Hybrid Persistency of Excitation in Adaptive Estimation for Hybrid Systems

A. Saoud M. Maghenem A. Loría R. G. Sanfelice

Abstract— **We propose a framework of stability analysis for a class of linear non-autonomous hybrid systems, with solutions evolving in continuous time governed by an ordinary differential equation and undergoing instantaneous changes governed by a difference equation. Furthermore, the jumps may also be triggered by exogeneous hybrid signals. The proposed framework builds upon a generalization of notions of *persistency of excitation* (PE) and *uniform observability* (UO), which we redefine to fit the realm of hybrid systems. Most remarkably, we propose for the first time in the literature a definition of hybrid persistency of excitation. Then, we establish conditions, under which hybrid PE is equivalent to hybrid UO and, in turn, uniform exponential stability (UES). Our proofs rely on an original statement for hybrid systems, expressed in terms of $\mathcal{L}_p$ bounds on the solutions. We illustrate the utility of our results on a generic adaptive estimation problem.**

I. Introduction

Persistency of excitation (PE), roughly speaking, is the property of a function of time that consists in the function's energy *never* vanishing. Over five decades, several mathematical definitions of PE have been proposed in various contexts (*e.g.*, depending on whether the said function evolves in continuous or in discrete time) to guarantee different stability properties. For linear continuous-time-varying systems, some PE properties guarantee uniform (in the initial time) exponential stability [1] or uniform global asymptotic stability [2]. For results concerning systems evolving in discrete time, we refer to [3], [4], where the concept of PE was originally introduced.

Furthermore, with a careful handling, that involves replacing some instance of the state with the system's solutions in the system's equations, PE-based statements tailored for linear systems may also apply to nonlinear systems [5]. In this case, a solution-dependent PE notion is necessary and sufficient to ensure uniform asymptotic stability. For particular classes of nonlinear non-autonomous systems, forms of solution-dependent PE conditions have been proposed in [6], [7], [8]. A non-solution-dependent PE condition tailored for nonlinear systems is provided in [9], where it is also showed to be necessary for uniform global asymptotic stability of generic nonlinear non-autonomous systems.

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The classes of systems where the PE property is used include, but are not restricted to, those appearing in problems of identification [10], adaptive control [11], learning-based identification [12], and state estimation [13], [14]. For instance, the so-called gradient systems, which appear in the context of gradient-descent estimation are among the linear time-varying systems, for which, PE is necessary and sufficient for UES of the origin. Moreover, convergence rate estimates [15], [5] and strict Lyapunov functions for gradient systems are available in the literature [16]. One of the landmark results in the study of PE is that it is equivalent to uniform observability (UO) for passive systems satisfying structural properties reminiscent of the Kalman-Yacubovich-Popov Lemma [17]. For nonlinear time-varying systems, there is an equivalence between PE along solutions and zero-state detectability [6]. To the best of our knowledge persistency of excitation has not been formally defined for hybrid systems, whose models combine differential and difference equations. Yet, the coexistence of continuous- and discrete-time phenomena (what we call *hybrid phenomena*) is unavoidable in some scenarios of control systems. This calls for Lyapunov-stability conditions stated in terms of a PE notion tailored for hybrid systems.

In this Note, which is the outgrowth of [18], we study this question for hybrid systems as considered in [19]. The class of systems to which our own framework applies covers impulsive systems, *i.e.,* a type of non-autonomous systems that experience jumps under the influence of a piece-wise continuous signal, and not only depending on whether the state trajectory is in the flow or the jump set at a given instant. Our main contribution is the formulation of a property of PE tailored for a class of hybrid systems, which we call hybrid persistency of excitation (HPE). We establish that HPE is equivalent to *hybrid uniform observability* (HUO) and we establish UES under HUO. The HPE property that we define captures the richness of time-varying piece-wise-continuous signals; richness that cannot be captured otherwise by classical definitions of PE, defined purely in continuous or discrete time. For illustration, we show that a hybrid version of the classical gradient-descent identification algorithm successfully estimates the unknown parameters of a *hybrid input-output plant* in cases where purely continuous- or purely discrete-time algorithms fail.

Our main results are presented in Sections III and IV, but we start our exposition with a brief recall of some definitions and notations that pertain to hybrid-systems. An illustrative example is provided in Section V.

II. Preliminaries on Hybrid systems

After [19], a hybrid dynamical system $\mathcal{H}$ is the combination of a constrained differential equation and a constrained difference equation given by

$$\mathcal{H}:\begin{cases} \dot{x} &= F(x) \quad x\in C \\ x^+ &= G(x) \quad x \in D, \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^{m_x}$ denotes the state variable, $\mathcal{X}$ the state space, $C \subseteq \mathcal{X}$ and $D \subseteq \mathcal{X}$ denote the flow and jump sets, respectively, and $F : C \to \mathbb{R}^{m_x}$ and $G : D \to \mathbb{R}^{m_x}$ correspond to the flow and jump maps. Solutions to (1) consist in functions with hybrid time domain defined as follows.

Definition 1 (hybrid signal and hybrid arc): A *hybrid signal* ϕ is a function defined on a hybrid time domain denoted $\operatorname{dom}\phi \subset \mathbb{R}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$. The hybrid signal ϕ is parameterized by ordinary time $t \in \mathbb{R}_{\geqslant 0}$ and a discrete counter $j \in \mathbb{Z}_{\geqslant 0}$. Its domain of definition is denoted $\operatorname{dom}\phi$ and is such that, for each $(T, J) \in \operatorname{dom}\phi$, $\operatorname{dom}\phi \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0}^{J}([t_j, t_{j+1}] \times \{j\})$ for a sequence $\{t_j\}_{j=0}^{J+1}$ such that $t_{j+1} \geqslant t_j$, $t_0 = 0$, and $t_{j+1} = T$. Moreover, if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j := \{t : (t, j) \in \operatorname{dom}\phi\}$, then the hybrid signal ϕ is said to be a *hybrid arc*. □

Definition 2 (Solution to $\mathcal{H}$): A hybrid arc $\phi : \operatorname{dom}\phi \to \mathbb{R}^{m_x}$ is a *solution* to $\mathcal{H}$ if $\phi(0, 0) \in \operatorname{cl}(C) \cup D$;

(S2) for all $j \in \mathbb{Z}_{\geqslant 0}$ such that $I_\phi^j = \{t : (t, j) \in \operatorname{dom}\phi\}$ has nonempty interior,

$$\begin{aligned} &\phi(t, j) \in C && \text{for all } t \in \operatorname{int}(I_\phi^j), \\ &\dot{\phi}(t, j) = F(\phi(t, j)) && \text{for almost all } t \in I_\phi^j; \end{aligned}$$

(S3) for all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j + 1) \in \operatorname{dom}\phi$,

$$\phi(t, j) \in D, \qquad \phi(t, j + 1) = G(\phi(t, j)).$$ □

A solution ϕ to $\mathcal{H}$ is said to be maximal if there is no solution ψ to $\mathcal{H}$ such that $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \operatorname{dom}\phi$ and $\operatorname{dom}\phi$ is a proper subset of $\operatorname{dom}\psi$. It is said to be nontrivial if $\operatorname{dom}\phi$ contains at least two points. It is said to be continuous if it is nontrivial and never jumps. It is said to be eventually discrete if $T := \sup_t \operatorname{dom}\phi < \infty$ and $\operatorname{dom}\phi \cap (\{T\} \times \mathbb{Z}_{\geqslant 0})$ contains at least two points. It is said to be eventually continuous if $J := \sup_j \operatorname{dom}\phi < \infty$ and $\operatorname{dom}\phi \cap (\mathbb{R}_{\geqslant 0} \times \{J\})$ contains at least two points. System $\mathcal{H}$ is said to be forward complete if the domain of each maximal solution is unbounded.

We are interested in sufficient conditions for uniform exponential stability (UES) of sets $\mathcal{A} \subset \mathcal{X}$ that are closed relative to $\mathcal{X}$, but not necessarily compact. Uniformity is considered with respect to all the initial conditions that are at an arbitrary given distance from the set $\mathcal{A}$. This distance is defined as $|x|_\mathcal{A} := \inf_{z\in\mathcal{A}} |x - z|$, where $|\cdot|$ denotes the Euclidean norm.

Definition 3 (UES): Let the closed subsets $(\mathcal{A}, \mathcal{D}) \subset \mathcal{X} \times \mathcal{X}$. The set $\mathcal{A}$ is said to be UES for $\mathcal{H}$ on $\mathcal{D}$ if there exist κ and $\lambda > 0$ such that, for each solution ϕ to $\mathcal{H}$ starting from $x_o \in \mathcal{D}$ at $(0, 0)$, we have

$$|\phi(t, j)|_\mathcal{A} \leqslant \kappa |x_o|_\mathcal{A} e^{-\lambda(t+j)} \quad \forall (t, j) \in \operatorname{dom}\phi. \tag{2}$$

If $\mathcal{D} = \mathcal{X}$, we say that the set $\mathcal{A}$ is UES for $\mathcal{H}$. □

III. Integral Characterization of UES

Our first statement is an original characterization of UES for hybrid systems, in terms of uniform $\mathcal{L}_p$-integrability of solutions. It is reminiscent of [8, Lemma 2] for continuous-time systems and [20] for discrete-time systems. As the solutions of hybrid systems may flow and jump, we first introduce certain notations related to integration over a hybrid time domain.

Hybrid Integral: Consider a function on a hybrid time domain $\phi : \operatorname{dom}\phi \to \mathbb{R}^{n\times n}$ and let $K \in \mathbb{R}_{>0} \cup \{+\infty\}$ and $(t, j) \in \operatorname{dom}\phi$. We use $E_{t,j,K}^\phi \subset \operatorname{dom}\phi$ to denote the shortest hybrid time domain, starting from (t, j), of length larger or equal than K and contained in $\operatorname{dom}\phi$. Note that if K is finite, then there exists a unique $(s_K, m_K) \in \operatorname{dom}\phi$, such that

$$K \leqslant (s_K - t) + (m_K - j) < K + 1, \tag{3}$$

and a unique non-decreasing sequence

$$\{t_j, t_{j+1}, ..., t_{m_K}, t_{m_K+1}\} \text{ with } t_j := t \text{ and } t_{m_K+1} := s_K,$$

such that

$$E_{t,j,K}^\phi := [t_j, t_{j+1}] \times \{j\} \cup \cdots \cup [t_{m_K}, t_{m_K+1}] \times \{m_K\}.$$

Thus, the hybrid integral of ϕ over the domain $E_{t,j,K}^\phi$ is defined as

$$\int_{E_{t,j,K}^\phi} \phi(s, i) d(s, i) := \sum_{i=j}^{m_K} \int_{t_i}^{t_{i+1}} \phi(s, i) ds + \sum_{i=j}^{m_K-1} \phi(t_{i+1}, i).$$

In particular, for $K = +\infty$, we have $s_\infty + m_\infty = +\infty$.

Akin to the case where signals evolve purely in continuous or discrete time—cf. [21], given a function ϕ on a hybrid time domain starting at $(t_o, j_o) \in \operatorname{dom}\phi$, we define the hybrid $\mathcal{L}_p$-norm, with $p \in [1, \infty)$, as

$$|\phi|_{\mathcal{A}p} := \left[\int_{E_{t_o,j_o,\infty}^\phi} |\phi(s, i)|_\mathcal{A}^p d(s, i)\right]^{\frac{1}{p}} \tag{4}$$

and the hybrid $\mathcal{L}_\infty$ norm,

$$|\phi|_{\mathcal{A}\infty} := \sup\left\{|\phi(t, j)|_\mathcal{A} : (t, j) \in E_{t_o,j_o,\infty}^\phi\right\}. \tag{5}$$

In the case that $\mathcal{A} = \{0\}$ we simply write $|\phi|_p$ and $|\phi|_\infty$.

Then, the following statement generalizes [5, Lemma 3] to the realm of hybrid systems.

Theorem 1 (Hybrid-integral characterization of UES): Consider the hybrid system $\mathcal{H} := (C, F, D, G)$, as defined in (1), and let $(\mathcal{A}, \mathcal{D}) \subset \mathcal{X} \times \mathcal{X}$ be closed subsets. The set $\mathcal{A}$ is UES on $\mathcal{D}$ if and only if there exist c and $p > 0$ such that, for each ϕ, solution to $\mathcal{H}$ starting from $x_o \in \mathcal{D}$, we have

$$\max\left\{|\phi|_{\mathcal{A}\infty},\ |\phi|_{\mathcal{A}p}\right\} \leqslant c|x_o|_\mathcal{A}. \tag{6}$$

Moreover, (2) holds with $\lambda := \frac{1}{pc^p}$ and $\kappa := c\exp(1/p)$ □

Proof of sufficiency: The solutions ϕ start at the hybrid time $(0, 0)$, so the $\mathcal{L}$-norms in (6) are to be considered on $E_{0,0,\infty}^\phi$. Now, following the proof lines of [5, Lemma 3], we note that condition (6) implies that, for all $(t, j) \in E_{0,0,\infty}^\phi$,

$$\sup\left\{ |\phi(s, i)|_\mathcal{A}^p : (s, i) \in E_{t,j,\infty}^\phi \right\} \leqslant c^p |\phi(t, j)|_\mathcal{A}^p \tag{7}$$

and

$$\int_{E_{t,j,\infty}^\phi} |\phi(s, i)|_\mathcal{A}^p d(s, i) \leqslant c^p |\phi(t, j)|_\mathcal{A}^p. \tag{8}$$

Next, we define the hybrid arc $v : \operatorname{dom}\phi \to \mathbb{R}_{\geqslant 0}$ given by

$$v(t,j) := \int_{E^{\phi}_{t,j,\infty}} |\phi(s,i)|^p_{\mathcal{A}} d(s,i), \tag{9}$$

and we distinguish the two following cases: for all t such that the solution flows, that is, if $t \in \operatorname{int}(I^j_\phi)$, with $I^j_\phi := \{t : (t,j) \in \operatorname{dom}\phi\}$ having a nonempty interior, we have

$$\dot{v}(t,j) = \frac{d}{dt}\left[\int_t^{t_{j+1}} |\phi(s,j)|^p_{\mathcal{A}} ds\right] = -|\phi(t,j)|^p_{\mathcal{A}} \leqslant -\frac{1}{c^p} v(t,j);$$

the last inequality follows from (8). If the solution jumps, that is, for all $(t,j) \in \operatorname{dom}\phi$ such that $(t,j+1) \in \operatorname{dom}\phi$, we have

$$v(t,j+1) - v(t,j) = -|\phi(t,j)|^p_{\mathcal{A}} \leqslant \frac{-1}{c^p} v(t,j);$$

again, the last inequality follows from (8). Then, using the comparison principle for hybrid systems—[22, Lemma 1], and replacing a therein by $\frac{1}{c^p}$, we obtain

$$v(t,j) \leqslant e^{-\frac{t+j}{c^p}} v(0,0). \tag{10}$$

Next, let $K > 0$ be arbitrarily fixed. For each $(t,j) \in E^{\phi}_{0,0,\infty}$,

$$\begin{aligned}
v(t,j) &\geqslant \sum_{i=j}^{m_K} \int_{t_i}^{t_{i+1}} |\phi(s,i)|^p_{\mathcal{A}} ds + \sum_{i=j}^{m_K-1} |\phi(t_{i+1},i)|^p_{\mathcal{A}} \\
&\geqslant \frac{1}{c^p}\left[\sum_{i=j}^{m_K} \int_{t_i}^{t_{i+1}} \sup\left\{|\phi(\tau,k)|^p_{\mathcal{A}} : (\tau,k) \in E^{\phi}_{t,j,K}\right\} ds\right] \\
&+ \frac{1}{c^p}\left[\sum_{i=j}^{m_K-1} \sup\left\{|\phi(\tau,k)|^p_{\mathcal{A}} : (\tau,k) \in E^{\phi}_{t,j,K}\right\}\right] \\
&\geqslant \frac{s_K - t + m_K - 1 - j}{c^p} \sup\left\{|\phi(\tau,k)|^p_{\mathcal{A}} : (\tau,k) \in E^{\phi}_{t,j,K}\right\} \\
&\geqslant \frac{K-1}{c^p} |\phi(s_K, m_K)|^p_{\mathcal{A}},
\end{aligned}$$

where to obtain the last inequality we used (3) and (7). Then, we define $K := c^p + 1$ and we use (7) and (10) to conclude that, for each $(t,j) \in E^{\phi}_{0,0,\infty}$,

$$|\phi(s_K, m_K)|^p_{\mathcal{A}} \leqslant v(t,j) \leqslant e^{-\frac{t+j}{c^p}} v(0,0) \leqslant c^p e^{-\frac{t+j}{c^p}} |\phi(0,0)|^p_{\mathcal{A}}.$$

The last inequality implies that, for each $(s,i) \in \operatorname{dom}\phi \backslash E^{\phi}_{0,0,K}$, $|\phi(s,i)|_{\mathcal{A}} \leqslant c e^{\frac{K}{pc^p}} e^{-\frac{s+i}{pc^p}} |\phi(0,0)|_{\mathcal{A}}$. On the other hand, for each $(s,i) \in E^{\phi}_{0,0,K}$,

$$|\phi(s,i)|_{\mathcal{A}} \leqslant c|\phi(0,0)|_{\mathcal{A}} \leqslant c e^{\frac{K}{pc^p}} e^{-\frac{s+i}{pc^p}} |\phi(0,0)|_{\mathcal{A}}.$$

The statement of sufficiency follows; necessity is trivial. ■

IV. UES FOR TIME-VARYING HYBRID SYSTEMS

Consider the non-autonomous hybrid system of the form

$$\begin{cases} \dot{\zeta} &= F'(\zeta,t,j) \quad t \in \operatorname{int}(I^j_A) \\ \zeta^+ &= G'(\zeta,t,j) \quad (t,j),(t,j+1) \in \operatorname{dom} A, \end{cases} \tag{11}$$

with state $\zeta \in \mathbb{R}^{m_\zeta}$, $F', G' : \mathcal{X} \to \mathbb{R}^{m_\zeta}$, $\mathcal{X} := \mathbb{R}^{m_\zeta} \times \operatorname{dom} A$, and such that A is a hybrid signal whose domain is $\operatorname{dom} A$ and $I^j_A := \{t : (t,j) \in \operatorname{dom} A\}$. The signal A may be an exogenous hybrid signal or may also depend on the system's hybrid trajectories—see [22] for examples. Then, the solutions to (11) are hybrid arcs whose domain is a subset of $\operatorname{dom} A$. That is, the solutions of (11) jump whenever A jumps.

To study the behavior of the solutions to (11), we recast it in the form of (1), by including the hybrid time as a bi-dimensional state variable. That is, defining $x := [\xi^\top \ p \ q]^\top$, system (11) can be rewritten as

$$\begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} F'(\xi,p,q) \\ 1 \\ 0 \end{bmatrix} & x \in C \\ \begin{bmatrix} \xi^+ \\ p^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} G'(\xi,p,q) \\ p \\ q+1 \end{bmatrix} & x \in D, \end{cases} \tag{12}$$

where the flow and jump sets are, respectively, defined as $C := \mathcal{X}$ and $D := \{x \in \mathcal{X} : (p, q+1) \in \operatorname{dom} A\}$. Then, a solution ζ to (11), starting from the initial condition $\zeta_o \in \mathbb{R}^{m_\zeta}$ at $(t_o, j_o) \in \operatorname{dom} A$, must coincide with a solution ϕ to (12), starting from the initial condition (ξ_o, t_o, j_o) at hybrid time $(0,0)$. In this case, we have $p(t,j) = t + t_o$ and $q(t,j) = j + j_o$ for all $(t,j) \in \operatorname{dom}\phi$. We use this fact in what follows of the paper to analyze time-varying hybrid systems in the form of (11).

Remark 1: If the set $\mathcal{A} := \{x \in \mathcal{X} : \xi = 0\}$ is UES for (12), as per Definition 3, then the origin $\{\zeta \in \mathbb{R}^{m_\zeta} : \zeta = 0\}$ is UES for (11), that is, every solution ζ, starting at (t_o, j_o) from ζ_o, satisfies

$$|\zeta(t,j)| \leqslant \kappa |\zeta_o| e^{-\lambda(t+j-t_o-j_o)} \quad \forall (t,j) \in \operatorname{dom}\zeta, \tag{13}$$

with κ and λ independent of (t_o, j_o). •

A. Problem formulation and standing hypotheses

In the sequel, we focus on perturbed non-autonomous hybrid systems of the form—cf. Eq. (11),

$$\mathcal{H}' : \begin{cases} \dot{\zeta} = -A(t,j)\zeta & t \in \operatorname{int}(I^j_A) \\ \zeta^+ = [I_{m_\zeta} - B(t,j)]\zeta & \\ & (t,j),(t,j+1) \in \operatorname{dom} A, \end{cases} \tag{14}$$

where A and B (are assumed to) have the same hybrid time domain, that is, A and $B : \operatorname{dom} A \to \mathbb{R}^{m_\zeta \times m_\zeta}$.

Remark 2: This class of systems is important as it covers a number of interesting cases that appear in adaptive estimation. For instance, when $A(t,j)$ and $B(t,j)$ are both symmetric and positive semidefinite, the model (14) generalizes that of the so-called *gradient system*, studied both in continuous and discrete time in the context of identification [1], [15] and multi-agent systems [23], [16]. The functions A and B may come from expressing outputs and inputs along solutions; namely, for a system $\dot{z} = A_z(y,u)z$, we let $A(t,j) := A_z(y(t,j), u(t,j))$. This artifice is commonly used to analyze some nonlinear observers [13], [14]. •

In what follows, we investigate sufficient conditions for the origin $\{\zeta \in \mathbb{R}^{m_\zeta} : \zeta = 0\}$ to be UES for $\mathcal{H}'$ in (14). We solve this problem under two standing hypotheses reminiscent of those commonly used in the context of continuous- or discrete-time systems. The first one essentially guarantees uniform stability of the origin $\{\zeta \in \mathbb{R}^{m_\zeta} : \zeta = 0\}$ and global boundedness of the solutions to (14). Roughly speaking, for this system we require the existence of a Lyapunov function with negative semidefinite derivative along flows and non-increasing over jumps. The second Assumption imposes uniform boundedness of the matrices A and B, which is common in the realm of non-autonomous systems.

Assumption 1 (Lyapunov (Non-Strict) Inequalities): There exists $P : \mathrm{dom}\, A \to \mathbb{R}^{m_\zeta \times m_\zeta}$ and $p_1,\ p_2 > 0$ such that $P(t,j) = P(t,j)^\top$, and $p_1 \leqslant |P|_\infty \leqslant p_2$, and there exist Q_c, $Q_d : \mathrm{dom}\, P \to \mathbb{R}^{m_\zeta \times m_\zeta}$ such that $Q_c(t,j)$, $Q_d(t,j)$ are symmetric positive semi-definite, for all $t \in \mathrm{int}(I_A^j)$,

$$\dot{P}(t,j) - A(t,j)^\top P(t,j) - P(t,j)A(t,j) \leqslant -Q_c(t,j), \quad (15)$$

and, for all $(t,j) \in \mathrm{dom}\, A$ such that $(t,j+1) \in \mathrm{dom}\, A$,

$$\begin{aligned}[I_{m_\zeta} - B(t,j)]^\top P(t,j+1)[I_{m_\zeta} - B(t,j)] - P(t,j) \\ \leqslant -Q_d(t,j).\end{aligned} \quad (16)$$

Assumption 2 (Uniform Boundedness): There exist $\bar{A}$, $\bar{B} > 0$ such that $|B|_\infty \leqslant \bar{B}$ and $|A|_\infty \leqslant \bar{A}$.

Next, we establish uniform exponential stability for (14) under Assumptions 1 and 2. We provide sufficient conditions in terms of hybrid uniform observability (HUO) and hybrid persistency of excitation (HPE).

B. UES under HUO

Consider the hybrid system $\mathcal{H}'$ in (14) with the hybrid output $y : \mathrm{dom}\, A \to \mathbb{R}^{m_y}$ given by

$$y(t,j) := \begin{cases} C_c(t,j)\zeta & \text{if } \ t \in \mathrm{int}(I_A^j) \\ C_d(t,j)\zeta & \text{otherwise,} \end{cases} \quad (17)$$

for some $C_{c,d} : \mathrm{dom}\, A \to \mathbb{R}^{m_y \times m_\zeta}$. Furthermore, we introduce the hybrid transition matrix $\mathcal{M} : \mathrm{dom}\, A \times \mathrm{dom}\, A \to \mathbb{R}^{m_\zeta \times m_\zeta}$ such that, for each $((t,j),(t_o,j_o)) \in \mathrm{dom}\, A \times \mathrm{dom}\, A$, the solution ζ starting from ζ_o at (t_o,j_o) satisfies

$$\zeta(t,j) = \mathcal{M}((t,j),(t_o,j_o))\zeta_o. \quad (18)$$

The hybrid transition matrix $\mathcal{M}$ is the solution to the hybrid system

$$\dot{\mathcal{M}}((t,j),(t_o,j_o)) = -A(t,j)\mathcal{M}((t,j),(t_o,j_o)) \quad \text{for almost all } t \in I_A^j, \quad (19a)$$

$$\mathcal{M}((t,j+1),(t_o,j_o)) = \left[I_{m_\zeta} - B(t,j)\right]\mathcal{M}((t,j),(t_o,j_o)) \quad (t,j),(t,j+1) \in \mathrm{dom}\, A, \quad (19b)$$

$$\mathcal{M}((t_o,j_o),(t_o,j_o)) = I_{m_\zeta}. \quad (19c)$$

Then, we define hybrid uniform observability as follows.

Definition 4 (HUO): The pair $\{(A,B),(C_c,C_d)\}$ is HUO if there exist K, $\mu > 0$ such that, for each $(t_o,j_o) \in \mathrm{dom}\, A$,

$$\int_{E^A_{t_o,j_o,K}} \mathcal{M}\left((s,j),(t_o,j_o)\right)^\top \Phi(s,j)\mathcal{M}\left((s,j),(t_o,j_o)\right) d(s,j) \geqslant \mu I_{m_\zeta}, \quad (20)$$

where $\Phi : \mathrm{dom}\, A \to \mathbb{R}^{m_\zeta \times m_\zeta}$ is given by

$$\Phi(t,j) := \begin{cases} C_c(t,j)^\top C_c(t,j) & \text{if } \ t \in \mathrm{int}(I_A^j) \\ C_d(t,j)^\top C_d(t,j) & \text{otherwise.} \end{cases} \quad (21)$$

□

Theorem 2 (HUO ⇔ UES): Consider the hybrid system $\mathcal{H}'$ in (14) such that Assumption 2 holds, and Assumption 1 holds for $Q_c := C_c^\top C_c$ and $Q_d := C_d^\top C_d$. Then, the origin $\{\zeta = 0\}$ is UES provided that the pair $\{(A,B),(C_c,C_d)\}$ is HUO. Moreover, if the origin $\{\zeta \in \mathbb{R}^{m_\zeta} : \zeta = 0\}$ is UES and (15)-(16) hold with equality, then the pair $\{(A,B),(C_c,C_d)\}$ is HUO. □

Proof: <u>HUO ⇒ UES:</u> The stability of the origin $\{\zeta \in \mathbb{R}^{m_\zeta} : \zeta = 0\}$ may be analyzed using the framework described in Section II. By rewriting the system as one that is time-invariant, *i.e.,* of the form (12) with flow and jump maps

$$F(x) := [-\xi^\top A(p,q)^\top \quad 1 \quad 0]^\top, \quad (22a)$$

$$G(x) := [\xi^\top [I_{m_\zeta} - B(p,q)]^\top \quad p \quad q+1]^\top, \quad (22b)$$

state $x := (\xi,p,q) \in \mathcal{X} := \mathbb{R}^{m_\zeta} \times \mathrm{dom}\, A$, and flow and jump sets defined by $C := \mathcal{X}$ and $D := \{x \in \mathcal{X} : (p,q+1) \in \mathrm{dom}\, A\}$, respectively. In particular, after Remark 1, the UES bound (13) holds for $\mathcal{H}'$ if the set

$$\mathcal{A} := \{x \in \mathcal{X} : \xi = 0\}, \quad (23)$$

which is closed relative to $\mathcal{X}$, is UES (as per Definition 3) for $\mathcal{H}$, defined by (12), (22), C, and D as defined above. Thus, to prove the first item we use Theorem 1 and this equivalent time-invariant representation of $\mathcal{H}'$. That is, we explicitly compute $c > 0$ such that, along each solution ϕ to the hybrid system (12) with data as in (22), and starting from $x_o := (\xi_o,t_o,j_o)$ at $(0,0)$, it holds that

$$\max\left\{|\phi|^2_{\mathcal{A}\infty}, |\phi|^2_{\mathcal{A}2}\right\} \leqslant c|x_o|^2_{\mathcal{A}}, \quad (24)$$

where $\mathcal{A}$ is defined in (23). To that end, we introduce the Lyapunov function candidate

$$V(x) := \xi^\top P(p,q)\xi, \quad (25)$$

where P is introduced in Assumption 1. Furthermore, after the latter, we have

$$\langle \nabla V(x), F(x) \rangle \leqslant -\xi^\top Q_c(p,q)\xi \qquad \forall x \in C,$$

while

$$V(G(x)) - V(x) \leqslant -\frac{1}{2}\xi^\top Q_d(p,q)\xi \qquad \forall x \in D.$$

Therefore, after (21), along the maximal solution ϕ, we have

$$\dot{V}(\phi(t,j)) \leqslant -\xi(t,j)^\top \Phi(p(t,j),q(t,j))\xi(t,j),$$

for all $t \in \mathrm{int}(I_A^j)$, while

$$\begin{aligned}V(\phi(t,j+1)) - V(\phi(t,j)) \\ \leqslant -\xi(t,j)^\top \Phi(p(t,j),q(t,j))\xi(t,j)\end{aligned}$$

for almost all $(t,j) \in \mathrm{dom}\, \phi$ such that $(t,j+1) \in \mathrm{dom}\, \phi$. Thus, using the fact that Q_c and Q_d are positive definite from Assumption 1, it follows that

$$V(\phi(t,j)) \leqslant V(\phi(0,0)) \qquad \forall (t,j) \in E^\phi_{0,0,\infty},$$

which implies that, for each $(t,j)\in E^{\phi}_{0,0,\infty}$, we have

$$p_1\big|\xi(t,j)\big|^2 \leqslant V(\phi(t,j)) \leqslant V(\phi(0,0)) \leqslant p_2\big|\xi_o\big|^2.$$

Finally, since $|\xi| = |\phi|_{\mathcal{A}}$, we conclude that $|\phi|^2_{\mathcal{A}\infty} \leqslant \frac{p_2}{p_1}|\phi(0,0)|^2_{\mathcal{A}}$. This establishes the first bound in (24).

Next, we compute the second bound. To that end, we follow the proof steps of [24, Proposition 1]. Let the HUO property generate $K>0$ and, for each $(t,j)\in \operatorname{dom}\phi$, a unique pair $(s_K,m_K)\in\operatorname{dom}\phi$ satisfying (3). We have

$$V(\phi(t,j)) - V(\phi(s_K,m_K)) \geqslant \int_{E^{\phi}_{t,j,K}} \xi(s,i)^\top \Phi(p(s,i),q(s,i))\xi(s,i)d(s,i) \quad (26)$$

—recall that $\xi(s,i)$ is a component of the solution $\phi(s,i)$ to (12). Now, because the latter is a time-invariant equivalent representation of (14), the hybrid arc $\xi(s,i)$, with $(s,i)\in E^{\phi}_{t,j,K}$, starting at (t,j) coincides with the solution to (14), ζ, starting at $(t+t_o, j+j_o)$. Therefore, the relation

$$\zeta(s+t_o,i+j_o) = \mathcal{M}\big((s+t_o,i+j_o),(t+t_o,j+j_o)\big)\zeta\big((t+t_o,j+j_o)\big),$$

which holds under (18), implies that

$$\xi(s,i) = \overline{\mathcal{M}}((s,i),(t,j))\xi(t,j), \quad (27)$$

where

$$\overline{\mathcal{M}}((s,i),(t,j)) := \mathcal{M}((s+t_o,i+j_o),(t+t_o,j+j_o)).$$

As a result, we obtain

$$V(\phi(t,j)) - V(\phi(s_K,m_K)) \geqslant \xi(t,j)^\top \times \int_{E^{\phi}_{t,j,K}} \overline{\mathcal{M}}((s,i),(t,j))^\top \Phi(p(s,i),q(s,i))\overline{\mathcal{M}}((s,i),(t,j))d(s,i) \times \xi(t,j) \geqslant \mu\big|\xi(t,j)\big|^2.$$

Next, since $\big|\xi(t,j)\big|^2 \geqslant \frac{p_1}{p_2}\big|\xi(s_K,m_K)\big|^2$, we obtain

$$V(\phi(t,j)) - V(\phi(s_K,m_K)) \geqslant \mu\frac{p_1}{p_2}\big|\xi(s_K,m_K)\big|^2.$$

Integrating on both sides over $E^{\phi}_{0,0,\infty}$, and using the fact that

$$|\xi(s,i)|^2 = |\phi(s,i)|^2_{\mathcal{A}} \qquad \forall (s,i)\in E^{\phi}_{0,0,\infty},$$

we obtain

$$\int_{E^{\phi}_{0,0,K}} V(\phi(s,i))d(s,i) \geqslant \frac{p_1\mu}{p_2}\int_{E^{\phi}_{0,0,\infty}} \big|\phi(s,i)\big|^2_{\mathcal{A}}d(s,i) - \frac{p_1\mu}{p_2}\int_{E^{\phi}_{0,0,K}} \big|\phi(s,i)\big|^2_{\mathcal{A}}d(s,i),$$

which, in turn, implies that

$$\begin{aligned}\int_{E^{\phi}_{0,0,\infty}} \big|\phi(s,i)\big|^2_{\mathcal{A}}d(s,i) &\leqslant \frac{p_2}{p_1\mu}\int_{E^{\phi}_{0,0,K}} V(\phi(s,i))d(s,i) \\ &\quad + \int_{E^{\phi}_{0,0,K}} \big|\phi(s,i)\big|^2_{\mathcal{A}}d(s,i) \\ &\leqslant (K+1)\left[\frac{p_2^2}{p_1\mu}+\frac{p_2}{p_1}\right]\big|\phi(0,0)\big|^2_{\mathcal{A}}.\end{aligned}$$

This completes the proof of UES.

UES $\Rightarrow$ HUO: We now assume uniform exponential stability of the origin $\{\zeta\in\mathbb{R}^{m_\zeta} : \zeta = 0\}$ for $\mathcal{H}'$. To verify HUO, we start rewriting $\mathcal{H}'$ in the form (12) and let $\phi := (\xi,p,q)$ be a corresponding solution generating a solution ζ to $\mathcal{H}'$ starting at $(t_o,j_o)\in\operatorname{dom}A$. Given $(t,j)\in\operatorname{dom}\phi$, we consider the unique pair $(s_K,m_K)\in\operatorname{dom}\phi$ satisfying (3) for some $K>0$ that we specify later. Using the properties of the matrix P in Assumption 1, we conclude the existence of $a_1,a_2>0$ such that

$$\begin{aligned}V(\phi(s_K,m_K)) &= \xi(s_K,m_K)^\top P(s_K,m_K)\xi(s_K,m_K) \\ &\leqslant a_1\exp^{-a_2K} V(\phi(t,j)).\end{aligned}$$

The latter implies that

$$-V(\phi(t,j)) + V(\phi(s_K,m_K)) \leqslant \big(a_1\exp^{-a_2K}-1\big)V(\phi(t,j)).$$

Now, we fix K sufficiently large such that

$$\mu := -\big(a_1\exp^{-a_2K}-1\big)p_2 > 0.$$

Furthermore, we note that (26) holds with an equality since we assumed that (15)-(16) hold with equality. Hence,

$$\int_{E^{\phi}_{t,j,K}} \xi(s,i)^\top\Phi(p(s,i),q(s,i))\xi(s,i)d(s,i) \geqslant \mu\xi(t,j)^\top\xi(t,j).$$

Finally, the proof is completed using (27) and the fact that $p(s,i) = s+t_o$, $q(s,i) = i+j_o$, and $E^{\phi}_{t,j,K} = E^{A}_{t_o+t,j_o+j,K}$. ■

C. UES Under HPE

The following is a relaxed PE property, which captures the richness of signals that may fail to be PE if considered as functions of purely continuous or purely discrete time.

Definition 5 (HPE): The pair (A,B) of hybrid arcs $A,B:\operatorname{dom}A\to\mathbb{R}^{m_\zeta\times m_\zeta}$, *i.e.*, with $\operatorname{dom}A=\operatorname{dom}B$, is said to be HPE if there exist K and $\mu>0$ such that

$$\int_{E^{A}_{t_o,j_o,K}} \Phi_{AB}(s,i)d(s,i) \geqslant \mu I_{m_\zeta} \qquad \forall (t_o,j_o)\in\operatorname{dom}A, \quad (28)$$

where $\Phi_{AB}:\operatorname{dom}A\to\mathbb{R}^{m_\zeta\times m_\zeta}$ is given by

$$\Phi_{AB}(t,j) := \begin{cases} A(t,j) & \text{if } \ t\in\operatorname{int}(I^j_A) \\ B(t,j) & \text{otherwise.}\end{cases}$$

□

Theorem 3, below, generalizes to the realm of hybrid systems, the well-known fact that PE is equivalent to UO—see [17]. Yet, Theorem 3 is not a direct extension since its proof approach is original. For instance, it differs from that used in [25] for continuous-time systems by being direct and not relying on many intermediate results.

The statement is formulated for systems satisfying the following structural property, which is reminiscent of so-called gradient systems, at the basis of classical identification schemes both in discrete and continuous time [26]. In Section V we briefly revisit a case-study for illustration.

Assumption 3 (Structural property): For each $(t,j)\in\operatorname{dom}A$, $A(t,j)=A(t,j)^\top\geqslant 0$, $B(t,j)=B(t,j)^\top\geqslant 0$, and $|B(t,j)|_\infty\leqslant 1$.

Theorem 3 (HPE ⇔ HUO): Consider the hybrid system $\mathcal{H}'$ under Assumptions 2 and 3. Let $C_c := \sqrt{A}$ and $C_d := \sqrt{B}$. Then, the pair (A, B) is HPE if and only if the pair $\{(A, B), (C_c, C_d)\}$ is HUO. □

Proof: HPE ⇒ HUO: Under Assumption 3, it follows that Assumption 1 holds with $P = I_{m_\theta}$, $Q_c(t,j) = A(t,j)$, and $Q_d(t,j) = B(t,j)$. Therefore, to verify the HUO property, it suffices to find $\mu_o > 0$ such that, for each $(t_o, j_o) \in \operatorname{dom} A$, we have

$$\int_{E^A_{t_o,j_o,K}} \Gamma_o(s,j)d(s,j) \geqslant \mu_o I_{m_\zeta}, \quad (29)$$

where we defined

$$\Gamma_o(s,j) := \mathcal{M}\left((s,j),(t_o,j_o)\right)^\top \Phi_{AB}(s,j)\mathcal{M}\left((s,j),(t_o,j_o)\right)$$

to compact the notation, K comes from the HPE of (A, B). Then, to establish (29), we show that, for each $\zeta_o \in \mathbb{R}^{m_\zeta}$,

$$\zeta_o^\top \int_{E^A_{t_o,j_o,K}} \Gamma_o(s,j)d(s,j)\, \zeta_o \geqslant \mu_o |\zeta_o|^2.$$

To that end, first we note that

$$\zeta_o^\top \int_{E^A_{t_o,j_o,K}} \Gamma_o(s,j)d(s,j)\, \zeta_o = \int_{E^A_{t_o,j_o,K}} \zeta(s,j)^\top \Phi_{AB}(s,j)\zeta(s,j)d(s,j)$$

and we proceed to find $\mu_o > 0$ such that

$$\tilde{V} := \int_{E^A_{t_o,j_o,K}} \zeta(s,j)^\top \Phi_{AB}(s,j)\zeta(s,j)d(s,j) \geqslant \mu_o |\zeta_o|^2. \quad (30)$$

So, to prove (30), we express $\tilde{V}$ as

$$\tilde{V} = \sum_{j=j_o}^{m_K} V_F(j) + \sum_{j=j_o}^{m_K-1} V_G(j), \quad (31)$$

$$V_F(j) := \int_{t_j}^{t_j+1} \zeta(s,j)^\top A(s,j)\zeta(s,j)ds, \quad (32)$$

$$V_G(j) := \zeta(t_{j+1},j)^\top B(t_{j+1},j)\zeta(t_{j+1},j), \quad (33)$$

and we compute suitable lower bounds for these functions. In [22] we show that, for each $\rho > 0$ and for each $j \in \{j_o, \ldots, m_K\}$,

$$V_F(j) \geqslant \frac{\rho}{1+\rho} \int_{t_j}^{t_{j+1}} \left| A(s,j)^{\frac{1}{2}} \zeta_o \right|^2 ds - \rho(\bar{A}^2 + 2\bar{A})(2(j-j_o)+1)(t_{j+1}-t_j)(t_{j+1}-t_{j_o}+1)\tilde{V}.$$

$$V_G(j) \geqslant \frac{-\rho}{2}(2(j-j_o)+1)(\bar{A}(t_{j+1}-t_{j_o})+2)\tilde{V} + \frac{\rho/2}{1+\rho}\left| B(t_{j+1},j)^{\frac{1}{2}} \zeta_o \right|^2.$$

Combining the latter two inequalities, we obtain the following upper bound on $\tilde{V}$ for each $\rho > 0$:

$$\tilde{V} \geqslant \frac{\rho}{1+\rho} \sum_{j=j_o}^{m_K-1} \left| B(t_{j+1},j)^{\frac{1}{2}} \zeta_o \right|^2 + \frac{2\rho}{1+\rho} \sum_{j=j_o}^{m_K} \int_{t_j}^{t_{j+1}} \left| A(s,j)^{\frac{1}{2}} \zeta_o \right|^2 ds - \frac{\rho}{2}\tilde{V} \sum_{j=j_o}^{m_K} (2(j-j_o)+1)(\bar{A}(t_{j+1}-t_{j_o})+2) - \rho\bar{A}(\bar{A}+2)\tilde{V}\sum_{j=j_o}^{m_K}(2(j-j_o)+1)(t_{j+1}-t_j)(t_{j+1}-t_{j_o}+1).$$

Hence,

$$\tilde{V} \geqslant \frac{2\rho|\zeta_o|^2}{1+\rho} \int_{E^A_{t_o,j_o,K}} \Phi_{AB}(s,i)d(s,i) - \rho(m_K - j_o + 1)^2 \tilde{V} \times \left[\frac{\bar{A}}{2}(s_K - t_{j_o}) + 2) + \bar{A}(\bar{A}+1)(s_K - t_{j_o})(s_K - t_{j_o} + 1)\right].$$

Finally, using the HPE of the pair (A, B), we conclude that

$$\tilde{V} \geqslant \frac{2\rho\mu}{1+\rho}|\zeta_o|^2 - \rho(K+2)^2 \times \left[\frac{\bar{A}}{2}(K+1) + 1 + \bar{A}(\bar{A}+2)(K+1)(K+2)\right]\tilde{V}. \quad (34)$$

Thus, (30) follows by choosing

$$\rho := \frac{1/(K+2)^2}{\frac{\bar{A}}{2}(K+1) + 1 + \bar{A}(\bar{A}+2)(K+1)(K+2)}.$$

HUO ⇒ HPE: Since we already showed that HUO implies UES of $\{\zeta \in \mathbb{R}^{m_\zeta} : \zeta = 0\}$ for $\mathcal{H}'$. As a result, by taking a solution ζ to $\mathcal{H}'$ starting at $(t,j) \in \operatorname{dom} A$, we conclude the existence of $K > 0$ and a unique pair $(s_K, m_K) \in \operatorname{dom} A$ such that (3) holds and

$$|I_{m_\zeta} - \mathcal{M}((s_K, m_K),(t,j))| \geqslant 1/2.$$

Next, we let $\mu := \exp^{-L(K+1)}/2$, $L := \max\{\bar{A}, \bar{B}\}$. On the other hand, we note that

$$\mathcal{M}((s_K, m_K),(t,j)) - I_{m_\zeta} = \int_{E^A_{t,j,K}} \Phi_{AB}(s,m)\left[\mathcal{M}((s,m),(t,j)) - I_{m_\zeta}\right]d(s,m) + \int_{E^A_{t,j,K}} \Phi_{AB}(s,m)d(s,m).$$

Now, if the HPE property is not verified, then we can find $(t,j) \in \operatorname{dom} A$ such that $\int_{E^A_{t,j,K}} \Phi_{AB}(s,m)d(s,m) < \mu I_{m_\zeta}$. As a result, we obtain

$$|\mathcal{M}((s_K, m_K),(t,j)) - I_{m_\zeta}| < L\int_{E^A_{t,j,K}} |\mathcal{M}((s,m),(t,j)) - I_{m_\zeta}|d(s,m) + \mu.$$

By letting $f : \operatorname{dom} A \to \mathbb{R}_{\geqslant 0}$ such that $f(t,j) = \mu$ and

$$\dot{f}(s,m) = Lf(s,m) \qquad \text{for a.a. } s \in I^m_A$$
$$f(s,m+1) = [1+L]\,f(s,m) \quad (s,m),(s,m+1) \in \operatorname{dom} A,$$

we conclude that $|\mathcal{M}((s_K, m_K),(t,j)) - I_{m_\zeta}| < f(s_K, m_K)$ and at the same time $f(s_K, m_K) \leqslant \mu \exp^{L(K+1)} \leqslant 1/2$. ■

We deduce the following consequence of Theorem 3.

Theorem 4 (UES under HPE): Consider the hybrid system $\mathcal{H}'$ in (14) under Assumptions 2 and 3, and let the pair (A, B) be HPE. Then, the origin $\{\zeta = 0\}$ is UES for $\mathcal{H}'$. □

V. Case-study: the hybrid gradient-descent algorithm

Consider the linear regression model

$$y=\psi^\top\theta, \tag{35}$$

where $\psi:\operatorname{dom}\psi\to\mathbb{R}^{m_\theta}$ is known as *regressor*, $\theta\in\mathbb{R}^{m_\theta}$ is a constant vector of unknown parameters, and $y:\operatorname{dom}y\to\mathbb{R}$ is the output. An estimate of θ, denoted $\hat\theta$, may be carried out dynamically, in function of the tracking error $e:=\hat y-y$, where $\hat y:=\psi^\top\hat\theta$. A well-known identification law is based on the minimization of the cost $J(e):=(1/2)e^2$.

In the continuous-time setting, *i.e.*, if $\operatorname{dom}\psi=[0,+\infty)$, the update law for $\hat\theta$ is given by $\dot{\hat\theta}=-\gamma\nabla_{\hat\theta}J(e)$, where $\nabla_{\hat\theta}J$ denotes the gradient of J with respect to $\hat\theta$. Hence,

$$\dot{\hat\theta}=-\gamma\psi(t)[\psi(t)^\top\hat\theta-y(t)],\quad \gamma>0 \tag{36}$$

—see [27]. In this case, it is well-known (see, *e.g.*, [11]) that, if ψ is bounded, UES of $\{\theta=\hat\theta\}$ is equivalent to:

(CPE) There exist $T>0$ and $\mu>0$ such that

$$\int_t^{t+T}\psi(s)\psi(s)^\top ds\geqslant\mu I_{m_\theta}\qquad\forall t\geqslant0. \tag{37}$$

In the discrete-time setting, *i.e.*, if the regressor's domain is $\operatorname{dom}\psi=\mathbb{Z}_{\geqslant0}$, the gradient algorithm is given by

$$\hat\theta(t+1)=\hat\theta(t)-\sigma(t)\nabla_{\hat\theta}J(e), \tag{38}$$

where $\sigma:\mathbb{Z}_{\geqslant0}\to[0,1]$ is given by $\sigma(t):=\frac{\gamma}{1+\gamma|\psi(t)|^2}$, and $\gamma>0$ is the adaptation rate [10]. In the this case, the discrete-time PE condition reads—cf. [4]:

(DPE) There exist $N>0$ and $\mu>0$ such that

$$\sum_{s=t}^{t+N}\psi(s)\psi(s)^\top\geqslant\mu I_{m_\theta}\quad\forall t\geqslant0. \tag{39}$$

When the data (ψ,y) of the linear regression model (35) is hybrid; namely, when it is allowed to exhibit both continuous- and discrete-time evolution, we write

$$y(t,j)=\psi(t,j)^\top\theta\qquad(t,j)\in\operatorname{dom}\psi. \tag{40}$$

In this case, we propose to design a hybrid gradient-descent algorithm in a way that whenever the data (ψ,y) *jump*, *i.e.*, undergo an instantaneous change, $\hat\theta$ is updated via (38); whenever the data (ψ,y) *flow*, *i.e.*, evolve continuously, $\hat\theta$ is updated via (36). More precisely:

(HG1) When ψ flows, that is, for all $t\in\operatorname{int}(I_\psi^j)$, with $I_\psi^j:=\{t:(t,j)\in\operatorname{dom}\psi\}$, $\hat\theta$ is updated by

$$\dot{\hat\theta}=-\gamma\psi(t,j)[\,\psi(t,j)^\top\hat\theta(t,j)-y(t,j)\,].$$

(HG2) When ψ jumps, that is, for all $(t,j)\in\operatorname{dom}\psi$ such that $(t,j+1)\in\operatorname{dom}\psi$, the estimate $\hat\theta$ is updated using

$$\hat\theta(t,j+1)=\hat\theta(t,j)-\frac{\gamma\psi(t,j)[\,\psi(t,j)^\top\hat\theta(t,j)-y(t,j)\,]}{1+\gamma|\psi(t,j)|^2}.$$

Then, the dynamics of the parameter estimation error $\tilde\theta=\hat\theta-\theta$ is governed by a hybrid system as in (14), with $\zeta:=\tilde\theta$,

$$A(t,j):=\gamma\psi(t,j)\psi(t,j)^\top, \tag{41}$$

$$B(t,j):=-\frac{\gamma\psi(t,j)\psi(t,j)^\top}{1+\gamma|\psi(t,j)|^2}, \tag{42}$$

which satisfy the structural properties in Assumption 3. Furthermore, it is assumed that, by design, there exists $\bar\psi>0$ such that $|\psi|_\infty\leqslant\bar\psi$ holds and the pair (A,B) is HPE.

In the following example, we illustrate a scenario, where the regressor ψ in (40) is a hybrid signal.

Example 1 (Gathering real-time and old data): Consider the continuous-time input-output model

$$y_1(t)=\psi_1(t)^\top\theta\qquad t\geqslant0, \tag{43}$$

where $\psi_1:\mathbb{R}_{\geqslant0}\to\mathbb{R}^{m_\theta}$ is the input and $y_1:\mathbb{R}_{\geqslant0}\to\mathbb{R}$ is the output. The pair (ψ_1,y_1) defines the *real-time* input-output data. On the other hand, we assume that we have a memory containing a pair of *old* input-output data, which we denote by (ψ_2,y_2). The old data needs to be treated at specific times defining the sequence $\{t_1,t_2,...,t_J\}\subset\mathbb{R}_{\geqslant0}$ with $t_j\leqslant t_{j+1}$. As a result, the old input-output data satisfy

$$y_2(t_j)=\psi_2(t_j)^\top\theta\qquad\forall j\in\{1,2,...,J\}. \tag{44}$$

The incorporation of old data can be done periodically, it can also be triggered by an external supervisory algorithm. As a result, we introduce the hybrid time domain

$$\operatorname{dom}\psi:=[0,t_1]\times\{0\}\cup[t_1,t_2]\times\{1\}\cup...\cup[t_J,+\infty)\times\{J\}.$$

Furthermore, we introduce the pair of hybrid input-output data, gathering both old and real-time data, given by

$$\psi(t,j):=\begin{cases}\psi_1(t) & \text{if } t\in\operatorname{int}(I_\psi^j)\\ \psi_2(t_{j+1}) & \text{otherwise,}\end{cases}$$

$$y(t,j):=\begin{cases}y_1(t) & \text{if } t\in\operatorname{int}(I_\psi^j)\\ y_2(t_{j+1}) & \text{otherwise.}\end{cases}$$

The pair of hybrid input-output data is related to the parameter θ according to (40).

The hybrid gradient algorithm in this context allows to continuously explore real-time data on the open intervals $\operatorname{int}(I_\psi^j)$, $j\in\{1,2,...,J\}$, and to discretely exploit old data over the sequence of times $\{t_j\}_{j=1}^\infty$. □

Remark 3: When the hybrid arc ψ is eventually continuous (respectively, eventually discrete or Zeno), HPE of the pair (A,B) reduces to CPE of $\psi\psi^\top$ (respectively, DPE). Furthermore, when the regressor ψ is scalar (i.e, $m_\theta=1$), HPE of the pair (A,B) implies that either CPE or DPE holds. However, in the general case that $m_\theta>1$, it is possible that HPE hold, but none of the conditions CPE and DPE be satisfied. In other words, let $\{*\text{PE}\}$ be the set of functions satisfying the *PE property. Then, $\{\text{CPE}\}\cup\{\text{DPE}\}\subsetneq\{\text{HPE}\}$. •

For illustration, let us consider (40) with

$$\psi(t):=\begin{cases}[\sin(t)\quad 0]^\top & \forall t\in(2j\pi,2(j+1)\pi),\ j\in\mathbb{Z}_{\geqslant0}\\ [0.5\quad 1]^\top & \text{otherwise,}\end{cases} \tag{45}$$

so, over successive continuous intervals of time,

$$\psi(t)\psi(t)^\top=\begin{bmatrix}\sin(t)^2 & 0\\ 0 & 0\end{bmatrix}\forall t\in(2j\pi,2(j+1)\pi),\ j\in\mathbb{Z}_{\geqslant0},$$

while at discrete instants,

$$\psi(t)\psi(t)^\top := \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix} \qquad \forall t \in \{2\pi, 4\pi, ...\}.$$

The function $\psi\psi^\top$ defined in (45) does not satisfy neither CPE nor DPE. However, the corresponding maps A and B (with $\gamma = 1$), defined on $\operatorname{dom} A = \operatorname{dom} B = \bigcup_{j=0}^{\infty}[2j\pi, 2(j+1)\pi]$, are given by

$$A(t,j) = \begin{bmatrix} \sin(t)^2 & 0 \\ 0 & 0 \end{bmatrix}, \; B(t,j) = \begin{bmatrix} 0.1111 & 0.2222 \\ 0.2222 & 0.4444 \end{bmatrix}.$$

The pair (A, B) is HPE, with $K = 2\pi + 1$ and $\mu = 0.21$.

HPE is less conservative than its counterparts CPE and DPE as it captures the fact that the richness of a signal may be enhanced by an appropriate mingling of exciting flows and jumps, which, otherwise, are insufficient to guarantee that neither (37) nor (39) hold. The latter being necessary, $\tilde{\theta} \nrightarrow 0$ in either case, but $\tilde{\theta} \to 0$ under the hybrid gradient-descent algorithm—see Fig. 1 below.

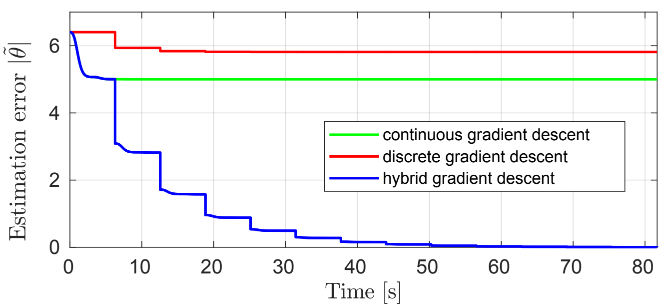

Fig. 1. Evolution of the norm of the parameter error $\tilde{\boldsymbol{\theta}}$ using continuous, discrete, and hybrid gradient algorithms

VI. Conclusion and Future Work

Persistency of excitation is a well-studied concept both in continuous and discrete-time. Yet, a proper mathematical setting for hybrid systems, that is, comprising both differential and difference equations had been missing up to this Technical Note. Through a simple example, one can see that persistency of excitation cannot be used as defined so far in textbooks, when it comes to analyzing systems that evolve in hybrid time. In addition, we have illustrated through a simple identification case-study that hybrid PE is a meaningful relaxed condition that still guarantees uniform global asymptotic stability. We believe that the proposed setting should serve to establish many extensions to concurrent learning and observer design. These lines of research are under study.

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