Supervising a Family of Hybrid Controllers for Robust Global Asymptotic Stabilization

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***Abstract*— This paper describes an algorithm for achieving robust, global asymptotic stabilization in nonlinear control systems by supervising the actions of a family of hybrid controllers. The family is such that the regions over which they operate cover the state space in an appropriate sense. Moreover, their behavior is such that they can be scheduled to move the state of the system toward a desirable region, whether it be an equilibrium point or a compact set. In establishing our main result, we use the concept of "events" for hybrid systems and show that, under mild assumptions, stability of a system without events is preserved when a finite number of events are incorporated. The algorithm is applied to robust, global stabilization problems involving vehicle orientation, position and orientation of a mobile robot, and the inverted configuration of a pendulum.**

I. Introduction

In certain control applications, control design tools that divide the problem into subproblems for which several control laws can be designed independently and then combined to solve the original problem are prevalent for many reasons. They reduce design and implementation time as well as add modularity and flexibility to the control system. They are also appropriate when a single, continuous stabilizing control law does not exist or when its design is not straightforward.

Such a "divide and conquer" approach to control design is also ubiquitous in control problems where precise control is desired nearby particular operating points while less stringent conditions need to be satisfied at other points. This includes the problem of uniting local and global controllers, in which two control laws are used: one that is supposed to work only locally, perhaps guaranteeing good performance, and another that is capable of steering the system trajectories to a neighborhood of the operating point, where the local control law works; see, e.g., [23], [15], and [4]. More recently, these ideas have been extended in [19] to allow for the combination of more than two state-feedback laws as well as open-loop control laws. Another asymptotically stabilizing control strategy that, rather than insisting on a single control law, uses multiple ones is patchy feedback control [1] for asymptotically controllable systems. It involves partitioning the state space into regions so that a state-feedback law can be designed to globally asymptotically stabilize the desired point. The hybrid patchy feedback control strategy in [16] provides a hysteresis-based implementation of the patchy feedback control in [1] that guarantees robust stability of asymptotically controllable nonlinear systems.

Control systems featuring multiple control laws employ a mechanism acting as a "supervisor", which selects the control law to be applied to the plant. This selection is typically performed in real time and involves the state, inputs, and outputs of the plant and controllers. Supervisory control has been addressed for linear systems in [13], [14], [8] and for several classes of discrete-time systems in [9], [10], [17]. In this paper, following the ideas outlined in [22], we design a supervisor for a family of hybrid controllers to achieve robust, global asymptotic stabilization of general nonlinear systems. Using hybrid controllers designed to operate in appropriately designed regions of the state space, which is a condition that we express in terms of a *pre-asymptotic stability* property, the supervisor chooses the value of a logic variable to schedule a hybrid controller so that the state of the plant is moved toward a desirable region, whether it be an equilibrium point or, more generally, a compact set. Under reasonable operating conditions of each hybrid controller and exploiting properties of certain jumps called "events", we show in Section III that such a hybrid supervisor can be constructed to render the desired compact set robustly, globally asymptotically stable. In Section IV, we apply this controller to stabilize the orientation of a vehicle orientation, the position and orientation of a mobile robot, and the state of a pendulum to the inverted configuration.

II. Preliminaries

Throughout the paper, $\mathbb{R}^n$ denotes n-dimensional Euclidean space; $\mathbb{R}$ denotes real numbers; $\mathbb{R}_{\geq 0}$ denotes nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0,\infty)$; $\mathbb{Z}$ denotes integers; and $\mathbb{N}$ denotes natural numbers including 0, i.e., $\mathbb{N} = \{0,1,\ldots\}$. Given a set S, $\overline{S}$ denotes its closure. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes Euclidean vector norm. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y\in S} |x-y|$. Given sets S_1, S_2 subsets of $\mathbb{R}^n$, $S_1 + S_2 := \{x_1 + x_2 \mid x_1 \in S_1, x_2 \in S_2\}$. S^1 denotes the unit circle, that is, $S^1 := \left\{x \in \mathbb{R}^2 \mid |x| = 1\right\}$. $R(\theta)$ denotes the rotation matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

A. Hybrid systems

Hybrid systems are dynamical systems with both continuous and discrete dynamics. Among several mathematical models, like those in [12], [11], [24], for the purposes of

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this paper we consider the framework outlined in [5] and further investigated in [6] and [21] from a dynamical systems viewpoint with an emphasis on robustness. A hybrid system $\mathcal{H}$ is defined by the following objects:

- A set $C \subset \mathbb{R}^n$ called the *flow set*.
- A set $D \subset \mathbb{R}^n$ called the *jump set*.
- A map $f : C \to \mathbb{R}^n$ called the *flow map*.
- A set-valued map $G : D \rightrightarrows \mathbb{R}^n$ called the *jump map*.

The flow map f defines the continuous dynamics on the flow set C, while the jump map G defines the discrete dynamics on the jump set D. These objects are referred to as the data of the hybrid system $\mathcal{H}$, which at times is explicitly denoted as $\mathcal{H} = (f, C, G, D)$ and written compactly as

$$\mathcal{H}: \quad x \in \mathbb{R}^n \quad \left\{ \begin{array}{rcll} \dot{x} & = & f(x) & x \in C \\ x^+ & \in & G(x) & x \in D\ . \end{array} \right. \tag{1}$$

Solutions are given on extended time domains by functions that satisfy the conditions suggested by (1). More precisely:

Definition 2.1 (hybrid time domain): A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ ... \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. △

Definition 2.2 (hybrid arc): A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and, for each $j \in \mathbb{N}$, $t \mapsto x(t, j)$ is locally absolutely continuous. △

Definition 2.3 (solution to $\mathcal{H}$): A hybrid arc $x : \operatorname{dom} x \mapsto \mathbb{R}^n$ is a *solution to the hybrid system $\mathcal{H}$* if $x(0, 0) \in C \cup D$;

(S1) $\forall j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) = f(x(t, j)); \text{ and}$$

(S2) $\forall (t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j))\ . \qquad △$$

Hybrid arcs, and solutions to $\mathcal{H}$ in particular, are parametrized by pairs (t, j), where t is the ordinary time component and j is the time component that keeps track of the number of jumps. A solution x is said to be *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0, 0)$, *maximal* if there does not exist another solution x' such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded. Solutions to $\mathcal{H}$ may not be unique, not only due to the jump dynamics being set-valued map, but also because when $C \cap D \neq \emptyset$, solutions from $C \cap D$ jump and, depending on the flow map, may be able to flow as well.

We say that a hybrid system $\mathcal{H} = (f, C, G, D)$ is *well-posed* if its data satisfies the following assumption.

Assumption 2.4 (hybrid basic conditions): Given a hybrid system $\mathcal{H} = (f, C, G, D)$, its data (f, C, G, D) satisfies:

(A1) C and D are closed subsets of $\mathbb{R}^n$.

(A2) $f : C \to \mathbb{R}^n$ is continuous.

(A3) $G : D \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, [1] and $G(x)$ is nonempty for all $x \in D$. △

While well-posedness usually refers to the uniqueness of a solution and its continuous dependence on parameters, for example, on initial conditions, these properties rarely hold for hybrid systems. On the other hand, hybrid systems $\mathcal{H}$ satisfying only the mild conditions in Assumption 2.4 have solution sets enjoying several structural and robustness properties. For example, for given a bounded sequence of solutions to $\mathcal{H}$ there exists a subsequence that converges to a solution to $\mathcal{H}$ [6, Theorem 4.4] and the set of solutions to $\mathcal{H}$ is equal to the set of solutions to $\mathcal{H}$ with vanishing state perturbations [21]. These properties are key in proving converse Lyapunov theorems [3] and invariance principles [20], and guarantee that nominal asymptotic stability of compact sets is robust [6]. Hence, the hybrid controllers designed here will insist on them so that the induced stability property is automatically robust.

The stability definitions below are generalizations of the standard stability concepts to the setting where completeness or even existence of solutions is not required. It is a natural stability notion for hybrid systems since often, the set $C \cup D$ does not cover $\mathbb{R}^n$ and because local existence of solutions is sometimes not guaranteed. For the problem of supervising hybrid controllers studied here, it allows to specify the effect of the individual controllers, which are not expected to operate on the entire state space.

Definition 2.5 (pre-asymptotic stability): Consider a hybrid system $\mathcal{H}$. Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. Then:

- $\mathcal{A}$ is *pre-stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t, j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} x$.
- $\mathcal{A}$ is *pre-attractive* for $\mathcal{H}$ if there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ is bounded and if it is complete then $x(t, j) \to \mathcal{A}$ as $t + j \to \infty$.
- $\mathcal{A}$ is *pre-asymptotically stable* if it is both pre-stable and pre-attractive.
- $\mathcal{A}$ is *asymptotically stable* if it is pre-asymptotically stable and there exists $\delta > 0$ such that any maximal solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ is complete.

The set from which all solutions are bounded and the complete ones converge to $\mathcal{A}$ is called the *basin of pre-attraction* of $\mathcal{A}$. $\mathcal{A}$ is *globally (pre-)asymptotically stable* when the basin of (pre-)attraction is equal to $\mathbb{R}^n$. △

By definition, the basin of pre-attraction contains a neighborhood of $\mathcal{A}$. Points in $\mathbb{R}^n \setminus (C \cup D)$ always belong to it since there are no solutions starting at such points.

B. Hybrid controllers for nonlinear systems

We consider nonlinear control systems of the form

$$\mathcal{P}: \quad \dot{x}_p = f_p(x_p, u_p), \quad y_p = h_p(x_p), \qquad x_p \in C^p, \tag{2}$$

[1] A set-valued map G defined on $S \subset \mathbb{R}^n$ is *outer semicontinuous* if for each sequence $x_i \in S$, $x_i \to x \in S$ and each sequence $y_i \in G(x_i)$ converging to a point y, $y \in G(x)$. It is *locally bounded* if, for each compact set $K \subset \mathbb{R}^n$ there exists $\mu > 0$ such that $G(K) := \cup_{x \in K} G(x) \subset \mu\mathbb{B}$.

where $x_p \in C^p$ is the state, $C^p \subset \mathbb{R}^{n_p}$ is a closed set where x_p evolves, $u_p \in \mathbb{R}^{m_p}$ is the input, and the functions $f_p : C^p \times \mathbb{R}^{m_p} \to \mathbb{R}^{n_p}$, $h_p : C^p \to \mathbb{R}^{n_p}$ are continuous. A well-posed hybrid controller $\mathcal{K} = (\kappa_c, f_c, C_c, G_c, D_c)$,

$$\mathcal{K}: \quad \left\{ \begin{array}{rcl} y_c & = & \kappa_c(u_c, x_c) \\ \dot{x}_c & = & f_c(u_c, x_c) \\ x_c^+ & \in & G_c(u_c, x_c) \end{array} \right. \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad (u_c, x_c) \in C_c \\ \quad\quad (u_c, x_c) \in D_c, \end{array} \tag{3}$$

where $u_c \in \mathbb{R}^{m_c}$ is the input, $y_c \in \mathbb{R}^{r_c}$ the output, $x_c \in \mathbb{R}^{n_c}$ the state, is such that the sets C_c and D_c are closed subsets of $\mathbb{R}^{m_c} \times \mathbb{R}^{n_c}$, $\kappa_c : C_c \to \mathbb{R}^{r_c}$ and $f_c : C_c \to \mathbb{R}^{n_c}$ are continuous, and $G_c : D_c \rightrightarrows \mathbb{R}^{n_c}$ is outer semicontinuous, locally bounded, and $G_c(u_c, x_c)$ is a nonempty subset of $\mathbb{R}^{n_c}$ for each $(u_c, x_c) \in D_c$. When $\mathcal{P}$ is controlled by $\mathcal{K}$ via the feedback interconnection $u_c = y_p$, $u_p = y_c$, the resulting system has state $x := [x_p^\top \ x_c^\top]^\top \in \mathbb{R}^n$:

$$\begin{aligned} \begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} &= f(x) := \begin{bmatrix} f_p(x_p, \kappa_c(h_p(x_p), x_c)) \\ f_c(h(x_p), x_c) \end{bmatrix} \quad (x_p, x_c) \in C \\ \begin{bmatrix} x_p^+ \\ x_c^+ \end{bmatrix} &\in G(x) := \begin{bmatrix} x_p \\ G_c(h_p(x_p), x_c) \end{bmatrix} \quad (x_p, x_c) \in D, \end{aligned} \tag{4}$$

where $C := \{(x_p, x_c) \mid x_p \in C^p, (h_p(x_p), x_c) \in C_c\}$ and $D := \{(x_p, x_c) \mid x_p \in C^p, (h_p(x_p), x_c) \in D_c\}$. The hybrid system (4) is well-posed.

III. Hybrid supervisors of hybrid controllers

Problem 1: Given a closed set $\Theta \subset C^p \times \mathbb{R}^{n_c - 1}$, a compact set $\mathcal{A} \subset \Theta$, a finite set $Q := \{1, \ldots, q_M\} \subset \mathbb{N}$, and a family of well-posed hybrid controllers $\mathcal{K}_q$, $q \in Q$, with state space $\mathbb{R}^{n_c - 1}$ and the properties in Assumption 3.1 below, design a well-posed hybrid supervisor $\mathcal{K} = (\kappa_c, f_c, C_c, G_c, D_c)$ with state $x_c \in \mathbb{R}^{n_c}$ for the hybrid controllers $\mathcal{K}_q$ so that the resulting feedback interconnection $\mathcal{H}_{cl}$ given in (4) satisfies:

1) $\mathcal{A} \times Q$ is globally pre-asymptotically stable.
2) $C \cup D = \Theta \times Q$.
3) All maximal solutions starting in $\Theta \times Q$ are complete.

Properties 1)-3) imply that $\mathcal{A} \times Q$ is stable and attractive from every point in $C \cup D$, which, in turn, implies that it is globally asymptotically stable for $\mathcal{H}_{cl}$.

Assumption 3.1: There exists a family of well-posed hybrid controllers $\mathcal{K}_q = (\kappa_{c,q}, f_{c,q}, C_{c,q}, G_{c,q}, D_{c,q})$, $q \in Q$:

$$\mathcal{K}_q: \left\{ \begin{array}{rcl} y_c & = & \kappa_{c,q}(u_c, \xi_c) \\ \dot{\xi}_c & = & f_{c,q}(u_c, \xi_c) \\ \xi_c^+ & \in & G_{c,q}(u_c, \xi_c) \end{array} \right. \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} (u_c, \xi_c) \in C_{c,q} \subset \Theta \\ \quad (u_c, \xi_c) \in D_{c,q} \subset \Theta, \end{array} \tag{5}$$

where $\xi_c \in \mathbb{R}^{n_c - 1}$ is the state. Moreover, there exists a collection of closed sets $\Psi_q \subset C_{c,q} \cup D_{c,q}$, $q \in Q$, satisfying:

1) $\cup_{q \in Q} \Psi_q = \Theta$.
2) $\forall q \in Q$, $\Phi_q := \cup_{i \in Q, i > q} \Psi_i$, the feedback interconnection of (2) with $\mathcal{K}_q$, denoted $\mathcal{H}_q$, is such that:
   a) The set $\mathcal{A}$ is globally pre-asymptotically stable.
   b) Each maximal solution is complete or ends in
   $$H_q := \Phi_q \cup \overline{\Theta \setminus (C_{c,q} \cup D_{c,q} \cup \Phi_q)} \ .$$
   c) No maximal solution starting in Ψ_q reaches
   $$\overline{\Theta \setminus (C_{c,q} \cup D_{c,q} \cup \Phi_q)} \setminus \mathcal{A} \ . \qquad \triangle$$

Remark 3.2: Item 2a is assuming that solutions with constant q are bounded, remain close to $\mathcal{A}$, and the complete ones converge to $\mathcal{A}$. If $(C_{c,q} \cup D_{c,q}) \cap \mathcal{A} = \emptyset$ then item 2a implies that $\mathcal{H}_q$ has no complete solutions. Item 2b implies that solutions with controller $\mathcal{K}_q$ that are not complete end at a point in a Ψ set of some controller with index different than q. This property permits a hybrid supervisor to guarantee that maximal solutions are complete. Item 2c combined with 2b imply that solutions from Ψ_q end at a Ψ set of some controller with index larger than q. Moreover, Item 2 for q_M implies that solutions to $\mathcal{H}_{q_M}$ from Ψ_{q_M} converge to $\mathcal{A}$ and the set in item 2c is empty for $q = 1$. $\triangle$

The individual hybrid controllers are combined into a single, well-posed hybrid controller $\mathcal{K} = (\kappa_c, f_c, C_c, G_c, D_c)$ of the form (3) with $x_c := [\xi_c^\top \ q]^\top$, $\kappa_c(u_c, x_c) := \kappa_{c,q}(u_c, \xi_c)$,

$$\begin{aligned} f_c(u_c, x_c) &:= \begin{bmatrix} f_{c,q}(u_c, \xi_c) \\ 0 \end{bmatrix}, C_c := \cup_{q \in Q}(C_{c,q} \times \{q\}) \\ G_c(u_c, x_c) &:= \begin{bmatrix} G_{c,q}(u_c, \xi_c) \\ q \end{bmatrix} \cup J_q(u_c, \xi_c) \\ D_c &:= \cup_{q \in Q}((D_{c,q} \cup H_q) \times \{q\}), \end{aligned} \tag{6}$$

where

$$J_q(u_c, \xi_c) := \left\{ \begin{array}{l} \begin{bmatrix} \{u_c\} \\ \{i \in Q \mid (u_c, \xi_c) \in \Psi_i\} \end{bmatrix} \\ \quad \text{if } (u_c, \xi_c) \in \overline{\Theta \setminus (C_{c,q} \cup D_{c,q} \cup \Phi_q)}, \\ \begin{bmatrix} \{u_c\} \\ \{i \in Q \mid i > q, \ (u_c, \xi_c) \in \Psi_i\} \end{bmatrix} \\ \quad \text{if } (u_c, \xi_c) \in H_q \setminus (\overline{\Theta \setminus (C_{c,q} \cup D_{c,q} \cup \Phi_q)}). \end{array} \right. \tag{7}$$

The feedback interconnection of $\mathcal{K}$ with $\mathcal{P}$ results in a hybrid system as in (4) with state x, which we denote $\mathcal{H}_{cl}$.

Theorem 3.3: Under Assumption 3.1, the hybrid controller $\mathcal{K} = (\kappa_c, f_c, C_c, G_c, D_c)$ defined in (6)-(7) *is well posed and solves Problem 1.*

The well-posedness property of the hybrid supervisor $\mathcal{K}$ in Theorem 3.3 combined with results in [6] imply that the nominal asymptotic stability induced by $\mathcal{K}$ is robust.

The argument in a proof of Theorem 3.3 is as follows. By construction, for every solution to $\mathcal{H}_{cl}$ starting from Θ, the number of jumps at which q changes value before the solution reaches $\mathcal{A} \times Q$ is finite. We call these jumps "events" to distinguish them from the jumps of the hybrid controllers $\mathcal{H}_q$, which do not alter q, and from any other jumps at $\mathcal{A} \times Q$. Note that the hybrid system resulting from removing the (finite number of) events, which we denote by $\mathcal{H}_{cl}^0$, is such that $\mathcal{A} \times Q$ is globally pre-asymptotically stable. This follows from the fact that, away from $\mathcal{A} \times Q$, q remains constant for each solution to $\mathcal{H}_{cl}^0$ and that each of the hybrid controllers $\mathcal{H}_q$ guarantees global pre-asymptotic stability of $\mathcal{A} \times Q$ for its interconnection with (2). Then, Theorem 3.3 follows from the fact that global pre-asymptotic stability of $\mathcal{A} \times Q$ for $\mathcal{H}_{cl}^0$ is preserved when the events are re-incorporated and that the construction of $\mathcal{K}$ guarantees completeness of maximal solutions from $\Theta \times Q$. In fact, the following general fact is true for general hybrid systems with finite number of events. To detect the events, we define an event counter to be an outer semicontinuous set-valued map $\mathcal{E} : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \{0, 1\}$, nonempty on $\cup_{x \in D}(G(x) \times \{x\})$, such that, at every event, is equal to one. For $\mathcal{H}_{cl}$, $\mathcal{E}$ can be defined so that at points $(x', x) \in \cup_{x \in D}(G(x) \times \{x\})$: if $x \in \mathcal{A} \times Q$ or $q' = q$ then $0 \in \mathcal{E}(x', x)$, otherwise, $\mathcal{E}(x', x) = 1$.

Theorem 3.4: Given $\mathcal{H} = (f, C, G, D)$, let $\mathcal{A} \subset \mathbb{R}^n$, $G(D \cap \mathcal{A}) \subset \mathcal{A}$, be compact and such that is globally pre-asymptotically stable for $\mathcal{H}^0 = (f, C, G^0, D^0)$, where

$$\begin{array}{rcl} G^0(x) & := & G(x) \cap \{x' \in \mathbb{R}^n \mid 0 \in \mathcal{E}(x, x')\}, \\ D^0 & := & D \cap \left\{x \in D \;\middle|\; G^0(x) \neq \emptyset\right\}, \end{array}$$

and $\mathcal{E} : \mathbb{R}^n \times \mathbb{R} \rightrightarrows \{0, 1\}$ is an outer semicontinuous set-valued map. Suppose that for each compact set $\mathcal{X} \subset \mathbb{R}^n$ there exists $N > 0$ such that each solution to $\mathcal{H} = (f, C, G, D)$ starting from $\mathcal{X}$ has no more than N events. Then, $\mathcal{A}$ is globally pre-asymptotically stable for $\mathcal{H}$.

Note that by construction, $\mathcal{H}^0$ in Theorem 3.4 is well posed and the solutions to $\mathcal{H}^0$ experience no events.

IV. Applications

A. Stabilization and tracking on the unit circle

We consider the problem of robustly globally stabilizing the point $\xi = \mathbf{1} := [1\ 0]^\top$ for the constrained system

$$\dot{\xi} = \omega R(-\pi/2)\xi\ , \quad \xi \in C^p := S^1, \tag{8}$$

where $\omega \in \mathbb{R}$. This model describes the evolution of a point on a circle as a function of the angular velocity, which is the control variable ω. We note that the (classical) feedback control $\omega = \xi_2$ would almost solve this problem. We would have $\dot{\xi}_1 = \xi_2^2 = 1 - \xi_1^2$ and the derivative of the function $V(\xi) := 1 - \xi_1$ would satisfy $\langle \nabla V(\xi), \xi_2 R(-\pi/2)\xi \rangle = -(1 - \xi_1^2)$. We note that the energy will remain constant if ξ starts at $\pm\mathbf{1}$. Instead, one could also consider the discontinuous feedback $\omega = \text{sgn}(\xi_2)$ where the function "sgn" is defined arbitrarily in the set $\{-1, 1\}$ when its argument is zero. This feedback is not robust to arbitrarily small measurement noise. From points in C^p nearby $-\mathbf{1}$ with $\xi_2 < 0$, it steers the solutions towards $\mathbf{1}$ counterclockwise while from points with $\xi_2 > 0$, it steers the solutions towards $\mathbf{1}$ clockwise. Then, from points in C^p arbitrarily close to $-\mathbf{1}$, there exists arbitrarily small measurement noise e appropriately changing sign so that $\text{sgn}(\xi_2 + e)$ is always pushing solutions towards $-\mathbf{1}$.

To achieve a robust, global asymptotic stability result, following Section III, we design a well-posed hybrid supervisor for the point $\mathbf{1}$. It uses the continuous-time controller $\omega = \xi_2$ when the state is not near $-\mathbf{1}$ and the continuous-time controller $\omega = \xi_1$, which drives the system away from $-\mathbf{1}$, when the state is near that point (it actually almost globally asymptotically stabilize the point $[0\ \ {-1}]^\top$ on C^p, with the only point not in the basin of attraction being $[0\ 1]^\top$). Let the domain of applicability for the controller $\omega = \xi_1 =: \kappa_{c,1}(\xi)$ be $C_{c,1} := \{\xi \in C^p \mid \xi_1 \leq -1/3\}$ and domain of applicability for the controller $\omega = \xi_2 =: \kappa_{c,2}(\xi)$ be $C_{c,2} := \{\xi \in C^p \mid \xi_1 \geq -2/3\}$. Notice that $C_{c,1} \cup C_{c,2} = C^p =: \Theta$. Since these are continuous-time controllers, $D_{c,q}$ and $G_{c,q}$ are empty for each $q \in Q$. Let us take

$$\Psi_1 := C_{c,1}, \qquad \Psi_2 := \overline{C^p \backslash C_{c,1}}.$$

Next, we check Assumption 3.1. (There is no state ξ_c in the controllers we are working with here.) By definition, $\Psi_1 \cup \Psi_2 = \Theta$. For each $q \in Q := \{1, 2\}$, the solutions of $\mathcal{H}_q$ (the system we get by using $\omega = \kappa_{c,q}(\xi)$ and restricting the flow to $C_{c,q}$), are such that the point $\mathbf{1}$ is globally pre-asymptotically stable. For $q = 1$, this is because there are no complete solutions and $\mathbf{1}$ does not belong to $C_{c,1}$. For $q = 2$, this is because $C_{c,2}$ is a subset of the basin of attraction for $\mathbf{1}$. We note that every maximal solution to $\mathcal{H}_1$ ends in Ψ_2. Every maximal solution to $\mathcal{H}_2$ is complete and every maximal solution to $\mathcal{H}_2$ starting in Ψ_2 does not reach $\overline{C^p \backslash C_{c,2}}$. Thus, Assumption 3.1 holds. Then, the hybrid supervisor $\mathcal{K} = (\kappa_c, f_c, C_c, G_c, D_c)$ given in (6)-(7) is completely determined using the above definitions. The proposed construction yields $H_1 = \Psi_2 = \overline{C^p \setminus C_{c,1}}$, $H_2 = \overline{C^p \setminus C_{c,2}}$, and $G_c(u_c, x_c) = 3 - q$.

Let us consider the problem of designing a controller so that the state of (8) robustly tracks the continuously differentiable signal $\zeta : \mathbb{R}_{\geq 0} \to S^1$. This problem can be recast as the point stabilization problem above via the change of coordinates $\xi = \begin{bmatrix} z_1\zeta_1 - z_2\zeta_2 \\ z_2\zeta_1 + z_1\zeta_2 \end{bmatrix}$, $z \in S^1$. It can be shown that $\xi = \zeta \iff z = \mathbf{1}$ and that the derivative of z satisfies $\dot{z} = \widetilde{\omega} R(-\pi/2) z$ when $\omega = \zeta_2 \dot{\zeta}_1 - \zeta_1 \dot{\zeta}_2 - \widetilde{\omega}$, where $\widetilde{\omega} \in \mathbb{R}$; see [22]. Then, tracking of ζ is accomplished when z is stabilized to $\mathbf{1}$. To achieve robust, global tracking of ζ, we apply the hybrid supervisor designed above to control $\widetilde{\omega}$.

B. Stabilization of a mobile robot

Consider the model of a unicycle or mobile robot

$$\dot{x} = \xi\vartheta,\ \dot{\xi} = \omega R(-\pi/2)\xi\ ,\ (x, \xi) \in C^p := \mathbb{R}^2 \times S^1, \tag{9}$$

where x denotes planar position from a reference point (in meters), ξ denotes orientation, $\vartheta \in \mathcal{V} := [-3, 30]$ denotes velocity (in meters per second), and $\omega \in [-4, 4]$ denotes angular velocity (in radians per second). Both ϑ and ω are control inputs. Due to the specification of the set $\mathcal{V}$, the vehicle is able to move more rapidly in the forward direction than in the backward direction. Our goal is to design a robust, global stabilizer for the point $\mathcal{A}_0$ given by $(x, \xi) = (0, \mathbf{1})$. It is well known that (9) fails Brockett's condition for robust local asymptotic stabilization by classical (even discontinuous) time-invariant feedback [2], [18], [7]. Nevertheless, a hybrid feedback stabilizer can be designed to accomplish the goal. We build three well-posed hybrid controllers and then combine them with the well-posed hybrid supervisor in Section III. The three controllers use a discrete state $p \in \mathcal{P} := \{-1, 1\}$ and are as follows:

- The first hybrid controller, $\mathcal{K}_1$, uses $\vartheta = \text{Proj}_{\mathcal{V}}(k_1 \xi^T x)$, where $k_1 < 0$ and $\text{Proj}_{\mathcal{V}}$ denotes the projection onto $\mathcal{V}$, while the feedback for ω is given by the hybrid controller in Section IV-A for tracking on the unit circle with reference signal for ξ given by $-x/|x|$. The two different values for q in that controller should be associated with the two values in $\mathcal{P}$. The particular association does not matter. The controller flow and jump sets are such that

$$C_{c,1} \cup D_{c,1} = \left\{ x \in \mathbb{R}^2 \;\middle|\; |x| \geq \varepsilon_{11} \right\} \times S^1 \times \mathcal{P}$$

where $\varepsilon_{11} > 0$, and $C_{c,1}, D_{c,1}$ are constructed from the hybrid controller in Section IV-A for tracking on the unit circle.

- The second hybrid controller, $\mathcal{K}_2$, uses $\vartheta = \mathrm{Proj}_{\mathcal{V}}(k_2 \xi^T x)$, $k_2 \leq 0$, while the feedback for ω is given as in Section IV-A for stabilization of the point $\mathbf{1}$ on the unit circle. Again, the q values of that controller should be associated with the values in $\mathcal{P}$ and the particular association does not matter. The controller flow and jump sets are such that

$$\begin{aligned} C_{c,2} \cup D_{c,2} &= \left(\left(\left\{ x \in \mathbb{R}^2 \;\middle|\; |x| \leq \varepsilon_{21} \right\} \times S^1 \right) \right. \\ &\left. \cap \left\{ (x,\xi) \in \mathbb{R}^2 \times S^1 \;\middle|\; 1 - \xi_1 \geq \varepsilon_{22} |x|^2 \right\} \right) \times \mathcal{P}, \end{aligned}$$

where $\varepsilon_{21} > \varepsilon_{11}$, $\varepsilon_{22} > 0$, and $C_{c,2}, D_{c,2}$ are constructed from the hybrid controller in Section IV-A for stabilization of the point $\mathbf{1}$ on the unit circle.

- The third hybrid controller, $\mathcal{K}_3$, uses $\vartheta = \mathrm{Proj}_{\mathcal{V}}(k_3 \xi^T x)$, $k_3 < 0$, while the feedback for ω is hybrid as defined below. The controller flow and jump sets are designed so that

$$\begin{aligned} C_{c,3} \cup D_{c,3} &= \left(\left\{ x \in \mathbb{R}^2 \;\middle|\; |x| \leq \varepsilon_{31} \right\} \times S^1 \right) \\ \cap \left\{ (x,\xi) \in \mathbb{R}^2 \times S^1 \;\middle|\; 1 - \xi_1 \leq \varepsilon_{32} |x|^2 \right\} &\times \mathcal{P} =: \Lambda_3, \end{aligned}$$

where $\varepsilon_{31} > \varepsilon_{21}$ and $\varepsilon_{32} > \varepsilon_{22}$. The control law for ω is given by $\omega = pk$, where $k > 0$ and the discrete state p has dynamics given by $\dot{p} = 0, p^+ = -p$. The flow and jump sets are given by

$$\begin{aligned} C_{c,3} &:= \Lambda_3 \cap \left(\{ (x,\xi,p) \mid \sigma(p)\xi_2 \leq 0 \} \right. \\ &\quad \left. \cup \left\{ (x,\xi,p) \mid \sigma(p)\xi_2 \geq 0, 1 - \xi_1 \leq \varepsilon_{22}|x|^2 \right\} \right) \\ D_{c,3} &:= \overline{\Lambda_3 \backslash C_{c,3}} \,. \end{aligned}$$

This design accomplishes the following: controller $\mathcal{K}_1$ makes ξ track $-x/|x|$ as long as $|x|$ is not too small, and thus the vehicle drives towards $x = 0$ eventually using only positive velocity; controller $\mathcal{K}_2$ drives ξ towards $\mathbf{1}$ to get the orientation of the vehicle correct; and controller $\mathcal{K}_3$ stabilizes ξ to $\mathbf{1}$ in a persistently exciting manner so that ϑ can be used to drive the vehicle to the origin.

Let $\Theta := \mathbb{R}^2 \times S^1 \times \mathcal{P}$ and $Q := \{1,2,3\}$. The control strategy above is coordinated by a hybrid supervisor with

$$\begin{aligned} &\Psi_1 := C_{c,1} \cup D_{c,1}, \;\; \Psi_2 := \left(\overline{\Theta \backslash \Psi_1} \right) \cap (C_{c,2} \cup D_{c,2}) \\ &\Psi_3 := \left(\left(\left\{ x \in \mathbb{R}^2 \;\middle|\; |x| \leq \varepsilon_{21} \right\} \times S^1 \right) \right. \\ &\qquad \left. \cap \left\{ (x,\xi) \in \mathbb{R}^2 \times S^1 \;\middle|\; 1 - \xi_1 \leq \varepsilon_{22}|x|^2 \right\} \right) \times \mathcal{P}. \end{aligned}$$

Proceeding as in Section IV-A, it can be verified that $\cup_{q \in Q} \Psi_q = \Theta$ and that Assumption 3.1 holds. By Theorem 3.3, the set $\mathcal{A} := \mathcal{A}_0 \times \mathcal{P}$ is globally asymptotically stable. Figure 1 depicts simulation results of the mobile robot with the hybrid controller proposed above for global asymptotic stabilization of $\mathcal{A} \times Q$.

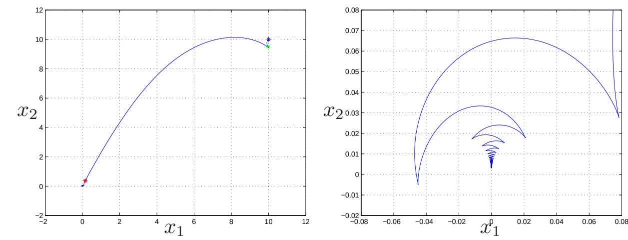

Fig. 1. Global stabilization of a mobile robot to the origin with orientation $\xi = \mathbf{1}$. Vehicle starts at $x(0,0) = (10,10)$ (in meters) and $\xi(0,0)$ corresponding to an angle of $\frac{\pi}{4}$ radians. (a) The vehicle is initially steered to a neighborhood of the origin with orientation $-x/|x|$. At about $1/5$ meters away from it, controller $\mathcal{K}_3$ is enabled to accomplish the stabilization task. (b) Zoomed version of the solution in (a) around the origin. Controller $\mathcal{K}_3$ steers the vehicle to $x = (0,0)$ and $\xi = \mathbf{1}$ by a sequence of "parking" maneuvers.

C. Swing up of a pendulum

Consider the task of robustly, globally asymptotically stabilizing the point $x^* := [0\ 1\ 0]^\top$ for the pendulum system with state $x := [\xi^\top\ z]^\top \in \mathbb{R}^3$ given by

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = f(x,u) := \begin{bmatrix} zR(-\pi/2)\xi \\ \xi_1 + \xi_2 u \end{bmatrix} \qquad (\xi, z) \in C^p := S^1 \times \mathbb{R},$$

where ξ denotes the angle of the pendulum and z corresponds to the angular velocity, with positive velocity in the clockwise direction. The point $\xi = [0\ 1]^\top$ corresponds to the upright position while $\xi = [0\ -1]^\top$ corresponds to the down position of the pendulum. This model was obtained after an input transformation from force to acceleration u and with ratio between the gravitational constant and the pendulum length equal to one. The cart dynamics are ignored to simplify the presentation; however, global asymptotic stabilization of the full cart/pendulum system can be addressed with the same tools used below. To accomplish the stabilization task, we combine three well-posed hybrid controllers with the hybrid supervisor in Section III. The first controller moves the system out of a neighborhood of the point $-x^*$. The second controller moves the system to a neighborhood of the point x^*. The third controller locally asymptotically stabilizes the point x^*. These are designed as follows:

- The third controller, $\mathcal{K}_3$, can be designed using the idea of partial feedback linearization with "output" ξ_1. This is possible since ξ_2 is positive and bounded away from zero in a neighborhood of x^*. Let $\kappa_{c,3} : C^p \to \mathbb{R}$ denote this local asymptotic stabilizer, let $C_{c,3}$ be a compact neighborhood of the point x^* that is also a subset of the basin of attraction for x^* for the system $\dot{x} = f(x, \kappa_{c,3}(x))$, $x \in C^p$, and let Ψ_3 be a compact neighborhood of the point x^* with the property that solutions of $\dot{x} = f(x, \kappa_{c,3}(x))$ starting in Ψ_3 do not reach the boundary of $C_{c,3}$. Then, redefine $C_{c,3}$ and Ψ_3 by intersecting the original choices with $S^1 \times \mathbb{R}$. The set Ψ_3 is indicated in green in Figure 2(d) while the set $C_{c,3}$ is the union of the green and yellow regions in the same figure.

- For the second controller, $\mathcal{K}_2$, let $0 < \delta < \varepsilon < 1$ and

$$\begin{array}{rcl} W(x) & := & \frac{1}{2}z^2 + 1 + \xi_2, \\ \Psi_2 & := & \overline{\{(\xi, z) \in S^1 \times \mathbb{R} \mid W(x) \geq \varepsilon\} \setminus \Psi_3}, \\ C_{c,2} & := & \overline{\{(\xi, z) \in S^1 \times \mathbb{R} \mid W(x) \geq \delta\} \setminus \Psi_3}, \\ \kappa_2(x) & := & -z\xi_2(W(x) - 2) \qquad \forall x \in C_{c,2}. \end{array}$$

  The set Ψ_2 is indicated in green in Figure 2(b) or, alternatively, in Figure 2(c). The set $C_{c,2}$ is the union of the green and yellow regions in the same figures.
- For the first controller, define $C_{c,1} := \overline{(S^1 \times \mathbb{R}) \setminus (\Psi_2 \cup \Psi_3)}$, $\Psi_1 := C_{c,1}$, and $\kappa_1(x) := k$ for all $x \in C_{c,1}$, where $k > \sqrt{\delta(2-\delta)}/(1-\delta)$. The set Ψ_1 is indicated in green in Figure 2(a).

Since each of the controllers is purely continuous, $D_{c,q}$ and $G_{c,q}$ are empty for each $q \in Q$.

It can be verified that the hybrid controllers above are well posed and that Assumption 3.1 is satisfies for $\mathcal{A} := \{x^*\}$, $Q := \{1, 2, 3\}$, and $\Theta := S^1 \times \mathbb{R}$. In turn, the hybrid supervisory control algorithm given in Section III robustly, globally asymptotically stabilizes the point x^* for the pendulum system.

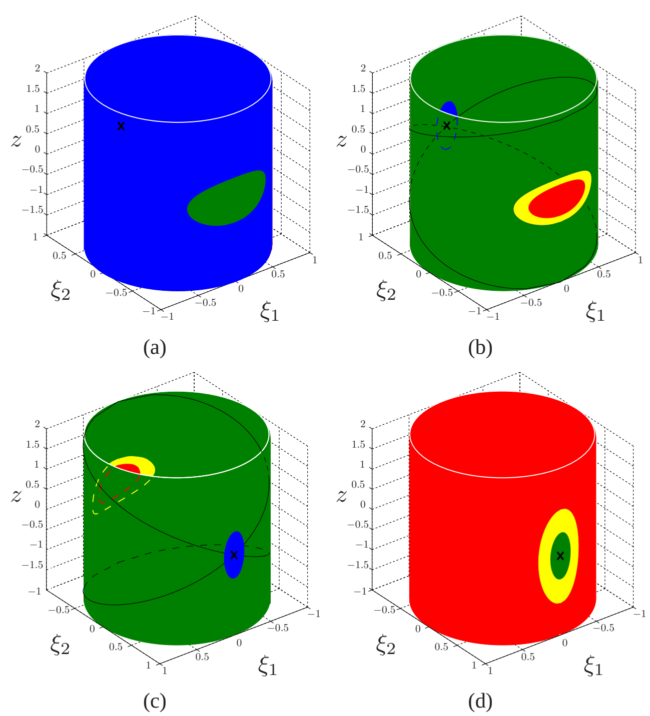

Fig. 2. Sets of the hybrid supervisor for the problem of swinging up a pendulum. The state $x = [\xi^\top, z^\top]^\top$ evolves on the cylinder $S^1 \times \mathbb{R} \subset \mathbb{R}^3$, while $q \in Q = \{1, 2, 3\}$. The inverted position $(\xi, z) = ((0, 1), 0)$ is indicated by a black $\mathbf{x}$. The black curve represents the set of points where $W(x) := 0.5z^2 + 1 + \xi_2 = 2$. Sets for $q = 1, q = 2$, and $q = 3$ are show in (a), (b), and (d), respectively. In (c), the sets for $q = 2$ are depicted with perspective rotated by 180 degrees.

V. Conclusion

We proposed a well-posed construction of general hybrid supervisors for robust, global asymptotic stabilization in nonlinear systems. The hybrid supervisor schedules an appropriate hybrid controller for every point in the region of operation to accomplish the desired task. We provided control applications that not only illustrate the design procedure of hybrid supervisory control but also provide motivation for the need of robust hybrid supervisors of hybrid controllers.

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