Analysis of hybrid systems resulting from relay-type hysteresis and saturation: a Lyapunov approach

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Abstract— **This paper studies a class of hybrid systems with linear (or linear plus saturated linear) continuous and discrete dynamics, which are determined by a flow map and jump map, and state-triggered jumps. One motivation for considering this class of systems is that they can model control systems with a relay-type hysteresis element. Based on Lyapunov theorems for hybrid systems, a Lyapunov function is constructed that effectively incorporates the feature of the jumps. Global asymptotic stability analysis is presented for the case when the flow map is linear, and local asymptotic stability analysis is presented for the case when the flow map is linear plus saturated linear. The stability conditions are derived as matrix inequalities. A numerical example is presented to illustrate the hybrid modeling process for a system experiencing hysteresis. Simulations confirm the effectiveness of the proposed analysis tools and demonstrate the potential of the Lyapunov function.**

Keywords: Hybrid systems, saturation, relay-type hysteresis, Lyapunov analysis, matrix inequalities

I. INTRODUCTION

Hybrid systems are a class of dynamical systems that display continuous behavior, or flows, which is usually represented by differential equations, and discontinuous behavior, or jumps, which is usually modeled by difference equations. Hybrid systems permit the modeling of a wide range of engineering systems and scientific processes. They are sometimes induced by system design, and other times, they appear as appropriate modeling abstractions. Notable references on hybrid systems include e.g., [1], [8], [21], [25].

In this paper, we study a class of hybrid systems with linear (or linear plus saturated linear) continuous and discrete dynamics, which are determined by a flow map and jump map, and state-triggered jumps. This class of hybrid systems arises in control systems with impulsive, state-dependent behavior in the same way as in impulsive hybrid systems and reset systems; see, e.g., [9], [14], [23], [17]. Those hybrid models have been shown useful in studying several problems in engineering and science. In particular, we consider systems with saturation and relay-type hysteresis nonlinearities (see e.g., [20]). Regarding the impulsive behavior, we are interested in the jumps that are state triggered and may or may not be persistent, whereas in [14] the jumps are persistent with an average dwell time condition.

Relay-type hysteresis is a nonlinearity that typically arises in closed-loop systems in engineering applications. It presents challenges for analysis and design of control systems, like in control design for pneumatic proportional valves with hysteresis in [15] and the limit cycle analysis of relaxation oscillators in feedback with relay hysteresis in [11]. Different types of hysteresis models can be found in, e.g., [13], [16]. In order to embed such a nonlinearity in hybrid systems with dynamics described above, we are focusing on the relay-type hysteresis.

The goal of this paper is to conduct stability analysis for this class of hybrid systems via Lyapunov functions. Our motivation is firstly to choose a *flow Lyapunov function* that will decrease along flows. The construction of Lyapunov functions has been extensively studied for the case of linear flow maps. When the flow is linear plus saturated linear, a number of Lyapunov techniques are also available in the literature. For example, quadratic Lyapunov functions are proposed in [5], [7], [10], Lure-type Lyapunov functions are proposed in [6], [12], and piecewise quadratic Lyapunov functions are proposed in [4], [22]. However, such a *flow Lyapunov function* may not decrease along jumps. To overcome this difficulty, we introduce a *jump Lyapunov term* in the Lyapunov function. The *jump Lyapunov term* is chosen so that it may increase during flows, but with a rate slower than the decreasing rate of the *flow Lyapunov function*, and it decreases during jumps, more than the increment of the *flow Lyapunov function*. By appropriately combining the *flow Lyapunov function* and *jump Lyapunov term*, the resulting Lyapunov function can be tailored to decrease along solutions. Based on a novel construction of the Lyapunov function as described above and the general stability properties of hybrid systems in [3] and [19], we derive global asymptotic stability conditions for the case of linear flow map, and local asymptotic stability conditions for the case of linear plus saturated linear flow map. Those conditions are cast into matrix inequalities. To handle the saturation nonlinearity, we rely partially on the analysis tools from the literature on systems with saturation (see e.g., [5], [7], [10]).

The rest of the paper is organized as follows. In Section II, we define the class of hybrid systems of interest as well as a hybrid model for relay-type hysteresis. In Section III, the sufficient conditions for asymptotic stability are given. In Section IV, an integral control example is used to illustrate the analysis results.

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Notation For compact description, we denote the saturation function as $\mathrm{sat}(u) := \frac{u}{\max\{1,|u|\}}$, and the deadzone function as $\mathrm{dz}(u) = u - \mathrm{sat}(u)$. For a square matrix X, we denote $\mathrm{He}X := X + X^T$. For $P \in \mathbb{R}^{n\times n}, P = P^T > 0$, denote $\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^T P x \leq 1\}$. For $H \in \mathbb{R}^{1\times n}$, $\mathcal{L}(H) := \{x \in \mathbb{R}^n : |Hx|_\infty \leq 1\}$. For a given function J, c^*-sublevel set is given by $L_J(c^*) := \{x : J(x) \leq c^*\}$. $|\cdot|$ denotes the Euclidean vector norm, and given a nonempty subset $\mathcal{A}$ of Euclidean space, $|x|_{\mathcal{A}} := \inf_{a\in\mathcal{A}} |x - a|$. The unit ball denotes $\mathbb{B} := \{x \in \mathbb{R}^n : |x| \leq 1\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class K_∞ if it is continuous, zero at zero, positive when its argument is positive, strictly increasing and unbounded. For a locally Lipschitz function V, $\partial V(x)$ denotes the Clarke generalized gradient of V at x.

II. General model and problem formulation

A. A class of hybrid systems

Following [18], a hybrid system $\mathcal{H}$ is given by five objects defining the *data*: the state space, two mappings that specify the continuous and discrete evolution, and two sets in the state space where the continuous and discrete evolution occur. In particular, the five objects are: the state space $\mathbb{R}^m$, the *flow set* $\mathcal{C} \subset \mathbb{R}^m$ where the continuous evolution occurs, the *jump set* $\mathcal{D} \subset \mathbb{R}^m$ where the discrete evolution occurs, the *flow map* $f : \mathbb{R}^m \to \mathbb{R}^m$, governing the continuous evolution, and the *jump map* $g : \mathbb{R}^m \to \mathbb{R}^m$, determining the discrete evolution.

In this paper, we are interested in hybrid systems with state space $\mathbb{R}^{n+1}$ and state given by

$$\eta = \begin{bmatrix}\xi \\ q\end{bmatrix}, \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the continuous state and q is a logic mode that remains constant along flows and is updated at jumps. The logic mode q takes values in $\mathcal{Q}' := \{1, -1\}$.

The flow map is given as

$$\begin{bmatrix}\dot{\xi} \\ \dot{q}\end{bmatrix} \quad := \quad f(\xi, q) = \begin{bmatrix}A\xi + B\sigma(\tilde{u}) \\ 0\end{bmatrix} \tag{2}$$

where $\tilde{u} = K\xi \in \mathbb{R}$ with $K \in \mathbb{R}^{1\times n}$. The function $\sigma(\cdot)$ is specified by an identity function $\sigma(\tilde{u}) = \tilde{u}$ if the flow map is linear, or a saturation function $\sigma(\tilde{u}) = \mathrm{sat}(\tilde{u})$ if the flow map is linear plus saturated linear.

With q taking values in $\mathcal{Q}'$, the jump map is given by

$$\begin{bmatrix}\xi^+ \\ q^+\end{bmatrix} \quad := \quad g(\xi, q) = \begin{bmatrix}\xi + 2\epsilon\Theta q \\ -q\end{bmatrix}, \tag{3}$$

where $\Theta \in \mathbb{R}^{n\times 1}$ and $\epsilon > 0$ scales the magnitude of the jump change in ξ. The parameter ϵ will correspond to the width of hysteresis in Section II-B. The update rule for the logic variable is defined in a way such that it is triggered when the control input reaches the threshold $\bar{u}$. The flow set is taken to be

$$\mathcal{C} := \Big\{ \begin{bmatrix}\xi \\ q\end{bmatrix} \in \mathbb{R}^{n+1} : K\xi q \leq \bar{u}, q \in \mathcal{Q}' \Big\}, \tag{4}$$

and the jump set is taken to be

$$\mathcal{D} := \Big\{ \begin{bmatrix}\xi \\ q\end{bmatrix} \in \mathbb{R}^{n+1} : K\xi q \geq \bar{u}, q \in \mathcal{Q}' \Big\}. \tag{5}$$

In the following subsection, we will describe the behavior of the system with hysteresis, which can be regarded as a special case of the aforementioned hybrid model.

B. Modeling hysteresis

Consider the input-output behavior of an actuator

$$v = u - \epsilon q, \;\; \forall (u, q) \in \{(u, q) \in \mathbb{R} \times \mathcal{Q}' : uq \leq a\}, \tag{6}$$

where u is the feedback control input, $\epsilon > 0$ is the width of the hysteresis, $a > 0$ is the threshold and $q \in \mathcal{Q}$ is the logic variable. Fig. 1 shows the input-output relationship.

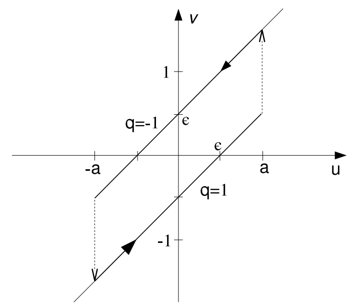

Fig. 1. Input-output behavior for the actuator with hysteresis

On the other hand, when the actuator is saturated, its input-output behavior is shown in Fig. 2 and the representation is given as

$$v = \mathrm{sat}(u - \epsilon q), \;\; \forall (u, q) \in \{(u, q) \in \mathbb{R} \times \mathcal{Q}' : uq \leq a\}, \tag{7}$$

where ϵ and a are defined in the same way as below (6).

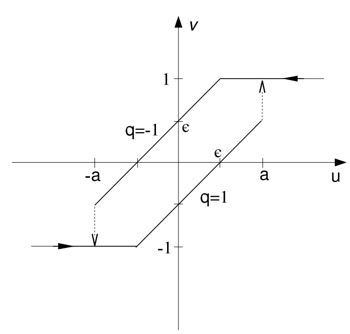

Fig. 2. Input-output behavior for the actuator with saturation and hysteresis

Assumption 1: $a > 0$, $\epsilon > 0$, $\epsilon < \min\{1, a\}$ and $1 - \epsilon < a \leq 1 + \epsilon$.

Hence, in Fig. 2, the threshold is such that switching happens in the linear section of each curve, and after switching, the output is saturated. When $a = \epsilon + 1$, two curves connect and the output does not jump.

To model the input-output behavior of the actuator in a control system, we define $\tilde{u} = u - \epsilon q$ such that $\sigma = v$. By appropriately choosing the control framework and defining the coordinates, we are able to embed the system with hysteresis into $\mathcal{H} = (f, \mathcal{C}, g, \mathcal{D})$ description and apply state-feedback laws. Later in Section IV, the modeling procedure will be illustrated by using a numerical example for a tracking problem. Please note that in such a case $\bar{u} = a - \epsilon$. In the following section, we will replace $\bar{u}$ in (4) and (5) with $a - \epsilon$ in the analysis.

III. Main Results

A. Preliminary results on hybrid systems

In this section, we summarize some preliminary results regarding the class of hybrid system $\mathcal{H}$ with data $(f, \mathcal{C}, g, \mathcal{D})$, or simply, $\mathcal{H} = (f, \mathcal{C}, g, \mathcal{D})$, defined in Section II.

Solutions to a hybrid system are given on hybrid time domains by hybrid arcs. A set E is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots J\})$ is a *compact hybrid time domain*, i.e., it can be written as $\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \ldots \leq t_J$. A hybrid arc η is a function defined on a hybrid time domain $\mathrm{dom}\ \eta$ mapping to $\mathbb{R}^m$ such that $\eta(t, j)$ is locally absolutely continuous in t for each j, $(t, j) \in \mathrm{dom}\ \eta$. A hybrid arc η is a solution to the hybrid system $\mathcal{H}$ if $\eta(0, 0) \in \mathcal{C} \cup \mathcal{D}$ and

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \mathrm{dom}\ \eta$,

$$\eta(t, j) \in \mathcal{C}, \quad \dot{\eta}(t, j) = f(\eta(t, j))$$

(S2) For all $(t, j) \in \mathrm{dom}\ \eta$ such that $(t, j+1) \in \mathrm{dom}\ \eta$,

$$\eta(t, j) \in \mathcal{D}, \quad \eta(t, j+1) = g(\eta(t, j)).$$

Conditions on the data $(f, \mathcal{C}, g, \mathcal{D})$, which we refer to as *hybrid basic conditions*, of a hybrid system $\mathcal{H}$ with state space $\mathbb{R}^m$ guaranteeing good structural properties of its solutions are continuity of the flow map f and jump map g, and closedness of the flow set $\mathcal{C}$ and jump set $\mathcal{D}$; see [19]. Now, we are ready to give the definition of stability for a hybrid system; also see [19, Section VI].

Definition 1: For a hybrid system $\mathcal{H}$ on a state space $\mathbb{R}^m$, a compact set $\mathcal{A} \subset \mathbb{R}^m$ is said to be: *stable* if for each $\bar{\epsilon} > 0$ there exists $\delta > 0$ such that each solution η to $\mathcal{H}$ starting at $\eta^0 \in \mathcal{A} + \delta\mathbb{B}$ satisfies $|\eta(t, j)|_{\mathcal{A}} \leq \bar{\epsilon}$ for all $(t, j) \in \mathrm{dom}\ \eta$; *attractive* if there exists $\mu > 0$ so that every maximal solution to $\mathcal{H}$ starting in $\mathcal{A} + \mu\mathbb{B}$ is complete and satisfies $\lim_{t+j\to\infty} |\eta(t, j)|_{\mathcal{A}} = 0$; and *asymptotically stable* if it is both stable and attractive. We denote the *domain of attraction* of $\mathcal{A}$, the set of all points from which all maximal solutions are complete and converge to $\mathcal{A}$, by $\mathcal{B}_{\mathcal{A}}$.

For hybrid systems satisfying *the hybrid basic conditions*, the conditions that guarantee *local existence of solutions* are:

(VC) For each point $\eta^0 \in \mathcal{C} \backslash \mathcal{D}$, there exists a solution with a nontrivial hybrid time domain, i.e., there exists a solution η and $t > 0$ such that $(t, 0) \in \mathrm{dom}\ \eta$.

(VD) $g(\mathcal{D}) \subset \mathcal{C} \cup \mathcal{D}$.

Condition (VC) guarantees that flow is possible from points in $\mathcal{C}$ while (VD) guarantees that jumps do not leave $\mathcal{C} \cup \mathcal{D}$; see [19, Proposition 2.1] for more details.

The following results are derived from the results in [24, Theorem 7.6 and Corollary 7.7].

Proposition 1: Given a hybrid system $\mathcal{H} = (f, \mathcal{C}, g, \mathcal{D})$, suppose it satisfies the *hybrid basic conditions*, (VC) and (VD). Let $\mathcal{A} \subset \mathbb{R}^m$ be compact. If there exists a locally Lipschitz function $V : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|\eta|_{\mathcal{A}}) \leq V(\eta) \leq \alpha_2(|\eta|_{\mathcal{A}}), \qquad \forall \eta \in \mathcal{C} \cup \mathcal{D} \tag{8a}$$

$$\max_{w \in \partial V(\eta)} \langle w, f(\eta) \rangle < 0, \qquad \forall \eta \in \mathcal{C} \setminus \mathcal{A} \tag{8b}$$

$$V(g(\eta)) - V(\eta) < 0, \qquad \forall \eta \in \mathcal{D} \setminus \mathcal{A} \tag{8c}$$

$$g(\mathcal{A} \cap \mathcal{D}) \subset \mathcal{A} \tag{8d}$$

then $\mathcal{A}$ is globally asymptotically stable.

Proposition 2: Given a hybrid system $\mathcal{H} = (f, \mathcal{C}, g, \mathcal{D})$, suppose it satisfies the *hybrid basic conditions*, (VC) and (VD). Let $\mathcal{A} \subset \mathcal{U} \subset \mathbb{R}^m$ be such that $\mathcal{A}$ is compact and contained in the interior of $\mathcal{U}$. If there exists a locally Lipschitz function $V : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|\eta|_{\mathcal{A}}) \leq V(\eta) \leq \alpha_2(|\eta|_{\mathcal{A}}), \forall \eta \in (\mathcal{C} \cup \mathcal{D}) \cap \mathcal{U} \tag{9a}$$

$$\max_{w \in \partial V(\eta)} \langle w, f(\eta) \rangle < 0,\ \forall \eta \in (\mathcal{C} \setminus \mathcal{A}) \cap \mathcal{U} \tag{9b}$$

$$V(g(\eta)) - V(\eta) < 0, \forall \eta \in (\mathcal{D} \setminus \mathcal{A}) \cap \mathcal{U} \tag{9c}$$

$$g(\mathcal{A} \cap \mathcal{D}) \subset \mathcal{A} \tag{9d}$$

then $\mathcal{A}$ is locally asymptotically stable. The domain of attraction contains every sublevel set of V that is a subset of $\mathcal{U}$.

B. The construction of Lyapunov function

Following the ideas of constructing Lyapunov function given in the Introduction, an initial guess of a Lyapunov function candidate for $\mathcal{H}$ would be a quadratic function

$$\tilde{\mathcal{V}}(\xi) = \xi^T Q^{-1} \xi \tag{10}$$

where $Q = Q^T > 0, Q \in \mathbb{R}^{n \times n}$. However, such a quadratic function does not necessary decrease along jumps for $\mathcal{H}$.

We consider a way to construct a Lyapunov function for $\mathcal{H}$ is to take the combination of a *jump Lyapunov term* and a *flow Lyapunov function*. We consider the locally Lipschitz Lyapunov function $V : \mathbb{R}^{n+1} \mapsto \mathbb{R}_{\geq 0}$ given by

$$V(\xi, q) = \rho(\xi, q)\sqrt{\xi^T Q^{-1} \xi}\, \exp(\sqrt{\xi^T Q^{-1} \xi}), \tag{11}$$

with

$$\rho(\xi, q) := \exp(-\lambda \cdot \max(0, \min(2a, a - \epsilon - K\xi q))) \tag{12}$$

where $\lambda > 0$ and $\rho : \mathbb{R}^{n+1} \mapsto [\exp(-2\lambda a), 1]$ is locally Lipschitz.

In the form of the Lyapunov function we construct, $\rho(\xi, q)$ is the *jump Lyapunov term*. This term may increase during flows, but more slowly than the *flow Lyapunov function* $\sqrt{\tilde{\mathcal{V}}(\xi)} \cdot \exp(\sqrt{\tilde{\mathcal{V}}(\xi)})$ decreases, and it decreases during jumps, more so than the *flow Lyapunov function* $\sqrt{\tilde{\mathcal{V}}(\xi)} \cdot \exp(\sqrt{\tilde{\mathcal{V}}(\xi)})$ increases. (The idea of constructing the $\rho(\cdot)$ function is proposed in [3], see the proof of [3, Theorem 7] for details).

For a fixed threshold a and parameter λ, the following inequality always holds

$$e^{-2\lambda a}\sqrt{\tilde{\mathcal{V}}(\xi)}\exp\sqrt{\tilde{\mathcal{V}}(\xi)} \leq V(\xi,q) \leq \sqrt{\tilde{\mathcal{V}}(\xi)}\exp\sqrt{\tilde{\mathcal{V}}(\xi)} \tag{13}$$

Furthermore, the quadratic function $\tilde{\mathcal{V}}(\xi)$ satisfies

$$\Sigma_{\min}(Q^{-1})|\xi|^2 \leq \tilde{\mathcal{V}}(\xi) \leq \Sigma_{\max}(Q^{-1})|\xi|^2$$

where $\Sigma_{\min}(Q^{-1})$ and $\Sigma_{\max}(Q^{-1})$ correspond to the minimum and maximum eigenvalues of Q^{-1}, respectively. We define

$$\alpha_1(s) := e^{-2\lambda a}\sqrt{\Sigma_{\min}(Q^{-1})}\ s\ e^{s\sqrt{\Sigma_{\min}(Q^{-1})}}$$
$$\alpha_2(s) := \sqrt{\Sigma_{\max}(Q^{-1})}\ s\ e^{s\sqrt{\Sigma_{\max}(Q^{-1})}}$$

so that (8a) and (9a) hold when $\mathcal{A} = \{(\xi,q) : \xi = 0, q \in \mathcal{Q}'\}$ is defined for the hybrid system in Section II.

C. Sufficient conditions on stability for the hybrid system

It is easy to verify that the *hybrid basic conditions* hold for system (1)-(5), since the flow map f and jump map g are continuous, and the flow set $\mathcal{C}$ and jump set $\mathcal{D}$ are closed in $\mathbb{R}^{n+1}$. The *local existence of solutions* follows from $\mathcal{C}\cup\mathcal{D} = \mathbb{R}^n \times \mathcal{Q}'$ and $g(\mathcal{D}) \subset \mathcal{C}\cup\mathcal{D}$. In the following content, we start by asserting that Proposition 1 and 2 can be applied to formulate statements about the stability of the compact set $\mathcal{A}$ for the hybrid system defined in Section II.

Based on Proposition 1 and Proposition 2, we are ready to state our main results by using the Lyapunov function (11).

Theorem 1: Let Assumption 1 hold for the hysteresis width ϵ and the threshold a. Consider the hybrid system $\mathcal{H}$ in (1)-(5) with $\sigma(K\xi) = K\xi$ in the flow map and $\bar{u} = a - \epsilon$ in the flow and jump sets. Then $\mathcal{A}$ is locally exponential stable if $A + BK$ is Hurwitz. Furthermore, given $V(\xi,q)$ in (11), if there exists matrix $Q = Q^T > 0$, and parameters $\beta, \lambda, b, c > 0$, such that the following matrix inequalities are feasible

$$(A+BK)Q + Q(A+BK)^T \leq -\beta Q \tag{14a}$$

$$\begin{bmatrix} \frac{\beta^2}{\lambda^2}Q & 2Q(A+BK)^T K^T \\ 2K(A+BK)Q & I \end{bmatrix} \geq 0 \tag{14b}$$

$$\begin{bmatrix} (b-1-2\epsilon)Q & \frac{\sqrt{4\epsilon^2+2\epsilon}}{|a-\epsilon|}QK^T\Theta^T \\ \frac{\sqrt{4\epsilon^2+2\epsilon}}{|a-\epsilon|}\Theta KQ & Q \end{bmatrix} \geq 0 \tag{14c}$$

$$\begin{bmatrix} c^2 Q & 2\sqrt{\epsilon}\Theta \\ 2\sqrt{\epsilon}\Theta^T & 1 \end{bmatrix} > 0, \begin{bmatrix} c^2 & 2\sqrt{\epsilon}\Theta^T \\ 2\sqrt{\epsilon}\Theta & Q \end{bmatrix} > 0 \tag{14d}$$

$$4a\lambda - 2\sqrt{\epsilon}c \geq \log_e b \tag{14e}$$

then $\mathcal{A}$ is globally asymptotically stable.

Remark 1: When $\epsilon = 0$ and $Q = Q^T > 0$ satisfies $(A + BK)Q + Q(A+BK)^T < 0$ then the matrix conditions (14a)-(14e) are guaranteed to be satisfied for this Q and some parameters $\beta, \lambda, b, c > 0$. In particular, β and λ are taken sufficiently small, b is taken bigger than but sufficiently close to one, and $c > 0$ can be arbitrary. ■

Theorem 2: Let Assumption 1 hold for the hysteresis width ϵ and the threshold a. Consider the hybrid system $\mathcal{H}$ in (1)-(5) with $\sigma(K\xi) = \mathrm{sat}(K\xi)$ in the flow map and $\bar{u} = a - \epsilon$ in the flow and jump sets. Then $\mathcal{A}$ is locally exponential stable if $A + BK$ is Hurwitz. Furthermore, given $V(\xi,q)$ in (11), if there exists matrix $Q = Q^T > 0$, diagonal matrices $T, U > 0$, matrices Y, Z with appropriate dimensions, and parameters $\beta, \lambda, b, c > 0$, such that the following matrix inequalities are feasible

$$\mathrm{He}\begin{bmatrix} (A+BK)Q + \frac{1}{2}\beta Q & -BT \\ KQ - Y & -T \end{bmatrix} \leq 0 \tag{15a}$$

$$\mathrm{He}\begin{bmatrix} -\frac{\beta^2}{2\lambda^2}Q & 0 & 0 \\ KQ - Z & -U & 0 \\ K(A+BK)Q & -KBU & -\frac{1}{2}I \end{bmatrix} < 0 \tag{15b}$$

$$\begin{bmatrix} 1 & Y \\ Y^T & Q \end{bmatrix} \geq 0, \begin{bmatrix} 1 & Z \\ Z^T & Q \end{bmatrix} \geq 0 \tag{15c}$$

$$(14c), (14d), (14e) \tag{15d}$$

then every trajectory starting from $L_V(c^*)$ converges to $\mathcal{A}$ where $c^* = \exp(-2\lambda a + 1)$ such that

$$L_V(c^*) \quad \subset \quad (\mathcal{E}(Q^{-1}) \times \mathcal{Q}'). \tag{16}$$

The set $L_V(c^*)$ gives an estimate of the domain of attraction. The domain of attraction contains points in $\mathcal{D}$ if $\mathcal{D} \cap L_V(c^*) \neq \emptyset$.

Remark 2: The estimate of the domain of attraction is derived from the structure of the Lyapunov function. Since $V(\xi,q)$ satisfies (13), then $V(\xi,q) \leq e^{-2\lambda a}e$ implies $\sqrt{\xi^T Q^{-1}\xi}e^{\sqrt{\xi^T Q^{-1}\xi}} \leq e$ for each ξ such that $\xi^T Q^{-1}\xi \leq 1$. Therefore, the level set $L_V(c^*) \subset (\mathcal{E}(Q^{-1}) \times \mathcal{Q}')$ gives an estimate of the domain of attraction $\mathcal{B}_{\mathcal{A}}$ of the hybrid system $\mathcal{H}$. Every trajectory starting from this set will converge to $\mathcal{A}$. ■

Remark 3: When $\epsilon = 0$, the matrix inequalities in (15a)-(15d) reduce to Theorem 3 in [10] for the same choice of β, λ, b, c in Remark 1, where Theorem 3 in [10] is derived by using a quadratic Lyapunov function as (10). ■

Remark 4: Theorem 1 and Theorem 2 are conservative especially for large ϵ. This partially results from the fact that the matrix inequalities (14c-14e) and (15c) are sufficient conditions for (8c) and (9c), correspondingly, not necessary ones. They impose unnecessary restrictions for the points which are outside of $\mathcal{D}$. When $\epsilon = 0$, those restrictions vanish for some trivial b, c, λ, but it becomes more difficult to choose b, c, λ, β and get feasible solutions when ϵ gets larger. We can image no feasible solutions would be found from (14) or (15) when the system is stable but with large enough ϵ. That will be further discussed in Section IV by using a numerical example. ■

Remark 5: The choice of parameters b, c, λ, β is normally by trial and error. In practice, we can always choose $c = \frac{4a\lambda - \log_e b}{2\sqrt{\epsilon}}$ to satisfy (14e). By observing the relationship among them, we can see larger ϵ implies larger c from (14d) so as larger λ, and also implies larger b for feasibility of (14c). When λ gets larger, β tends to be larger followed from (14b), so that the volume of $\mathcal{E}(Q^{-1})$ tends to be smaller from (14a). Overall, larger ϵ requires larger b, λ, β and results in a smaller domain of attraction estimate $L_V(c^*)$. ■

IV. Example

Consider a type I system, $H(s) = \frac{1}{s(s+0.5)}$, with the state-space description $\dot{x}_1 = x_2$, $\dot{x}_2 = -0.5x_2$, and $y = x_1$.

The control objective is to achieve exact tracking of constant references $r = 2/3$ and the closed-loop poles are all located at $s = -1$.

To implement the integral control for exact tracking (see e.g., [2, page 552]), denote $e = x_1 - r$ and $x_I = \int edt$. Then $y = e + r$ and the system is

$$\begin{bmatrix} \dot{e} \\ \dot{x}_2 \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ x_2 \\ x_I \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u. \qquad (17)$$

The control law is given by $u = -3e - 2.5x_2 - x_I$ so as to stabilize the system (17) and achieve exact tracking for the original system. However, if such a system is subject to actuator nonlinearity as shown in Fig. 1, and Fig. 2, then the closed-loop performance will be affected. As in Section II, we describe this system with a hybrid model as follows.

States: $[\xi_1 := e,\ \xi_2 := x_2,\ \xi_3 := x_I + \epsilon q,\ q\ ]^T$. The flow map is given as:

$$f(\xi, q) = [\xi_2,\ -0.5\xi_2 + \sigma(\tilde{u}),\ \xi_1,\ 0]^T,$$

where $\tilde{u} = u - \epsilon q = \begin{bmatrix} -3 & -2.5 & -1 \end{bmatrix} \xi$. In section IV-A, we will discuss the case $\sigma(\tilde{u}) = \tilde{u}$, while in section IV-B, we will discuss the case $\sigma(\tilde{u}) = \text{sat}(\tilde{u})$.

The jump map is:

$$g(\xi, q) = [\xi_1,\ \xi_2,\ \xi_3 - 2\epsilon q,\ -q]^T.$$

The flow set and the jump set are:

$$\mathcal{C} := \left\{ \begin{bmatrix} \xi \\ q \end{bmatrix} : [-3\ \ -2.5\ \ -1]\, \xi q \le 1 - \epsilon, q \in \mathcal{Q}' \right\},$$

$$\mathcal{D} := \left\{ \begin{bmatrix} \xi \\ q \end{bmatrix} : [-3\ \ -2.5\ \ -1]\, \xi q \ge 1 - \epsilon, q \in \mathcal{Q}' \right\},$$

and

$$\mathcal{A} := \left\{ \begin{bmatrix} \xi \\ q \end{bmatrix} : \xi = 0, q \in \mathcal{Q}' \right\}.$$

A. *Simulation for linear system with hysteresis*

Table 1: Global stability test for ϵ

Stability	$\epsilon \le 0.24$	$0.25 \le \epsilon \le 0.8$	$\epsilon \ge 0.9$
Theorem 1	Yes	No	No
Simulation	Yes	Yes	No

As shown in the table above, we use the method in Theorem 1 to test the global stability of the system with hysteresis. In this table, we present the stability results confirmed from Theorem 1 by comparing with the stability results we observe from the simulation. By trial and error, we can find feasible solutions from Theorem 1 for the hysteresis width ϵ up to 0.24 which indicate $\mathcal{A}$ is globally asymptotic stable, while the simulations show the stability for ϵ up to 0.8. We observe the oscillation in the simulation when ϵ is larger than 0.9. All simulations are generated with randomly chosen initial conditions. The simulations indicate that Theorem 1 can be conservative for large ϵ as pointed out in Remark 4.

If the hysteresis width is $\epsilon = 0.1$, we can find a feasible solution from Theorem 1, which confirms $\mathcal{A}$ is globally asymptotically stable. The system response is shown in Fig. 3 based on the initial condition $\xi_0 = \begin{bmatrix} -1 & 40 & 55 \end{bmatrix}^T$ and $q_0 = -1$. In Fig. 3, the two subplots on the left show the system output y and the state ξ_2 which has a fix amount of shift along jumps. In the two subplots on the right, we observe that the system is stabilized after two jumps, and along jumps the Lyapunov function is decreasing. The evolution of the Lyapunov function is shown in the form of natural logarithm $\log_e(V)$, where V is explicitly determined by Theorem 1.

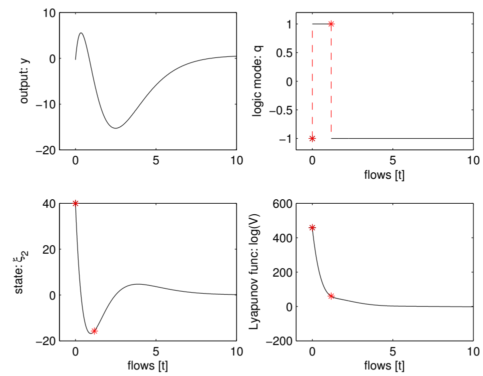

Fig. 3. The response of the linear system with hysteresis: globally stable with $\epsilon = 0.1$; $*$ shows the status right before jumps.

B. *Simulation for saturated system with hysteresis*

Table 2: Test on the domain of attraction

	$L_V(c^*) \cap \mathcal{D} \neq \emptyset$	$L_V(c^*) \cap \mathcal{D} = \emptyset$
Theorem 2	$0 \le \epsilon \le 0.21$	$0.21 < \epsilon \le 0.9$
Simulation	$0 \le \epsilon \le 0.9$	$0.9 < \epsilon \le 1$

In Table 2, Theorem 2 is used to test whether or not the domain of attraction contains points in $\mathcal{D}$, namely, if trajectories starting from the domain of attraction have jumps. By trial and error, we can find feasible solutions from Theorem 2 such that $L_V(c^*) \cap \mathcal{D} \neq \emptyset$ for the hysteresis width ϵ up to 0.21. According to (16), for feasible solutions of matrix inequalities (15) we have the set $\mathcal{E}(Q^{-1})$, and the estimate of the domain of attraction $L_V(c^*)$ is a subset of $\mathcal{E}(Q^{-1}) \times \mathcal{Q}'$. We also observe that the larger ϵ is, the smaller $\mathcal{E}(Q^{-1}) \times \mathcal{Q}'$ and so would be the estimate of the domain of attraction.

However, the simulation shows that the domain of attraction contains points in $\mathcal{D}$ for $0 \le \epsilon \le 0.9$. When ϵ is larger than 0.9, the system has oscillations if starting from points in $\mathcal{D}$. From Table 2, we can see Theorem 2 is conservative especially for larger ϵ, due to the reason explained in Remark 4.

Furthermore, to better illustrate how Theorem 2 works in the analysis of saturated system with hysteresis, we use $\epsilon = 0.2$ as the hysteresis width and do the following simulations.

For hysteresis width given as $\epsilon = 0.2$, we find $\beta = 0.89$, $\lambda = 0.41$, and $b = 2.67$ by trial and error, such that the matrix inequalities (15) give a feasible solution for the variable Q. Please note that we set $c = \frac{4\lambda - \log_e b}{2\sqrt{\epsilon}}$.

By using the Lyapunov function $V(\xi, q)$ in (11) and (16), an estimate of the domain of attraction is given as

$$L_V(1.197) = \big\{(\xi, q)|\ V(\xi, q) \le 1.197,\ q \in \mathcal{Q}'\big\}, \qquad (18)$$

where $Q^{-1} = \begin{bmatrix} 4.9017 & 3.6617 & 1.7744 \\ 3.6617 & 2.9504 & 1.2528 \\ 1.7744 & 1.2528 & 0.6711 \end{bmatrix}$.

Case 1: Choose $\xi_0 = \begin{bmatrix} -1.5404 & 1.5268 & 1.6365 \end{bmatrix}^T$ and $q_0 = -1$ such that $(\xi_0, q_0) \in L_V(1.197)$. Hence, the initial condition is in the domain of attraction. The response of the hybrid system is shown in the Fig. 4 for tracking the reference $r = 2/3$. In Fig. 4, we observe that the system jumps at $t = 0$, and along this jump the Lyapunov function decreases with $V^+ = 0.88$ (right after jump) and $V = 1.182$ (right before jump), while the function $\tilde{\mathcal{V}}(\xi)$ does not decrease since $\tilde{\mathcal{V}}^+ = 0.7262$ and $\tilde{\mathcal{V}} = 0.3966$.

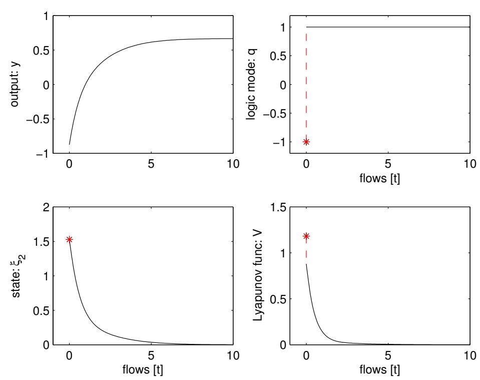

Fig. 4. The response of the saturated system with hysteresis: the initial condition in the domain of attraction; $*$ shows the status right before jumps.

Case 2: Choose $\xi_0 = \begin{bmatrix} -2 & -2 & -1 \end{bmatrix}^T$ and $q_0 = -1$ such that $(\xi_0, q_0) \notin L_V(1.197)$. Since the initial condition is outside of the domain of attraction, the Lyapunov function fails to verify stability, even in the case that the simulation shows the system is stable along flows and jumps. The function V increases along the flow in a certain time period, while the response of the system still converges to desired values. This case shows that Theorem 2 is conservative in some situations.

V. Conclusion

In this paper, we study a class of hybrid systems where the flow and jump maps are linear (or linear plus saturated linear) and jumps are state triggered. One motivation for considering this class of systems is that they model control systems with a relay-type hysteresis element. Motivated by the Lyapunov theorems for hybrid systems, a Lyapunov function is constructed that effectively incorporates the feature of the jumps. Global stability analysis is conducted when the flow map is linear, and local stability condition is conducted when the flow map is linear plus saturated linear. The stability conditions are derived as matrix inequalities. A numerical example is presented to illustrate the hybrid modeling process for a system experiencing hysteresis. Simulations confirm the effectiveness of the analysis and show some interesting simulation results.

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