Parameter Estimation for Hybrid Dynamical Systems with Delayed Jump Detection

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***Abstract*— We consider the problem of estimating a vector of unknown constant parameters for a class of hybrid dynamical systems with bounded delays in the detection of jumps. Using a hybrid systems framework, we propose an algorithm that estimates the jump times of the trajectories and uses stored data to update the parameter estimate at jumps. We show that the algorithm guarantees convergence of the parameter estimate to the true value, except possibly on the intervals wherein detection of jumps is delayed. Simulation results show the merits of the proposed approach.**

I. Introduction

Estimating the unknown parameters of a system is crucial in many engineering applications. Hybrid systems – or systems whose state variables may evolve continuously (or *flow*) and, at times, evolve discretely (or *jump*) – present a unique challenge for estimation algorithms due to the combination of continuous and discrete dynamics. The recent works [1]–[3] propose algorithms for hybrid linear regression, and [4]–[6] study parameter estimation for classes of hybrid dynamical systems. These works assume that jumps of the hybrid system – henceforth called the plant – are detected instantaneously, which allows the jumps of the estimators to be synchronized with the jumps in the plant state. In such cases, under appropriate excitation conditions, these estimation algorithms ensure convergence of the parameter estimation error to zero. However, in practice, jump detection is often delayed due to sensing, signal transmission, and computation delays. Moreover, if algorithms such as those in [1]–[6] are employed under delays in jump detection, the estimation error may fail to converge to zero. This finding motivates the development of algorithms for estimating unknown parameters in hybrid systems under bounded delays in the detection of jumps in the plant state. Note that several recent works proposed state observers for hybrid systems with jump times that are unknown [7], [8] or known only approximately [9]–[11].

In this paper, given a hybrid parameter estimation algorithm that is designed to jump synchronously with jumps in the plant state, we propose a new algorithm for estimating unknown parameters under delays in the detection of jumps in the plant state. The proposed algorithm, described in Section III, uses data stored during flows to estimate the jump times of the plant state. In Section IV, we show that the new algorithm preserves the stability bounds of the original estimation algorithm (with instantaneous detection of jumps), except possibly on the intervals wherein detection of jumps is delayed. Examples are in Section V and concluding remarks are in Section VI. Proofs are sketched due to space constraints.

A. Notation

We denote the set of real, nonnegative real, and positive real numbers as $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$, respectively. We denote the set of natural numbers (including zero) as $\mathbb{N}$. The matrix I denotes the identity matrix of appropriate dimension. For $x, y \in \mathbb{R}^n$, we write $[x^\top \;\; y^\top]^\top$ as (x, y). The Euclidean norm of vectors and the associated induced matrix norm is denoted by $|\cdot|$. The distance of a point x to a nonempty set S is denoted by $|x|_S = \inf_{y\in S} |y - x|$. The closed unit ball centered at the origin of appropriate dimension (in the Euclidean norm) is denoted by $\mathbb{B}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$.

B. Hybrid Dynamical Systems

In this paper, a hybrid system $\mathcal{H}$ is modeled as [12]

$$\mathcal{H} : \begin{cases} \dot{x} = F(x) & x \in C \\ x^+ = G(x) & x \in D \end{cases}$$

where $x \in \mathbb{R}^n$ is the state, $F : C \to \mathbb{R}^n$ is the flow map defining a differential equation capturing the continuous dynamics, and $C \subset \mathbb{R}^n$ defines the flow set on which flow is permitted. The mapping $G : D \to \mathbb{R}^n$ is the jump map defining the law resetting x at jumps, and $D \subset \mathbb{R}^n$ is the jump set on which jumps are permitted.

A solution x to $\mathcal{H}$ is a *hybrid arc* [12] that is parameterized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the elapsed ordinary time and j is the number of jumps that have occurred. The domain of x, denoted by $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, is a *hybrid time domain*, in the sense that for every $(T, J) \in \operatorname{dom} x$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ with $t_0 = 0$ such that $\operatorname{dom} x \cap ([0, T] \times \{0, 1, \ldots, J\}) = \bigcup_{j=0}^{J} ([t_j, t_{j+1}], \{j\})$. The operations $\sup_t \operatorname{dom} x$ and $\sup_j \operatorname{dom} x$ return the supremum of the t and j coordinates, respectively, of points in $\operatorname{dom} x$. A hybrid arc x is said to be

- *nontrivial* if $\operatorname{dom} x$ contains more than one point;
- *complete* if $\operatorname{dom} x$ is unbounded;
- *continuous* if nontrivial and $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \{0\}$;
- *discrete* if nontrivial and $\operatorname{dom} x \subset \{0\} \times \mathbb{N}$;
- *Zeno* if it is complete and $\sup_t \operatorname{dom} x < \infty$.

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A solution x to $\mathcal{H}$ is called *maximal* if it cannot be extended – that is, if there does not exist another solution x' to $\mathcal{H}$ such that $\operatorname{dom} x$ is a proper subset of $\operatorname{dom} x'$ and $x(t,j) = x'(t,j)$ for all $(t,j) \in \operatorname{dom} x$.

II. Problem Statement

Consider a hybrid plant, denoted by $\mathcal{H}_{\mathrm{P}}$, with dynamics

$$\mathcal{H}_{\mathrm{P}} : \begin{cases} \dot{x} = F_{\mathrm{P}}(x,\theta) & x \in C_{\mathrm{P}} \\ x^+ = G_{\mathrm{P}}(x,\theta) & x \in D_{\mathrm{P}} \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, F_{P} is the flow map, G_{P} is the jump map, $C_{\mathrm{P}} \subset \mathbb{R}^n$ is the flow set, $D_{\mathrm{P}} \subset \mathbb{R}^n$ is the jump set, $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, and $n, p \in \mathbb{N}$. If jumps in x are detected instantaneously, then a hybrid estimator can be designed to estimate θ for certain classes of hybrid plants [1]–[6]. We denote such an estimation algorithm as $\mathcal{H}_{\mathrm{E}}$ with state $z \in \mathbb{R}^m$, input $x \in \mathbb{R}^n$,[1] and dynamics

$$\mathcal{H}_{\mathrm{E}} : \begin{cases} \dot{z} = F_{\mathrm{E}}(x,z) & \text{when } \mathcal{H}_{\mathrm{P}} \text{ flows} \\ z^+ = G_{\mathrm{E}}(x,z) & \text{when } \mathcal{H}_{\mathrm{P}} \text{ jumps} \end{cases} \tag{2}$$

where F_{E} is the flow map, G_{E} is the jump map, and $m \in \mathbb{N}$. The state z of $\mathcal{H}_{\mathrm{E}}$ is partitioned as $z := (\hat{\theta}, \sigma)$, where $\hat{\theta} \in \mathbb{R}^p$ is an estimate of θ in (1) and $\sigma \in \mathbb{R}^{m-p}$ collects any auxiliary state variables needed by the algorithm. We denote the interconnection of $\mathcal{H}_{\mathrm{P}}$ and $\mathcal{H}_{\mathrm{E}}$ as $\widetilde{\mathcal{H}}$, with dynamics

$$\widetilde{\mathcal{H}} : \begin{cases} \left.\begin{aligned} \dot{x} &= F_{\mathrm{P}}(x,\theta) \\ \dot{z} &= F_{\mathrm{E}}(x,z) \end{aligned}\right\} =: \widetilde{F}(x,z,\theta) \quad (x,z) \in \widetilde{C} \\ \left.\begin{aligned} x^+ &= G_{\mathrm{P}}(x,\theta) \\ z^+ &= G_{\mathrm{E}}(x,z) \end{aligned}\right\} =: \widetilde{G}(x,z,\theta) \quad (x,z) \in \widetilde{D} \end{cases} \tag{3}$$

where $\widetilde{C} := C_{\mathrm{P}} \times \mathbb{R}^m$ and $\widetilde{D} := D_{\mathrm{P}} \times \mathbb{R}^m$.

The flow map F_{E} and jump map G_{E} of $\mathcal{H}_{\mathrm{E}}$ are designed so that, under appropriate persistence of excitation conditions, each complete solution (x,z) to $\widetilde{\mathcal{H}}$ converges (in distance) to the set

$$\mathcal{A} := \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : \hat{\theta} = \theta\}. \tag{4}$$

In practice, exact synchronization between a plant and an estimator is difficult to achieve due to delays in sensing, signal transmission, and computation. Moreover, if detection of jumps in the plant state is delayed, resetting z based on G_{E} may result in divergence of the parameter estimate – see the example in Section V. This motivates the design of a new estimation algorithm that accounts for such delays. We denote this algorithm as $\hat{\mathcal{H}}_{\mathrm{E}}$, with dynamics

$$\hat{\mathcal{H}}_{\mathrm{E}} : \begin{cases} \dot{z} = F_{\mathrm{E}}(x,z) & \text{otherwise} \\ z^+ = \hat{G}_{\mathrm{E}}(x,z) & \text{when a jump in } x \text{ is detected} \end{cases} \tag{5}$$

where F_{E} is given in (2) and $\hat{G}_{\mathrm{E}}$ is to be designed to solve the following problem.

[1]See [13] for details on hybrid systems with inputs.

Problem Statement: Given a hybrid plant $\mathcal{H}_{\mathrm{P}}$ as in (1) and a hybrid estimator $\mathcal{H}_{\mathrm{E}}$ as in (2) providing convergence to $\mathcal{A}$ in (4), design the jump map $\hat{G}_E$ of $\hat{\mathcal{H}}_{\mathrm{E}}$ in (5) so that, under delays in detection of jumps in the plant state, the parameter estimate $\hat{\theta}$ converges to the unknown parameter vector θ in (1), except possibly on the delay intervals.

III. Problem Solution

A. Assumptions

To enable our design of $\hat{G}_{\mathrm{E}}$, we make the following assumption on the data of $\mathcal{H}_{\mathrm{P}}$ and $\mathcal{H}_{\mathrm{E}}$.

Assumption 3.1: Given the hybrid systems $\mathcal{H}_{\mathrm{P}}$ in (1) and $\mathcal{H}_{\mathrm{E}}$ in (2),

1. $\theta \in \Theta$, where $\Theta \subset \mathbb{R}^p$ is a known compact set;
2. $\operatorname{dom} F_{\mathrm{P}} \supset \mathbb{R}^n \times \Theta$;
3. $\operatorname{dom} F_{\mathrm{E}} = \mathbb{R}^n \times \mathbb{R}^m$;
4. $\operatorname{dom} G_{\mathrm{E}} = \mathbb{R}^n \times \mathbb{R}^m$.

We impose the following local Lipschitz continuity condition.

Assumption 3.2: For each $(x,z,\theta) \in \mathbb{R}^n \times \mathbb{R}^m \times \Theta$, there exist $\delta_x, \delta_z > 0$ and $L \geq 0$ such that

$$|\widetilde{F}(x_1,z_1,\theta) - \widetilde{F}(x_2,z_2,\theta)| \leq L(|x_1 - x_2| + |z_1 - z_2|)$$

for each $x_1, x_2 \in x + \delta_x \mathbb{B}$ and each $z_1, z_2 \in z + \delta_z \mathbb{B}$.

We make the following assumption on solutions to $\mathcal{H}_{\mathrm{P}}$.

Assumption 3.3: Given the hybrid system $\mathcal{H}_{\mathrm{P}}$ in (1), there exist $\tau^* > 0$, $\Delta_1 \in [0, \tau^*)$, $\varepsilon > 0$, and a closed set $\mathcal{X} \subset \mathbb{R}^n$ such that the following conditions hold:

1. Each maximal solution x to $\mathcal{H}_{\mathrm{P}}$ from $x(0,0) \in \mathcal{X}$ satisfies $t_j - t_{j-1} \geq \tau^*$ for all $j \in \{1, 2, \cdots, \sup_j \operatorname{dom} x\}$, where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom} x}$ is the sequence defining $\operatorname{dom} x$ as in Section I-B.[2]
2. Each solution x to $\mathcal{H}_{\mathrm{P}}$ from $x(0,0) \in \mathcal{X}$ satisfies $|x(t,j+1) - x(t,j)| \geq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$.
3. Following each jump in x, there is a delay of at most Δ_1 seconds before the jump can be detected.

Finally, we make the following assumption on $\hat{\mathcal{H}}_{\mathrm{E}}$.

Assumption 3.4: Given the hybrid system $\hat{\mathcal{H}}_{\mathrm{E}}$ in (5) and τ^*, Δ_1 from Assumption 3.3, there exists $\Delta_2 \in (0, \tau^* - \Delta_1]$, such that, for each $(x,z) \in \operatorname{dom} \hat{G}_{\mathrm{E}}$, the computation of the jump map $\hat{G}_{\mathrm{E}}(x,z)$ takes at most Δ_2 seconds.

Item 1 of Assumption 3.3 means that each maximal solution x to $\mathcal{H}_{\mathrm{P}}$ from $x(0,0) \in \mathcal{X}$ has a dwell time of at least τ^* seconds. Item 2 imposes a lower bound on the change in x at jumps. Item 3 of Assumption 3.3 and Assumption 3.4 mean that the sum of the delay in detecting a jump in x and the time required to compute $\hat{G}_{\mathrm{E}}$ is at most $\Delta_1 + \Delta_2$ seconds. Since $\Delta_1 + \Delta_2 < \tau^*$, this ensures that each time x jumps, it does not jump again in the (hybrid) time required for the

[2]If x is continuous, we define $t_1 := \sup_t \operatorname{dom} x$.

estimation algorithm $\hat{\mathcal{H}}_{\rm E}$ to detect the jump and compute the jump map. In Section V we show that, for the bouncing ball system [12, Example 1.1] with coefficient of restitution equal to one, each solution with initial conditions in a closed set $\mathcal{X}$ that excludes the origin satisfies Assumption 3.3.

Remark 3.5: According to Assumptions 3.3 and 3.4, the time required to detect a jump in x and compute $\hat{G}_{\rm E}$ depends on the hybrid time at which the jump occurred. In other words, the delay is not necessarily the same for each jump in x. Such a property holds for Zeno solutions as well; however, the dwell time condition imposed by item 1 of Assumption 3.3 prohibits solutions with intervals of flow that have vanishing length as hybrid time evolves.

B. Hybrid Model of the Interconnection with Delay

Inspired by [11], we model the interconnection of $\mathcal{H}_{\rm P}$ and $\hat{\mathcal{H}}_{\rm E}$ as[3]

$$\mathcal{H}: \begin{cases} \left.\begin{aligned} \dot{x} &= F_{\rm P}(x,\theta) \\ \dot{z} &= F_{\rm E}(x,z) \\ \dot{\tau}_\delta &= -\min\{\tau_\delta+1, 1\} \end{aligned}\right\} \xi \in C \\ \left.\begin{aligned} x^+ &= G_{\rm P}(x,\theta) \\ z^+ &= z \\ \tau_\delta^+ &\in [0,\Delta] \end{aligned}\right\} \xi \in D_{\text{-}1} \\ \left.\begin{aligned} x^+ &= x \\ z^+ &= \hat{G}_{\rm E}(x,z) \\ \tau_\delta^+ &= -1 \end{aligned}\right\} \xi \in D_0 \end{cases} \tag{6}$$

with state $\xi := (x, z, \tau_\delta) \in \mathcal{Z} := \mathbb{R}^n \times \mathbb{R}^m \times \big(\{-1\} \cup [0,\Delta]\big)$, flow set

$$C := \widetilde{C} \times \big(\{-1\} \cup [0,\Delta]\big),$$

and jump set $D := D_{\text{-}1} \cup D_0$, where

$$D_{\text{-}1} := \widetilde{D} \times \{-1\}, \quad D_0 := \big(\widetilde{C} \cup \widetilde{D} \cup \widetilde{G}(\widetilde{D})\big) \times \{0\}$$

with $\widetilde{C}$, $\widetilde{D}$, $\widetilde{G}$ as in (3) and $\Delta := \Delta_1 + \Delta_2$.

Compared to $\widetilde{\mathcal{H}}$ in (3), $\mathcal{H}$ contains a new state component $\tau_\delta \in \{-1\} \cup [0,\Delta]$ that models the delay between the jumps of the plant and the jumps of the estimator. When $\tau_\delta = -1$ and x does not jump, $\mathcal{H}$ flows with x and z flowing according to $F_{\rm P}$ and $F_{\rm E}$, respectively, and τ_δ remains equal to -1. When the plant state x jumps, τ_δ is reset to a value in $[0,\Delta]$ thus starting a delay period. Then, $\mathcal{H}$ flows and τ_δ decreases until it reaches 0, at which point a delay interval of length smaller than or equal to Δ has elapsed. Once τ_δ reaches zero, the estimator state is reset based on $\hat{G}_{\rm E}$, and τ_δ is reset to -1.

We design the jump map $\hat{G}_{\rm E}$ of $\mathcal{H}$ so that the dynamics of the $\hat{\theta}$ component of solutions to $\mathcal{H}$ are equivalent to the dynamics of the $\hat{\theta}$ component of solutions to $\widetilde{\mathcal{H}}$, except perhaps on the delay intervals. The algorithm we propose requires sampling of solutions to $\mathcal{H}$, which we describe in the following section.

[3]Note that the jump map in (6) encodes a sequential execution of jumps. That is, it does not allow solutions to jump due to the state component x reaching $D_{\rm P}$ during a delay interval. This modeling decision is justified since Assumptions 3.3 and 3.4 impose that x flows for the duration of each delay interval.

C. Sampling of Solutions to $\mathcal{H}$

Let $\xi = (x, z, \tau_\delta)$ be a solution to $\mathcal{H}$ in (6). During flows, we sample x and z at hybrid time instants $\{(\tilde{t}_k, \tilde{j}_k)\}_{k=0}^{S(t)} \in \operatorname{dom}\xi$, where $t \mapsto S(t) \in \mathbb{N}\setminus\{0\}$ indicates a time-dependent number of samples that is to be designed, and $(\tilde{t}_0, \tilde{j}_0) := (0,0)$. We also record the time t at which each sample is taken. We store the samples of x, z, and t in time-varying matrices X, Z, and T, respectively, defined as[4]

$$\begin{aligned} \mathrm{X}(t,j) &:= \begin{bmatrix} x_1(t,j) & x_2(t,j) & \cdots & x_{N(t)}(t,j) \end{bmatrix} \in \mathbb{R}^{n\times N(t)} \\ \mathrm{Z}(t,j) &:= \begin{bmatrix} z_1(t,j) & z_2(t,j) & \cdots & z_{N(t)}(t,j) \end{bmatrix} \in \mathbb{R}^{m\times N(t)} \\ \mathrm{T}(t,j) &:= \begin{bmatrix} \tau_1(t,j) & \tau_2(t,j) & \cdots & \tau_{N(t)}(t,j) \end{bmatrix} \in \mathbb{R}^{1\times N(t)} \end{aligned} \tag{7}$$

for all $(t,j) \in \operatorname{dom}\xi$, where $t \mapsto N(t) \in \mathbb{N}\setminus\{0\}$ indicates that the number of columns of X, Z, and T is time dependent. The matrices X, Z, and T are initialized as $\mathrm{X}(0,0) = x(0,0)$, $\mathrm{Z} = z(0,0)$, and $\mathrm{T}(0,0) = 0$, respectively, with $N(0) = 1$. Each time a sample of x (resp., z and t) is stored, we append a new column to the right of the last column of X (resp., Z and T), thereby increasing the value of $N(t)$ by one, and store the sample in the new column. Samples are stored whenever the current value of t or x differs sufficiently from the value stored in the last column of T or X, respectively. In particular, when

$$|t - \tau_{N(t)}(t,j)| \geq \alpha_t \tag{8a}$$

or

$$|x(t,j) - x_{N(t)}(t,j)| \geq \alpha_x \tag{8b}$$

where $\alpha_t \in (0, (\tau^* - \Delta_1)/3]$ and $\alpha_x \in (0, \varepsilon/2]$ are design parameters, with $\tau^* > 0$, $\Delta_1 \in [0,\tau^*)$, and $\varepsilon > 0$ from Assumption 3.3. Since, by item 1 of Assumption 3.3, jumps in x occur at most every τ^* seconds, we remove column ℓ of X, Z, and T when $\tau_{\ell+1}(t,j) \leq t - \tau^*$, for each $(t,j) \in \operatorname{dom}\xi$ and each $\ell \in \{1, 2, \cdots, N(t)-1\}$, to ensure that there is at most one jump among the samples stored in X. Each time a column of X, Z, and T is removed, the value of $N(t)$ is decreased by one. Thus, the elements of X, Z, and T are piecewise constant right-continuous signals, with values changing only at the sample times.

Omitting the arguments of X, Z, T, and N for readability, suppose that a jump in x is detected at hybrid time $(t^*, j^*) \in \operatorname{dom}\xi$. Since each solution ξ to $\mathcal{H}$ in (6) jumps each time the plant state x jumps, it follows that the jump in x occurred at hybrid time $(t^* - \delta_1, j^* - 1)$, where δ_1 is unknown and satisfies $\delta_1 \in [0, \Delta_1]$, with $\Delta_1 \in [0, \tau^*)$ from Assumption 3.3. Note that, due to Assumption 3.3, (8a) guarantees that, at hybrid time (t^*, j^*), there are at least three samples of x (resp., z and t) stored in X (resp., Z and T) from the interval of flow prior to the jump at hybrid time $(t^* - \delta_1, j^* - 1)$. Furthermore, (8b) guarantees that a sample of x (resp., z and t) is stored in X (resp., Z and T) immediately after the jump at hybrid time $(t^* - \delta_1, j^* - 1)$. In the next section, we propose a method of determining the index of the sample right before the jump.

[4]We denote the columns of T as τ_i, rather than t_i, to avoid confusion with the sequence of times $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\xi}$ that define $\operatorname{dom}\xi$ as in Section I-B.

D. Jump Index Determination

To find the index of the sample of x that was stored right before the jump, we propose the following approach that we explain first in words before formally defining an algorithm.

Pick $\ell \in \{1, 2, \cdots, N-1\}$ and $\vartheta \in \Theta$, with $\Theta \subset \mathbb{R}^p$ from item 1 of Assumption 3.1, where ℓ is a candidate for the index of the sample of x that was stored right before the jump, and ϑ is a candidate for the value of the unknown parameter vector θ. Using F_{P} in (1), we compute two solutions, denoted by $t \mapsto \hat{x}_1(t)$ with initial condition x_1, and $t \mapsto \hat{x}_{\ell+1}(t)$, with initial condition $x_{\ell+1}$. In particular, we solve

$$\begin{aligned} \dot{\hat{x}}_1 &= F_{\mathrm{P}}(\hat{x}_1, \vartheta), & \hat{x}_1(\tau_1) &= x_1 & \forall t &\in [\tau_1, \tau_\ell], \\ \dot{\hat{x}}_{\ell+1} &= F_{\mathrm{P}}(\hat{x}_{\ell+1}, \vartheta), & \hat{x}_{\ell+1}(\tau_{\ell+1}) &= x_{\ell+1} & \forall t &\in [\tau_{\ell+1}, \tau_N], \end{aligned} \tag{9}$$

where x_i and τ_i, $i \in \{1, 2, \cdots, N\}$, are columns of X and T, respectively, in (7). Note that, due to item 2 of Assumption 3.1, the systems in (9) are well defined for all $\hat{x}_1, \hat{x}_{\ell+1} \in \mathbb{R}^n$ and all $\vartheta \in \Theta$. Finally, we compare the solutions $\hat{x}_1$ and $\hat{x}_{\ell+1}$ against the samples stored in X by computing

$$\alpha(\ell, \vartheta) := \sum_{i=1}^{\ell} |x_i - \hat{x}_1(\tau_i)|^2 + \sum_{i=\ell+1}^{N} |x_i - \hat{x}_{\ell+1}(\tau_i)|^2. \tag{10}$$

Let $\vartheta \in \Theta$ be such that the value of $\alpha(\ell, \vartheta)$ is minimized. If the jump in x occurred between the samples x_ℓ and $x_{\ell+1}$, then $\alpha(\ell, \vartheta)$ will be small. On the other hand, if the jump did not occur between x_ℓ and $x_{\ell+1}$, then $\alpha(\ell, \vartheta)$ may be large.

Remark 3.6: Note that, for each $\ell \in \{1, 2, \cdots, N\}$, multiple distinct values of $\vartheta \in \Theta$ may yield the same minimum value of $\alpha(\ell, \vartheta)$. In particular, if the flow map F_{P} in (1) does not depend on components of θ, then α can be minimized for any values of the corresponding components of ϑ. An example of such a case is described in Section V, where F_{P} does not depend on the second component of θ.

Formally, we determine the index of the sample of x that was stored right before the jump by solving the following:

$$\begin{aligned} &\text{minimize} && \alpha(\ell, \vartheta) \\ &\text{subject to} && \ell \in \{1, 2, \cdots, N-1\}, \quad \vartheta \in \Theta, \\ &&& \dot{\hat{x}}_1 = F_{\mathrm{P}}(\hat{x}_1, \vartheta), \quad \hat{x}_1(\tau_1) = x_1, \\ &&& \dot{\hat{x}}_{\ell+1} = F_{\mathrm{P}}(\hat{x}_{\ell+1}, \vartheta), \quad \hat{x}_{\ell+1}(\tau_{\ell+1}) = x_{\ell+1}. \end{aligned} \tag{11}$$

Note that the optimization problem in (11) can be solved in parallel for each $\ell \in \{1, 2, \cdots, N-1\}$.

E. Design of $\hat{G}_{\mathrm{E}}$

Let $\ell \in \{1, 2, \cdots, N-1\}$ and $\vartheta \in \Theta$ be the result of (11). Since, by (8b) and item 2 of Assumption 3.3, a sample of x is stored in X immediately after each jump, it follows that the jump in x occurred at hybrid time $(\tau_{\ell+1}, j^* - 1)$. We design the jump map $\hat{G}_{\mathrm{E}}$ in (5) to reset the estimator state, z, using the data stored in Z, based on the knowledge that a jump in x occurred at hybrid time $(\tau_{\ell+1}, j^* - 1)$. To do so, we first compute solutions to the system

$$\begin{aligned} \dot{\hat{x}} &= F_{\mathrm{P}}(\hat{x}, \vartheta), & \hat{x}(\tau_\ell) &= x_\ell \\ \dot{\mu} &= F_{\mathrm{E}}(\hat{x}, \mu), & \mu(\tau_\ell) &= z_\ell \end{aligned} \tag{12}$$

for all $t \in [\tau_\ell, \tau_{\ell+1}]$. Note that, due to items 2 and 3 of Assumption 3.1, the system in (12) is well defined for all $(\hat{x}, \mu, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^m \times \Theta$ and, due to Assumption 3.2, a solution to (12) from a given initial condition is unique. Then, we reset column ℓ of Z in (7) using the value of the jump map G_{E} in (2) evaluated at $(\hat{x}(\tau_{\ell+1}),\ \mu(\tau_{\ell+1}))$, which are obtained from computing the solution to (12). That is,

$$z_{\ell+1} = G_{\mathrm{E}}(\hat{x}(\tau_{\ell+1}), \mu(\tau_{\ell+1})) \tag{13}$$

which is well defined due to item 4 of Assumption 3.1. Next, we forward propagate the system in (12) from time $\tau_{\ell+1}$ up to the current time t^* by solving

$$\begin{aligned} \dot{\hat{x}} &= F_{\mathrm{P}}(\hat{x}, \vartheta), & \hat{x}(\tau_{\ell+1}) &= x_{\ell+1} \\ \dot{\mu} &= F_{\mathrm{E}}(\hat{x}, \mu), & \mu(\tau_{\ell+1}) &= z_{\ell+1} \end{aligned} \tag{14}$$

for all $t \in [\tau_{\ell+1}, t^*]$. Given the solution $t \mapsto (x(t), \mu(t))$ to (14), we reset columns $\ell+2$ through N of Z in (7) as $z_i = \mu(\tau_i)$ for all $i \in \{\ell+2, \ell+3, \cdots, N\}$. Finally, we reset the estimator state z as

$$z^+ = \mu(t^*). \tag{15}$$

Given a solution ξ to $\mathcal{H}$ and X, Z, T as in (7), we implement (11)–(15) using Algorithm 1. Recall that, from Assumption 3.4, we assume that an output from this algorithm can be numerically generated in at most Δ_2 seconds.

Algorithm 1 Algorithm for computing $\hat{G}_{\mathrm{E}}$ in (5)

Require: $(t^*, j^*) \in \operatorname{dom} \xi$
  Create an empty vector $Q \in \mathbb{R}^{N-1}$
  for $\ell = 1$ to $N-1$ **do**
    Solve for $\vartheta \in \Theta$ that minimizes $\alpha(\ell, \vartheta)$ in (10)
    Store $\alpha(\ell, \vartheta)$ in row ℓ of Q
  end for
  Find $\min Q$ and set ℓ as the corresponding row index
  Compute the solution $(\hat{x}, \mu)$ to (12) for all $t \in [\tau_\ell, \tau_{\ell+1}]$
  Set $z_{\ell+1} = G_{\mathrm{E}}(\hat{x}(\tau_{\ell+1}), \mu(\tau_{\ell+1}))$ as in (13)
  Compute the solution $(\hat{x}, \mu)$ to (14) for all $t \in [\tau_{\ell+1}, t^*]$
  for $i = \ell+2$ to N **do**
    Set $z_i = \mu(\tau_i)$
  end for
  Set $z^+ = \mu(t^*)$

IV. Stability Analysis

To establish the stability properties induced by the proposed algorithm, we first make the following assumption regarding the parameter estimation error for $\widetilde{\mathcal{H}}$ in (3).

Assumption 4.1: Given a closed set $\mathcal{X} \subset \mathbb{R}^n$, for each compact set $K \subset \mathbb{R}^m$, there exists $\beta \in \mathcal{KL}$ such that the parameter estimation error $(t, j) \mapsto \hat{\theta}(t, j) - \theta$ for each solution (x, z) to the hybrid system $\widetilde{\mathcal{H}}$ in (3) from $(x(0,0), z(0,0)) \in \mathcal{X} \times K$ satisfies, for all $(t, j) \in \operatorname{dom}(x, z)$,

$$|\hat{\theta}(t, j) - \theta| \le \beta(|\hat{\theta}(0, 0) - \theta|, t + j). \tag{16}$$

In words, Assumption 4.1 states that the set $\mathcal{A}$ in (4) is $\mathcal{KL}$ *pre-asymptotically stable*[5] on $\mathcal{X}\times K$ for $\widetilde{\mathcal{H}}$. The recent works [1]–[4] proposed hybrid parameter estimation algorithms that, under appropriate excitation conditions, satisfy this assumption. We now establish our main stability result.

Theorem 4.2: Given the hybrid systems $\mathcal{H}_{\rm P}$ in (1) *and $\mathcal{H}_{\rm E}$ in* (2)*, suppose that the resulting interconnection $\widetilde{\mathcal{H}}$ in* (3) *satisfies Assumption 4.1 for a given compact set $K\subset\mathbb{R}^m$ and for $\mathcal{X}\subset\mathbb{R}^n$ satisfying the conditions in Assumption 3.3. Then, with $\mathcal{H}$ defined in* (6)*, using $F_{\rm P}$, $G_{\rm P}$ from $\mathcal{H}_{\rm P}$, $F_{\rm E}$ from $\mathcal{H}_{\rm E}$, and $\hat{G}_{\rm E}$ from Algorithm 1, the parameter estimation error $(t,j)\mapsto\hat{\theta}(t,j)-\theta$ for each solution ξ to $\mathcal{H}$ from $\xi(0,0)\in\mathcal{Z}_0:=\{\xi\in\mathcal{Z}:x\in\mathcal{X},z\in K,\tau_\delta=-1\}$ satisfies*

$$|\hat{\theta}(t,j)-\theta|\leq\beta\big(|\hat{\theta}(0,0)-\theta|,t+\eta(j)\big) \tag{17}$$

for all $(t,j)\in\operatorname{dom}\xi$ such that $\tau_\delta(t,j)=-1$, with $\beta\in\mathcal{KL}$ from Assumption 4.1 and $\eta(j):=j-\lfloor j/2\rfloor$.

Theorem 4.2 provides an upper bound on the norm of the parameter estimation error for $\mathcal{H}$, except possibly during the delay intervals, namely, for $(t,j)'s$ such that $\tau_\delta(t,j)\in[0,\Delta]$.

Remark 4.3: Note that Assumptions 3.1, 3.2, and 3.4 are not included in Theorem 4.2. These assumptions are used to justify the hybrid model of $\mathcal{H}$ in (6) and to design the jump map $\hat{G}_{\rm E}$ of $\hat{\mathcal{H}}_{\rm E}$ in Sections III-C–III-E. However, given $\mathcal{H}$, to prove Theorem 4.2, we need only Assumption 4.1 and a set $\mathcal{X}$ satisfying the conditions in Assumption 3.3.

Sketch of Proof: We rewrite $\mathcal{H}$ in (6) to incorporate our design for the jump map $\hat{G}_{\rm E}$. To do so, we augment the state vector of $\mathcal{H}$ with a new component, μ, that evolves based on the dynamics of μ in (12)–(14). The resulting hybrid system, denoted by $\mathcal{H}'$, has dynamics

$$\mathcal{H}':\begin{cases}\left.\begin{array}{l}\dot{x}=F_{\rm P}(x,\theta)\\ \dot{z}=F_{\rm E}(x,z)\\ \dot{\mu}=F_{\rm E}(x,\mu)\\ \dot{\tau}_\delta=-\min\{\tau_\delta+1,1\}\end{array}\right\}=:F'(\xi,\theta) & \xi'\in C'\\ \left.\begin{array}{l}x^+=G_{\rm P}(x,\theta)\\ z^+=z\\ \mu^+=G_{\rm E}(x,\mu)\\ \tau_\delta^+\in[0,\Delta]\end{array}\right\}=:G'_{-1}(\xi,\theta) & \xi'\in D'_{-1}\\ \left.\begin{array}{l}x^+=x\\ z^+=\mu\\ \mu^+=\mu\\ \tau_\delta^+=-1\end{array}\right\}=:G'_0(\xi,\theta) & \xi'\in D'_0\end{cases} \tag{18}$$

with state $\xi':=(x,z,\mu,\tau_\delta)\in\mathcal{Z}':=\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times\big(\{-1\}\cup[0,\Delta]\big)$, flow set

$$C':=\widetilde{C}\times\mathbb{R}^m\times\big(\{-1\}\cup[0,\Delta]\big)$$

[5]The term "pre-asymptotic," as opposed to "asymptotic," indicates the possibility of maximal solutions that are not complete. This allows for separating the conditions for completeness from the conditions for stability and attractivity.

and jump set $D':=D'_{-1}\cup D'_0$, where

$$D'_{-1}:=\widetilde{D}\times\mathbb{R}^m\times\{-1\},\quad D'_0:=\big(\widetilde{C}\cup\widetilde{D}\cup\widetilde{G}(\widetilde{D})\big)\times\mathbb{R}^m\times\{0\}$$

with $\widetilde{C}$, $\widetilde{D}$, $\widetilde{G}$ as in (3).

Compared to $\mathcal{H}$ in (6), $\mathcal{H}'$ contains a new state component $\mu\in\mathbb{R}^m$. When $\tau_\delta=-1$ and x does not jump, μ flows per the flow map $F_{\rm E}$, as in (12). When the plant state x jumps, μ is reset to the value of $G_{\rm E}(x,\mu)$, and then continues flowing per $F_{\rm E}$ until τ_δ reaches zero, as in (13)–(14). At the end of the delay interval, the estimator state z is reset to the value of μ, as in (15). The recent work [11] establishes a general framework for modeling hybrid systems with delayed jumps.

Pick a solution $\xi'=(x,z,\mu,\tau_\delta)$ to $\mathcal{H}'$ from $\xi'(0,0)\in\mathcal{Z}'_0:=\{\xi'\in\mathcal{Z}':x\in\mathcal{X},z\in K,\mu=z,\tau_\delta=-1\}$. We define x^r and μ^r as j-reparamaterizations [14] of x and μ, respectively, that remove the trivial jumps from x and μ. In particular, $x^r(t,\eta(j)):=x(t,j)$ and $\mu^r(t,\eta(j)):=\mu(t,j)$ for all $(t,j)\in\operatorname{dom}\xi'$, with η as in Theorem 4.2. Using [12, Definition 2.6], it can be shown that $\xi^r:=(x^r,\mu^r)$ is a solution to the hybrid system $\widetilde{\mathcal{H}}$ in (3). Then, (17) follows from the fact that, by Assumption 4.1, each solution to $\widetilde{\mathcal{H}}$ in (3) from $(x(0,0),z(0,0))\in\mathcal{X}\times K$ satisfies (16). $\square$

V. Example

Consider the problem of estimating the acceleration due to gravity and the restitution coefficient for a bouncing ball. The ball has state $x=(x_1,x_2)\in\mathbb{R}^2$, where x_1 is the height above the ground and x_2 is the vertical velocity. The bouncing ball system has dynamics [12, Example 1.1]

$$\mathcal{H}_{\rm P}:\begin{cases}\dot{x}=\begin{bmatrix}x_2\\-\gamma\end{bmatrix}=F_{\rm P}(x,\theta) & x\in C_{\rm P}\\ x^+=\begin{bmatrix}x_1\\-\lambda x_2\end{bmatrix}=G_{\rm P}(x,\theta) & x\in D_{\rm P}\end{cases} \tag{19}$$

with flow set $C_{\rm P}:=\{x\in\mathbb{R}^2:x_1\geq 0\}$, jump set $D_{\rm P}:=\{x\in\mathbb{R}^2:x_1=0,x_2\leq 0\}$, and $\theta:=(\gamma,\lambda)\in\Theta:=[0,10]\times[0,1]$, where γ is the acceleration due to gravity and λ is the restitution coefficient.

For the estimation algorithm $\mathcal{H}_{\rm E}$, we employ the algorithm proposed in [4, Chapter 4]. This algorithm has state $z=(\hat{\theta},\psi,\alpha)\in\mathbb{R}^2\times\mathbb{R}^{2\times 2}\times\mathbb{R}^2$, input $x\in\mathbb{R}^2$, and data

$$\begin{aligned}F_{\rm E}(x,z)&=\begin{bmatrix}\gamma_c\psi^\top(y-\psi\hat{\theta})\\-\lambda_c\psi+\phi_c(x)\\-\lambda_c(x+\alpha)-f_c(x)\end{bmatrix}\\ G_{\rm E}(x,z)&=\begin{bmatrix}\hat{\theta}+\frac{\psi^{+\top}}{\gamma_d+|\psi^+|^2}(y^+-\psi^+\hat{\theta})\\(1-\lambda_d)\psi+\phi_d(x)\\(1-\lambda_d)(x+\alpha)-g_d(x)\end{bmatrix}\end{aligned} \tag{20}$$

where $\gamma_c,\lambda_c,\gamma_d>0$, $\lambda_d\in(0,2)$ are design parameters, $y:=x+\alpha$, $f_c(x):=\left[\begin{smallmatrix}x_2\\0\end{smallmatrix}\right]$, $\phi_c(x):=\left[\begin{smallmatrix}0&0\\-1&0\end{smallmatrix}\right]$, $g_d(x):=\left[\begin{smallmatrix}x_1\\0\end{smallmatrix}\right]$, and $\phi_d(x):=\left[\begin{smallmatrix}0&0\\0&-x_2\end{smallmatrix}\right]$.

The interconnection of $\mathcal{H}_{\rm P}$ and $\mathcal{H}_{\rm E}$ is simulated with $\theta=(9.81,1)$, $\gamma_c=1.5$, $\lambda_c=0.01$, $\gamma_d=0.5$, and $\lambda_d=1.99$ from the initial conditions $x(0,0)=(4.91,0)$, $\hat{\theta}(0,0)=(0,0)$, $\psi(0,0)=\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$, and $\alpha(0,0)=-x(0,0)$.

It can be shown numerically that the trajectory of the plant state x is sufficiently exciting to ensure convergence of $\hat{\theta}$ to θ for $\mathcal{H}_{\mathrm{E}}$ (see [4, Thereom 4.7] for details). Hence, if jumps in x are detected instantaneously and the jump map is computed instantly, the parameter estimate $\hat{\theta}$ converges exponentially to θ as shown in blue in Figure 1.[6] However, when jumps of $\mathcal{H}_{\mathrm{E}}$ are delayed, the parameter estimation error fails to converge to zero, as shown in green in Figure 1.

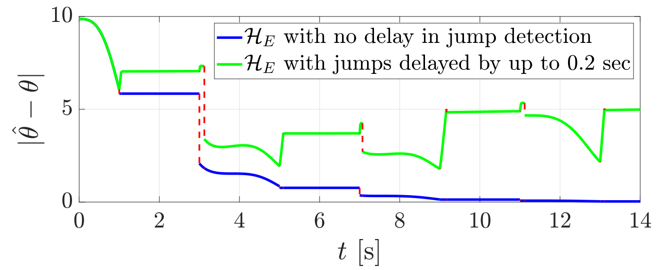

Fig. 1: The projection onto t of the estimation error for $\mathcal{H}_{\mathrm{E}}$ with no delay in the jump detection (blue) and with a delay of up to 0.2 seconds (green). When jumps of $\mathcal{H}_{\mathrm{E}}$ are delayed, the parameter estimate fails to converge.

To estimate θ in the presence of delays in jump detection and jump map computation, we employ the proposed algorithm $\mathcal{H}$ in (6), with $\hat{G}_{\mathrm{E}}$ computed using Algorithm 1, $\alpha_t = 0.28$, and $\alpha_x = 9.81$. The maps F_{P} in (19) and $F_{\mathrm{E}}, G_{\mathrm{E}}$ in (20) satisfy Assumptions 3.1 and 3.2. Furthermore, with $\theta = (9.81, 1)$, for each maximal solution x to (19) from $x(0,0) \in [4.91, \infty) \times [0, \infty)$, the initial interval of flow in $\operatorname{dom} x$ has length of at least 1 second, and x flows for at least 2 seconds after each jump. Suppose that detection of jumps in x is delayed by at most 0.15 seconds and Algorithm 1 takes at most 0.05 seconds to compute. Then, Assumption 3.3 holds with $\mathcal{X} = [4.91, \infty) \times [0, \infty)$, $\tau^* = 1$, $\Delta_1 = 0.15$, and $\varepsilon = 19.62$, and Assumption 3.4 holds with $\Delta_2 = 0.05$. Since, for each compact set $K \subset \mathbb{R}^m$, the $\hat{\theta}$ component of each solution to $\widetilde{\mathcal{H}}$ from $\mathcal{X} \times K$ converges exponentially to θ, it follows that Assumption 4.1 holds and the conditions of Theorem 4.2 are satisfied. The parameter estimation error for $\mathcal{H}$, shown in green in Figure 2, converges to zero and is equal to the estimation error for $\mathcal{H}_{\mathrm{E}}$ (with no delay in jump detection) shown in blue, except possibly on the delay intervals in accordance with Theorem 4.2.

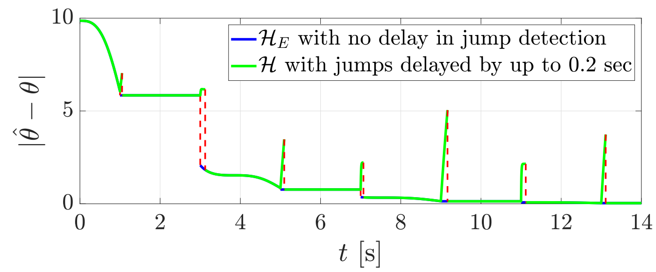

Fig. 2: The projection onto t of the estimation error for $\mathcal{H}_{\mathrm{E}}$ with no delay in jump detection (blue) and for $\mathcal{H}$ with a delay of up to 0.2 seconds (green). The estimation error for $\mathcal{H}$ converges to zero, except possibly on the delay intervals.

VI. Conclusion

In this paper, given a hybrid parameter estimation algorithm that is designed to jump synchronously with jumps in the plant state, we proposed a new algorithm for estimating unknown parameters in a class of hybrid dynamical systems under delays in the detection of jumps in the plant state. We showed that the proposed algorithm preserves the stability bounds of the original version, except possibly during the delays in detection of jumps. Future work on this topic includes analyzing the parameter estimation error for the proposed algorithm during the delay intervals, and studying the robustness properties induced by the algorithm. Moreover, since the optimization that we employ to determine the jump times of the plant state may be nonconvex, future work includes studying methods to convexify this problem to guarantee unique solutions. Analyzing the computational complexity of the optimization is also of interest.

[6] Code at `https://github.com/HybridSystemsLab/ApproximatelyKnownJumpTimes_BouncingBall`

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