A Robust and Global Hybrid Complementary Filter on $\mathrm{SO}(3)$ using Morse Functions on $\mathbb{RP}^3$

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***Abstract*— This paper addresses the problem of global attitude filtering on the special orthogonal group $\mathrm{SO}(3)$. A hysteresis-based hybrid switching strategy is used to switch between two filters that operate in different regions of $\mathrm{SO}(3)$. The first filter is the passive complementary filter, whereas the second filter is designed using an appropriately chosen Morse function. To this end, a novel approach to design Morse functions on $\mathrm{SO}(3)$ is proposed. The proposed hybrid filter is shown to be input-to-state stable with respect to measurement noise. Simulations validate the stability properties of the hybrid filter.**

I. Introduction

The attitude of a rigid body is represented as an element of the special orthogonal group of dimension three, denoted by $\mathrm{SO}(3)$. The problem of attitude estimation/filtering involves using noisy measurements of the attitude and the angular velocity to obtain a filtered attitude estimate. As $\mathrm{SO}(3)$ is a non-Euclidean Lie group, standard estimation methods like the extended Kalman filter fail to ensure that the attitude estimate remains on $\mathrm{SO}(3)$. To address this issue, the multiplicative extended Kalman filter (MEKF) was proposed in [1], where the update step of the Kalman filter satisfies the constraints posed by the structure of the Lie group $\mathrm{SO}(3)$.

While MEKF works with quaternion parametrization of the attitude, [2] proposes a version of the complementary filter for attitude estimation without requiring attitude parametrization. Notably, a passive complementary filter and an explicit complementary filter are proposed in [2]. The former assumes that the measurements of the attitude lie on $\mathrm{SO}(3)$, whereas the latter relaxes this assumption by considering vector measurements that are obtained from the IMU sensor. These filters render the identity element of $\mathrm{SO}(3)$ almost globally asymptotically stable for the filter error dynamics. Indeed, it is shown in [3] that a continuous feedback on $\mathrm{SO}(3)$ cannot render a point globally asymptotically stable. Furthermore, [4] proves that a discontinuous feedback on $\mathrm{SO}(3)$, while guaranteeing global asymptotic stability, is not robust to arbitrarily small measurement noise. Unfortunately, these issues also affect filtering algorithms.

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Motivated by these challenges, hybrid systems theory [5], [6] has been used to design attitude filters that ensure robust and global asymptotic stability of the identity element for the filter error dynamics. In particular, using ideas from synergistic hybrid control proposed in [7], a hybrid filter motivated by the explicit complementary filter is proposed in [8] that renders the identity globally asymptotically stable for the filter error dynamics. Hybrid attitude estimators that utilize synergistic potential functions are also proposed in [9] and [10] to obtain global stability results, where, notably, the filter in [10] results in global exponential stability of the identity for the error dynamics.

In this paper, we propose a global hybrid attitude filter on $\mathrm{SO}(3)$. While leveraging the passive complementary filter, we design a filter on $\mathrm{SO}(3)$ that causes the filter error to evolve in the direction of the negative gradient of an appropriately chosen Morse function. We design the Morse function with desired critical points as a function on the real projective space of dimension three, denoted by $\mathbb{RP}^3$. Since $\mathbb{RP}^3$ is diffeomorphic to $\mathrm{SO}(3)$, we obtain a Morse function on $\mathrm{SO}(3)$. Then, an appropriate hysteresis-based switching mechanism ensures robust and global convergence of the attitude estimate to its true value. A special case of the results in this paper can be found in [11], where a global attitude filter on $\mathrm{SO}(2)$ was developed. The Morse function-based design on $\mathbb{RP}^3$ is the key difference between the hybrid filter proposed in this paper and the ones in [8]–[10].

The remainder of this paper is organized as follows. Section II introduces notation and preliminaries. Section III sets up the problem formulation. The passive complementary filter and the Morse function-based filter are designed in Section IV. Section V describes the switching strategy. Section VI provides simulation results and Section VII concludes the paper. Due to space constraints, proofs are omitted and will be published elsewhere.

II. Preliminaries

A. Notation

The set of real and nonnegative numbers is denoted by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, respectively. The special orthogonal group of dimension n is defined as $\mathrm{SO}(n) := \{R \in \mathbb{R}^{n\times n} : R^\top R = RR^\top = I, \det R = 1\}$. The Lie algebra of $\mathrm{SO}(n)$ is defined as $\mathfrak{so}(n) := \{X \in \mathbb{R}^{n\times n} : X + X^\top = 0\}$. The n-sphere is defined as $\mathbb{S}^n := \{v \in \mathbb{R}^{n+1} : v^\top v = 1\}$. We define the *cross* map $\cdot_\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ such that $v_\times w = v \times w$ for each $v, w \in \mathbb{R}^3$. The inverse of the cross map is defined by $\mathrm{vex} : \mathfrak{so}(3) \to \mathbb{R}^3$, $v_\times \mapsto \mathrm{vex}(v_\times) := v$. The distance from a point $R \in \mathrm{SO}(3)$ to another point $X \in \mathrm{SO}(3)$ is defined as $|R|_X^2 := \mathrm{tr}\big(I - R^\top X\big)/4$. The distance from a point $R \in \mathrm{SO}(3)$ to a nonempty, compact set $\mathcal{A} \subset \mathrm{SO}(3)$ is defined as $|R|_\mathcal{A} := \min_{X\in\mathcal{A}} |R|_X$. For each $A \in \mathbb{R}^{n\times n}$, $\mathcal{E}(A)$ denotes the set of eigenvectors of A. Given functions $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, their composition is denoted by $g \circ f : \mathcal{A} \to \mathcal{C}$.

The set of all lines in $\mathbb{R}^{n+1}$ passing through the origin is defined by the real projective space

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\})/\sim,$$

where the equivalence relation is $x \sim \lambda x$ for each $\lambda \in \mathbb{R} \setminus \{0\}$. Given an equivalence relation $x \sim y$ on a set $\mathcal{X}$, the equivalence class of $x \in \mathcal{X}$ is defined as $[x] := \{y \in \mathcal{X} : y \sim x\}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds. The tangent space to $\mathcal{M}$ at $x \in \mathcal{M}$ and the tangent bundle of $\mathcal{M}$ are denoted by $\mathrm{T}_x\mathcal{M}$ and $\mathrm{T}\mathcal{M}$, respectively. For a smooth map $f : \mathcal{M} \to \mathcal{N}$, the differential of f at $x \in \mathcal{M}$ acting on $v \in \mathrm{T}_x\mathcal{M}$ is denoted by $\mathrm{d}f_x(v)$. The set of all critical points of f is $\mathrm{crit}\, f := \{x \in \mathcal{M} : \mathrm{d}f_x(v) = 0 \ \ \forall v \in \mathrm{T}_x\mathcal{M}\}$. If $(\mathcal{M}, g_\mathcal{M})$ is a Riemannian manifold and $f : \mathcal{M} \to \mathbb{R}$ is a smooth function, the gradient of f at $x \in \mathcal{M}$, denoted by $\mathrm{grad}\, f(x)$, is the unique element in $\mathrm{T}_x\mathcal{M}$ satisfying $\mathrm{d}f_x(v) = g^x_\mathcal{M}(\mathrm{grad}\ f(x), v)$ for each $v \in \mathrm{T}_x\mathcal{M}$, where $g^x_\mathcal{M} : \mathrm{T}_x\mathcal{M} \times \mathrm{T}_x\mathcal{M} \to \mathbb{R}$ denotes the evaluation of the Riemannian metric $g_\mathcal{M}$ at x.

B. Nonautonomous Hybrid Systems

In this paper, we design an estimation algorithm for attitude kinematics on $\mathrm{SO}(3)$ that involves a logic variable. Such a system can be modeled as a hybrid system $\mathcal{H}$ with data (C, F, D, G, ζ) as follows [6]:

$$\mathcal{H} : \begin{cases} \dot{x} &= F(x, v, w) \quad (x, v, w) \in C \\ x^+ &= G(x, v, w) \quad (x, v, w) \in D \\ \zeta &= H(x, v, w) \end{cases} \tag{1}$$

where $x \in \mathcal{X}$ is the state of the system which accounts for the attitude as well as the logic variable, $v \in \mathcal{V}$ is the input, $w \in \mathcal{W}$ is the disturbance acting on the system, and ζ is the output. We let $\mathrm{id} \in \mathcal{W}$ denote the zero disturbance. The set $C \subset \mathcal{X}\times\mathcal{V}\times\mathcal{W}$ is the flow set on which flows are permitted, and $D \subset \mathcal{X} \times \mathcal{V} \times \mathcal{W}$ is the jump set on which jumps are permitted. The function $F : C \to \mathrm{T}\mathcal{X}$ denotes the flow map and $G : D \to \mathcal{X}$ denotes the jump map.

A solution (x, v, w) to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes the ordinary time that has passed, and j denotes the number of times the solution has jumped. The domain of the solution, denoted by $\mathrm{dom}\,(x, v, w)$, is a hybrid time domain [5, Definition 2.3]. With some abuse of notation, $\mathrm{dom}\,(x, v, w)$ represents $\mathrm{dom}\, x \cap \mathrm{dom}\, v \cap \mathrm{dom}\, w$. For simplicity, in this paper, we assume $\mathrm{dom}\,(x, v, w) = \mathrm{dom}\, x = \mathrm{dom}\, v = \mathrm{dom}\, w$. The input-disturbance pair (v, w) is a hybrid input to $\mathcal{H}$ (see [6, Definition 2.27]). The notion of a solution to $\mathcal{H}$ is adapted from [6, Definition 2.29]. A solution is maximal if it cannot be extended, and it is complete if its domain is unbounded.

Now, we define the following stability notion for (1).

Definition 1 (Input-to-state stability)**.** A nonempty set $\mathcal{A}$ is said to be *locally input-to-state stable* (LISS) for a system $\mathcal{H}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, and constants $\delta \in (0, \sup_{x\in\mathcal{X}} |x|_\mathcal{A})$ and $k_w > 0$ such that every solution $(t, j) \mapsto (x(t, j), v(t, j), w(t, j))$ to $\mathcal{H}$ with $|x(0, 0)|_\mathcal{A} \leq \delta$ and $|w_\#|_{\mathrm{id}} \leq k_w$ satisfies

$$|x(t, j)|_\mathcal{A} \leq \beta(|x(0, 0)|_\mathcal{A}, t + j) + \gamma(|w|_{\mathrm{id}}) \tag{2}$$

for all $(t, j) \in \mathrm{dom}\,(x, v, w)$, where $|w_\#|_{\mathrm{id}} := \sup_{(t,j)\in\mathrm{dom}\,(x,v,w)} |w(t, j)|_{\mathrm{id}}$. If (2) holds for all $x(0, 0) \in \mathcal{X}$, then $\mathcal{A}$ is *input-to-state stable* (ISS) for $\mathcal{H}$.

Now, we provide conditions for $\mathcal{H}$ to be well-posed. In particular, $\mathcal{H}$ is well-posed if it satisfies the following assumption [5].

Assumption 1 (Hybrid basic conditions)**.** For the hybrid system $\mathcal{H}$,

(A1) C and D are closed subsets of $\mathcal{X} \times \mathcal{V} \times \mathcal{W}$,
(A2) $F : C \to \mathrm{T}\mathcal{X}$ is continuous,
(A3) $G : D \to G(D)$ is continuous.

III. Problem Formulation and Outline of the Proposed Solution

The rotational kinematics of a rigid body on $\mathrm{SO}(3)$ are given by

$$\dot{R} = R\Omega_\times \tag{3}$$

where $t \mapsto \Omega(t) \in \mathbb{R}^3$ represents the angular velocity of the rigid body in the body fixed frame. We employ the following measurement model:

$$R^y = NR, \qquad \Omega^y = \Omega + \eta. \tag{4}$$

where $t \mapsto N(t) \in \mathrm{SO}(3)$ and $t \mapsto \eta(t) \in \mathbb{R}^3$ are signals of time denoting bounded noise in attitude and angular velocity measurements, respectively, with the bounds given by

$$\theta_R := \sup_{t\geq 0} |N(t)|_I, \qquad \overline{\eta} := \sup_{t\geq 0} |\eta(t)|. \tag{5}$$

These bounds on the noise define the set of admissible measurement noise (N, η) as follows:

$$\mathcal{W} := \{X \in \mathrm{SO}(3) : |X|_I \leq \theta_R\} \times \{x \in \mathbb{R}^3 : |x| \leq \overline{\eta}\}. \tag{6}$$

The objective of this paper is to design an attitude filter that has output $\hat{R} \in \mathrm{SO}(3)$ and uses noisy measurements of R and Ω to ensure that $\hat{R}$ asymptotically converges to some neighborhood of R, or, alternatively, $\widetilde{R} := \hat{R}R^\top$ *practically* stabilizes to $I \in \mathrm{SO}(3)$. The size of the said neighborhood depends on the size of the noise. As any continuous feedback on $\mathrm{SO}(3)$ cannot be globally stabilizing [3], we are motivated to design a hybrid filter that uses two attitude filters, which will operate in overlapping regions on $\mathrm{SO}(3)$, such that appropriate switching between the two filters results in convergence of $\hat{R}$ to a neighborhood of R with arbitrary initial estimate $\hat{R}(0)$. This is achieved as follows:

- The first filter that we will design operates in a neighborhood of the identity. We call this the *local filter* and denote its region of operation by $C_0 \subset \mathrm{SO}(3)$.
- The second filter, which we call the *global filter*, is designed such that its region of operation, denoted by $C_1 \subset \mathrm{SO}(3)$, contains the portion of $\mathrm{SO}(3)$ that is not included in the region of operation of the local filter. In particular, C_1 is such that $\mathrm{SO}(3) \setminus C_0 \subset C_1$. We also require that the global filter practically stabilize $\widetilde{R}$ to a point in the interior of the set C_0.

We introduce a logic variable $q \in Q := \{0, 1\}$ that decides which filter is being operated, with $q = 0$ corresponding to the local filter and $q = 1$ to the global filter.

IV. The Local and Global filters

A. The Local Filter

With the measurement model in (4) and (5), we use the passive complementary filter [2] as the local filter and obtain the following result.

Theorem 1. *Given the rotational kinematics* (3) *with a locally absolutely continuous signal* $t \mapsto \Omega(t)$, *a constant* $k_p > 0$, *and measurements* R^y *and* Ω^y *satisfying* (4), (5), *and* $\theta_R \leq \sqrt{(5+2\sqrt{5})/10}$, *the filter given by*

$$\dot{\hat{R}} = \hat{R}\Big(\Omega^y + k_p \omega(\overline{R}^y)\Big)_\times, \quad \hat{R}(0) \in \mathrm{SO}(3), \tag{7}$$

where $\omega(\overline{R}^y) := \mathrm{vex}(\overline{R}^y - \overline{R}^{y^\top})/2$ *and* $\overline{R}^y := \hat{R}^\top R^y$ *is such that the set* $\{I\} \subset \mathrm{SO}(3)$ *is LISS for the filter error dynamics of* $\widetilde{R} = \hat{R}R^\top$, *which are given by*

$$\dot{\widetilde{R}} = F_0(\widetilde{R}, v_c, w_c) := \hat{R}\eta_\times R^\top + k_p \hat{R}\omega(\overline{R}^y)_\times R^\top \tag{8}$$

where $t \mapsto v_c(t) := (R(t), \Omega(t)) \in \mathrm{SO}(3) \times \mathbb{R}^3$ *and* $t \mapsto w_c(t) := (N(t), \eta(t)) \in \mathcal{W}$.

We refer the reader to [11, Theorem 1] for the explicit $\mathcal{KL}$-bound, similar to (2), for the above result.

In the absence of measurement noise, as a consequence of Theorem 1, the identity is almost globally exponentially stable for (8) and the basin of attraction is $\mathrm{SO}(3) \setminus \mathbb{U}_0$, where the set $\mathbb{U}_0 := \{X \in \mathrm{SO}(3) : |X|_I = 1\}$ is the set of Lebesgue measure zero in $\mathrm{SO}(3)$ that represents all rotations by $180°$ from the identity orientation. Therefore, if $\widetilde{R}(0) \in \mathbb{U}_0$, then appropriately chosen arbitrarily small measurement noise can prevent the convergence of $\widetilde{R}$ to a *small* neighborhood of the identity. This motivates the need for a hysteresis-based hybrid filter that ensures robustness of the filter with respect to bounded measurement noise.

In the remainder of this paper, we refer to the set $\mathbb{U}_0$ as the unstable and forward invariant set for (8) as that indeed holds true when $\theta_R = \overline{\eta} = 0$.

B. The Global Filter Design Approach

Firstly, we design a Morse function[1] on $\mathrm{SO}(3)$ with a unique local minimum such that all of its critical points lie away from $\mathbb{U}_0$. This is achieved by exploiting the fact that the set $\mathbb{RP}^3$ is diffeomorphic to $\mathrm{SO}(3)$ [12, Proposition 9.2.10]. Therefore, it suffices to design a Morse function with desired critical points on $\mathbb{RP}^3$, which can then be *pulled* onto $\mathrm{SO}(3)$ via the diffeomorphism to obtain a Morse function on $\mathrm{SO}(3)$. Then, following [13], we design the global filter such that its error dynamics, defined by the negative gradient of this Morse function, cause the value of the Morse function to decrease to its minimum value almost everywhere. Using the Morse function as a Lyapunov function candidate, we certify almost global asymptotic stability of the stable critical point of the Morse function for the global filter error dynamics.

[1]Given a smooth manifold $\mathcal{M}$, a function $f : \mathcal{M} \to \mathbb{R}$ is a Morse function if all of its critical points are nondegenerate, i.e., the Hessian, evaluated at the critical points, is nonsingular.

C. Morse Functions on $\mathbb{RP}^3$

Given an invertible matrix $B \in \mathbb{R}^{4\times4}$ and a matrix $A = A^\top \in \mathbb{R}^{4\times4}$ with distinct eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, consider the function $f : \mathbb{RP}^3 \to \mathbb{R}$ such that

$$f([v]) := \frac{v^\top B^\top A B v}{v^\top B^\top B v} \qquad \forall v \in \mathbb{R}^4 \setminus \{0\}. \tag{9}$$

Since $f([v]) = f([\lambda v])$ for each $\lambda \in \mathbb{R} \setminus \{0\}$ and each $v \in \mathbb{R}^4 \setminus \{0\}$, f is well defined on $\mathbb{RP}^3$.

D. Global Filter Design on $\mathrm{SO}(3)$

As $\mathrm{SO}(3)$ and $\mathbb{RP}^3$ are diffeomorphic, let $\varphi : \mathrm{SO}(3) \to \mathbb{RP}^3$ be a diffeomorphism. Consequently, $f \circ \varphi : \mathrm{SO}(3) \to \mathbb{R}$ is a Morse function on $\mathrm{SO}(3)$. Since the gradient computation of $f \circ \varphi$ requires a Riemannian metric on $\mathrm{SO}(3)$, let $g_{\mathrm{SO}(3)}$ be the Frobenius inner product, i.e. for each $R \in \mathrm{SO}(3)$

$$g^R_{\mathrm{SO}(3)}(X, Y) := \mathrm{tr}\big(X^\top Y\big) \quad \forall X, Y \in \mathrm{T}_R\mathrm{SO}(3). \tag{10}$$

Then, $g_{\mathrm{SO}(3)}$ induces a Riemannian metric $g_{\mathbb{RP}^3}$ on $\mathbb{RP}^3$ due to the diffeomorphism φ; see [14, p. 332]. Using the resulting metric $g_{\mathbb{RP}^3}$, we obtain the following result.

Lemma 1. *Given an invertible matrix* $B \in \mathbb{R}^{4\times4}$ *and a matrix* $A = A^\top \in \mathbb{R}^{4\times4}$ *with distinct eigenvalues* λ_i *and corresponding eigenvector* ξ_i *for each* $i \in \{1, 2, 3, 4\}$, *the function* $f : \mathbb{RP}^3 \to \mathbb{R}$, *as defined in* (9), *is a Morse function,*

$$\mathrm{grad}\, f([v]) = 2\frac{B^\top A B v - f([v]) B^\top B v}{v^\top B^\top B v} \quad \forall v \in \mathbb{R}^4 \setminus \{0\},$$
$$\mathrm{crit}\, f = \big\{[B^{-1}\xi_1], [B^{-1}\xi_2], [B^{-1}\xi_3], [B^{-1}\xi_4]\big\},$$
$$f([B^{-1}\xi_i]) = \lambda_i \quad \forall i \in \{1, 2, 3, 4\}.$$

With the measurement model in (4), we follow ideas from [13] to design a gradient-like filter on $\mathrm{SO}(3)$ using the Morse function $f \circ \varphi$ and obtain the following result.

Theorem 2. *Given the rotational kinematics* (3) *with a locally absolutely continuous signal* $t \mapsto \Omega(t)$, *measurements* R^y *and* Ω^y *satisfying* (4), (5), *and* $\theta_R \ll 1$, *an invertible matrix* $B \in \mathbb{R}^{4\times4}$, *a matrix* $A = A^\top \in \mathbb{R}^{4\times4}$ *with distinct eigenvalues* λ_i *and corresponding eigenvector* ξ_i *for each* $i \in \{1, 2, 3, 4\}$ *with* $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, *a Morse function* $f : \mathbb{RP}^3 \to \mathbb{R}$ *as in* (9), *a diffeomorphism* $\varphi : \mathrm{SO}(3) \to \mathbb{RP}^3$, *the Riemannian metric* $g_{\mathrm{SO}(3)}$ *as defined in* (10), *an induced Riemannian metric* $g_{\mathbb{RP}^3}$, *and a constant* $\overline{k}_p > 0$, *the filter given by*

$$\dot{\hat{R}} = \hat{R}\Omega^y_\times - \overline{k}_p\Big((d\varphi_{\widetilde{R}^y})^{-1} \circ \mathrm{grad}\, f \circ \varphi(\widetilde{R}^y)\Big)R^y \tag{11}$$

with $\hat{R}(0) \in \mathrm{SO}(3)$, where $\widetilde{R}^y := \hat{R}R^{y\top}$, is such that the singleton set $\{\varphi^{-1}([B^{-1}\xi_1])\} \subset \mathrm{SO}(3)$ is LISS for the filter error dynamics of $\widetilde{R}$, given by

$$\begin{aligned}\dot{\widetilde{R}} &= F_1(\widetilde{R}, v_c, w_c) \\ &:= \hat{R}\eta_\times R^\top - \bar{k}_p \mathrm{grad}\,(f \circ \varphi)(\widetilde{R}^y) R^y R^\top,\end{aligned} \tag{12}$$

where we recall v_c and w_c from (8).

To unite the local and the global filter so as to obtain a hybrid filter, the unstable and forward invariant sets for the filter error dynamics of the local and the global filter must be disjoint. Since the forward invariant set for the filter error dynamics of the global filter is simply crit $(f \circ \varphi)$, the following holds:

$$\mathbb{U}_0 \cap \mathrm{crit}\,(f \circ \varphi) = \varnothing. \tag{13}$$

Then, using Lemma 1, we can design matrices A and B that define the Morse function f on $\mathbb{RP}^3$ and a diffeomorphism $\varphi : \mathrm{SO}(3) \to \mathbb{RP}^3$ such that (13) holds.

V. Uniting Local and Global Filters

In this section, we construct appropriate regions of operations for the local and global filters according to the outline mentioned in Section III, and formulate the hybrid filter. This construction follows closely to [11].

We define the sets $C_0 \subset \mathrm{SO}(3)$ and $C_1 \subset \mathrm{SO}(3)$, which are the regions of operation of the local and global filters as explained in Section III, using positive constants c_0 and c_1, respectively. In particular, pick constants c_0, c_1 satisfying $0 < c_1 < c_0 < 1$. Then, we define the region of operation of the local and global filter as

$$C_0 := \{X \in \mathrm{SO}(3) : |X|_I \le c_0\}, \tag{14}$$

$$C_1 := \{X \in \mathrm{SO}(3) : |X|_I \ge c_1\}. \tag{15}$$

Note that, since $c_1 < c_0$, the nonempty set $C_0 \cap C_1$ is the hysteresis region.

Recall from Section III that the proposed hybrid filter employs a logic variable $q \in Q = \{0, 1\}$. We define the sets where $\widetilde{R}^y$ can lie to trigger a switch in the value of q as follows:

$$D_0 := \overline{\mathrm{SO}(3) \setminus C_0}, \qquad D_1 := \overline{\mathrm{SO}(3) \setminus C_1}. \tag{16}$$

In particular, if $q = 1$ and $\widetilde{R}^y \in D_1$, i.e., $\widetilde{R}^y$ does not lie in the interior of the region of operation of the global filter, the value of q is reset to zero, and since $D_1 \subset C_0$, $\widetilde{R}^y$ now belongs in the region of operation of the local filter. Thus, operating the local filter then ensures convergence of $\widetilde{R}$ to a neighborhood of I.

Similarly, if $q = 0$ and $\widetilde{R}^y \in D_0$, i.e., $\widetilde{R}^y$ does not lie in the interior of the region of operation of the local filter, the value of q is reset to one. Since $D_0 \subset C_1$, $\widetilde{R}^y$ now lies in the region of operation of the global filter. Then, if the global filter can bring the filter error in the interior of D_1 in finite time, a jump in the value of q can be triggered, causing the local filter to ensure practical stabilization of $\widetilde{R}$ to I.

For the global filter to bring the filter error in the interior of D_1 in finite time for any choice of a matrix $A = A^\top \in \mathbb{R}^{4\times4}$ with distinct eigenvalues λ_i and corresponding eigenvectors ξ_i for each $i \in \{1, 2, 3, 4\}$, we design an invertible matrix $B \in \mathbb{R}^{4\times4}$ such that crit $(f \circ \varphi) \subset \mathrm{int}\, D_1$. In particular, we design the invertible matrix $B \in \mathbb{R}^{4\times4}$ such that for each $i \in \{1, 2, 3, 4\}$,

$$\varphi^{-1}([B^{-1}\xi_i]) \in \mathrm{int}\, D_1 \implies |\varphi^{-1}([B^{-1}\xi_i])|_I < c_1. \tag{17}$$

The existence of such a matrix B is guaranteed by Lemma **??**. Note that, due to the choice of the constant c_1, every Morse function $f \circ \varphi$ satisfying (17) also satisfies (13).

A. Hybrid Filter Design

We formalize the above logic by modeling the filter as a hybrid system $\hat{\mathcal{H}}$ of the form (1), with state $\hat{x} := (\hat{R}, q) \in \mathcal{X} := \mathrm{SO}(3) \times Q$, output $\zeta := \hat{R} \in \mathrm{SO}(3)$, and data $(\hat{C}, \hat{F}, \hat{D}, \hat{G}, \zeta)$. Recall that $Q = \{0, 1\}$. The input u to the hybrid filter consists of measurements R^y and Ω^y, i.e., $u := (R^y, \Omega^y) \in \mathcal{U} := \mathrm{SO}(3) \times \mathbb{R}^3$. However, since R^y and Ω^y are signals of time and u is supposed to be a hybrid input to $\hat{\mathcal{H}}$, we define u on the hybrid time domain of a solution $\hat{x}$, i.e., $\mathrm{dom}\,\hat{x} = \mathrm{dom}\, u$, so that u flows when $\hat{x}$ flows, and jumps to the same value when $\hat{x}$ jumps. More formally, given a signal $t \mapsto (R^y(t), \Omega^y(t))$ and a solution $\hat{x}$, the hybrid input $(t, j) \mapsto u(t, j)$ is defined as

$$u(t, j) := (R^y(t), \Omega^y(t)) \quad \forall (t, j) \in \mathrm{dom}\,\hat{x}. \tag{18}$$

The flow set for $\hat{\mathcal{H}}$ is defined to be the points where, for each $q \in Q$, the state-input pair causes $\widetilde{R}^y$ to lie in the region of operation of the corresponding filter. The flow map defines the filter dynamics according to (7) and (11). Similarly, the jump set is defined to be the values of $(\hat{x}, u)$ that trigger a jump in q, as explained below (16). The jump map resets q from zero to one and vice versa, keeping $\hat{R}$ unchanged.

Therefore, the data $(\hat{C}, \hat{F}, \hat{D}, \hat{G}, \zeta)$ of the hybrid filter $\hat{\mathcal{H}}$ is given as

$$\begin{aligned}
&\hat{C} := \left\{ (\hat{x}, u) \in \mathcal{X} \times \mathcal{U} : (\hat{R}R^{y\top}, q) \in \bigcup_{q \in Q} (C_q \times \{q\}) \right\} \\
&\hat{F}(\hat{x}, u) := \begin{pmatrix} (1-q)\hat{F}_0(\hat{x}, u) + q\hat{F}_1(\hat{x}, u) \\ 0 \end{pmatrix} \quad \forall (\hat{x}, u) \in \hat{C} \\
&\hat{D} := \left\{ (\hat{x}, u) \in \mathcal{X} \times \mathcal{U} : (\hat{R}R^{y\top}, q) \in \bigcup_{q \in Q} (D_q \times \{q\}) \right\} \\
&\hat{G}(\hat{x}, u) := \begin{pmatrix} \hat{R} \\ 1 - q \end{pmatrix} \quad \forall (\hat{x}, u) \in \hat{D} \\
&\zeta := \hat{R}
\end{aligned}$$

where, for each $(\hat{x}, u) \in \hat{C}$, the maps $\hat{F}_0(\hat{x}, u)$ and $\hat{F}_1(\hat{x}, u)$ are defined using (7) and (11) as follows:

$$\begin{aligned}
&\hat{F}_0(\hat{x}, u) := \hat{R}\Big(\Omega^y + k_p \omega(\overline{R}^y)\Big)_\times, \\
&\hat{F}_1(\hat{x}, u) := \hat{R}\Omega^y_\times - \bar{k}_p \Big((d\varphi_{\widetilde{R}^y})^{-1} \circ \mathrm{grad}\, f \circ \varphi(\widetilde{R}^y)\Big) R^y.
\end{aligned}$$

Note that, since D_0 and D_1 are disjoint, the jump map $\hat{G}$ is well defined.

B. Hybrid Filter Error Dynamics

Since the desired reference trajectory for the attitude estimate $\hat{R}$ is the true attitude R, the filter error is defined as $\widetilde{R} := \hat{R}R^\top$ so that $\widetilde{R}$ approaches I as $\hat{R}$ approaches R. We compute the dynamics of $\widetilde{R}$ as follows:

$$\dot{\widetilde{R}} = \dot{\hat{R}}R^\top + \hat{R}\dot{R}^\top,$$

where, for $q = 0$ (resp., $q = 1$), $\dot{\hat{R}}$ is given as in (7) (resp., (11)). This results in the following filter error dynamics:

$$\dot{\widetilde{R}} = (1-q)F_0(\widetilde{R}, v_c, w_c) + qF_1(\widetilde{R}, v_c, w_c),$$

where the functions F_0 and F_1 are as defined in (8) and (12), respectively. For its jump dynamics, we see that $\widetilde{R}$ jumps when $\hat{R}$ jumps, resulting in $\widetilde{R}^+ = \hat{R}^+R^\top = \hat{R}R^\top = \widetilde{R}$.

We use the following result from [11, Lemma 1] and characterize the flow and the jump sets of $\widetilde{R}$ as follows.

Lemma 2. *Consider constants c_0 and c_1 satisfying $0 < c_1 < c_0 < 1$. Given $\widetilde{R} \in \mathrm{SO}(3)$ and a signal $t \mapsto N(t)$ satisfying (5) with $\theta_R \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}$, the following holds:*

1) $\widetilde{R}^y \in C_0 \implies \widetilde{R} \in C_{0,w} := \{X \in \mathrm{SO}(3) : |X|_I^2 \leq c_0^2 + \mathfrak{F}(\theta_R)\}$,
2) $\widetilde{R}^y \in D_0 \implies \widetilde{R} \in D_{0,w} := \{X \in \mathrm{SO}(3) : |X|_I^2 \geq c_0^2 - \mathfrak{F}(\theta_R)\}$,
3) $\widetilde{R}^y \in C_1 \implies \widetilde{R} \in C_{1,w} := \{X \in \mathrm{SO}(3) : |X|_I^2 \geq c_1^2 - \mathfrak{F}(\theta_R)\}$,
4) $\widetilde{R}^y \in D_1 \implies \widetilde{R} \in D_{1,w} := \{X \in \mathrm{SO}(3) : |X|_I^2 \leq c_1^2 + \mathfrak{F}(\theta_R)\}$,

where $\mathfrak{F}(s) := s^2 + \sqrt{s^2(1-s^2)}$ for all $s \in [0,1]$.

For each $q \in Q$, the set $C_{q,w}$ denotes the flow set and $D_{q,w}$ denotes the jump set for the hybrid filter error dynamics. These sets depend on measurement noise, which is made evident from the subscript w.

The choice of constants c_0 and c_1, and the matrices A and B determines the bound on the admissible measurement noise. In particular, the noise should be small enough to ensure $D_{0,w} \cap D_{1,w} = \varnothing$ so that persistent jumping of a solution to the filter error dynamics is prohibited. Furthermore, to ensure that the global filter steers $\widetilde{R}^y$ to the interior of $D_{1,w}$ in finite time, it is desired that $\mathrm{crit}\,(f \circ \varphi) \subset \mathrm{int}\, D_{1,w}$. Similarly, for the local filter, it is desired that $\mathbb{U}_0 \subset D_{0,w}$. These conditions are captured by the bound $\mathfrak{F}(\theta_R) \leq \rho_{\max}$, where

$$\mathfrak{F}(\theta_R) \leq \rho_{\max} := \min\left\{\frac{c_0^2 - c_1^2}{2}, 1 - c_0^2, c_1^2 - \overline{c}^2\right\} \tag{19}$$

and $\overline{c}^2 := \max_{i \in \{1,2,3,4\}} |\varphi^{-1}([B^{-1}\xi_i])|_I^2$. We define the admissible measurement noise so that (5) and (19) hold. In particular, for each $\rho \in [0, \rho_{\max})$, we define the set of admissible measurement noise as

$$\mathcal{W}_\rho := \{(X, x) \in \mathcal{W} : \mathfrak{F}(|X|_I) \leq \rho\} \subset \mathrm{SO}(3) \times \mathbb{R}^3, \tag{20}$$

where $\mathcal{W}$ is defined in (6). As $\mathcal{W}_\rho$ is compact, the measurement noise is bounded.

The hybrid filter error dynamics are defined as a hybrid system $\widetilde{\mathcal{H}} = (\widetilde{C}, \widetilde{F}, \widetilde{D}, \widetilde{G})$. The state of $\widetilde{\mathcal{H}}$ is defined as $\widetilde{x} := (\widetilde{R}, q) \in \mathcal{X}$. The input signal $(t,j) \mapsto \widetilde{v}(t,j) := (R(t,j), \Omega(t,j)) \in \mathcal{U}$ to $\widetilde{\mathcal{H}}$ is the hybrid signal obtained from the continuous-time signal $t \mapsto v_c(t)$ according to (18). Similarly, the hybrid disturbance signal $(t,j) \mapsto \widetilde{w}(t,j) := (N(t,j), \eta(t,j)) \in \mathcal{W}_\rho$ acting on $\widetilde{\mathcal{H}}$ is obtained from the measurement noise signal $t \mapsto w_c(t)$.

With this, the data $(\widetilde{C}, \widetilde{F}, \widetilde{D}, \widetilde{G})$ of the hybrid system $\widetilde{\mathcal{H}}$ is defined as follows:

$$\widetilde{C} := \bigcup_{q \in Q} (C_{q,w} \times \{q\}) \times \mathcal{U} \times \mathcal{W}_\rho,$$

$$\widetilde{F}(\widetilde{x}, \widetilde{v}, \widetilde{w}) := \begin{pmatrix} (1-q)F_0(\widetilde{R}, \widetilde{v}, \widetilde{w}) + qF_1(\widetilde{R}, \widetilde{v}, \widetilde{w}) \\ 0 \end{pmatrix} \quad \forall (\widetilde{x}, \widetilde{v}, \widetilde{w}) \in \widetilde{C},$$

$$\widetilde{D} := \bigcup_{q \in Q} (D_{q,w} \times \{q\}) \times \mathcal{U} \times \mathcal{W}_\rho,$$

$$\widetilde{G}(\widetilde{x}, \widetilde{v}, \widetilde{w}) := \begin{pmatrix} \widetilde{R} \\ 1-q \end{pmatrix} \quad \forall (\widetilde{x}, \widetilde{v}, \widetilde{w}) \in \widetilde{D}.$$

Theorem 3. *Suppose that the input signal $t \mapsto \Omega(t) \in \mathbb{R}^3$ is locally absolutely continuous in $\mathrm{dom}\,\Omega$, and the noise signal $t \mapsto w_c(t) := (N(t), \eta(t)) \in \mathrm{SO}(3) \times \mathbb{R}^3$ is Lebesgue measurable and locally essentially bounded in $\mathrm{dom}\, w_c$. Then, for each constants c_0 and c_1 satisfying $0 < c_1 < c_0 < 1$, each diffeomorphism $\varphi : \mathrm{SO}(3) \to \mathbb{RP}^3$, each matrix $A = A^\top \in \mathbb{R}^{4\times4}$ with distinct eigenvalues , each invertible matrix $B \in \mathbb{R}^{4\times4}$ satisfying (17), a Morse function f on $\mathbb{RP}^3$ as in (9), the Riemannian metrics on $\mathrm{SO}(3)$ and $\mathbb{RP}^3$ as defined in (10), each constants $k_p, \bar{k}_p > 0$, and each $\rho \in [0, \rho_{\max})$ the following holds:*

i) the hybrid systems $\hat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$ are well-posed,

ii) if $t \mapsto w_c(t)$ satisfies (5), $\theta_R \ll 1$, $w_c(t) \in \mathcal{W}_\rho$ for each $t \in \mathrm{dom}\, w_c$, and $\mathrm{dom}\,\Omega = \mathrm{dom}\, w_c = \mathbb{R}_{\geq 0}$, then every maximal solution to $\widetilde{\mathcal{H}}$ is complete and exhibits no more than two jumps,

iii) if $t \mapsto w_c(t)$ satisfies (5), $\theta_R \ll 1$, and $w_c(t) \in \mathcal{W}_\rho$ for all $t \in \mathrm{dom}\, w_c$, then $\mathcal{A} := \{I\} \times \{0\} \in \mathcal{X}$ is ISS for $\widetilde{\mathcal{H}}$.

Remark 1. Item iii of Theorem 3 implies that, in the absence of measurement noise, the set $\mathcal{A}$ is globally asymptotically stable for $\widetilde{\mathcal{H}}$. As the hybrid error system $\widetilde{\mathcal{H}}$ is well-posed, it follows from [5] that the global asymptotic stability property of $\mathcal{A}$ for $\widetilde{\mathcal{H}}$ is robust to arbitrarily small measurement noise. However, since $\mathcal{A}$ is ISS for $\widetilde{\mathcal{H}}$, robustness is guaranteed even when the noise is not arbitrarily small.

VI. Numerical Example and Simulation Results

A. Choosing a Diffeomorphism φ

We note that $\mathbb{RP}^3 = \mathbb{S}^3/{\sim_{\mathbb{S}^3}}$, where the equivalence relation is such that $x \sim_{\mathbb{S}^3} -x$. The corresponding equivalence class is defined as $[x]_{\mathbb{S}^3} := \{y \in \mathbb{S}^3 : y \sim_{\mathbb{S}^3} x\}$ for each $x \in \mathbb{S}^3$. Furthermore, for each $q \in \mathbb{S}^3$ and each $\chi \in \mathrm{T}_q\mathbb{S}^3$, we have $\mathrm{T}_{[q]_{\mathbb{S}^3}}\mathbb{RP}^3 = [\chi]_{\mathrm{T}\mathbb{S}^3}$, where the equivalence class $[\cdot]_{\mathrm{T}\mathbb{S}^3}$ is defined by the equivalence relation $\mathrm{T}_q\mathbb{S}^3 \ni \chi \sim_{\mathrm{T}\mathbb{S}^3} -\chi \in \mathrm{T}_{-q}\mathbb{S}^3$.

Consider the set-valued map $\psi : \mathrm{SO}(3) \rightrightarrows \mathbb{S}^3$ that maps each $R \in \mathrm{SO}(3)$ to the corresponding unit quaternion as $\psi(R) := \pm[\cos\frac{\theta}{2} \quad v^\top \sin\frac{\theta}{2}]^\top$, where $(v, \theta) \in \mathbb{S}^2 \times [0, \pi]$

denotes the axis and the angle of rotation corresponding to the rotation matrix R. Note that ψ maps each $R \in \mathrm{SO}(3)$ to two values in $\mathbb{S}^3$ that are negations of each other. Due to the equivalence relation $\sim_{\mathbb{S}^3}$, we have $[q]_{\mathbb{S}^3} = [-q]_{\mathbb{S}^3}$ for each $q \in \mathbb{S}^3$. Then, we define the diffeomorphism $\varphi : \mathrm{SO}(3) \to \mathbb{RP}^3$ as follows:

$$\varphi(R) := [\psi(R)]_{\mathbb{S}^3} \quad \forall R \in \mathrm{SO}(3). \tag{21}$$

We note from [3], [15] that for each continuous curve $t \mapsto R(t)$ that is a solution to (3) and each $q(0) \in \mathbb{S}^3$ such that $\varphi(R(0)) = [q(0)]_{\mathbb{S}^3}$, there exists a unique, continuous curve $t \mapsto q(t)$, with $\operatorname{dom} q = \operatorname{dom} R$, that is a solution to

$$\dot{q} = \frac{1}{2}\Pi(q)\Omega, \tag{22}$$

where $q = \begin{bmatrix} s & \varepsilon^\top \end{bmatrix}^\top$ and $\Pi(q) := \begin{bmatrix} -\varepsilon & sI - \varepsilon_\times \end{bmatrix}^\top$. Therefore, using $\Pi(q)^\top \Pi(q) = I$ for all $q \in \mathbb{S}^3$, it follows from (3) and (22) that, for each $[q]_{\mathbb{S}^3} \in \mathbb{RP}^3$ and each $\chi \in \mathrm{T}_{[q]_{\mathbb{S}^3}} \mathbb{RP}^3$, the following holds:

$$(\mathrm{d}\varphi_{\varphi^{-1}([q]_{\mathbb{S}^3})})^{-1}(\chi) = 2\varphi^{-1}([q]_{\mathbb{S}^3})(\Pi(q)^\top \chi)_\times \tag{23}$$

The above equation, together with (21) and Lemma 1, defines the global filter according to (11).

B. Simulation Results

We set the parameters $c_0 = 0.866$, $c_1 = 0.5$, $k_p = \bar{k}_p = 1$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 0.882 & 0.571 & -2.130 & -2.130 \\ 0.597 & -2.227 & 1.172 & -1.441 \\ 0.597 & -2.227 & -1.442 & 1.172 \\ -1.034 & 3.857 & 2.495 & 2.495 \end{bmatrix}.$$

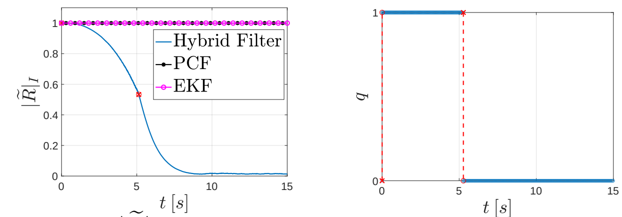
This choice of c_0, c_1, A, and B satisfies (17).

With the angular velocity signal $t \mapsto \Omega(t) = \begin{bmatrix} \sin t & \cos t & \sin 2t \end{bmatrix}^\top$, the true kinematics are initialized as $R(0,0) = I$, and the hybrid filter is initialized as $\hat{R}(0,0) = \mathtt{diag}(-1, 1, -1)$. This initialization results in $\widetilde{R}(0,0) \in \mathbb{U}_0$; in particular, $(\widetilde{R}(0,0), q(0,0)) \in D_0 \times \{0\}$. Let $\bar{\theta}$ denote the angle of rotation of $\widetilde{R}$ and let $\overline{R} := \hat{R}^\top R$. For each $X \in \mathrm{SO}(3)$, let $\mathrm{axis}(X) \in \mathbb{S}^2$ denote its axis of rotation. We consider a bounded noise signal $\eta(t) = \sin(\bar{\theta})\mathrm{axis}(\overline{R})$ if $|\widetilde{R}|_I > 0.999$ and $\eta(t) = -\mathrm{sign}(\sin(\bar{\theta}))\mathrm{axis}(\overline{R})$ otherwise. The noise in attitude measurements $N(t) \in \mathrm{SO}(3)$ is such that $\mathrm{axis}(N(t))$ is sampled from a uniform distribution over $\mathbb{S}^2$, and its angle of rotation is sampled from a uniform distribution over $[-\pi/18, \pi/18]$. In this setting, the simulations[2] show that the passive complementary filter (PCF) and the extended Kalman filter (EKF) fail to ensure convergence of the filter error $\widetilde{R}$ to a neighborhood of the identity, as seen in Figure 1a. The hybrid filter, however, is successful in ensuring said convergence, and the solution jumps at $t = 0\,\mathrm{sec}$ and at $t = 5.32\,\mathrm{sec}$, as seen from Figures 1a and 1b.

VII. Conclusion

We have proposed a hybrid global attitude filter using the following two filters: i) the passive complementary filter, and ii) a gradient-like filter obtained from an appropriately

[2]The simulation files can be found at https://github.com/HybridSystemsLab/HybridComplementaryFilterOnSO3.



(a) $|\widetilde{R}|_I$ vs time. (b) q vs time.

Fig. 1: Comparison of the hybrid filter with PCF and EKF.

chosen Morse function. The proposed hybrid filter is input-to-state stable to bounded measurement noise. In the process of designing this filter, we have proposed a novel, constructive approach to construct Morse functions on $\mathrm{SO}(3)$ with desired critical points. Simulations illustrate the validity of the proposed hybrid filter.

References

[1] E. Lefferts, F. Markley, and M. Shuster, "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control, and Dynamics*, vol. 5, no. 5, pp. 417–429, 1982.

[2] R. Mahony, T. Hamel, and J. Pflimlin, "Nonlinear Complementary Filters on the Special Orthogonal Group," *IEEE Trans. Autom. Control*, vol. 53, no. 5, pp. 1203–1218, 2008.

[3] S. P. Bhat and D. S. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Systems & Control Letters*, vol. 39, no. 1, pp. 63–70, 2000.

[4] C. G. Mayhew and A. R. Teel, "On the topological structure of attraction basins for differential inclusions," *Systems & Control Letters*, vol. 60, no. 12, pp. 1045–1050, 2011.

[5] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. New Jersey: Princeton University Press, 2012.

[6] R. G. Sanfelice, *Hybrid Feedback Control*. New Jersey: Princeton University Press, 2021.

[7] C. G. Mayhew and A. R. Teel, "Synergistic potential functions for hybrid control of rigid-body attitude," in *Proc. of the 2011 Amer. Control Conf.*, 2011, pp. 875–880.

[8] T. Wu, E. Kaufman, and T. Lee, "Globally Asymptotically Stable Attitude Observer on $\mathrm{SO}(3)$," in *2015 54th IEEE Conf. on Decision and Control (CDC)*, 2015, pp. 2164–2168.

[9] S. Berkane and A. Tayebi, "Construction of Synergistic Potential Functions on $\mathrm{SO}(3)$ With Application to Velocity-Free Hybrid Attitude Stabilization," *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 495–501, 2017.

[10] S. Berkane, A. Abdessameud, and A. Tayebi, "Global hybrid attitude estimation on the Special Orthogonal group $\mathrm{SO}(3)$," in *2016 American Control Conf. (ACC)*, 2016, pp. 113–118.

[11] P. P. Jirwankar and R. G. Sanfelice, "Robust Global Hybrid Passive Complementary Filter on $\mathrm{SO}(2)$," in *Proceedings of the 63rd IEEE Conf. on Decision and Control*, Dec. 2024.

[12] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems* (Texts in Applied Mathematics), en. New York, NY: Springer, 1999.

[13] C. Lageman, J. Trumpf, and R. Mahony, "Gradient-Like Observers for Invariant Dynamics on a Lie Group," *IEEE Trans. Autom. Control*, vol. 55, no. 2, pp. 367–377, 2010.

[14] J. M. Lee, *Introduction to Smooth Manifolds* (Graduate Texts in Mathematics). Springer New York, 2012.

[15] P. Casau, R. G. Sanfelice, R. Cunha, D. Cabecinhas, and C. Silvestre, "Robust global trajectory tracking for a class of underactuated vehicles," *Automatica*, vol. 58, pp. 90–98, 2015.