency oscillations earned Paul the Nobel prize in 1989 for the Paul Trap. See [10]. R. M. is grateful to Mark Levi for pointing out this connection.

[5] We use the standard notation of differential topology. The vector field $f(\theta)\mathrm{d}\theta$ implements the differential equation $\dot{\theta} = f(\theta)$.

topological reasons this is impossible within the standard framework of smooth feedback systems on M. We can however, using a bit of Morse theory, make m_0 into an "almost global attractor" for the gradient vector field of a function ϕ to be designed below:

$$\dot{z} = -\nabla\phi(z) \qquad z \in M. \tag{6.2}$$

When we say "almost global attractor" we mean that the basin of attraction for m_0 is an open dense subset of M.

We take $\phi : M \to \mathbb{R}$ to be a Morse function whose only local minimum is m_0. Consequently m_0 is the global minimizer of ϕ. [6] The hybrid strategy, following the circle example, is to understand where and how the gradient flow gets hung up and misses limiting to m_0. We then use another vector field Y – called a "breeze" below– to nudge the system away from these bad sticking points. The sticking points are exactly the other critical points of ϕ. Finally, we use the idea of hybrid feedback to switch back and forth between these two vector fields at judicious locations of the manifold with the help of an auxiliary bit $q \in \{0, 1\}$ which allows the introduction of memory in the feedback control algorithm.

6.2.2 Morse theory

We recall the relevant definitions and basic properties around Morse functions. A critical point of a smooth function $\phi : M \to \mathbb{R}$ is a point p where the differential $d\phi(p) = \sum \mathrm{d}\phi x^i|_p dx^i$ of the function ϕ vanishes. Here, the $x^i, i = 1, \ldots, n$, are coordinates near p and n is the dimension of M. At a critical point p, we can form the Hessian of ϕ:

$$\mathrm{Hess}(\phi) = \sum \frac{\partial^2\phi}{\partial x_i \partial x_j} dx^i dx^j$$

which is understood as a bilinear symmetric form on the tangent space. The Hessian is independent of coordinates, but, unlike the Euclidean space setting, the Hessian is undefined if p is not a critical point. (The associated quadratic form obtained by using the formula at a non-critical point is coordinate dependent, its value changing as we change coordinates.)

Definition 1. *A critical point p of a function ϕ is called non-degenerate if the Hessian of ϕ is non-degenerate (i.e. the matrix of the Hessian is invertible) at p.*

Definition 2. *A smooth function is called a Morse function if all its critical points are non-degenerate.*

Lemma 1 (Morse lemma). *If p is a non-degenerate critical point of the smooth function ϕ on the n-dimensional manifold M then there exists a smooth coordinate system $x_1, \ldots, x_c, y_1, \ldots y_k$ on M centered at p such that, in these coordinates,*

$$\phi(x_1, \ldots x_c, y_1, \ldots, y_k) = \phi(p) + \sum_{a=1}^{c} x_a^2 - \sum_{b=1}^{k} y_b^2. \tag{6.3}$$

Here $k + c = n$ and k is the index of the critical point p.

[6] In order to define the gradient we need an auxiliary Riemannian metric on M. In tensor notation $-\nabla\phi(z) = \sum g^{ij}(z)\mathrm{d}\phi x^i \mathrm{d}x^j$ where the metric is $\sum g_{ij}(z) dx^i dx^j$.

We have employed two standard definitions:

Definition 3. *A coordinate system* [7] $x : M \dashrightarrow \mathbb{R}^n$ *is centered at* p *if* $x(p) = 0$.

Definition 4. *The index* k *of a nondegenerate critical point* p *for* ϕ *is the index of its Hessian: the largest possible dimension of a subspace* $S \subset T_pM$ *such that the restriction of* $\mathrm{Hess}(\phi)_p$ *to* S *is negative definite.*

Lemma 2 (Sard-Morse). *Every manifold admits Morse functions. Moreover, the space of Morse functions is open and dense within the space of all smooth functions on* M *endowed with the Whitney* C^2*-topology*

We refer the reader to Guillemin-Pollack [8], or Milnor [15] for proofs of the Morse lemma and the Sard-Morse theorem.

We need a special case of the "handle slide procedure" in order to guarantee only one local minimum for ϕ.

Proposition 1. *If* ϕ_0 *is a Morse function on the connected manifold* M *having* $m_0 \in M$ *as a local minimum, then we can deform* ϕ_0 *into another Morse function* ϕ_1 *which has* m_0 *as its only local minimum and is such that the critical values* c_i *of* ϕ_1 *are all distinct.*

This deformation is a homotopy, i.e., a path $\phi_t, 0 \le t \le 1$ of smooth functions all of which have m_0 as a local minimum. Except for a finite number of times t, each ϕ_t is Morse. The critical points of all the ϕ_t can be taken to be isolated. For a proof see [18, p. 143, Proposition 5.4.1] and the discussion in the paragraph preceding this proposition.

6.2.3 Hang Ups

For the same reasons that the gradient descent method works in Euclidean space, ϕ decreases strictly monotonically along any non-equilibrium trajectory to the gradient flow (6.2). It follows that almost all trajectories converge to m_0, it being the only minimum of ϕ. Some trajectories will get hung up on saddle points, that is to say, limit to an unstable equilibrium of the gradient flow. The equilibria of our gradient flow are exactly the critical points of ϕ. All trajectories which are not equilibria converge to one of these critical points.

By the Morse lemma the critical points are isolated, and hence finite in number. *We write* $N + 1$ *for this number, with* m_0 *counted amongst the critical points.* Consequently, there are N critical points, which are saddles or local maxima. We write the non-minimal critical points as $p_i, i = 1, \dots, N$, and their critical values as $c_i = \phi(p_i)$.

Recall that the stable manifold of an equilibrium point p_i is the set of initial conditions z for which the trajectory of (6.2) through z converges to p_i in the limit as time approaches infinity. We denote this manifold by $W^+(p_i)$. It is a smooth embedded [8] manifold passing through p_i and whose dimension is the *co-index* $c = n - k$ of the critical point p_i.

[7] The broken arrow notation here is used to simply denote that the domain of x is an open subset of M and not all of M.

[8] Stable manifolds for general smooth vector fields are immersed, not embedded submanifolds. To wit: heteroclinic tangles and Hamiltonian chaos. However the stable manifolds of gradient flows are embedded submanifolds. See for example Corollary 7.4.1 in [Jost, Riemannian geometry and geometric analysis].

By assumption, the only critical point of index 0 is our target point m_0. See Proposition 1. Take the union of all the stable manifolds *except* for m_0's: That is, consider

$$\Omega = \bigcup_{i=1}^{N} W^+(p_i).$$

We have that $W^+(m_0) = M \setminus \Omega$: the basin of attraction of m_0 equals the complement of Ω. Away from Ω all trajectories of the gradient flow in (6.2) converge to m_0. Note that Ω has measure zero in M, being the finite union of submanifolds all of which have codimension at least 1. Consequently, the basin of attraction for m_0 is an open dense set of full measure – a non-linear counterpart of the complement of a finite collection of proper linear subspaces in a Euclidean space.

6.2.4 A Steady Breeze

One strategy for finding a hybrid feedback stabilizer to bring all points to m_0 would be to find a nonzero vector field Y transverse to each stratum $W^+(p_i)$ of Ω. Think of Y as a "strong wind, blowing past Ω." When we get close to Ω turn off the gradient flow and "let this wind blow." The flow of Y, being transverse to Ω, will push us back into the basin of attraction of m_0.

Finding such a Y is hard. It requires global knowledge of the stable manifolds $W^+(p_i)$ of our unstable critical points. *We can make due instead with the local knowledge provided by the Morse lemma* and, in essence, construct a collection of local winds or "breezes" Y_i , one for each unstable critical point p_i. The flow of Y_i will push all points sufficiently near p_i into a region collecting points p' such that $\phi(p') < \phi(p_i) - K$, where $K > 0$ is a constant. Once in this region we revert to the gradient vector field whose flow decreases ϕ, pushing points way from p_i and further decreasing ϕ either all the way down to its global minimum at m_0 or, with bad luck, near another unstable critical point p_j, one with $\phi(p_j) < \phi(p_i)$. Once near to this p_j, we can repeat the process, invoking the local breeze Y_j. Cycles between neighborhoods of different critical points cannot occur since we will insist that these neighborhoods do not intersect and between them ϕ strictly decreases since we use the gradient flow.

Consider a vector field $Y : M \to TM$ satisfying the property that

$$\mathrm{Hess}(\phi)_{p_i}(Y(p_i), Y(p_i)) = -2 \tag{6.4}$$

The existence of such a Y is straightforward. Since the Hessian has negative directions $y_1, \ldots, y_k$ at each p_i finding a vector $v_i \in T_{p_i}M$ with $\mathrm{Hess}(\phi)_{p_i}(v_i, v_i) = -2$ is easily done. Indeed, $v_i = \mathrm{d}y_1$ works, where (x, y) are Morse coordinates. Now all smooth manifolds M share a number of basic extension properties, one which is as follows. Given a vector $v \in T_pM$ at a point p, we can always find a vector field $Y : M \to TM$ with $Y(p) = v$. This extension property holds for any finite number $v_1, \ldots, v_N$ of vectors attached at distinct points of M. Consequently we have the existence of our Y.

Lemma 3 (steady breeze lemma). *[See Figure 6.2.] Associated to our vector field Y there are neighborhoods V_i of each non-minimal critical point p_i of our Morse function ϕ, and positive constants k_1, k_2 with the following property. Any trajectory for Y crossing into or starting in V_i leaves V_i within a time k_1, exiting at a point p of ∂V_i with $\phi(p) < \phi(p_i) - k_2$.*

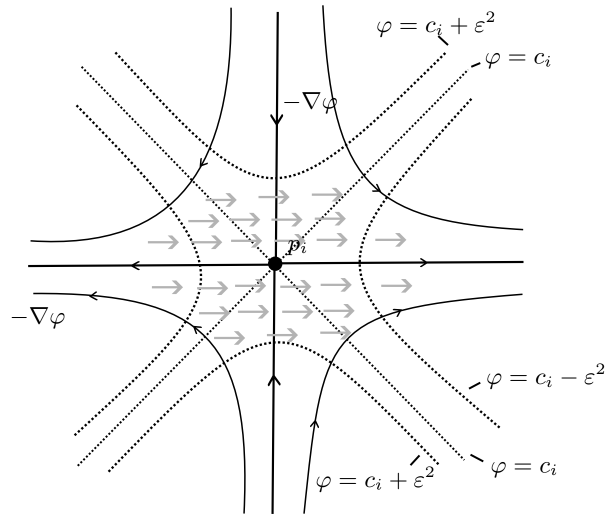


Fig. 6.2: The level sets of the Morse function ϕ near the critical point p_i are dashed. The vector field $\nabla\phi$ for gradient flow is indicated by its solid trajectories. The vector field Y for the breeze that blows past p_i is indicated by the short solid (brown) horizontal arrows.

PROOF.

To begin with, take Y to be the constant vector fields $\mathrm{d}y_1$ in the coordinates of the Morse Lemma, (Lemma 1 above). The flow Ψ_t of Y in our Morse coordinates is the translational flow $(x, y) \mapsto (x, y + te_1) = \Psi_t(x, y)$, where e_1 denotes the vector in $\mathbb{R}^k$ (of $\mathbb{R}^n = \mathbb{R}^c \times \mathbb{R}^k$) whose only 1 corresponds to the choice of index $a = 1$, i.e., e_1 is the coordinate representation of $\mathrm{d}y_1$. Rewrite the Morse normal form as

$$\phi(x, y) - c_i = |x|^2 - |y|^2, \text{ where } \phi(p_i) = c_i$$

where the norms are the standard coordinate norms on the corresponding x and y coordinate spaces. Then

$$\phi(\Psi_t(x, y)) - c_i = (\phi(x, y) - c_i) - 2ty_1 - t^2.$$

View this as a quadratic expression in t. Imposing the conditions that $|x|^2 + |y|^2$ and hence y_1 are very small, the constant term $(\phi(x, y) - c_i)$ and the coefficient of the linear term $-2ty_1$ can be made arbitrarily small, so that the quadratic term eventually beats them. We view the conditions on $|x|^2 + |y|^2$ and y_1 as initial conditions for solving for the flow of Y. For V_i we can take a flow box of the form $|y_1| < A$, $|x|^2 + \sum_{a>1} y_a^2 < \delta$. The lemma follows immediately for this case.

There are at least two routes in to the general case. For one of these routes, use the symmetry group $SO(n-k, k)$ of the quadratic form $|x|^2 - |y|^2$ to "rotate" coordinates so that $Y(p_i) = \mathrm{d}y_1$. Then, argue that $Y(p)$ does not deviate far from $Y(p_i)$ as long as we stay in a small enough neighborhood of p_i. For the other route, use the 2nd order Taylor series with error estimates for the trajectory $\gamma_*(t)$ of Y passing through p_i to get that $\phi(\gamma_*(t)) < c_i - \frac{3}{4}t^2$ for all sufficiently small t, and then argue by uniform convergence of the flow $\Psi_t(p)$ of Y that the "far side" of the Taylor estimates, $k_2/2 < t < k_2$, hold for k_2 small and p close to p_i. We leave the details up to the reader.

QED

Remark. There are points p' arbitrarily close to p_i for which $\phi(p') > c_i$. Since $\Psi_0(p') = p'$ the inequality $\phi(\Psi_t(p')) - c_i < -\frac{1}{2}t^2$ must fail for t in an interval about 0 for these p'. For such p' we need to flow a non-zero amount of time before $\phi - c_i$ begins to become negative and then for a bit longer until our inequality holds.

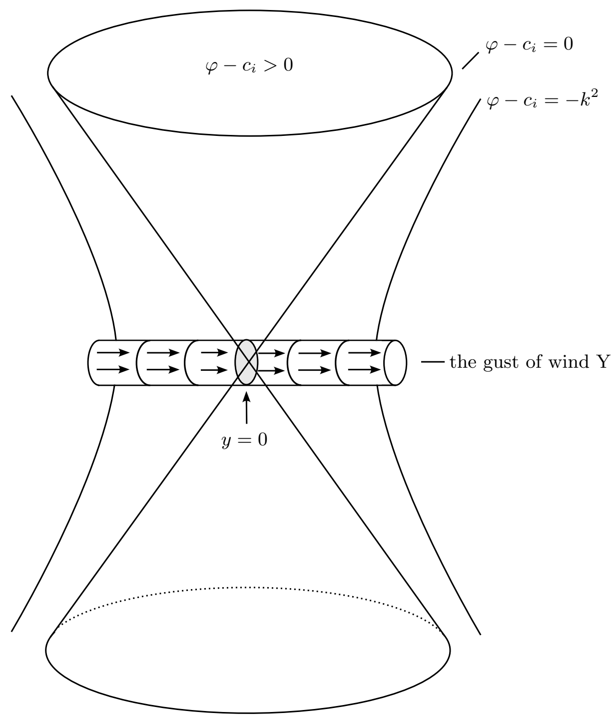


Fig. 6.3: A flowtube for the breeze flow of Y and its relation to the level sets of ϕ.

6.2.5 Topology

Morse theory relates critical points and their indices to the topology of the manifold. A basic topological invariant of a manifold is its "Betti numbers" $b_k = b_k(M)$, $k = 0, 1, \ldots$, which are popularly described as the "number of k-dimensional holes" in M. We have $b_j = 0, j > n$. Stated more carefully, for each choice of field F there are integers $b_k(M, F)$ that are equal to $\dim_F H_k(M, F)$, where $H_k(M, F)$ is the k-th homology group of M with coefficients in the field F. The Betti number we are talking about is the maximum over all fields of the $b_k(M, F)$.

Write m_k for the number of index k critical points of our Morse function ϕ. Then

$$m_k \geq b_k.$$

In particular

$$N + 1 \geq \sum b_k.$$

since $\sum m_k = N + 1$ where $N + 1$ is the number of critical points $m_0, p_1, \ldots, p_N$.

Example 1. Take $M = SO(3)$. It is well-known that $SO(3) = \mathbb{RP}^3$. Its Betti numbers are $b_0 = b_1 = b_2 = b_3 = 1$. If we work over the field F of rational or real numbers we will find that $b_1(M, F) = b_2(M, F) = 0$. However, over the field $F = \mathbb{Z}_2$ of two elements we have that $b_1(M, \mathbb{Z}_2) = b_2(M, \mathbb{Z}_2) = 1$. For all fields $b_0(M, F) = b_3(M, F) = 1$. From the Morse inequalities it follows that any Morse function on $SO(3)$ has at least 4 critical points.

6.3 Errors, Robustness, Hybridization

In this section, we propose a simple way to switch between Y and $-\nabla\phi$ so as to arrive to a feedback law that globally asymptotically stabilizes m_0. Then, we shoot down this law on grounds of robustness. Measurement errors can make discontinuous feedback laws induce undesired behavior, for example, it can "stabilize" the system to one of the unstable fixed points p_i of the gradient flow. Through the study of robustness (or lack of) to measurement noise of such a feedback law, a hybrid control feedback law is discovered. The intuition is that if we carry a bit $q \in \{0, 1\}$ in our pocket (it does not have to be a qubit!) as we travel around M, taking measurements of ϕ and $\|\nabla\phi\|$ as we travel, and switching bits appropriately, we can build a robustly globally stabilizing hybrid feedback law.

6.3.1 A Discontinuous Stabilizer

Let us return to the circle example in Section 6.2. Modify our feedback law near $\theta = \pi$ using a discontinuous control law having a discontinuity at $\theta = \pi$. One way to achieve this is to add to $u = \sin(\theta)$ any function of the form $g(\theta) := \beta(\theta)\mathrm{sgn}(\theta - \pi)$, where $\mathrm{sgn}(x)$ is the sign function $\mathrm{sgn}(x) = -1$ if $x < 0, \mathrm{sgn}(x) = +1$ if $x > 0$, and where $\beta(\theta)$ is a bump function supported in a small neighborhood of $\theta = \pi$ and such that $\beta(\pi) = 1$. Choose either -1 or $+1$ for the value of $\beta(\theta)\mathrm{sgn}(\pi - \theta)$ at $\theta = \pi$. Thus, we are investigating the flow of the discontinuous vector field $\dot{\theta} = \sin(\theta) + g(\theta)$. Declare a solution to be an absolutely continuous curve $t \mapsto \theta(t)$ that satisfies $\dot{\theta} = g(\theta)$ almost everywhere. Then, for every initial condition $\theta_0 \in M$ passes a unique solution and this solution converges to m_0, which we recall is the point $\theta = 0$.

We can copy this example onto our manifold. Recall the neighborhoods V_i of the steady breeze lemma (Lemma 3). They can be taken to be balls or tubes with smooth boundaries, and so that $-\nabla\phi$ and Y are transverse to ∂V_i at all but a finite number of points. Set

$$V = \bigcup_{i=1}^{N} V_i \tag{6.5}$$

Define the discontinuous vector field

$$F(z) := \begin{cases} -\nabla\phi(z) & \text{if } z \notin V \\ Y(z) & \text{if } z \in V \end{cases}$$

Under this discontinuous vector field, every (maximally defined) trajectory of $\dot{z} = F(z)$ converges to m_0.

6.3.2 ...Gets Ruined in the Presence of Measurement Noise

Hermes [9] made a basic observation linking noise and uncertainty to the Fillipov lemma. As a consequence we can establish (or design) arbitrarily small noise or measurement uncertainties, applied near the discontionuities, which stabilize the system there!

Imagine we are working in a single Morse chart and that Y is straightened out so as to be the constant vector field $e_1 = \mathrm{d}y_1$. Suppose that we are near the discontinuity of F at ∂V as defined above. Rewrite our system as a control system

$$\dot{z} = -u\nabla\phi(z) + (1-u)Y(z) \tag{6.6}$$

where u is only allowed to be 0 or 1. We have been choosing the possibility of 0 or 1 depending on whether or not we are in V or outside of V. We suppose that the measurements of z are not exact, namely, we do not know the value of z with infinite precision. Imagine, for example, imposing one possibility or another depending on some very noisy and highly oscillatory imprecision as to where the boundary of V_i lies.

Now recall the Fillipov lemma. [9] The lemma asserts that the accessible set for a control system with only binary off-on ("bang-bang") controls as above, agrees with the accessible set for the convex hull of the two vector fields. In particular, at points z_* where Y and $\nabla\phi$ are linearly dependent and pointing in the same direction, we can choose a system of controls which turns this z_* into a fixed point. Now model this control with uncertainty on the measurements of z. A bit more work turns the new fixed point z_* into a stable fixed point under the effect of such uncertainty.

Are there really points z_* so that we can write $0 = -u\nabla\phi(z_*) + (1-u)Y(z_*)$ for some u, $0 \leq u \leq 1$? The degree of $-\nabla\phi$ at p_i is $(-1)^{k_i}$ and, in particular, is nonzero. It follows that $\nabla\phi/\|\nabla\phi\|$ sweeps out all possible points $e \in \S^{n-1}$ of the unit sphere as z varies over a small sphere surrounding p_i. By elementary topology (∂V_i is homologous to the boundary of this small sphere) the same is true as z varies over ∂V_i. In particular, there will be points $z \in \partial V_i$ where $Y = e_1$ and $\nabla\phi$ point in the same direction [10] and we can then use u to scale accordingly. We have our u and our point z_*.

We have indicated how arbitrarily small uncertainty, such as measurement noise, can render our previously unstable fixed point p_i for the gradient flow into a stable fixed point if we try to implement our above discontinuous alteration of gradient flow. This is not a good solution if we want to achieve our goal.

6.3.3 Robustness and Measurement

Measurements come with uncertainties. So do control forces or torques. The environment in which our controlled object moves has noise, wind, uneven terrain, etc. And our physical analog model of our system, the way we encode it as an ODE, will be imprecise. It turns out that measurement noise can wreak havoc with discontinuous vector fields, rendering previously unstable locations stable and inadvertently hanging us up indefinitely near one of the unstable equilibria p_i. The goal of robust

[9] We do not mean to trivialize the result or Hermes discussion of it. There is a deep and non-trivial discussion of what is meant by a "Fillipov solution" and the consequent measure theory around it in [9] and in the subsequent literature.

[10] If, as in the figure, our ∂V_i has corners, use the usual subdifferential style tangent space at the corners a la Clarke and this argument still works.

control is to guarantee that we arrive within a pre-specified window of our desired goal m_0 in the presence of appropriately bounded uncertainty.

Suppose our state space is a real vector space, say $\mathbb{R}^n$, and on it we have an expected or "nominal" [11] vector field $z \mapsto F(z)$. We imagine this vector field arriving to us after *implementing some feedback control loop.* So we expect that the system evolves according to

$$\dot{z} = F(z) \qquad z \in \mathbb{R}^d. \tag{6.7}$$

Measurement uncertainty corresponds to not knowing exactly where we are. So replace the state variable z at which we evaluate the vector field F by $z + \eta_m(z,t)$ where $(z,t) \mapsto \eta_m(z,t)$ represents measurement noise. We allow η_m to depend on time since measurement noise could be time dependent. We want to compare the end results of our nominal ODE in (6.7) with that of its "nearby cousins"

$$\dot{z} = F(z + \eta_m(z,t)). \tag{6.8}$$

Suppose that the nominal system has the origin as a global attractor. Do the cousins continue to have the origin as global attractor? This is too much to hope for, since it would require that the noise vanish at the origin.

Definition 5 (Robustness to measurement error). *Suppose the nominal vector field (6.7) – imagined to arise from a feedback stabilization control scheme – has the origin as a global attractor. Then, we say this control scheme (or its vector field) is "robust" to measurement errors if, given any $\delta > 0$ sufficiently small we can find $\epsilon > 0$ such that all trajectories to all the noisy cousins (6.8) to the nominal control with $\|\eta_m\| < \epsilon$ converge to a δ-ball of the origin as time tends to infinity.*

Remark 1. Of course the norm used to measure $\|\eta_m\|$ will matter! We use the sup norm.

Measurement Noise on Manifolds

We are in a decidedly vector space setting in this formulation of robustness since we cannot add points on manifolds! If $F : M \to TM$ is a vector field on a manifold the expression $F(z + \eta_m(z,t))$ does not make sense! We cannot add points on a manifold. Even if we could, $F(z + \eta)$ would be a vector in the tangent space to M at $z + \eta$, not to the tangent space of M at z, so it would not represent a vector field. Rather than follow these lines to try to make sense of measurement noise and robustness on a manifold, we return to the control theory drawing board and look into where F comes from. Notably, we introduce the control-theoretic idea of a "measurement" in addition to "control" and "feedback" Rewrite our original system in the traditional form

$$\dot{z} = f(z,u) \qquad z \in M, u \in \mathbb{R}^m$$

where the controls u take values in a convex subset of $\mathbb{R}^m$. Naturally,

$$f : M \times \mathbb{R}^m \to TM$$

with

[11] Dictionaries give multiple definitions of "nominal." Here, by "nominal" we mean that the system is operating without perturbations, namely, the system under study has state z that is precisely governed by (6.7).

$$f(z, u) \in T_z M \qquad \forall z \in M$$

uniformly on u. See also Brockett [6] who takes the u's to vary within an auxiliary vector bundle over M. We want to implement a feedback law $u = \kappa(z)$ in a way which allows the modeling of measurement noise. To do this we introduce the *intermediary of a measurement.*

Definition 6. *A measurement on M is a vector valued map*

$$h : M \to \mathbb{R}^\ell, \qquad z \mapsto y = h(z)$$

meant to model the sampling and recording of partial information regarding the state $z \in M$.

We insist that our feedback laws depend only on what we measure, that is,

$$u = \kappa(h(z)),$$

where, now

$$\kappa : \mathbb{R}^\ell \to \mathbb{R}^m$$

represents our feedback law. Since h takes values in a vector space, we can simply add time-dependent measurement uncertainty $\eta_m : M \times \mathbb{R} \to \mathbb{R}^\ell$ to our measurements by

$$h \mapsto h + \eta_m; \qquad \eta_m : M \times \mathbb{R} \to \mathbb{R}^\ell$$

thus replacing the feedback law $z \mapsto \kappa(h(z))$ by its nearby noisy cousins given by

$$\kappa(h(z) + \eta_m(z, t)).$$

We have set things up now so that we can define "robustness to measurement error" in essentially a way identical to our earlier definition. We merely replace the feedback law in $F(z, \kappa(h(z)))$ by its perturbation $F(z, \kappa(h(z) + \eta_m(z, t))$.

Remark 2. Modeling environmental noise, control noise, and uncertainty in the model are all straightforward on a manifold. They correspond to the perturbations $F(z, u) + \eta_{env}$, $F(z, u) \to F(z, u + \delta u)$, and $F(z, u) \to F(z, u) + (\delta F)(z, u)$, respectively.

We can summarize what we have done using a commutative diagram where the dotted arrow of "feedback" closes the loop. In the case of our example of stabilizing to $m_0 \in M$, we will see next that we need two feedback control laws and two measurements, so $k = \ell = 2$.

$$\begin{tikzcd} & M \arrow[dr, "\text{measurement}"] & \\ \mathbb{R}^2 \arrow[ur, "\text{control}"] & & \mathbb{R}^2 \arrow[ll, dotted, "\text{feedback}"] \end{tikzcd}$$

6.3.4 Our New Setup

To put our "gradient flow / breeze system" into this framework introduce two controls u_1, u_2 so as to encode our system as the control system:

$$\dot{z} = -u_1 \nabla \phi(z) + u_2 Y(z). \tag{6.9}$$

If $u_1 = 1$ and $u_2 = 0$ we have pure gradient flow. If $u_1 = 0$ and $u_2 = 1$ we have pure steady breeze. Introduce measurements $y : M \to \mathbb{R}^2$ where $y = (y_1, y_2)$ is the function

$$y_1 = \|\nabla \phi(z)\|, \qquad y_2 = \phi(z) \tag{6.10}$$

We will be continuously monitoring y_1. Whenever y_1 is sufficiently small, we are in a Morse neighborhood U_i of one of the p_i or perhaps of m_0. We can decide which point p_i or m_0 z is closest to (and closed to which neighborhood U_i), by measuring y_2 and comparing it to the possible (known) critical values of ϕ.

Preparing the Morse Function for Hybridization

Recall that our goal point m_0 is the globaly minimum of ϕ and its only local minimum. Translating $\phi \mapsto \phi - \phi(m_0)$ insures that $\phi(m_0) = 0$ so that $\phi(z) > 0$ for each $z \neq m_0$. We have also assumed (by a wiggling of ϕ) that the critical values $\phi(p_i)$ of ϕ are all distinct. (See Proposition 1 above.) Scaling ϕ by a (possibly large) scalar $K > 0$, we can separate the critical values so they are all at least a unit apart

$$p_i \neq p_i \implies \phi(p_i) - \phi(p_j) \geq 1$$

and

$$\phi(p_i) \geq 1.$$

This scaling of ϕ can be used to ensure that the rescaled ϕ also enjoys the property that $\{z : \|\nabla \phi(z)\| < 1\}$ consists of $N+1$ topological (open) balls $W_1, W_2, \ldots, W_N$, one for each critical point p_i, and one, say W_0 for m_0, and that each of these balls is contained in a Morse neighborhood U_i of the critical point. This scaling and translating of ϕ does not change the location of the critical points p_i or their index.

Note that scaling ϕ by K rescales both the Morse coordinates y, x and the breeze Y by $1/\sqrt{K}$.

For each $i = 1, \ldots, N$, we may take the breeze neighborhoods V_i on which the flow of Y is well controlled and brings us to $\phi < c_i - k_2$ so that $V_i \subset W_i$. Note that the intersections of W_i with $\{z : \|\nabla \phi(z)\| < r\}$ form a family of nested balls converging to p_i as $r \to 0$. Now choose k_3 small enough so that

$$B_i := \{z : \|\nabla \phi(z)\| < k_3\} \cap W_i \subset V_i.$$

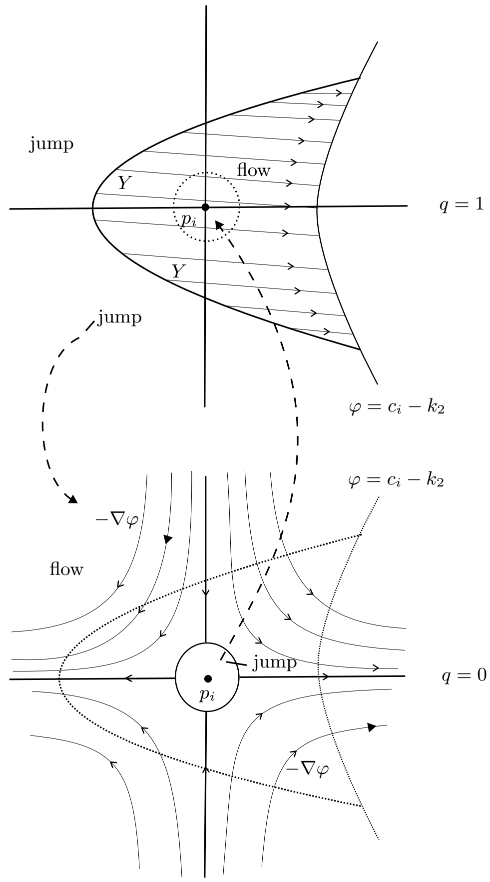
and that the boundary of B_i and of V_i are disjoint. See Figure 6.4. Since $\|\nabla \phi\|$ acts as a measure of distance from p_i, we have that $k(i) > k_3$ for each i, where

$$k(i) = \min_{p \in \partial V_i} \|\nabla \phi(p)\|.$$

Set

$$k_V = \min_i k(i).$$

Our "margin of robustness" – the measurement tolerance we need to guarantee for $y_1 := \|\nabla \phi\|$ to ensure that our control scheme will stabilize z to m_0 – is some fraction of the minimum of $k_V - k_3$ and k_3. With such a measurement area we can be sure to distinguish between being inside B_i and leaving V_i, a)



b)

Fig. 6.4: The disc B_i centered about the unstable equilibrium p_i of the gradient flow forms a connected component of the jump set for $q = 0$. The set B_i is contained in the parabolic V_i which is a component of the flow set for $q = 1$, whose flow is that of Y. The complement of the union of the V_i forms the jump set for $q = 1$. The exterior of the union of the B_i is the flow set for $q = 0$ for the gradient flow. The $q = 1$ flow lines in V_i terminate when $\phi \leq c_i - k_2$. Sample jumps from $q = 0$ to $q = 1$, and vice versa, are marked with dashed arrows.

6.3.5 Hybridizing

Let us introduce the discrete variable

$$q \in \{0, 1\}$$

which will toggled *on* or *off* to define a hybrid feedback control law depending on the measurements. The role of q is select whether $-\nabla\phi$ or Y should update z during flows when the state is in the so-called *flow set*, which we denote by C. The toggles of q occur when the state is in the so-called *jump set*, which we denote as D. Specifically, we define the state of the closed-loop system with the hybrid feedback controller as (z, q), whose goal is to globally and robustly asymptotically stabilize

$$M \times \{0\}.$$

The flow set C is defined as the union of the sets $C_0 \times \{0\}$ and $C_1 \times \{1\}$, and the jump set D as the union of the sets $D_0 \times \{0\}$ and $D_1 \times \{1\}$, where the sets C_0, C_1, D_0, and D_1 are defined next. Set

$$D_0 := \bigcup_{i=1}^{N} B_i, \qquad C_1 := \bigcup_{i=1}^{N} V_i,$$

the index of the disjoint union running from 1 to N, the labels of the nonminimal critical points p_i. Since $B_i \subset V_i$, we have that $D_0 \subset C_1$ – in fact, C_1 contains a neighborhood of D_0. Use these sets to define two partitions of M, namely

$$C_0 := M \setminus D_0, \qquad D_1 := M \setminus C_1.$$

To properly selects the among the two feedback laws, we define the *jump map* as the map that keeps z constant and toggles q from 0 to 1 or from 1 to 0 when the state z is in the jump set. More precisely, we denote the jump map as

$$G : M \times \{0,1\} \to M \times \{0,1\}$$

and define it as

$$G(z,1) := (z,0), \qquad G(z,0) := (z,1).$$

The state z is updated continuously according to the *flow map* obtained from (9.31), which is

$$F(z,u) := (-u_1 \nabla\phi(z) + u_2 Y(z), 0)$$

where $u = (u_1, u_2)$ and, conveniently, we apply the feedback law

$$\kappa(z,0) := (1,0), \qquad \kappa(z,1) := (0,1).$$

Then, during flow – that is, when $(z,q) \in C$, the state (z,q) is governed by

$$(\dot{z}, \dot{q}) = F(z,q) = (-\kappa(z,q)\nabla\phi(z) + \kappa(z,q)Y(z), 0)$$

while at jumps, which occur when $(z,q) \in D$, the state (z,q) is updated by

$$(z^+, q^+) = G(z,q) = (z, 1-q)$$

The flow and jump dynamics described above lead to the hybrid closed-loop system given as

$$\mathscr{H} \; : \; \begin{cases} (\dot{z}, \dot{q}) = F(z,q) & (z,q) \in C \\ (z^+, q^+) = G(z,q) & (z,q) \in D \end{cases} \tag{6.11}$$

Our hybrid stabilization scheme operates as follows:

- If $q = 0$ and $z \in C_0$, z flows according to the first component of $F(z,0)$, namely, $-\nabla\phi(z)$. As we do so, the controller measures $y_1 = \|\nabla\phi(z)\|$ and $y_2 = \phi$. If y_1 ever crosses below the threshold value k_3 while $y_2 = \phi$ is greater than c_1, the smallest nonzero critical value of ϕ, then z entered $D_0 = \bigcup_i B_i$. If $z \in D_0$, then the jump map is applied to reset q to 1 – note that z remains at the same point in B_i. Since $B_i \subset V_i$, z can flow with $q = 1$.

- IF $q = 1$ and $z \in C_1$, z flows according to the first component of $F(z, 1)$, namely, $Y(z)$ while measuring $y_2 = \phi(z)$. The value of y_2 will be close to some critical value c_i. Eventually, y_2 crosses below $c_i - k_2$, which means that z leaves C_1 and enters D_1. If $z \in D_1$ with $q = 1$, then the jump map is applied to reset q to zero, and z remains unchanged. Since $B_i \subset V_i$, z is outside B_i and so in the flow regime for $q = 0$.

Since $C_0 \cup D_0 = M$ and $C_1 \cup D_1 = M$, the above rules cover all possibilities for points $(z, q) \in M \times \{0, 1\}$. Teaders can convince themselves that this scheme provides a global feedback stabilization law to m_0.

Robustness of the scheme follows from the strict containment $C_0 \subset D_1$. Specifically, the scheme we just described is robust to measurement errors in y_1 provided these errors are small enough to allow us to distinguish between being inside a B_i and leaving a V_i. We can quantify the error bounds by recalling that $y_1 := \|\nabla\phi(z)\|$, $y_1 = k_3$ on ∂B_i and $k(i) = \min_{p \in \partial V_i} \|\nabla\phi(p)\| > k_3$. Set $k_V = \min_i k(i)$ and $k_* = \frac{1}{2}\min\{k_3, k_V - k_3\}$. If our error bars on measuring y_1 are less than k_* then by evaluating y_1 we can guarantee whether or not we are in B_i or have left V_i with sufficient accuracy as to know whether we should be flowing or jumping. We call k_* the "margin of robustness" for this scheme.

Theorem 1 (Theorem). *On any compact connected n-dimensional manifold M, and for any chosen point m_0 of that manifold, we can design a hybrid control system whose logic part consists of a single bit $q \in \{0, 1\}$ as in* (6.11) *and which has $\{m_0\} \times \{0\}$, as a global attracting and stable set, this property being robust with respect to measurement and all other errors in the system.*

WHY THE PARABOLIC-SHAPED V_i?

In Figure 6.4, we have made V_i so as to have a parabolic boundary capped by a level set of ϕ. We did this to guarantee that the vector field Y is transverse to the boundary ∂V_i everywhere except at the points where the cap joins the parabola. Being transverse is "robust" (unchanged by perturbations) whereas tangency is easily destroyed by perturbations. This is why we prefer the parabolic profile for the boundary.

Here is how to make such a parabolic neighborhood. Begin with a standard flox-box for Y. In flow-box coordinates, the flow-box is a cylinder of the form tube has the form $I \times B$, where $I = [-T, T] \subset \mathbb{R}$ is in the $Y = \mathrm{d}y_1$ direction and B is a solid unit ball in $\mathbb{R}^{n-1}$. For simplicity of notation, label the coordinates of $\mathbb{R}^{n-1}$ as x_a instead of the old $(x_a, y_b), b > 1$. Then, the flow tube can be expresses as $\rho \leq 1, -T \leq y_1 \leq T$, where

$$\rho = \sqrt{\sum_a x_a^2}.$$

Now take any smooth strictly monotonic increasing function $g : [-T, T] \to [0, 1]$, $g = g(y_1)$, which starts out either with $g(-T) = 0$ and increases strictly monotonically to $g(T) = 1$. (For a standard parabola take $g(y) = \frac{1}{4T^2}(y + T)^2$.) Our neighborhood is given by $\{(y_1, \rho) : \rho \leq g(y_1)\}$. This parabolic neighborhood has the property that all trajectories of Y enter into it through the parabolic bottom and leave it along the cap with $\phi = c_i - k_2$. Since transversality cannot be changed by small perturbations, a perturbed $\tilde{Y} = Y + w$ will continue to have these nice entrance and exit properties.

6.3.6 Solutions to Hybrid systems

Some words are in order regarding what we mean by a "solution" to this system which is a combination of continuous flow and discrete jump. The instantaneous state of our hybrid system is a $(z,q) \in M \times \{0,1\}$, where the index q indicates that we should think of $z \in M_q$.

HYBRID TIME In the hybrid literature one keeps track of jumps by introducing a discrete integer time $j \in \mathbb{N}$ as well as the continuous time. Solutions are parameterized by "stair steps" $E \subset \mathbb{R} \times \mathbb{N}$. These stair steps are graphs of piecewise constant monotone functions taking integer values with jumps of 1. In other words, $E = \bigcup_{j=1,N} I_j \times \{j\}$, where, for the particular construction in (6.11) $I_j \subset \mathbb{R}$ are the intervals whose endpoints are where the jumps in $q \to \bar{q}$ occur. So, in this case, the right endpoint of I_j equals the left endpoint of I_{j+1}. In the open part of each interval, (z,q) flows according to F. The continuous variable z flows on the flat part of each step, i.e., on the interior of the I_j's. At jumps, $j \to j+1$ from one step to the next, (z,q) is reset by the jump map, which keeps z constant and flips q. Using this language, one expresses solutions as maps $x : S \to M \times \{0,1\}$ by writing $x(t,j) = (z(t,j), q(t,j))$. For (6.11), the discrete variable is constant on each open interval $int(I_j) \times \{j\}$. It makes a jump at the transition from one interval (step) to the next. (In the general case, solutions may be such that z exhibits jumps: $z(t,j) \to z(t,j+1) = G(z(t,j), q(t,j))$ according to some pre-specified collection of maps $G(\cdot, q)$ whose domains and ranges may depend on q.)

HOW MANY JUMPS? If a solution to (6.11) starts with $q = 0$ then typically we expect that there will be no jump at all. The initial z would lie in the basin of attraction of m_0 and its trajectory would avoid all of the B_i, so the gradient flow would take it all the way down to m_0. Similarly, if a solution starts with $q = 1$ and in C_1, we expect that there will be a single jump, followed by a gradient flow all the way to m_0.

In the worst case, if the solution starts with $q = 0$ with z sitting at the global maximum for ϕ, there could be as many as $2N$ jumps, with two jumps per critical point until z enters a ball about m_0. There are two jumps per close encounter with a critical point p_j, one upon entering B_j from 0 to 1 to turn on the breeze, and then one upon leaving V_j from 1 to 0 to turn back on the gradient flow. We can insure fewer jumps if the gradient flow is *Morse-Smale*. Let $\beta \le n+1$ be the number of indices k such that the kth Betti number $b_k(M)$ is nonzero. (Here n is the dimension of M.) To be Morse-Smale [12] means that the stable and unstable manifolds of all critical points intersect transversally and implies that whenever a trajectory connects one critical point p_i to another one p_j then the index of p_i is larger than that of p_j. If the balls B_j are sufficiently small and $-\nabla\phi$ is Morse-Smale, then trajectories of the gradient flow will only enter at most β balls as they travel down to m_0. We do not need to count the final ball about m_0 since solutions do not jump upon entering it. In this way, we get the worst-case scenario count of $2(\beta - 1) \le 2n$ jumps total.

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