A Tutorial on Hybrid Feedback Control

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***Abstract*— This tutorial paper introduces hybrid feedback control through a self-contained examination of hybrid control systems modeled by the combination of differential and difference equations with constraints. Using multiple examples, it illustrates the power of hybrid feedback control, which stems from the integration of continuous and discrete dynamics, where state variables update instantaneously at specific events while flowing continuously otherwise. The paper defines hybrid closed-loop systems as interconnected hybrid plants and controllers with designated inputs and outputs, and formalizes their solutions. It summarizes key properties of hybrid systems and reviews various control strategies, including supervisory control with logic variables to select feedback controllers, event-triggered control to minimize control input updates, and strategies using multiple Lyapunov-like functions for stabilization. Pointers to further reading and other strategies in the literature are provided.**

I. INTRODUCTION

Control theory provides powerful tools for the design of feedback control algorithms that assure the provable satisfaction of key dynamical properties, such as stability, attractivity, invariance, optimality, and robustness, to just list a few. The classical setting for the system to control, usually called *the plant*, is for it be given in terms of a continuous-time system or of a discrete-time system. Differential equations effectively capture the evolution of plants with continuously evolving variables with finite dimension. In such continuous-time setting, the control algorithms resulting from using control theory tools are usually of continuous-time nature, given in terms of static maps (e.g., state-feedback laws) or differential equations. When the variables evolve in discrete time, difference equations are a suitable modeling framework, naturally leading to discrete-time control algorithms. An emerging control theoretical approach that exploits the capabilities of continuous-time and discrete-time control is *hybrid feedback control* [1], [2], [3], [4], [5], [6]. Hybrid feedback control can lead to control algorithms that outperform the capabilities of purely continuous-time and discrete-time controllers due to allowing

- Variables that *flow* continuously over ordinary time; and
- Variables that instantaneously *jump* to new values upon events.

As argued in [6], control algorithms with such hybrid dynamics can implement feedback strategies that combine behavior that is typical of continuous-time controllers and of discrete-time controllers. In addition, a hybrid control algorithm can orchestrate multiple controllers to solve a complex problem, by using each controller to solve a small piece of the whole problem [2], [7], [8], [9], [10], [11]. Conveniently, a hybrid control algorithm has the capability of resetting its variables when certain events occur. For instance, upon communication or sampling events, memory states in the algorithm can be reset so as to store the new information [12], [13], [14], [15]. Another example is when events are associated to faults, upon which a hybrid control algorithm can reconfigure itself to cope with a faulty system. The power of hybrid control stems from its state allowing for the combination of logic variables, timers, and memory states, along with the logic-based conditions updating these variables so as to make the proper decisions that would lead to the desired behavior of the overall hybrid system [6].

This tutorial paper presents a self-contained introduction to hybrid feedback control. Hybrid dynamical systems are modeled in terms of *hybrid equations/inclusions*. These models combine differential equations and difference equations with constraints. Motivated by several examples arguing the need of hybrid models, this general modeling framework is introduced in Section III. This section motivates the combination of continuous and discrete dynamics using a sample-and-hold control architecture, in which sampling and hold events lead to instantaneous updates of state variables while, in between such events, the state variables flow continuously. Similarly, the problem of robustly and globally asymptotically stabilizing a point on the unit circle is used to motivate the need for a control algorithm that implements hysteresis-based switching to update a logic variable selecting the feedback law to use. The conditions triggering the events are captured by sets that constraint the state variables during flows and at jumps, giving rise to the so-called hybrid equations/inclusions model.

Also, in Section III, a hybrid closed-loop system is defined as the interconnection of a hybrid plant and a hybrid controller, both modeled within the same framework, including inputs and outputs that are properly assigned to define the interconnection. In addition, Section III introduces a concept of solution that formalizes state trajectories for such models. This notion is introduced in a tutorial manner,

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building from solution notions for continuous-time systems and for discrete-time systems. The concepts employed in their definition, specifically, hybrid time, hybrid time domain, and hybrid arc are introduced and exercised in the sample-and-hold control problem. With the modeling framework laid out, This section provides an overview of asymptotic stability and its robustness.

The sections that follow introduce several hybrid feedback control strategies. Section IV introduces a supervisory control strategy that features multiple feedback controllers and involves a logic variable, along with a properly defined logic, to determine which one of the feedback controllers is to be used under the current conditions. In Section V, a hybrid control strategy that updates the control input upon events is introduced. This event-triggered control strategy can be designed to minimize the rate of control input updates, hence, saving computational resources. Section VI presents a hybrid control strategy that exploits the availability of multiple Lyapunov functions, and associated state-feedback laws, for the asymptotic stabilization of a set. This strategy synergisticaly steers the state of the plant by using the value of the Lyapunov-like functions to select the state-feedback law to employ. Section VII provides a list of references related to hybrid feedback control that the reader might be interested in to further explore this fascinating field.

II. Notation

Throughout this paper, we use $\mathbb{R}$ to represent real numbers and $\mathbb{R}_{\geq 0}$ its nonnegative subset. The set of natural numbers is denoted $\mathbb{N}_{>0}$; namely, $\mathbb{N}_{>0} = \{1, 2, \ldots\}$. The set of naturals including zero is denoted $\mathbb{N}$. The notation $S_1 \subset S_2$ indicates S_1 is a subset of S_2, not necessarily proper. Given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, the notation $[x^\top y^\top]^\top$ is equivalent to the convenient notation (x, y). Given $x \in \mathbb{R}^n$, its Euclidean norm is denoted $|x|$. The distance from $x \in \mathbb{R}^n$ to a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is denoted $|x|_\mathcal{A} := \inf_{y \in \mathcal{A}} |x - y|$. We denote by $\mathcal{A} + \delta\mathbb{B}$ the set of all $x \in \mathbb{R}^n$ such that $|x - y| \leq \delta$ for some $y \in \mathcal{A}$. The closure of a set $S \subset \mathbb{R}^n$ is denoted $\overline{S}$. A strictly increasing continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ is said to be a class-$\mathcal{K}$ function. An unbounded class-$\mathcal{K}$ function is said to be a class-$\mathcal{K}_\infty$ function. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \searrow 0} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. The notation $L_V(c)$ stands for the c-sublevel set of the function $V\colon \operatorname{dom} V \to \mathbb{R}$.

III. Hybrid Feedback Control Systems

This tutorial paper introduces *hybrid feedback control systems* modeled as the combination of differential equations, difference equations, and constraints. These elements govern the evolution of a finite-dimensional state. Such a model leads to a system that has state trajectories that may evolve continuously and also exhibit jumps. Due to combination of continuous and discrete behavior, the state of such a system can involve continuous-valued variables – for example, physical quantities like position and velocity – as well as discrete-valued variables – for instance, logic variables that determine the (discrete) mode of operation of the system. In this section, we introduce a general model of a hybrid feedback control system that involves two key systems: a system to control, called *the hybrid plant* ($\mathcal{H}_P$), and a control algorithm, called *the hybrid controller* ($\mathcal{H}_K$).

We arrive to this model by fixing the ideas with concrete applications.

Example 1 (Sample-and-Hold Control)**.** *Consider a continuous-time control system with state ξ, input $\widetilde{u}$, and dynamics*

$$\dot{\xi} = \widetilde{f}(\xi, \widetilde{u})$$

where $\widetilde{f}$ is the right-hand side. When the state ξ is measured, a static state-feedback control law that might be able to stabilize a desired setpoint ξ^* is given by

$$\widetilde{u} = \kappa_c(\xi)$$

A sample-and-hold implementation of this feedback performs the following tasks:

1) *Every T^* seconds, measure the state ξ, calculate $\kappa_c(\xi)$, and update the input $\widetilde{u}$ to the result of the calculation;*
2) *In between such events, keep the input $\widetilde{u}$ constant, equal to the value obtained at the previous calculation.*

See Figure 1 for a schematic representation of the closed-loop system.

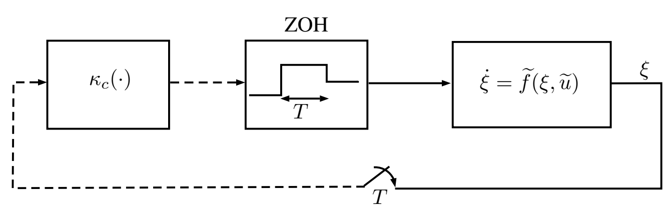

Fig. 1. A schematic representation of the sampled-data feedback control system in Example 1.

To capture this implementation in a mathematical model, we employ the following state variables:

- *A timer state τ that triggers the sampling and hold events when τ reaches T^*; and*
- *A memory state ℓ_v that updates $\widetilde{u}$ using a zero-order hold (ZOH) mechanism:*
  - *At each event, the memory state ℓ_v is reset to the value obtained from calculating $\kappa_c(\xi)$ using the current value of ξ;*
  - *In between events, the value of the memory state ℓ_v is kept constant.*

The task in item 1 can be captured by the condition

$$\tau = T^*$$

To trigger such events every T^* seconds, the timer is reset to zero after each such event. This mechanism can be captured *by the difference equation*

$$\tau^+ = 0$$

Since the memory state ℓ_v is to be updated to the result of computing $\kappa_c(\xi)$ for the current value of the state, the memory state is reset via

$$\ell_v^+ = \kappa_c(\xi)$$

which is also a difference equation. At such events, the physical state ξ does not change, so it evolves according to

$$\xi^+ = \xi$$

The (trivial) differential equation

$$\dot{\ell}_v = 0$$

keeps the memory state constant in between events, while the differential equation

$$\dot{\tau} = 1 \tag{1}$$

makes the timer count the amount of time elapsed since the last event.

Putting the equations and conditions above, the state variables ξ, ℓ_v, and τ are updated via the difference equations

$$\xi^+ = \xi, \quad \ell_v^+ = \kappa_c(\xi), \quad \tau^+ = 0 \tag{2}$$

at the events, which corresponds to

$$\tau = T^* \tag{3}$$

In between events, these state variables are updated via the differential equations

$$\dot{\xi} = \widetilde{f}(\xi, \kappa_c(\xi)), \quad \dot{\ell}_v = 0, \quad \dot{\tau} = 1 \tag{4}$$

The condition indicating that there is no event – namely, that the state variables should evolve continuously – is simply

$$\tau \in [0, T^*) \tag{5}$$

Note that including $\tau = T^*$ in this condition has no effect on the evolution of the state since from such a point, the timer cannot increase further continuously while satisfying the condition in (5)*; hence, even though the conditions* (3) *and* (5) *would overlap, the only possibility for the trajectory to continue is for τ to get reset to zero.*

The system resulting from the model developed above is a hybrid system due to combining differential equations, namely, (4)*, difference equations, that is,* (2)*, and constraints –* (5) *and* (3)*. The first constraint indicates when continuous evolution – called* flow *– of the state variables is possible according to the differential equations, and the second constraint determined when discrete evolution – called* jump *– of those variables is possible using the difference equations.*

In several applications, sampling may occur aperiodically. Aperiodic sampling can be used to model packet dropouts in networked systems, jitter in digital devices, or even denial of service attacks. In these contexts, a typical assumption consists of supposing that the sampling time varies in a bounded interval $[\underline{T}, \overline{T}]$, where $0 < \underline{T} < \overline{T}$. To capture this behavior, we modify the hybrid system obtained by combining (2)–(4) *with* (3)–(5) *as follows. We let the timer state τ flow according to* (1) *as long as*

$$\tau \in [0, \overline{T}] \tag{6}$$

and trigger a sampling event whenever

$$\tau \in [\underline{T}, \overline{T}] \tag{7}$$

the flow and jump dynamics of the states ξ and ℓ_v are unchanged. As a consequence, the time elapsed in between events is no smaller than $\underline{T}$ and no larger than $\overline{T}$.

Example 2 (Global and Robust Control on the Unit Circle)**.** *Consider the problem of globally and robustly asymptotically stabilizing a point-mass evolving on the unit circle to a desired point on the circle. Denoting the unit circle by $\mathbb{S}^1$, the evolution of the position of the point-mass, denoted $\xi = (\xi_1, \xi_2) \in \mathbb{S}^1$, is given by*

$$\dot{\xi} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xi \qquad \xi \in \mathbb{S}^1 , \tag{8}$$

where $u \in \mathbb{R}$ is the control input. Without loss of generality, let

$$\xi^* := (1, 0)$$

be the setpoint of interest. This point corresponds to the intersection between the unit circle and the horizontal positive semi axis.

To design a feedback law that accomplishes the desired goal, a suitable (energy-like) quantity to consider initially is

$$V(\xi) := 1 - \xi_1 \qquad \forall \xi \in \mathbb{S}^1$$

since it vanishes at ξ^* and is positive everywhere else. The static state-feedback law

$$u = \kappa_0(\xi) := -\xi_2 \tag{9}$$

leads to the following continuous change of V, typically denoted as $\dot{V}$:

$$\left\langle \nabla V(\xi), \kappa_0(\xi) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xi \right\rangle = -(1 - \xi_1^2)$$

Along solutions to the closed-loop system resulting from using (9) *in* (8)*, with initial condition ξ such that ξ_1 is not equal to -1, solutions approach the setpoint ξ^*. However, the solution from $(-1, 0)$ remains at its initial condition for all time. Consequently, the feedback law in* (9) *does not induce global asymptotic stability – formally, this feedback ensures that the setpoint ξ^* is almost globally asymptotically stable, with basin of attraction equal to $\mathbb{S}^1 \setminus \{-\xi^*\}$.*

A way to globally asymptotically stabilize the desired setpoint is via a state-feedback law that is discontinuous. To this end, consider the feedback

$$u = -\text{sgn}(\xi_2) =: \kappa_1(\xi) \tag{10}$$

where sgn *is equal to* 1 *if its argument is positive, -1 if it is negative, and arbitrarily in the set $\{-1, 1\}$ when* *its argument is zero. With this feedback, the solutions to the resulting closed-loop system converge to ξ^* from each initial condition in $\mathbb{S}^1$. However, this feedback is not robust to measurement noise. In fact, there exists arbitrarily small measurement noise that, for initial conditions nearby $-\xi^*$, solutions remain nearby $-\xi^*$. Specifically:*

- *For initial conditions $\xi(0)$ with $\xi_2(0) < 0$, this feedback leads to solutions that evolve towards ξ^* counterclockwise;*
- *On the other hand, for initial conditions with $\xi_2(0) > 0$, this feedback leads to solutions that evolve towards ξ^* clockwise.*

As a consequence, from initial conditions arbitrarily close to $-\xi^*$, there exists an arbitrarily small measurement noise signal $t \mapsto m(t)$ that changes sign appropriately so that $-\mathrm{sgn}(\xi_2 + m)$ is always keeping trajectories around $-\xi^*$.

Fortunately, robust and global asymptotic stability can be obtained by using a hysteresis-based logic to combine the state-feedback law in (9) with a feedback law that drives the system away from $-\xi^*$. To this end, a logic variable can be used to select which feedback law should be used, depending on the location of the state ξ, so as to guarantee global asymptotic stability in the presence of arbitrarily small measurement noise. An overview of the logic is as follows: denoting the logic variable by q, and its possible values being $q = 0$ when the law κ_0 in (9) is used and $q = 1$ when the other law, κ_1, is used,

1) *If $q = 0$, then apply κ_0 unless ξ gets to a neighborhood of $-\xi^*$ of size $\epsilon_0 > 0$, in which case q is reset to 1;*
2) *If $q = 1$, then apply κ_1 until ξ leaves a neighborhood of $-\xi^*$ of size $\epsilon_1 > \epsilon_0$, in which case q is reset to 0.*

The discrete update of q at each reset event is given by

$$q^+ = 1 - q \tag{11}$$

In between such resets, q remains constant – hence,

$$\dot{q} = 0 \tag{12}$$

Using this algorithm, the input to (8) is given by

$$u = \kappa_q(\xi)$$

where $q \in \{0, 1\}$ is dynamically updated using the hysteresis-based logic outlined above, leading to a hybrid feedback controller. The interconnection between (8) and this hybrid controller results in a hybrid closed-loop system that combines the differential equations in (8) and (12), capturing the flow. The jumps of the hybrid system are captured by the difference equation (11) modeling the resets of q, the trivial update $\xi^+ = \xi$ for the state ξ (as it should not exhibit changes at reset times), and the conditions in the logic triggering the updates.

Example 3. *Consider the problem of robustly and globally asymptotically stabilizing the compact set $\mathcal{A}_P := \{0, 6\}$ for the continuous-time plant*

$$\dot{z} = u \quad u \in [-1, 1].$$

The main difficulty in achieving this goal is that the set $\mathcal{A}_P$ is not connected. A possible state-feedback law ensuring global asymptotic stability of $\mathcal{A}_P$ is given by

$$\kappa(z) := \begin{cases} -sat(z) & \text{if } z \leq 3 \\ -sat(z-6) & \text{if } z > 3 \end{cases} \tag{13}$$

where sat is the unitary saturation function. With this feedback law, solutions initialized in $(-\infty, 3]$ converge to 0, while solutions initialized in $(3, \infty)$ converge to 6. Moreover, since the closed-loop system behaves linearly around 0 and 6, stability of the set $\mathcal{A}_P$ follows. However, notice that (13) is discontinuous at $z = 3$, leading to lack of robustness to vanishing perturbations. In particular, arbitrarily small measurement noise can trigger arbitrarily fast switching and prevent solutions that start from 3 to approach the set $\mathcal{A}_P$. A possible strategy to overcome this drawback consists of introducing a hybrid controller preventing the feedback law (13) from switching too fast, thereby ensuring global asymptotic stability of $\mathcal{A}_P$ robustly in the presence of measurement noise. This is illustrated in the forthcoming Example 6.

A. Hybrid Equations/Inclusions

For simplicity, we start by formulating a model of a hybrid closed-loop system that does not have inputs. The model capturing the dynamics of sample-and-hold control in Example 1 is one such example. For such *closed* hybrid dynamical system, the state of the system is denoted $x \in \mathbb{R}^n$. The flow of the state is governed by a differential equation of the form

$$\dot{x} = F(x) \tag{14}$$

when

$$x \in C \tag{15}$$

The function F is called the *flow map* and the set C, subset of $\mathbb{R}^n$, is called the *flow set*. Jumps of the state are determined by the difference equation

$$x^+ = G(x) \tag{16}$$

when

$$x \in D \tag{17}$$

The function G is called the *jump map* and the set D, also a subset of $\mathbb{R}^n$, is called the *jump set*.

The sample-and-hold control system in Example 1 can be written as in (14)-(17). The state of the system is given by

$$x = (\xi, \ell_v, \tau)$$

Suppose that the dimension of ξ is n_ξ and that the dimension of ℓ_v is m_ξ. Since the timer is a scalar quantity, the state x takes values from $\mathbb{R}^n$ with $n := n_\xi + m_\xi + 1$. Then, from (4), the differential equation governing x during flows is[1]

$$\dot{x} = (\dot{\xi}, \dot{\ell}_v, \dot{\tau}) = (\widetilde{f}(\xi, \ell_v), 0, 1)$$

[1]Note that here we use the equivalent notation $[x^\top y^\top]^\top = (x, y)$.

from where the function F in (14) is given by

$$F(x) := (\widetilde{f}(\xi, \ell_v), 0, 1)$$

Note that 0 in F has dimension $m_\xi \times 1$. According to (5) and the discussion below it, flows are allowed when $\tau \in [0, T^*]$. Hence, the flow set C is given by

$$C := \{x \in \mathbb{R}^n : \tau \in [0, T^*]\} = \mathbb{R}^{n_\xi} \times \mathbb{R}^{m_\xi} \times [0, T^*]$$

Following (2), the update of x at jumps is given by

$$x^+ = (\xi, \kappa_c(\xi), 0)$$

from where the jump map is given by

$$G(x) := (\xi, \kappa_c(\xi), 0)$$

Such updates should only be allowed when (3) holds. This condition is captured by the jump set

$$D := \{x \in \mathbb{R}^n : \tau = T^*\} = \mathbb{R}^{n_\xi} \times \mathbb{R}^{m_\xi} \times \{T^*\}$$

The combination of equations and constraints in (14)-(17) leads to a hybrid closed-loop system, which is denoted $\mathcal{H}$ and has data (C, F, D, G). This system can be conveniently written as

$$\mathcal{H} \;:\; \begin{cases} x \in C & \dot{x} \;= F(x) \\ x \in D & x^+ = G(x) \end{cases}$$

which we refer to as a *hybrid equation*. At times, F might be set valued, in the sense that given x, $F(x)$ returns more than one value, namely, a set; similarly for G. Set valuedness in these maps allow to conveniently capture uncertainty.

For example, if the continuous-time system in Example 1 includes uncertainty, such as measurement noise w and additive disturbance d, then the flow of ξ under the effect of the sample-and-hold controller would be given by

$$\dot{\xi} = \widetilde{f}(\xi, \ell_v) + d$$

and the jumps of ℓ_v by

$$\ell_v^+ = \kappa_c(\xi + w)$$

When these disturbances are bounded as $|w| \leq \rho_w$ and $|d| \leq \rho_d$, their effect can be characterized by analyzing the set-valued dynamics

$$\dot{\xi} \in \widetilde{f}(\xi, \ell_v) + \rho_d\mathbb{B}$$

and

$$\ell_v^+ \in \kappa_c(\xi + \rho_w\mathbb{B})$$

respectively, where $\widetilde{f}(\xi, \ell_v) + \rho_d\mathbb{B}$ collects all the sums between points in $\widetilde{f}(\xi, \ell_v)$ and those in $\rho_d\mathbb{B}$. In this case, the resulting flow and jump maps are set valued, so $\mathcal{H}$ is written as

$$\mathcal{H} \;:\; \begin{cases} x \in C & \dot{x} \;\in F(x) \\ x \in D & x^+ \in G(x) \end{cases} \tag{18}$$

which we refer to as a *hybrid inclusion*.

B. Hybrid on Hybrid: Hybrid Plant and Hybrid Controller

The hybrid equation/inclusion formulated in the previous section might be hybrid due to the plant or the control algorithm being truly hybrid. For instance, the model of sample-and-hold control outlined in Example 1 is hybrid due to the controller being hybrid. However, there are numerous examples of plants that have hybrid dynamics, such as spiking neurons [16], walking robots [17], networked systems [12], [13], to just list a few. In general, $\mathcal{H}$ could be the result of interconnecting a hybrid plant and a hybrid controller. Each one of these systems can be modeled independently, as a hybrid equation or inclusion with inputs and outputs. Hence, a hybrid control system is partitioned into two main components:

- *a hybrid plant*, denoted $\mathcal{H}_P$, capturing the dynamics of the system to be controlled and, if needed, dynamics of other relevant mechanisms; e.g., signal conditioners, sensors, interfaces to algorithms, etc.; and
- *a hybrid controller*, denoted $\mathcal{H}_K$, capturing the dynamics of the algorithms used for communication and control, as well as dynamics of mechanisms that are needed to define a complete model of the hybrid control system.

Following the model of $\mathcal{H}$ in (18), a hybrid plant $\mathcal{H}_P$ is given by

$$\mathcal{H}_P \;:\; \begin{cases} (z,u) \in C_P & \dot{z} \;\in\; F_P(z,u) \\ (z,u) \in D_P & z^+ \;\in\; G_P(z,u) \\ & y \;=\; h(z) \end{cases} \tag{19}$$

where z is the state, u the input, and y the output. Similar to the data of $\mathcal{H}$ in (18), the data of $\mathcal{H}_P$ is given by (C_P, F_P, D_P, G_P, h). In the same spirit, a hybrid controller $\mathcal{H}_K$ is given by

$$\mathcal{H}_K \;:\; \begin{cases} (v,\eta) \in C_K & \dot{\eta} \;\in\; F_K(v,\eta) \\ (v,\eta) \in D_K & \eta^+ \;\in\; G_K(v,\eta) \\ & \zeta \;=\; \kappa(v,\eta) \end{cases} \tag{20}$$

where η is the state, v the input, and ζ the output. If the controller $\mathcal{H}_K$ involves continuous dynamics only, then the jump set is empty (and the jump map is arbitrary), in which case the controller reduces to

$$\mathcal{H}_K \;:\; \begin{cases} (v,\eta) \in C_K & \dot{\eta} \;\in\; F_K(v,\eta) \\ & \zeta \;=\; \kappa(v,\eta) \end{cases} \tag{21}$$

In a simpler setting, when $\mathcal{H}_K$ is a static control law, for example, a proportional controller, a neural network, or a look-up table, it reduces to

$$\mathcal{H}_K \;:\; \qquad \zeta \;=\; \kappa(v) \tag{22}$$

where $v = z$ or $v = y$.

These constructions will be employed in the upcoming sections introducing hybrid control strategies.

C. Notion of Solution

A notion of solution for a dynamical system defines the properties required for a function of time to qualify as a solution to the system. For instance, for the closed continuous-time system

$$\dot{x} = F(x)$$

a notion of solution characterizes the properties that functions of the form $t \mapsto x(t)$, typically defined for ordinary time t in subsets of $\mathbb{R}_{\geq 0}$, for it to be a solution to the system. If the initial condition is $x_\circ$, then $t \mapsto x(t)$ needs to satisfy $x(0) = x_\circ$. Furthermore, it needs to be smooth enough for

$$\frac{d}{dt}x(t) = F(x(t))$$

to hold over its domain of definition, namely, $\operatorname{dom} x \subset \mathbb{R}_{\geq 0}$. If the state is constrained to the set C, then the function x needs to further satisfy $x(t) \in C$, at least for t's in the interior of $\operatorname{dom} x$.

Similarly, for the (open) continuous-time system

$$\dot{x} = F(x, u)$$

a solution is given by a pair

$$t \mapsto (x(t), u(t))$$

such that $x(0) = x_\circ$, the functions x and u are defined over the same domain, and satisfy

$$\frac{d}{dt}x(t) = F(x(t), u(t))$$

over $\operatorname{dom} x = \operatorname{dom} u = \operatorname{dom}(x, u)$, where the function u should be regular enough for the integral in

$$x(t) = x_\circ + \int_0^t F(x(s), u(s))ds$$

to be well defined for all $t \in \operatorname{dom}(x, u)$. Due to having state variables that may evolve continuously and discretely, a notion of solution to a hybrid system needs to allow for intervals of flow and jumps. We start by introducing this notion for the case of closed hybrid dynamical systems $\mathcal{H}$ given as in (18), for which we go back to the sample-and-hold control system in Example 1.

The timer state τ in the sample-and-hold control system triggers events when $\tau = T^*$. When the initial value of the timer is zero, then a solution to this system evolves continuously until it reaches T^*, upon which it gets reset to zero. This cycle repeats indefinitely leading to infinitely many events over time. A convenient way to parameterize the evolution of the timer state is by using two time parameters:

- A continuous-valued parameter $t \in \mathbb{R}_{\geq 0}$ that counts time, and
- A discrete-valued parameter $j \in \mathbb{N}$ that counts the number of jumps in the solution so as to parameterize the jumps.

The parameter t plays the role of ordinary time t. The parameter j plays a role similar to that of the discrete parameter – usually denoted as k – used in discrete-time systems. However, no discretization is involved in the notion of time used in this paper. With this hybrid time notion, the timer starting from zero is defined over intervals of flow

$$[0, T^*], \quad [T^*, 2T^*], \quad \ldots \quad , \quad [jT^*, (j+1)T^*], \quad \ldots$$

where $j \in \mathbb{N}$. As each interval is associated to a different value of the jump counter j, these intervals can be indexed by j. Then, the domain of definition of the solution would be

$$\operatorname{dom} x = \bigcup_{j \in \mathbb{N}} ([jT^*, (j+1)T^*] \times \{j\})$$

One advantage of using closed intervals indexed by the jump counter is that the value of the timer right before and right after the event are included in the function defining the solution. Specifically, the timer component of the solution would then be parameterized by t and j, as

$$\tau(t, j) = t - jT^* \qquad \forall (t, j) \in \operatorname{dom} x$$

Note that this function is such that, for each $j \in \mathbb{N}$,

$$t \mapsto \tau(t, j)$$

is continuously differentiable and satisfies

$$\frac{d}{dt}\tau(t, j) = 1, \quad \tau(t, j) \in [0, T^*]$$

over the interior of the intervals of flow, namely, $(jT^*, (j+1)T^*)$. Moreover, for each jump time, namely, for each $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $t = (j+1)T^*$, it follows that

$$\tau(t, j) = T^*$$

and

$$\tau(t, j+1) = 0$$

Hence, the function τ defined above qualifies as the timer component of a solution to the hybrid dynamical system modeling the sample-and-hold control system defined in Example 1.

In general, the combination of the parameters t and j gives rise to the notion of *hybrid time* and *hybrid time domain*. Following the discussion for the example above, a solution to $\mathcal{H}$ in (18) is defined on a hybrid time domain. A hybrid time domain is the union of intervals of the form

$$[t_j, t_{j+1}] \times \{j\}$$

possibly with the last interval being open to the right, for some nondecreasing sequence t_j. To define a solution to (18), functions

$$x : \operatorname{dom} x \to \mathbb{R}^n$$

with $\operatorname{dom} x$ being a hybrid time domain and, for each $j \in \mathbb{N}$,

$$t \mapsto x(t, j)$$

being locally absolutely continuous, are considered. Such functions are called *hybrid arcs*.

A hybrid arc x defines a solution to $\mathcal{H} = (C, F, D, G)$ if

(S0) $x(0, 0) \in \overline{C}$ or $x(0, 0) \in D$;

(S1) For each $j \in \mathbb{N}$ such that $I_x^j := \{t \in \mathbb{R}_{\geq 0} : (t, j) \in \operatorname{dom} x\}$ has a nonempty interior

$\mathrm{int}(I_x^j)$, x satisfies

$$x(t,j) \in C \qquad \text{for all } t \in \mathrm{int}(I_x^j)$$

and

$$\frac{d}{dt}x(t,j) \in F(x(t,j)) \qquad \text{for almost all } t \in I_x^j$$

(S2) For each $(t,j) \in \mathrm{dom}\, x$ such that $(t,j+1) \in \mathrm{dom}\, x$, x satisfies

$$x(t,j) \in D$$

and

$$x(t,j+1) \in G(x(t,j))$$

A solution x to $\mathcal{H}$ is said to be nontrivial if $\mathrm{dom}\, x$ contains at least two points; complete if $\mathrm{dom}\, x$ is unbounded; bounded if the set $\mathrm{rge}\, x := \{x(t,j) : (t,j) \in \mathrm{dom}\, x\}$ is bounded; precompact if complete and $\mathrm{rge}\, z$ is bounded; Zeno if it is complete and the projection of $\mathrm{dom}\, x$ onto $\mathbb{R}_{\geq 0}$ is bounded; discrete if nontrivial and $\mathrm{dom}\, x \subset \{0\} \times \mathbb{N}$; continuous if nontrivial and $\mathrm{dom}\, x \subset \mathbb{R}_{\geq 0} \times \{0\}$; maximal if there does not exist another solution x' such that x is a truncation of x' to some proper subset of $\mathrm{dom}\, x'$.

The following example illustrates the definition of solution above.

Example 4 (Resettable Timer)**.** *The dynamics of the timer state τ in the sample-and-hold control system in Example 1 do not depend on the other state variables. Hence, its dynamics are captured by the hybrid system*

$$\begin{cases} \tau \in C := [0, T^*] & \dot{\tau} \ = F(\tau) := 1 \\ \tau \in \ D := \{T^*\} & \tau^+ = G(\tau) := 0 \end{cases}$$

Given an initial condition $\tau_\circ \in C$, the function

$$\tau(t,j) := t - jT^* + \tau_\circ \qquad \forall (t,j) \in \mathrm{dom}\, \tau$$

is a hybrid arc, where $\mathrm{dom}\, \tau$ is given by

$$([0, T^* - \tau_\circ] \times \{0\}) \cup \bigcup_{j \in \mathbb{N} \setminus \{0\}} ([jT^* - \tau_\circ, (j+1)T^* - \tau_\circ] \times \{j\})$$

Moreover, it is also a solution to it. In particular, item (S0) holds since $\tau_\circ \in [0, T^*]$. Moreover, for $j = 0$, $I_\tau^0 = [0, T^* - \tau_\circ]$. If $\tau_\circ < T^*$, then I_τ^0 has a nonempty interior,

$$\tau(t,0) \in C \qquad \forall t \in I_\tau^0$$

and

$$\frac{d}{dt}\tau(t,j) = 1 \qquad \text{for all } t \in I_\tau^0$$

Hence, (S1) holds for $j = 0$. Proceeding similarly, (S1) holds for each $j \in \mathbb{N} \setminus \{0\}$. Finally, (S2) holds at each jump. In fact, at each jump time in $\mathrm{dom}\, \tau$, namely, at each $(t,j) \in \mathrm{dom}\, \tau$ with $t_{j+1} = (j+1)T^* - \tau_\circ$, $j \in \mathbb{N}$, it follows that $(t, j+1) \in \mathrm{dom}\, \tau$,

$$\tau(t_{j+1}, j) = T^* \in D$$

and

$$\tau(t_{j+1}, j+1) = 0$$

Furthermore, the solution is nontrivial *since its domain has at least two points,* maximal *since it cannot be further extended, and* complete *due to its domain being unbounded. See [18], [6] for more details.*

Unlike continuous-time systems or discrete-time systems, existence of solutions to hybrid dynamical systems requires that the data of $\mathcal{H}$ plays well nicely to allow either flow or jumps. For starters, since jumps are always possible from points in the jump set D, there exist at least one solution that jumps from each point in D. From points in the flow set C that are not in the jump set D, namely, points in the set $C \setminus D$, flow is possible if the flow map generates solutions that stay in C – see [18, Proposition 2.10]. Under the following mild assumptions, known as the *hybrid basic conditions*, flow from $C \setminus D$ can be guaranteed using the tangent cone to the set C, denoted T_C.

The hybrid closed-loop system $\mathcal{H} = (C, F, D, G)$ in (18) satisfies the hybrid basic conditions if

(A1) C and D are closed subsets of $\mathbb{R}^n$;

(A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C, $C \subset \mathrm{dom}\, F$, and $F(x)$ is convex for each $x \in C$;

(A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, and $D \subset \mathrm{dom}\, G$.

Note that a map F is outer semicontinuous if its graph is closed and is locally bounded if, for each compact set $K \subset \mathrm{dom}\, F$, there exists a compact set K' such that $F(K) \subset K'$. For more details, see [18, Chapter 5].

The following existence result follows – see [18, Proposition 6.10].

Let $\mathcal{H} = (C, F, D, G)$ as in (18) satisfy the hybrid basic conditions. Take an arbitrary $x_\circ \in C \cup D$. If $x_\circ \in D$ or

(VC) there exists a neighborhood U of $x_\circ$ such that for every $x \in U \cap C$,

$$F(x) \cap T_C(x) \neq \emptyset,$$

then there exists a nontrivial solution x to $\mathcal{H}$ with $x(0,0) = x_\circ$. If (VC) holds for every point in $C \setminus D$, then there exists a nontrivial solution to $\mathcal{H}$ from every initial point in $C \cup D$, and every maximal solution x satisfies exactly one of the following conditions:

(a) x is complete;

(b) $\mathrm{dom}\, x$ is bounded and the interval I^J, where $J = \sup\{j : (t,j) \in \mathrm{dom}\, x\}$, has nonempty interior and $t \mapsto x(t,J)$ is a maximal solution to $\dot{z} \in F(z)$, in fact $\lim_{t \to T} |x(t,J)| = \infty$, where $T = \sup\{t : (t,j) \in \mathrm{dom}\, x\}$;

(c) $x(T, J) \notin C \cup D$, where $(T, J) = \sup \operatorname{dom} x$.

Furthermore, if $G(D) \subset C \cup D$, then (c) above does not occur.

D. Notions and Analysis Tools

Tools for the analysis of asymptotic stability for hybrid dynamical systems relying on Lyapunov methods are available in the literature [18]. Asymptotic stability for the (closed) hybrid inclusion in (18) is most useful when defined relative to a set, rather than just a point. For instance, suppose that the goal of the sample-and-hold controller in Example 1 is to guarantee stable convergence of ξ to a setpoint ξ^*. Since the hybrid system model of the resulting closed-loop system given in Section III-A includes the memory state storing the values of the input and the timer state triggering the events, the values to which those components should converge to should be specified. A particular choice for ℓ_v is for it to converge to a constant denoted ℓ_v^* that depends on the setpoint ξ^* – for instance, when the setpoint ξ^* is zero and the feedback law κ_c is linear, then ℓ_v should converge to zero. Regarding the timer state, it should simply remain in the allowed range $[0, T^*]$. Then, the set of interest for this system is

$$\mathcal{A} := \{\xi^*\} \times \{\ell_v^*\} \times [0, T^*]$$

Note that this set is a subset of the state space of the hybrid closed-loop system provided in Section III-A. For the stabilization problem in Example 2, the set to stabilize is given by the point $\{(1, 0)\} \times \{0, 1\}$ in the space $\mathbb{S}^1 \times \{0, 1\}$ resulting from the state of the closed loop resulting from using the hybrid controller therein featuring a logic variable q taking values from $\{0, 1\}$.

Denoting by $\mathcal{A}$ the set of interest for the analysis of $\mathcal{H}$ as in (18), asymptotic stability and its robustness are key properties of interest. Asymptotic stability is defined as the property of solutions that start close to the set, stay close to the set – called *stability* – and the property that solutions converge to the set – called *attractivity*. These notions are defined in a way that completeness of maximal solutions is not required. In light of this generality, attractivity is referred to as *pre-attractivity*.

Specifically, the set $\mathcal{A}$ is *stable* for $\mathcal{H}$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that each solution x to $\mathcal{H}$ with

$$|x(0, 0)|_{\mathcal{A}} \leq \delta$$

satisfies

$$|x(t, j)|_{\mathcal{A}} \leq \varepsilon \qquad \forall (t, j) \in \operatorname{dom} x$$

The set $\mathcal{A}$ is *pre-attractive* for $\mathcal{H}$ if there exists $\mu > 0$ such that every solution x to $\mathcal{H}$ with

$$|x(0, 0)|_{\mathcal{A}} \leq \mu$$

is such that $(t, j) \mapsto |x(t, j)|_{\mathcal{A}}$ is bounded and if x is complete then

$$\lim_{(t,j) \in \operatorname{dom} x,\ t+j \to \infty} |x(t, j)|_{\mathcal{A}} = 0$$

If μ can be selected arbitrarily large, we say that $\mathcal{A}$ is *globally pre-attractive* for $\mathcal{H}$. Then, when $\mathcal{A}$ is both stable and pre-attractive for $\mathcal{H}$, we say that $\mathcal{A}$ is *pre-asymptotically stable* for $\mathcal{H}$. In the case that every maximal solution is complete, pre-attractivity can simply be called *attractivity* – in such case, a pre-asymptotically stable set $\mathcal{A}$ is said to be *asymptotically stable*.

It can be shown that, when $\mathcal{H}$ satisfies the hybrid basic conditions and $\mathcal{A}$ is compact, pre-asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$ can be characterized by the following bound: there exists a class-$\mathcal{KL}$ function β such that, for each solution x to $\mathcal{H}$,

$$|x(t, j)|_{\mathcal{A}} \leq \beta(x(0, 0), t + j) \qquad \forall (t, j) \in \operatorname{dom} x \tag{23}$$

Remarkably, under the said assumptions, this property is preserved, semiglobally and practically, under sufficiently small perturbations to $\mathcal{H}$. In fact, the following property, stated loosely, holds: for each compact set of initial conditions K and each $\varepsilon > 0$, for perturbed solutions x_δ to $\mathcal{H}$ from K under the effect of small enough perturbations satisfies [18]

$$|x_\delta(t, j)|_{\mathcal{A}} \leq \beta(x_\delta(0, 0), t + j) + \varepsilon \qquad \forall (t, j) \in \operatorname{dom} x_\delta \tag{24}$$

Perturbations may affect the measurements, the feedback law, or the models of the continuous and discrete dynamics. This robust pre-asymptotic stability property implies that perturbed solutions converge to an ε-neighborhood of $\mathcal{A}$ (when complete).

Tools for the certification of these properties are available in the literature. In particular, methods for establishing asymptotic stability and robustness are well documented in [18] – see Chapters 3, 7, and 8 and the references therein.

Remark 1 (About Other Hybrid Systems Frameworks)**.** *The framework introduced is general enough to capture the differential automata model introduced in [19], the hybrid automata considered in [20], [21], and the hybrid system models in [1], [22], [23], which explicitly divide the state into a continuous-valued state component and a discrete-valued component. In particular, similar to the control strategy in Example 2, the discrete state component describes the mode of operation of the system (e.g., "on or off" or "high or low"). The framework introduced in this paper can also model impulsive systems [24] and switched systems [25] for specific switching signals.*

IV. Supervisory and Uniting Control

A. Motivation

In some applications, it is very difficult (or even impossible) to design a single controller to accomplish a desired task. This is typically the case when the dynamics of the plant are too complex for a single controller to handle it globally [26] or when good performance for multiple operating modes is hard to ensure [27]. Another relevant scenario in which combining multiple controllers is paramount is when topological constraints rule out the existence of a smooth global stabilizer [28], [29]. This last issue is made more concrete via the example given next, which appeared in [6, Section 1.2.3].

Example 5 (Revisiting Example 2). *Consider the system evolving in the unit circle given in* (8)*. This system models a particle traveling on the unit circle centered at the origin of the plane; z denotes the position of the particle on the plane. The sign of the control input u determines the direction of motion of the particle. In particular, when $u > 0$ the particle moves counterclockwise and it moves clockwise otherwise. Suppose one wants to design a feedback law rendering the setpoint $z^\star := (1, 0) \in \mathbb{S}^1$ globally asymptotically stable. A natural feedback law that one might consider to enforce this property is as follows*

$$\kappa(z) = -z_2.$$

The main rational behind this feedback law is that when $z_2 > 0$, $u < 0$ (the particle moves clockwise) and when $z_2 < 0$, $u > 0$ (the particle moves counterclockwise). It can be easily observed that by using the above feedback controller, solutions to the closed-loop system starting away from $-z^\star$ converges to $z^\star$ and solutions starting close to $z^\star$ stay close to it. However, solutions starting from $-z^\star$ are stuck and never approach $z^\star$. Unfortunately, it turns out that asymptotic convergence to $z^\star$ cannot be extended to the entire set $\mathbb{S}^1$ by preserving continuity of the state feedback law. This is a major issue since discontinuous feedback laws are overly sensitive to small perturbations such as measurement noise. A possible approach to overcome this limitation is to use two different control laws depending on the value of the state z. In particular, in a connected subset of $\mathbb{S}^1$ not including $-z^\star$ but including $z^\star$ in its interior, say $\mathcal{V}$, one selects $\kappa(z) = -z_2$. Away from $\mathcal{V}$, κ can be any law driving solutions to $\mathcal{V}$. These two controllers can be patched together via a supervisory algorithm. In Example 7, we show how the tools introduced in this section can be adopted to build a hybrid controller ensuring global asymptotic stability of the point $z^\star$, while avoiding the use of discontinuous feedback laws.

An effective approach to coordinate multiple controllers consists of relying on a supervisory control paradigm. In this setting, a specific object, called the *supervisor*, is employed to make a decision about which controller needs to be adopted. In particular, the supervisor assigns each controller to a specific region of the state space and decides when to operate a switch. Next, we make this architecture more precise and show how it can be modeled as a hybrid dynamical system.

B. Modeling

For simplicity, we consider a continuous-time plant defined as

$$\mathcal{H}_P\colon \quad (z, u) \in C_P = \mathbb{R}^{n_P} \times \mathbb{R}^{m_P} \qquad \dot{z} \;\in\; F_P(z, u) \tag{25}$$

and assume that the plant state z can be measured. In this setting, we suppose that there exists a family of continuous-time controllers

$$\mathcal{H}_{K,q} \qquad q \in Q := \{0, 1, \ldots, q_{\max}\}$$

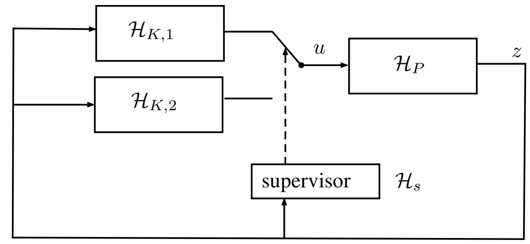

Fig. 2. A schematic representation of a supervisory control-based closed-loop system.

with dynamics

$$\mathcal{H}_{K,q} \;:\; \begin{cases} (\xi, v) \in C_{K,q} & \dot{\xi} \;\in\; F_{K,q}(\xi, v) \\ & \zeta \;=\; \kappa_q(\xi, v). \end{cases} \tag{26}$$

The case when the controllers $\mathcal{H}_{K,q}$ are genuinely hybrid can be dealt similarly, yet with a larger notational burden; see [6, Chapter 8]. The selection of the controller $\mathcal{H}_{K,q}$ from the given family is determined by the following supervisory algorithm:

$$\mathcal{H}_s \;\begin{cases} (q, v_s) \in C_s & \dot{q} \;=\; 0 \\ (q, v_s) \in D_s & q^+ \;\in\; G_s(q, v_s). \end{cases} \tag{27}$$

The state q is a logic variable that defines which controller $\mathcal{H}_{K,q}$ is active. In particular, when $q = q^\star \in Q$ and $(q^\star, v_s) \in C_s$, flow is possible and $\mathcal{H}_{K,q^\star}$ controls $\mathcal{H}_P$. When $q = q^\star \in Q$ and $(q^\star, v_s) \in D_s$, a jump occurs and q is reset to a point in $G_s(q^\star, v_s)$, which determines the new controller $\mathcal{H}_{K,q}$ to be used. A schematic representation of the proposed supervisory control architecture is depicted in Fig. 2. The definition of C_s, D_s, and G_s should guarantee that $(G_s(q, v_s) \times \{v_s\}) \subset C_s \cup D_s$ for each $(q, v_s) \in D_s$. Note that the resulting controller is hybrid and can be written as $\mathcal{H}_K$ in (20) with state $\eta = (\xi, q)$.

An example of application of the supervisory control paradigm is given next.

Example 6 (Example 3 Revisited). *We now show how a supervisory controller can be employed to come up with a robust feedback law for the stabilization problem considered in Example 3. To this end, the plant in Example 3 is modeled as the hybrid plant $\mathcal{H}_P$ defined by*

$$F_P(z, u) = u, \quad C_P = \mathbb{R} \times [-1, 1]$$

$D_P = \emptyset$ and G_P can be selected arbitrarily since no jumps in the state z occur. As pointed out in Example 3, lack of robustness is related to (13) *being discontinuous at $z = 3$. A possible strategy to overcome this drawback consists of introducing a supervisor to handle the inherent "switching" mechanism implemented* (13)*. In particular, by suitably designing the supervisor, one can prevent u from switching too often. In this specific case, the controllers $\mathcal{H}_{K,q}$ are static controllers and $Q = \{0, 1\}$. More in particular*

$$\kappa_0(z) = -sat(z), \qquad \kappa_1(z) = -sat(z - 6).$$

The logic to be implemented by the supervisory algorithm is *as follows*

- *If $q = 0$ (κ_0 is active) and $z \leq 4$, then do not switch*
- *If $q = 0$ (κ_0 is active) and $z > 4$, then switch to κ_1*
- *If $q = 1$ (κ_1 is active) and $z \geq 2$, then do not switch*
- *If $q = 1$ (κ_1 is active) and $z < 2$, then switch to κ_0.*

This logic leads to the supervisor defined by the following data

$$\begin{aligned} &G_s(q, v_s) = 1 - q \\ &C_s = (\{1\} \times (-\infty, 4]) \cup (\{2\} \times [2, \infty)) \\ &D_s = (\{1\} \times (4, \infty)) \cup (\{2\} \times (-\infty, 2)). \end{aligned}$$

It can be verified that the proposed supervisory controller ensures that the set $\mathcal{A} := \mathcal{A}_P \times Q$ is GAS for the closed-loop system. Moreover, it turns out that maximal solutions to the closed-loop system converge to a neighborhood of $\mathcal{A}$ in the presence of small perturbations.

The interconnection of supervisor $\mathcal{H}_s$ with controllers $\mathcal{H}_{K,q}$, $q \in Q$, can be thought as a hybrid controller $\mathcal{H}_K = (C_K, F_K, D_K, G_K, \kappa)$ with state $\eta := (q, \xi) \in Q \times \mathbb{R}^{n_K}$, input $v \in \mathbb{R}$, and data

$$\begin{aligned} C_K &= C_s \\ D_K &= D_s \\ F_K(\eta, v) &= \begin{bmatrix} 0 \\ F_{K,q}(\eta, v) \end{bmatrix} \\ G_K(\eta, v) &= (G_s(q, \xi, v), \xi) \\ \zeta = \kappa(\eta, v) &:= \kappa_q(\xi, v). \end{aligned} \tag{28}$$

The interconnection between the plant $\mathcal{H}_P$ and the controller $\mathcal{H}_K$ is defined via the following relationships

$$v = z, \quad u = \zeta.$$

This results into the hybrid closed-loop system $\mathcal{H} = (C, F, D, G)$ with state $x := (z, \eta) \in \mathbb{R}^{n_P} \times Q \times \mathbb{R}^{n_K}$, where

$$\begin{aligned} C &:= \{x \in \mathbb{R}^{n_P} \times Q \times \mathbb{R}^{n_K} : (\eta, z) \in C_K\} \\ F(x) &:= (F_P(z, \kappa_q(\eta, z)), F_K(\eta, z)) \\ G(x) &:= (z, G_K(\eta, z)) \\ D &:= \{x \in \mathbb{R}^{n_P} \times Q \times \mathbb{R}^{n_K} : (\eta, z) \in D_K\}. \end{aligned}$$

We are interested in the following control problem.

Problem 1. *Given a compact set $\mathcal{A}_P \subset \mathbb{R}^{n_P} \times \mathbb{R}^{n_K}$ and a closed set $X \subset \mathbb{R}^{n_P} \times \mathbb{R}^{n_K}$, design C_s, D_s, and G_s such that $C \cup D = X \times Q$ and the set $\mathcal{A} := \mathcal{A}_P \times Q$ is globally asymptotically stable (GAS) for the hybrid closed-loop system $\mathcal{H}$.*

The set X can be interpreted as a region of operation assigned for the (z, ξ)-component of the closed-loop system state. The set $\mathcal{A}_P$ may represent a desired setpoint or a region defined in the (z, ξ)-coordinates that one wants to asymptotically stabilize; this can be seen as the main control task.

In the next subsection, we show how, under some reasonable assumptions on the controller and plant data, a supervisory control algorithm solving Problem 1 can be designed.

C. Construction of the Supervisory Algorithm

To solve Problem 1, we consider the following assumption [9], [5].

Assumption 1. *There exists a collection of closed sets $\{\Psi_q\}_{q \in Q}$, where for all $q \in Q$, $\Psi_q \subset C_{K,q}$, such that:*

1) $\bigcup_{q \in Q} \Psi_q = X$
2) *For all $q \in Q$, let*

$$\mathcal{H}_q : \quad (z, \xi) \in C_{K,q} \quad \begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} \in \begin{bmatrix} F_P(z, \kappa_q(\xi, z)) \\ F_{K,q}(\xi, z) \end{bmatrix}$$

the following properties hold

(a) *The set $\mathcal{A}_P$ is GpAS for $\mathcal{H}_q$*

(b) *Each maximal solution to $\mathcal{H}_q$ is complete or ends in*

$$H_q := \Phi_q \cup \overline{X \setminus (C_{K,q} \cup \Phi_q)}$$

where, for all $q \in Q$,

$$\Phi_q := \bigcup_{i \in Q, i > q} \Psi_i$$

(c) *No maximal solution to $\mathcal{H}_q$ starting in Ψ_q reaches*

$$\overline{X \setminus (C_{K,q} \cup \Phi_q)} \setminus \mathcal{A}_P.$$

The set

$$\Upsilon_q := \overline{X \setminus (C_{K,q} \cup \Phi_q)} \tag{29}$$

appearing in items 2b and 2c represents the set from which a controller with index smaller than q can be activated.

Item 1 is instrumental to guarantee that the sets C_s and D_s can be designed such that $C \cup D = X$. In particular, item 1 assures that for any $(z, \xi) \in X$, there exists $q^\star \in Q$ such that $(z, \xi) \in \Psi_{q^\star} \subset C_{K,q^\star}$. Namely, from any initial condition in X, there exists at least a controller in the family (26) that can be activated. Item 2a ensures that complete maximal solutions with constant q approach the set $\mathcal{A}_P$. Notice that this is a milder property than the global asymptotic stability of $\mathcal{A}_P$ required in Problem 1. Indeed, item 2a only ensures that any maximal solution is bounded and that complete solutions approach $\mathcal{A}_P$. Item 2b guarantees that each maximal solution either converges to $\mathcal{A}_P$ or ends in a set where a controller with an index different than q can be activated. This property enables the hybrid supervisor to ensure completeness of maximal solutions of the closed-loop system. Item 2c combined with item 2b ensures that maximal solutions end in a set where a controller with a larger index can be activated. This prevents the supervisor to switch back and forth from different controllers. Moreover, for $q = q_{\max}$, item 2 implies that maximal solutions converge to $\mathcal{A}_P$ and that the set in item 2c is empty for $q = 0$.

Based on Assumption 1, we select the following data for the supervisory algorithm

$$C_s := \bigcup_{q\in Q} \left(\{q\} \times C_{K,q}\right), \quad D_s := \bigcup_{q\in Q} \left(\{q\} \times H_q\right),$$
$$G_s(q, v_s) := \begin{cases} \{i \in Q \colon v_s \in \Psi_i\} & \text{if } v_s \in \Phi_q \cup \Upsilon_q \\ \{i \in Q \colon i > q \colon v_s \in \Psi_i\} & \text{if } v_s \in H_q \setminus (\Phi_q \cup \Upsilon_q) \end{cases} \tag{30}$$

where Υ_q is defined in (29). The rational behind the proposed construction is as follows. When $(z, \eta) \in C_{K,q^\star}$, for some $q^\star \in Q$, then controller $\mathcal{H}_{K,q^\star}$ controls $\mathcal{H}_P$ and flow is possible. When $(z, \eta) \in H_{q^\star}$, jumping is possible. In particular, if $(z, \eta) \in \Phi_{q^\star} \cup \Upsilon_{q^\star}$, then q jumps to a value $i \in Q$ such that $(z, \eta) \in \Psi_i$. This ensures that after the jump, only flowing is possible. Moreover, based on Assumption 1 items 2.b and 2.c, the resulting solution flows until it reaches the set Ψ_i. Differently, if $(z, \eta) \in H_q \setminus (\Phi_q \cup \Upsilon_q)$, then q may jump to a value $i \in Q$, with $i > q$, such that $(z, \eta) \in \Psi_i$. This, again, ensures that after the jump, only flowing is possible and the above rationals based on Assumption 1 follow. Assumption 1 leads to the following result.

Let Assumption 1 hold. Let the data of the supervisor $\mathcal{H}_s$ (27) be defined as in (30). Then, the set

$$\mathcal{A} := \mathcal{A}_P \times Q$$

is GAS for the hybrid closed-loop system $\mathcal{H}$.

Example 7 (Example 5 Revisited)**.** *We now show how the proposed supervisory control paradigm can be used to deal with the global stabilization problem dealt in Example 5. In particular, based on the rationale outlined in Example 5, we consider the following family of static feedback controllers*

$$\begin{aligned} z \in C_{K,0} &= \left\{z \in \mathbb{S}^1 \colon z_1 \le c_1\right\}, & \kappa_0(z) &= -z_1 \\ z \in C_{K,1} &= \left\{z \in \mathbb{S}^1 \colon z_1 \ge c_0\right\}, & \kappa_1(z) &= -z_2 \end{aligned} \tag{31}$$

where $c_0 \in (-1, 0)$ and $c_1 \in (c_0, 0)$. In this case, $X = \mathbb{S}^1$. In particular, the "almost" globally stabilizing feedback law κ_1 introduced in Example 5 is used when in $C_{K,1}$, while when in $C_{K,0}$, κ_0 is used. Next we illustrate how the proposed selection of the family of controllers (31) *enables to fulfill all the items in Assumption 1. This in turn ensures that the set $\mathcal{A} := \{(1, 0)\} \times \{0, 1\}$ is GAS for the closed-loop system.*

Item 1 is fulfilled with $\Psi_0 := C_{K,0}$ and $\Psi_1 := C_{K,1}$. Concerning item 2a, notice that having restricted κ_1 to operate only within the set $C_{S,1}$ it ensures GpAS of the set $\mathcal{A}_P := \{(1, 0)\}$ for $\mathcal{H}_1$. On the other hand, maximal solutions to $\mathcal{H}_0$ are not complete (and bounded), which in turn ensures that the set $\mathcal{A}_P$ is GpAS for $\mathcal{H}_0$. Thus, item 2a is fulfilled. Now observe that maximal solutions to $\mathcal{H}_0$ converge in finite hybrid-time in the set Ψ_1. Furthermore, maximal solutions to $\mathcal{H}_1$ are complete and converge to $\mathcal{A}_P$. This ensures the satisfaction of item 2b. Finally, notice that item 2c holds since for $q = 0$ the set in item 2c is empty and solutions to $\mathcal{H}_1$ starting in Ψ_1 converge to $\mathcal{A}_P$. A simulation showing the effectiveness of the proposed control strategy in stabilizing the set $\mathcal{A}$ is shown in Fig. 3. In this simulation,

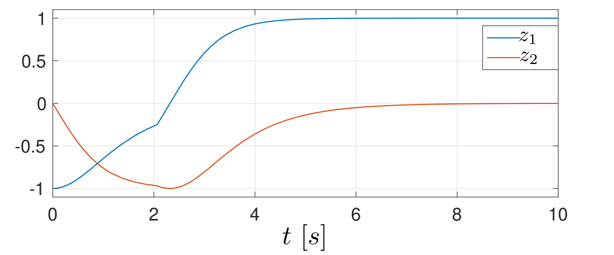

Fig. 3. Simulation results for the system in Example 7 with initial condition $z(0, 0) = (-1, 0)$ and $q(0, 0) = 1$. The solution converges to $\mathcal{A}$ with two jumps: one at the initial condition and another when z is on the boundary of $C_{K,0}$.

$c_0 = -0.75$ *and* $c_1 = -0.25$.

D. Application to the Global/Local Uniting Problem

In some applications, the design of a single controller ensuring global asymptotic stability and a local level of performance is hard to perform. Indeed, although several tools to design (globally) asymptotically stabilizing feedback controllers exist, the design of global controllers performing "optimally" is less obvious. However, the design of controllers ensuring local asymptotic stability and a specified level of performance locally can be addressed by relying on linearized models. In these situations, an effective solution consists of uniting the local controller with a global controller. This problem, commonly called "the uniting problem" has received the attention of researchers over the last twenty years. Fundamental results about the uniting problem of continuous-time controllers can be found, e.g., in [30], [31]. The application of this technique to a practically relevant problem has been developed in [27]. The problem of uniting two output feedback hybrid controllers has been explored in [32]. A complete overview about the uniting problem can be found in [6, Chapter 4].

The task of uniting a global and a local controller can be achieved (robustly) by suitably designing a supervisor that selects the most appropriate controller depending on the value of the plant state. This is illustrated next. To this end, we consider again the continuous-time plant (25). For simplicity, we suppose the two controllers are static state feedback laws, the case of dynamic feedback controllers can be worked out similarly. In particular, we assume that there exist two static state feedback controllers $\kappa_1, \kappa_2 \colon \mathbb{R}^{n_P} \to \mathbb{R}^{m_P}$ such that:

- κ_1 locally stabilizes the origin of $\mathcal{H}_P$ and produces efficient transient responses;
- κ_2 globally stabilizes the origin of $\mathcal{H}_P$ but with unsatisfactory performance.

Our goal is to globally stabilize the origin of $\mathcal{H}_P$ while using κ_2 far from the origin and κ_1 close to the origin. In particular, suppose that κ_1 is used when $z \in C_{S,1}$ and that κ_2 is used when $z \in \overline{C_{S,2}}$, where $C_{S,1}$ and $C_{S,2}$ are selected later. Let $D_{S,1} := \overline{\mathbb{R}^{n_P} \setminus C_{S,1}}$ and $D_{S,2} := \overline{\mathbb{R}^{n_P} \setminus C_{S,2}}$; see Fig. 4 for a pictorial representation of the sets $C_{S,q}, D_{S,q}$. Then,

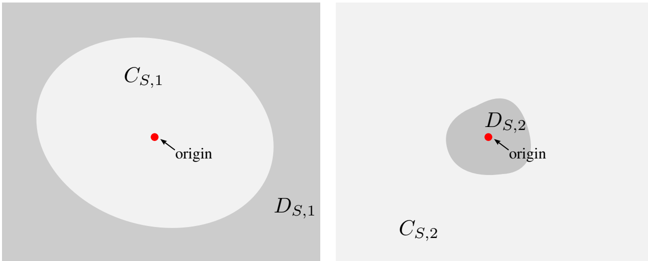

Fig. 4. Sets for the uniting hybrid controller.

the switching policy to be implemented by the supervisory algorithm is as follows.

- If κ_1 is active and $z \in C_{S,1}$ do not switch
- If κ_1 is active and $z \in D_{S,1}$ switch to κ_2
- If κ_2 is active and $z \in C_{S,2}$ do not switch
- If κ_2 is active and $z \in D_{S,2}$ switch to κ_1.

The proposed feedback law can be thought as a hybrid controller $\mathcal{H}_K = (C_K, F_K, D_K, G_K, \kappa)$ with state $q \in Q := \{1,2\}$, input $v \in \mathbb{R}^{n_P}$, and data

$$\begin{aligned} C_K &:= (C_{S,1} \times \{1\}) \cup (C_{S,2} \times \{2\}) \\ D_K &:= (D_{S,1} \times \{1\}) \cup (D_{S,2} \times \{2\}) \\ F_K(q,v) &:= 0 \\ G_K(q,v) &:= 3 - q \\ \kappa(q,v) &:= \kappa_q(v). \end{aligned} \tag{32}$$

The interconnection of (32) with (25) is obtained by selecting $v = z$ and $u = \kappa$ and leads to the hybrid closed-loop system $\mathcal{H} = (C, F, D, G)$ with state $x := (z, q) \in \mathbb{R}^{n_P} \times \{1,2\}$ and data

$$\begin{aligned} C &:= C_K \\ D &:= D_K \\ F(x) &:= (F_P(z, \kappa_q(z)), F_K(q)) \\ G(x) &:= (z, G_K(q)). \end{aligned} \tag{33}$$

For the hybrid controller to work as intended, there needs to be a relationship between $C_{S,1}$ and $D_{S,2}$. In particular, solutions to

$$\dot{z} \in F_P(z, \kappa_1(z)) \tag{34}$$

starting in $D_{S,2}$ need to remain in a closed set that is contained in the interior of $C_{S,1}$. Moreover, any solution to (34) starting in $C_{S,1}$ and remaining therein need to converge to the origin. Since κ_1 locally stabilizes the origin, these two properties can be guaranteed by selecting $C_{S,1}$ as a closed neighborhood of the origin contained in the basin of attraction of κ_1 and $D_{S,2}$ as a sufficiently small neighborhood of the origin strictly contained in $C_{S,1}$.

Next we provide sufficient conditions for the compact set

$$\mathcal{A} = \{0\} \times \{1,2\} \tag{35}$$

to be GAS for the hybrid closed-loop system $\mathcal{H}$.

Assume that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, positive definite functions ρ_1, ρ_2, a closed neighborhood $\mathcal{U} \subset \mathbb{R}^{n_P}$ of the origin, and two continuously differentiable functions $W_1, W_2 \colon \mathbb{R}^{n_P} \to \mathbb{R}_{\geq 0}$ such that

1) $\alpha_1(|z|) \leq W_i(z) \leq \alpha_2(|z|), \quad \forall i \in \{1,2\}, z \in \mathbb{R}^{n_P}$
2) $\langle \nabla W_1(z), f_p \rangle \leq -\rho_1(|z|), \quad \forall z \in \mathcal{U}, f_p \in F_P(z, \kappa_1(z))$
3) $\langle \nabla W_2(z), f_p \rangle \leq -\rho_2(|z|), \quad \forall z \in \mathbb{R}^{n_P}, f_p \in F_P(z, \kappa_2(z))$.

Let $C_{S,1} = \mathcal{U}$ and $D_{S,1} = \overline{\mathbb{R}^{n_P} \setminus C_{S,1}}$. Select $D_{S,2} \subset L_{W_1}(c)$ compact and containing the origin in its interior, where $c > 0$ is such that

$$L_{W_1}(c) \subset \mathcal{U}.$$

Finally, let $C_{S,2} = \overline{\mathbb{R}^{n_P} \setminus D_{S,2}}$. Then, the set $\mathcal{A}$ defined in (35) is GAS for the hybrid closed-loop system $\mathcal{H}$ with data defined in (33).

E. Further Reading

We presented the main ideas about supervisory control and showed how those can be used to solve challenging control problems by relying on a systematic design approach. The generalization of the architecture we presented to the case of hybrid controllers can be found in [28]. For a complete overview on supervisory control, the reader is referred to [6, Chapter 8]. Supervisory control has a long history that is difficult to summarize in a short literature review. First results about this topic can be traced back in the work by S. Morse; see, e.g., [33] and [34]. The use of this paradigm has been explored in several fields of application such as aerospace [35], power electronics [36], [37], and optimization [38], just to mention a few. Further results on the global/local uniting problem can be found in [39], [40], [41]. A uniting local-global strategy for fixed-time state estimation is presented in [42].

V. Event-triggered Control

A. Motivation

Event-Triggered Control (ETC) is best described as a "deliberate, opportunistic aperiodic" sampling strategy whose aim is to increase the average sampling time of the closed-loop system without compromising control performance (cf. [43]). The goal of the present section is not to provide a comprehensive overview of ETC, but rather to highlight the importance of modeling ETC systems in the framework of hybrid dynamical systems.

To this end, we present hybrid system models for the event-triggered controllers described in [44] and [45]. Even though these controllers were not initially conceived under the framework of hybrid dynamical systems, we show that modeling the resulting closed-loop systems under that framework is helpful in identifying potential pathological solutions, such as solutions with arbitrarily small intersampling time. We also present two standard approaches to remove pathological solutions, but we show that they require a trade-off between control performance and sampling frequency. Finally, we present some simulation results that underscore the potential of Model-Based ETC in increasing the average sampling time without compromising performance.

B. Send-On-Delta (SoD)

In this section, we present the send-on-delta approach as developed by [44] from the perspective of hybrid dynamical systems. Consider the problem of sampled-data control of a continuous-time plant with state $z \in \mathbb{R}^n$ and dynamics

$$\dot{z} = F_P(z, u), \tag{36}$$

where $u \in \mathbb{R}^m$ is the input. While the standard approach is to use periodic sampling, it was shown in [44] that there is some benefit to using a more opportunistic sampling strategy. Send-on-Delta (SoD) constitutes one such strategy, where sampling events are triggered when the difference between the current value of the state z and the value at the previous sampling event η exceeds a threshold δ, and the actuation signal is kept constant in between events. The hybrid closed-loop system that represents the interconnection between the SoD controller and the continuous-time system (36) is given by

$$\begin{aligned} &(z,\eta) \in C \quad \begin{cases} \dot{z} = F_P(z, \kappa(\eta)) \\ \dot{\eta} = 0 \end{cases} \\ &(z,\eta) \in D \quad \begin{cases} z^+ = z \\ \eta^+ = z \end{cases} \end{aligned} \tag{37}$$

where κ is a feedback law and

$$\begin{aligned} C &= \{(z,\eta) \in \mathbb{R}^{2n} : |z - \eta| \leq \delta\} \\ D &= \{(z,\eta) \in \mathbb{R}^{2n} : |z - \eta| \geq \delta\}. \end{aligned} \tag{38}$$

Anticipating the controller design in the next section, assume that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\underline{\alpha}(|z|) \leq V(z) \leq \overline{\alpha}(|z|) \tag{39}$$

$$\nabla V(z)^\top F_P(z, \kappa(\eta)) \leq -\alpha(|z|) + \gamma(|\eta - z|) \tag{40}$$

for each $(z,\eta) \in \mathbb{R}^{2n}$, where $\gamma, \alpha, \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$. It follows from [46, Theorem 1] that there exists a class-$\mathcal{K}_\infty$ function θ such that the set $\mathcal{A} := V^{-1}([0,\epsilon]) \times \mathbb{R}^n$ with $\epsilon := \theta(\delta)$ is stable for (37), and each precompact solution to (37) approaches $V^{-1}([0,\epsilon]) \times (V^{-1}([0,\epsilon]) + \delta\mathbb{B})$.

While the application of continuous-time feedback to the system (36) would render the origin asymptotically stable under assumption (39), its SoD implementation will, at best, render the origin practically stable, in the sense that solutions do not necessarily converge to the origin but rather to a neighborhood of the origin. This neighborhood can be made arbitrarily small by tuning of the controller parameter δ.

C. Lyapunov-Based ETC

Similarly to the previous section, suppose that we are given a feedback law $z \mapsto \kappa(z)$ that asymptotically stabilizes the origin for the plant (36), and a Lyapunov function $z \mapsto V(z)$ satisfying (39). Lyapunov-based ETC revolves around the idea of sampling the output of a plant only if some upper bound to the derivative of the given Lyapunov function is violated. There are a multitude of possible Lyapunov-based ETC strategies, but we will focus our attention on the event-triggered controller in [45], which has shaped much of the research on ETC since its publication.

Following the controller design in [45], we derive the hybrid closed-loop system:

$$\begin{aligned} &(z,\eta) \in C \quad \begin{cases} \dot{z} = F_P(z, \kappa(\eta)) \\ \dot{\eta} = 0 \end{cases} \\ &(z,\eta) \in D \quad \begin{cases} z^+ = z \\ \eta^+ = z \end{cases} \end{aligned} \tag{41}$$

where

$$\begin{aligned} C &= \{(z,\eta) \in \mathbb{R}^{2n} : \gamma(|\eta - z|) \leq \sigma\alpha(|z|)\} \\ D &= \{(z,\eta) \in \mathbb{R}^{2n} : \gamma(|\eta - z|) \geq \sigma\alpha(|z|)\}. \end{aligned}$$

and $\sigma \in (0,1)$ is a controller parameter. In this case, [46, Theorem 1] can be used to show that the set $\mathcal{A} := \{0\} \times \mathbb{R}^n$ is stable and each precompact solution converges to $\{0\} \times \{0\}$. The key problem with this implementation is that it has a complete discrete solution at $\{0\} \times \{0\}$, which means the existence of solutions with arbitrarily fast sampling under the influence of state perturbations cannot be ruled out, as discussed in the next section.

D. Minimum Intersampling Time

In ETC, one forgoes direct control over sampling events with the hope that, by sampling "only if needed", it is possible to increase the average sampling time without compromising performance. However, there is the risk that events happen too frequently. For example, the existence of Zeno solutions in ETC is particularly pathological as there is no hardware in a sampled-data control system that is able to handle arbitrarily fast sampling. Therefore, the design of event-triggered controllers is often coupled with some guarantees on the existence of a lower bound to the intersampling time.

One of the key advantages in modeling event-triggered controllers using the hybrid systems framework is that it helps with the identification of potentially pathological solutions. For example, even though [45, Theorem III.1] proves that there is a uniform lower bound to the intersampling time, it does so by excluding solutions starting from the origin. It was shown in [47] that, if the Krasovskii regularization of a system has a complete discrete solution, then there exists a solution with arbitrarily small intersampling time under the influence of arbitrarily small state perturbations. On the other hand, notice that the hybrid system (37) satisfies the condition $G(D) \cap D = \emptyset$, where $G(z,\eta) = (z,z)$ denotes the jump map of (41), and, for that reason, each maximal solution has a lower bound to the intersampling time (cf. [48]). The way to address the existence of complete discrete solutions in (41) is to remove them, either by temporal regularization, or by spatial regularization.

1) Temporal regularization: Temporal regularization refers to the implementation of a timer within the controller that blocks sampling events until a certain time $T > 0$ has passed. Following the strategy in [46], we extend the state variable in (41) by including a timer variable τ. The hybrid closed-loop system under temporal regularization becomes:

$$\begin{aligned} \widetilde{x} \in \widetilde{C} \quad & \begin{cases} \dot{z} = F_P(z, \kappa(\eta)) \\ \dot{\eta} = 0 \\ \dot{\tau} = \rho(\tau) \end{cases} \\ \widetilde{x} \in \widetilde{D} \quad & \begin{cases} z^+ = z \\ \eta^+ = z \\ \tau^+ = 0 \end{cases} \end{aligned} \tag{42}$$

where $\widetilde{x} = (z, \eta, \tau)$,

$$\begin{aligned} \widetilde{C} &= (C \times \mathbb{R}_{\geq 0}) \cup \mathbb{R}^{2n} \times [0, T], \\ \widetilde{D} &= D \times [T, +\infty), \end{aligned}$$

and, for each $\tau \geq 0$,

$$\rho(\tau) = \begin{cases} [0,1] & \text{if } \tau = T^\star \\ 1 & \text{if } \tau \in [0, T^\star) \\ -\tau + T & \text{if } \tau > T^\star \end{cases}$$

with $T^\star > T$. It follows from [46, Theorem 3] that, if $\mathcal{A}$ is pre-asymptotically stable for (41), then $\mathcal{A} \times [0, T^\star]$ is semiglobally practically stable, in the sense that, for each compact set of initial conditions and each $\epsilon > 0$, there exists $T > 0$ such that each solution to (42) approaches $(\mathcal{A} \times [0, T^\star]) + \epsilon\mathbb{B}$.

2) Spatial Regularization: In spatial regularization, one defines a neighborhood around complete discrete solutions that turns off events. Following a similar approach to [49], the hybrid closed-loop system (41) is modified as follows:

$$\begin{aligned} (z, \eta) \in \widetilde{C} \quad & \begin{cases} \dot{z} = F_P(z, \kappa(\eta)) \\ \dot{\eta} = 0 \end{cases} \\ (z, \eta) \in \widetilde{D} \quad & \begin{cases} z^+ = z \\ \eta^+ = z \end{cases} \end{aligned} \tag{43}$$

where

$$\begin{aligned} \widetilde{C} &= \{(z, \eta) \in \gamma(|\eta - z|) \leq \sigma\alpha(|z|) + \nu\} \\ \widetilde{D} &= \{(z, \eta) \in \gamma(|\eta - z|) \geq \sigma\alpha(|z|) + \nu\}, \end{aligned}$$

where $\nu > 0$. In this case, it follows from [46, Theorem 1] that there exists a class-$\mathcal{K}_\infty$ function θ such that the set $\widetilde{\mathcal{A}} := V^{-1}([0, \epsilon]) \times \mathbb{R}^n$ with $\epsilon := \theta(\nu)$ is stable and each precompact solution to (43) approaches $\widetilde{\mathcal{A}} \cap \widetilde{C}$.

We conclude that, in general, in order to prevent arbitrarily fast sampling in the presence of perturbations it might be necessary to trade asymptotic stability for practical stability. The event-triggered controller in [50] is a notable exception to this trade-off. However, if one finds that practical stability is an acceptable solution for a given control problem, then it might be worth to consider a model-based ETC solution.

E. Model-Based ETC (MB-ETC)

Model-based ETC is an adaptation of the SoD approach which leverages knowledge of the plant dynamics in order to maximize the intersampling time. Instead of keeping the value of the last sample stored in memory, a holding function is used to propagate the value of the state from the last sample. In this way, the closed-loop system runs mostly in open-loop, since events are triggered only if the value of the state deviates from the value of the predicted state by an amount δ. Following the Model-based ETC approach, the hybrid closed-loop system (37) becomes

$$\begin{aligned} (z, \eta) \in C \quad & \begin{cases} \dot{z} = F_P(z, \kappa(\eta)) \\ \dot{\eta} = F_P(\eta, \kappa(\eta)) \end{cases} \\ (z, \eta) \in D \quad & \begin{cases} z^+ = z \\ \eta^+ = z \end{cases} \end{aligned} \tag{44}$$

where C and D are as in (37). Under the assumptions given in Section V-B, the stability result does not change, but one can expect the average intersampling time to increase significantly.

Example 8 (Comparison between SoD and MB-ETC)**.** *To illustrate the differences between the Send-on-Delta controller* (37) *and its model-based implementation* (44)*, we borrow the example in [49, Section VI.A] of a jet engine compressor. The dynamics are given by*

$$\begin{aligned} \dot{z}_1 &= -z_2 - \frac{3}{2}z_1^2 - \frac{z_1^2}{2} \\ \dot{z}_2 &= u \end{aligned}$$

where z_1 represents the mass flows, z_2 represents the pressure rise and u is the throttle input. A feedback law that stabilizes the origin for this system is given by

$$\kappa(z) = 4z_1 - 4z_2 - \frac{9}{2}z_1^2 - \frac{3}{2}z_1^3.$$

Figure 5 represents the evolution of both the plant state z as well as the controller variable η for a particular execution of the hybrid closed-loop systems (37) *and* (44) *under the influence of an actuator fault that increases by 10% the output signal relative to a given command u. We do this because, otherwise, the system* (44) *would operate solely in open-loop. We have set $\delta = 0.1$ which is a fairly large value for two reasons: it becomes more apparent that the origin is not asymptotically stable, as the system undergoes oscillatory behavior in a neighborhood of the origin, and sampling events are sufficiently spread apart so as not to overcrowd the figures. It is possible to verify that the model-based ETC only has two sampling events, whereas the pure SoD approach exhibits much more frequent sampling. In addition, the MB-ETC solutions does not oscillate as much as SoD.*

F. Further Reading

The literature on ETC is vast as it spans more than two decades worth of contributions. Therefore, rather than an exhaustive list of works on ETC, we point the reader

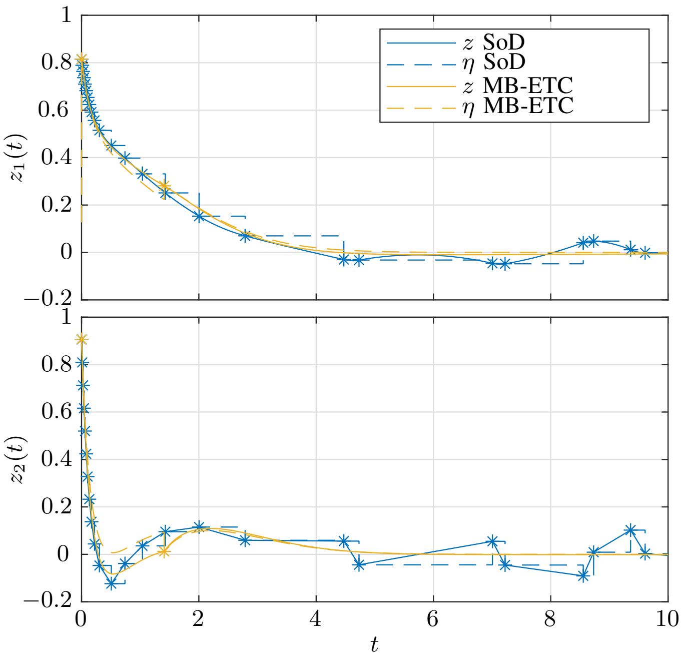

Fig. 5. Representation of the evolution of both plant state and controller state with time for the hybrid closed-loop systems (37) and (44).

to a few key results. The work in [51] provides a good starting point to anyone interested in ETC. An important dynamic event-triggered controller has been proposed in [52] which is shown to increase the average intersampling time relative to static event-triggering mechanisms. The issue of robustness in event separation has also been studied in [53]. A different approach to ETC where one does not have to worry about event separation is Periodic ETC, which is thoroughly studied in [54] and [55]. More details on model-based ETC can be found in [56] and [43].

VI. Synergistic Hybrid Feedback

A. Motivation

We have seen in Example 5 that there are dynamical systems for which global asymptotic stabilization of a setpoint is not possible using continuous feedback, and we have devised a Supervisory Hybrid Control approach in Example 7 which solves that problem. Synergistic hybrid feedback is a hybrid control solution that tackles the same problem but using a different approach [57].

Synergistic hybrid feedback can be seen as a supervisory control strategy in which the supervisor triggers controller switching when the difference between the current value of a Lyapunov function $V_q(z)$ and the lowest possible value among a collection of Lyapunov functions satisfies

$$\mu(z,\eta) := V_\eta(z) - \min\{V_\eta(z) : \eta \in \mathcal{Q}\} > \delta(z,\eta), \quad (45)$$

for a given positive function δ. The function (45) is known as the synergy gap, and $\mathcal{Q}$ is the set of all possible values of η. If $V(z,\eta) := V_\eta(z)$ is positive definite with respect to a compact set $\mathcal{A}$ and nonincreasing during flows, by switching the current controller η to the minimizer of $V_\eta(z)$ we guarantee the decrease of the Lyapunov function during jumps. Unlike supervisory hybrid control, the switching regions are implicitly defined as the set of points for which a decrease of the Lyapunov is guaranteed. To illustrate the design principles of synergistic hybrid feedback controllers, let us revisit Example 7 next.

Example 9 (Example 7 Revisited)**.** *Consider a logic variable $\eta \in \mathcal{Q} := \{0,1\}$, and the following pairs of Lyapunov functions and feedback laws on $\mathbb{S}^1$:*

$$\begin{aligned} V_0(z) &:= 1 - z_1, & \kappa_0(z) &:= -z_2, \\ V_1(z) &:= \alpha + \beta(1 - z_2), & \kappa_1(z) &:= z_1. \end{aligned} \quad (46)$$

for each $z := (z_1, z_2) \in \mathbb{S}^1$, where α and β are positive constants. We define a hybrid controller as follows:

$$\begin{aligned} (z,\eta) &\in C & \dot{\eta} &= 0, \\ (z,\eta) &\in D & \eta^+ &\in \arg\min_{q\in\mathcal{Q}} V_q(z), \end{aligned} \quad (47)$$

where $\delta > 0$ is a controller parameter and

$$C := \left\{ (z,\eta) \in \mathbb{S}^1 \times \mathcal{Q} : V_\eta(z) - \min_{q\in\mathcal{Q}} V_q(z) \le \delta \right\},$$
$$D := \left\{ (z,\eta) \in \mathbb{S}^1 \times \mathcal{Q} : V_\eta(z) - \min_{q\in\mathcal{Q}} V_q(z) \ge \delta \right\}.$$

The hybrid closed-loop system resulting from the interconnection between (8) *and* (47) *is given by*

$$\begin{aligned} (z,\eta) \in C \quad & \begin{cases} \dot{z} = \kappa_\eta(z) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} z \\ \dot{\eta} = 0 \end{cases} \\ (z,\eta) \in D \quad & \begin{cases} z^+ = z \\ \eta^+ \in \arg\min_{q\in\mathcal{Q}} V_q(z) \end{cases} \end{aligned} \quad (48)$$

Given that κ_0 alone is not able to globally asymptotically stabilize $(1,0)$, the question is then: are there values of α and β such that

$$\mathcal{A} := \{(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q} : z = (1,0),\ \eta = 0\}$$

is globally asymptotically stable for the hybrid closed-loop system (48)*?*

Firstly, note that $V(z,\eta) := V_\eta(z)$ is positive definite relative to $\mathcal{A}$, thus V is a Lyapunov function candidate with respect to $\mathcal{A}$. Secondly, the change in the value of V during jumps is upper bounded by $-\delta$ by construction. Finally, the change in the value of V during flows is given by

$$\dot{V}(z,\eta) = \begin{cases} -z_2^2 & \text{if } \eta = 0, \\ -z_1^2 & \text{if } \eta = 1. \end{cases} \quad (49)$$

In order to achieve global asymptotic stability of $\mathcal{A}$, we need to ensure that $\dot{V}(z,\eta) < 0$ for all $(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q} \setminus \mathcal{A}$. This can be achieved by selecting α and β such that

$$\dot{V}(z,\eta) = 0 \implies (z,\eta) \in D \setminus C$$

for all $(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q} \setminus \mathcal{A}$. There exists $\delta > 0$ such that this *condition can be achieved if and only if*

$$1 < \alpha < 2, \qquad 0 < \beta < 2 - \alpha. \tag{50}$$

As can be seen in Example 9, the key to global asymptotic stability by synergistic hybrid feedback is guaranteeing that μ is greater than zero for undesired equilibria of the hybrid closed-loop system.

A strong motivation to pursue the development of hybrid controllers in general – and synergistic hybrid feedback in particular – stems from the fact that there is no continuous feedback law that can globally asymptotically stabilize a setpoint on a closed manifold, i.e., a compact manifold without boundary (cf. [58]). Furthermore, it was proved in [29] that, if there is no continuous feedback law that can globally asymptotically stabilize a setpoint, then there is no discontinuous feedback that can robustly globally asymptotically stabilize it either.

Since most robotic systems have rotational degrees of freedom, the aforementioned results suggest that the development of hybrid controllers is essential for the control of robotic systems. In this context, the development of synergistic hybrid feedback is particularly relevant because it provides a systematic way to design hybrid controllers that can robustly asymptotically stabilize a setpoint on a closed manifold.

B. Properties of the Hybrid Closed-Loop System

In this section, we provide a more formal definition of synergistic hybrid feedback which is taken from [59] and presented here for completeness. In this direction, let us consider the problem of globally asymptotically stabilizing a compact subset $\mathcal{A}$ of $\mathcal{Z} \times \mathcal{Q}$ for a system resulting from the interconnection between

$$\mathcal{H}_P : \begin{cases} (z, u) \in C_P & \dot{z} \in F_P(z, u) \\ & y = z \end{cases} \tag{51}$$

and

$$\mathcal{H}_K : \begin{cases} (v, \eta) \in C_K & \dot{\eta} \in F_K(v, \eta) \\ (v, \eta) \in D_K & \eta^+ \in G_K(v, \eta) \\ & \zeta = \kappa(v, \eta) \end{cases} \tag{52}$$

by setting $v = y$ and $u = \zeta$, where $C_P := \mathcal{Z} \times \mathcal{U}$ and $u \in \mathcal{U}$ is the control input. The sets $\mathcal{Z}$, $\mathcal{Q}$ and $\mathcal{U}$ are closed subsets of some Euclidean space, and F_P is outer semicontinuous, locally bounded, and convex-valued.

Definition 1. *Given a compact subset $\mathcal{A}$ of $\mathcal{Z} \times \mathcal{Q}$, a feedback law $(z, \eta) \mapsto \kappa(z, \eta)$, a continuous function $(z, \eta) \mapsto V(z, \eta)$, and set-valued maps $(z, \eta) \rightrightarrows Q(z, \eta)$ and $(x, \eta) \rightrightarrows F_K(x, \eta)$, we say that the hybrid controller (κ, V, Q, F_K) is a synergistic candidate relative to $\mathcal{A}$ for* (51) *if the following conditions hold:*

(C1) For each $(z, \eta) \in \mathcal{Z} \times \mathcal{Q}$, there exists $g \in Q(z, \eta)$ such that $V(x, g) < +\infty$.

(C2) F_K is outer semicontinuous, locally bounded, and convex-valued.

(C3) The set-valued map Q is outer semicontinuous, lower semicontinuous, and locally bounded;

(C4) The function κ is continuous and

$$\{(z, \eta) \in \mathcal{Z} \times \mathcal{Q} : V(z, \eta) < +\infty\} \subset \operatorname{dom} \kappa.$$

(C5) V is continuous, positive definite relative to $\mathcal{A}$, and $V^{-1}([0, c])$ is compact for each $c \in \mathbb{R}_{\geq 0}$.

The synergistic hybrid feedback controller is derived from the data (κ, V, Q, F_K) as follows:

$$\begin{aligned} (z, \eta) \in C_K \quad & \dot{q} \in F_K(z, \eta), \\ (z, \eta) \in D_K \quad & q^+ \in \rho_V(z, \eta), \end{aligned} \tag{53}$$

with $\rho_V(z, \eta) := \arg\min\{V(z, q) : q \in Q(z, \eta)\}$ and

$$\begin{aligned} C_K &:= \{(z, \eta) \in \mathcal{Z} \times \mathcal{Q} : \mu(z, \eta) \leq \delta(z, \eta)\} \\ D_K &:= \{(z, \eta) \in \mathcal{Z} \times \mathcal{Q} : \mu(z, \eta) \geq \delta(z, \eta)\} \end{aligned}$$

where $(z, \eta) \mapsto \delta(z, \eta)$ is a continuous function, and μ is defined in (45).

The conditions used to specify synergistic candidates guarantee that the hybrid closed-loop system resulting from the interconnection between (51) and (53), given by

$$\begin{aligned} (z, \eta) \in C \quad & \begin{cases} \dot{z} \in F_P(z, \kappa(z, \eta)) \\ \dot{\eta} \in F_K(z, \eta) \end{cases} \\ (z, \eta) \in D \quad & \begin{cases} z^+ = z \\ \eta^+ \in \rho_V(\eta) \end{cases} \end{aligned} \tag{54}$$

with $C = C_K \cap C_P$ and $D = D_K$ satisfies the hybrid basic conditions and, consequently, establish the well-posedness of (54). In addition, Condition (C5) sets the preliminary assumptions for the stability analysis that follows.

Definition 2. *Given a compact subset $\mathcal{A}$ of $\mathcal{Z} \times \mathcal{Q}$, we say that a synergistic candidate relative to $\mathcal{A}$ for* (51) *with data (κ, V, Q, F_K), is synergistic relative to $\mathcal{A}$ for* (51) *if:*

(C6) The function V is Lipschitz continuous on a neighborhood of C and

$$\dot{V}(z, \eta) \leq u_c(z, \eta) \leq 0 \qquad \forall (z, \eta) \in \mathcal{Z} \times \mathcal{Q}$$

for some function u_c;

(C7) Let $u_c^{-1}(0) := \{(z, \eta) \in \mathcal{Z} \times \mathcal{Q} : u_c(z, \eta) = 0\}$. The largest weakly invariant[2] subset of

$$\dot{z} \in F_P(z, \kappa(z, \eta)), \quad \dot{\eta} \in F_K(z, \eta) \tag{55}$$

in $\overline{u_c^{-1}(0)}$, denoted by Ψ, is such that

$$\inf\{\mu_V(z, \eta) : (z, \eta) \in \Psi \setminus \mathcal{A}\} > 0. \tag{56}$$

[2]A set $\mathcal{A}$ is said to be *weakly invariant* for $\mathcal{H}$ if it is *weakly backward invariant* and *weakly forward invariant* for $\mathcal{H}$, which are properties defined as follows: $\mathcal{A}$ is weakly backward invariant for $\mathcal{H}$ if for each $x_0' \in \mathcal{A}$ and each $T > 0$ there exist $x_0 \in K$ and at least one solution x to $\mathcal{H}$ from x_0 such that for some $(t^*, j^*) \in \operatorname{dom} x$ with $t^* + j^* \geq T$, x satisfies $x(t^*, j^*) = x_0$ and $x(t, j) \in \mathcal{A}$ for all $(t, j) \in \operatorname{dom} x$ such that $t + j \leq t^* + j^*$, and it is weakly forward invariant for $\mathcal{H}$ if for each $x_0' in \mathcal{A}$ there exist at least one solution x to $\mathcal{H}$ from x_0 such that $x(t, j) \in \mathcal{A}$ for all $(t, j) \in \operatorname{dom} x$.

If Condition (C7) holds, then it is possible to select a positive function $(z,\eta) \mapsto \delta(z,\eta)$ satisfying $\mu(z,\eta) > \delta(z,\eta)$ for each $(z,\eta) \in \Psi \setminus \mathcal{A}$, in which case we say that (κ, V, Q, F_K) has synergy gap exceeding δ. The following theorem holds the key to most applications of synergistic hybrid feedback.

Given a compact subset $\mathcal{A}$ of $\mathcal{Z} \times \mathcal{Q}$ and a continuous function $\delta : \mathcal{Z} \times \mathcal{Q} \to \mathbb{R}_{>0}$, if (κ, V, Q, F_K) is synergistic relative to $\mathcal{A}$ for (51) with synergy gap exceeding δ, then the set $\mathcal{A}$ is globally pre-asymptotically stable for (54). If each maximal solution to (54) is complete, then $\mathcal{A}$ is globally asymptotically stable for (54).

In the following sections, we discuss more advanced concepts of synergistic hybrid feedback and its applications.

C. Central vs Noncentral Synergism

The synergistic controller of Example 9 is said to be noncentral, because it exhibits a preference over the value of the logic variable, in the sense that, the controller always switches to $\eta = 0$ in a neighborhood of $(1,0)$. However, if one is not careful with the controller design, it is possible for the state of the plant to move away from the point $(1,0)$ when the controller state η is equal to 1. It is possible to prevent such behavior with the design of a centrally synergistic controller, as the one in the following example.

Example 10 ([60]). *Let $\mathcal{Q} = \{q \in \mathbb{S}^1 : q^\top x \leq \gamma\}$ for some $\gamma \in (-1,1)$, and consider the controller data (κ, V, Q, F_K) given by*

$$\kappa(z,\eta) := z^\top \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla_z V(z,\eta)$$
$$V(z,\eta) := \frac{1 - r^\top z}{1 - r^\top z + k(1 - \eta^\top x)}$$

for each $(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q}$, where $r \in \mathbb{S}^1$ is the desired setpoint, $k > 0$ is a controller parameter, and $Q(z,\eta) = \mathcal{Q}$ and $F_K(z,\eta) = 0$. The derivative of V along flows of the hybrid closed-loop system is given by

$$\dot{V}(z,\eta) = -(\Pi(z)\nabla_z V(z,\eta))^2$$

for each $(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q}$, where $\Pi(z) := I_2 - zz^\top$. It is shown in [60, Corollary A.1] that the largest weakly invariant subset in $\{(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q} : \dot{V}(z,\eta) = 0\}$ is $\Psi = \{(r,\eta)\}$ and in [60, Corollary 1] that

$$\inf\{\mu(z,\eta) : (z,\eta) \in \Psi \setminus \mathcal{A}\} = \frac{1+\gamma}{2/k + 1 + \gamma}.$$

Hence, (κ, V, Q, F_K) is synergistic relative to $\mathcal{A} := \{r\} \times \mathcal{Q}$ with synergy gap exceeding δ for any positive continuous function δ satisfying $\delta(z,\eta) < \frac{1+\gamma}{2/k+1+\gamma}$.

The controller in Example 10 has a few features that are important to mention: 1) it allows for global exponential stabilization of a setpoint on a circle, which is something that cannot be achieved with continuous feedback; 2) it enables global asymptotic stabilization of a setpoint on $\mathbb{S}^n$ without preference over the logic variable associated with that setpoint; 3) the set $\mathcal{Q}$ is not discrete, which opens the possibility of having nontrivial controller state dynamics. Even though the vast majority of synergistic controllers have a controller state that is constant during flows, there are some advantages to having a dynamically changing controller state, as shown in the following example.

Example 11 (Dynamic vs. Static Controller States)**.** *Consider the controller data κ, V, and Q of Example 10. However, instead of F_K being identically zero, let us consider*

$$F_K(z,\eta) := -\Gamma(\eta)\Pi(\eta)\nabla_\eta V(z,\eta)$$

for each $(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q}$, where $\Pi(\eta) := I_2 - \eta\eta^\top$ for each $\eta \in \mathbb{S}^1$ is the projection operator that maps vectors in $\mathbb{R}^2$ to the tangent space to $\mathbb{S}^1$ at η, and Γ is a continuous nonnegative function on $\mathbb{S}^1$ such that $\Gamma(\eta) = 0$ for each $\eta \in \mathcal{Q}$ satisfying $\eta^\top r \geq \gamma$. With these dynamics, the derivative of V is given by

$$\dot{V}(z,\eta) = -(\Pi(z)\nabla_z V(z,\eta))^2 - |\Pi(q)\nabla_q V(z,\eta)|^2$$

for each $(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q}$. Hence, for any given (z,η) the given controller dynamics guarantee that the derivative of V is less than or equal to the derivative of V when the controller variable remains static.

D. Synergistic Controllers with Arbitrarily Large Synergy Gap

Definitions 1 and 2 accommodate the possibility that V is not necessarily finite. While this is not standard within the literature on synergistic hybrid feedback, it is necessary in order to encompass the results in [61], which we present in this section.

Example 12. *Let us consider once again the dynamical system* (8)*. The collection $\mathscr{A} := \{(\phi_\eta, U_\eta)\}_{\eta \in \mathcal{Q}}$ with $\mathcal{Q} := \{-1, 1\}$,*

$$\phi_\eta(z) := \frac{z_1}{1 - \eta z_2}, \ \forall z := (z_1, z_2) \in U_\eta$$

and $U_\eta := \{x \in \mathbb{S}^1 : \eta x_2 \neq 1\}$ defines a maximal atlas for the manifold $\mathbb{S}^1$. The collection $\mathscr{A}$ is a maximal atlas for $\mathbb{S}^1$ because the domains of the charts cover the entire manifold and the transition maps are smooth. In fact, ϕ_q is the well known stereographic projection. One can use these coordinate representations to asymptotically stabilize a given setpoint $r \in \mathbb{S}^1$ almost globally by finding the representation of the setpoint in the given coordinates and then follow a straight path (in coordinates) from almost any initial condition to the setpoint. The singularity of ϕ_η at $x = (0,\eta)$ is handled by switching to the other chart. The key assumption in this controller design is that the setpoint r is not at the singularity of any of the charts.

Under this assumption, the synergistic controller

(κ, V, Q, F_K) data is as follows:

$$
\begin{aligned}
V(z,\eta) &:= \begin{cases} \frac{1}{2}|\phi_\eta(z) - \phi_\eta(r)|^2 & \text{if } z_2 \neq \eta \\ +\infty & \text{otherwise} \end{cases}, \\
\kappa(z,\eta) &:= x^\top \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla_z V(z,\eta), \\
Q(z,\eta) &:= \mathcal{Q}, \\
F_K(z,\eta) &:= 0.
\end{aligned}
$$

We have that $V(z,\eta) = +\infty$ when $x = (0,\eta)$, and in such cases, it follows that $V(x,g) < +\infty$ for $g = -\eta$, thus Condition (C1) is satisfied. F_K and Q satisfy (C2) and (C3), respectively. The domain of κ is precisely the set $\{(z,\eta) \in \mathbb{S}^1 \times \mathcal{Q} : V(z,\eta) < +\infty\}$, hence condition (C4) is satisfied. V is continuous and positive definite relative to $\mathcal{A} := \{r\} \times \mathcal{Q}$ and it is radially unbounded, thus condition (C5) is satisfied. We conclude that (κ, V, Q, F_K) is a synergistic candidate relative to $\mathcal{A}$. The change of V along flows is

$$
\begin{aligned}
\dot{V}(z,\eta) &= -|\Pi(z)\nabla_z V(z,\eta)|^2 \\
&= -|\Pi(z)\nabla\phi_\eta(z)(\phi_\eta(z) - \phi_\eta(z))|^2
\end{aligned}
$$

which is equal to 0 if and only if $z = r$. Notably, for this controller we have $\Psi = \mathcal{A}$, hence

$$
\inf\{\mu(z,\eta) : (z,\eta) \in \Psi \setminus \mathcal{A}\} = +\infty,
$$

thus, not only is (κ, V, Q, F_K) synergistic relative to $\mathcal{A}$, but it has synergy gap exceeding δ for any bounded δ.

The work in [61] not only provides the construction of synergistic controllers for global asymptotic stabilization on a smooth manifold, but also allows for arbitrarily large synergy gap. The main disadvantage is that increasingly large values of δ require increasingly large actuation signals.

E. Further Reading

We have covered the definition and the main result of synergistic hybrid feedback, and illustrated key features by means of examples of global asymptotic stabilization on $\mathbb{S}^1$. However, we are leaving out of this tutorial many other more interesting applications of synergistic hybrid feedback which we will summarize next. Synergistic control first came to prominence in [62], where the problem of global attitude tracking was tackled through unit-quaternion feedback. However, unit-quaternion feedback requires consistent reconstruction as in [63]. Another possibility is to use directly the rotation matrix in the feedback law, as done in [64], [65] or [66]. An application of quaternion-based synergistic hybrid feedback to quadrotor control can be found in [67]. To find out more about central and noncentral synergistic controllers we refer the reader to [68] for the definitions, and to [60] for its application to quadrotor control. More recent developments on robust synergistic hybrid feedback can be found in [59] where the application of synergistic controllers for obstacle avoidance is also explored. Finally, there has also been an ongoing effort to develop synergistic hybrid observers, see e.g. [69].

VII. Conclusion

In addition to the references listed earlier in the further reading sections, the following resources might be useful to further learn about hybrid dynamical systems:

1) An earlier tutorial on hybrid dynamical systems published in IEEE Control Systems Magazine, including material on hybrid feedback control [5].
2) The Hybrid Equations Toolbox in [70] introduces a software package for Matlab/Simulink for the computation of approximations of trajectories to hybrid equations.
3) The book [18] introduces the hybrid dynamical systems and presents a comprehensive theory for robust asymptotic stability of sets.
4) The textbook [6] provides a more in-depth exposition to hybrid feedback control of the strategies covered by this tutorial paper. It also includes a detailed presentation of the Hybrid Equations Toolbox in [70], with pointers to implementations of the simulator in Octave and Python.

The hybrid control strategies presented in this tutorial paper are just a small sample of those available in the literature such as reset integrators [71], passivity-based control [72], throw-and-catch control [73], patchy control Lyapunov functions [74], invariance-based control [75], [76], energy-based control [77], model predictive control [78], [79], and many others that can be found within the references listed above.

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