Switched-Gain Observers for Linear Systems by Using Max/Min Lyapunov Functions

A. Alessandri, R.G. Sanfelice

***Abstract*—A new switched-gain observer for linear systems subject to bounded disturbances is proposed. Specifically, we formulate the problem of constructing bimodal switched-gain observers within a hybrid systems framework, and establish novel conditions based on max/min Lyapunov functions to ensure positive invariance of the estimation error. The invariant sets are described by sublevel sets of such Lyapunov functions, which can be found by solving conditions given in terms of bilinear matrix inequalities. The observer design stems from solving such conditions by using grid search methods based on linear matrix inequalities. Numerical results are reported to show improvements with respect to previous design methods.**

***Index Terms*—Observers for linear systems, switched systems, stability of hybrid systems, Lyapunov methods.**

I. Introduction

In this paper, we examine switched-gain observers, initially introduced in [1], [2]. More recent studies, such as [3]–[5], have highlighted the benefits of incorporating gains commutations within the observer structure. Unlike these works, switched-gain observers were explored in [6] within the context of hybrid systems [7], [8], likewise experimented in [9], [10]. Building on [6], we enhance the results for bimodal switched-gain observers by employing a “high” gain L_1 during the transient phase to ensure rapid convergence, and a “low” gain L_2 during the steady state to mitigate performance degradation in the presence of significant measurement noise, as shown Fig. 1. Based on the output error, the switching logic is responsible for gain switching, under stability guarantees ensured through a suitable design, which is the topic of this paper.

Since the design approach in [6], which is based on a standard quadratic Lyapunov function, failed by producing an L_2 gain larger than L_1, we have explicitly investigated the design of switched-gain observers using Lyapunov functions such as max/min Lyapunov functions derived from the maximum or minimum of quadratics [11], [12]. These Lyapunov functions possess a richer structure compared to the quadratic Lyapunov functions used in [6], allowing for a greater number of parameters to appropriately assign the gains L_1 and L_2. Specifically, a “high” gain L_1 is employed with a large output error during the transient phase to ensure rapid convergence, while a “low” gain L_2 is utilized during the steady state to reduce the impact of measurement noise. The conditions ensuring invariance properties for the estimation error with switched-gain observers are provided by bilinear matrix inequalities (BMI). These BMIs can be solved using iterative algorithms that involve solving linear matrix inequalities (LMIs) [13], [14], driven by grid search methods [15]. This represents a qualifying aspect of the present paper.

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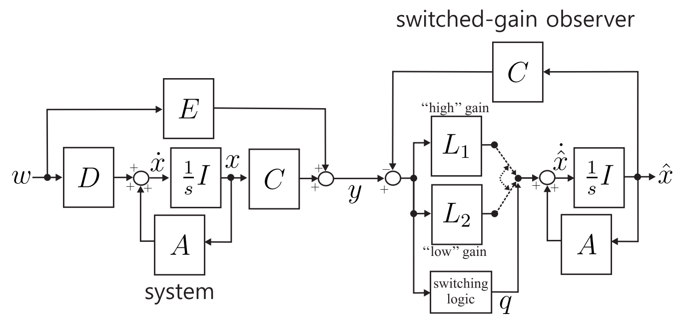

Fig. 1. Sketch of a bimodal switched-gain observer for a linear system subject to disturbances and with $q \in \{1, 2\}$ as discrete state, which selects the gain L_q.

The paper is organized as follows. In Section II, we present switched-gain state observers for linear systems and the basic definitions of invariance based on quadratically boundedness (QB) [16]. Section III illustrates two new design approaches based on max/min Lyapunov functions. Simulation comparisons with previous design methods are presented in Section IV. The conclusions are drawn in Section V.

Notation: Let us define $(x, y) := [x^\top, y^\top]^\top$, where x and y are column vectors. $\mathbb{R}_{>0} := (0, +\infty)$, $\mathbb{R}_{\geq 0} := [0, +\infty)$. The minimum and maximum eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively; if, in addition, $P > 0$ ($P < 0$) we intend that it is positive (negative) definite. For a generic matrix $M \in \mathbb{R}^{n\times m}$, $|M| := \big(\lambda_{\max}(M^\top M)\big)^{1/2} = \big(\lambda_{\max}(MM^\top)\big)^{1/2}$ denotes its norm. As a consequence, $|v| := (v^\top v)^{1/2}$ is just the Euclidean norm of a vector $v \in \mathbb{R}^n$. The diagonal matrix with $v_1, v_2, \ldots, v_n$ on the diagonal is denoted by $\mathrm{diag}(v) \in \mathbb{R}^{n\times n}$. The symbol $1^\top$ is a row vector of appropriate dimension with every element equal to 1. The symbol $\star$ in a block matrix means completion of the matrix through symmetry. For a locally Lipschitz function $V(\cdot)$, we define $\dot{V}(t) := \limsup_{h\to 0^+} (V(t + h) - V(t))/h$. The Clarke generalized gradient of $V : \mathbb{R}^n \to \mathbb{R}$ is defined at $x \in \mathbb{R}^n$ as follows: $\partial V(x) := \overline{\mathrm{co}}\big\{v \in \mathbb{R}^n : \exists x_k \to$

$x\,,\, x_k \notin \mathcal{N}_V \cup \mathcal{S},\; v = \lim_{k\to+\infty} \nabla V(x_k)\}$, where $\mathcal{N}_V :=$ $\{x \in \mathbb{R}^n : \nabla V(x)$ does not exists$\}$, $\mathcal{S}$ is the union of the sets of zero Lebesgue measure, and $\overline{\mathrm{co}}K$ is the convex hull of $K \subset \mathbb{R}^n$.

II. Switched-Gain Observers and Invariance

We consider continuous-time LTI systems subject to external disturbances as follows

$$\begin{aligned} \dot{x} &= A\,x + B\,u + D\,w \\ y &= C\,x + E\,w \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the control input, $y \in \mathbb{R}^m$ is the output; $w \in \mathbb{R}^s$ is the vector of system and measurement noises; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{n\times s}$, and $E \in \mathbb{R}^{m\times s}$. Moreover, we assume bounded disturbance, in that $t \mapsto w_i(t) \in \mathbb{R}$ is Lebesgue measurable, essentially bounded, and such that $|w_i(t)| \le 1$, $i = 1, 2, \ldots, s$, for almost all $t \ge 0$, without loss of generality. [1]

Luenberger observers are studied in [6], [17] for what concerns the invariance property of the estimation error $e := x - \hat{x} \in \mathbb{R}^n$. More specifically, if the pair (A, C) is detectable the Luenberger observer

$$\dot{\hat{x}} = A\,\hat{x} + B\,u + L\,(y - C\,\hat{x}) \tag{2}$$

with $\hat{x} \in \mathbb{R}^n$ denoting the estimate of x and a gain $L \in \mathbb{R}^{n\times m}$ provides an exponentially stable estimation error in a noise-free setting if we choose L such that $A - LC$ is Hurwitz. In the presence of noise, from (1) and (2) we obtain

$$\dot{e} = (A - LC)\,e + (D - LE)\,w \tag{3}$$

Generally speaking, we say that the estimation error is invariant with Lyapunov function $V : \mathbb{R}^n \to [0, +\infty)$ if

$$V(e) > 1 \Rightarrow \dot{V}(e, w) < 0\ \forall w \in [-1, 1]^s\,. \tag{4}$$

In other words, the sublevel set $\mathcal{E}_V := \{e \in \mathbb{R}^n : V(e) \le 1\}$ is positively invariant and attractive.

If we adopt a quadratic Lyapunov function $V(e) = e^\top P e$ with $P \in \mathbb{R}^{n\times n}$ symmetric and $P > 0$ [16], quadratically boundedness (QB) is said to hold if

$$\begin{aligned} e^\top P e > 1 \Rightarrow 2\,e^\top P((A - LC)\,e& \\ + (D - LE)\,w) < 0\ \forall w \in [-1, 1]^s\,.& \end{aligned} \tag{5}$$

Thus, $\mathcal{E}_P := \left\{e \in \mathbb{R}^n : e^\top P e \le 1\right\}$ is positively invariant and attractive. Moreover, $t \mapsto e(t)$ can be bounded from above as follows:

$$|e(t)|^2 \le \frac{1}{\lambda_{\min}(P)} \max\left\{e(0)^\top P e(0), 1\right\} \tag{6}$$

for all $t \ge 0$ and

$$\limsup_{t\to+\infty} |e(t)| \le \frac{1}{\sqrt{\lambda_{\min}(P)}}. \tag{7}$$

For brevity, the right-hand side of (7) will henceforth be referred to as the ultimate asymptotic bound (UAB). The following result (see [17, Theorem 1, p. 5353]) holds for (2).

[1] The assumption on the boundedness of the noise does not entail loss of generality since we can scale D and/or E in such a way as to make $|w_i(t)| \le 1$ be satisfied.

Theorem 1: If there exist a symmetric matrix $P > 0$, $Y \in \mathbb{R}^{n\times m}$, $\alpha \in \mathbb{R}^s_{\ge 0}$, and a scalar $\beta \ge 0$ such that

$$\begin{pmatrix} A^\top P - C^\top Y^\top + PA - YC + \beta P & PD - YE \\ \star & -\mathrm{diag}(\alpha) \end{pmatrix} < 0 \tag{8}$$

$$\sum_{i=1}^{s} \alpha_i - \beta \le 0\,. \tag{9}$$

the estimation error given by the observer (2) with gain $L = P^{-1}Y$ is quadratically bounded.

□

The upper bound (7) suggests to maximize $\lambda_{\min}(P)$ in such a way to get the smallest invariant set for the estimation error. Thus, we consider

$$\max_{P>0;Y;\alpha\in\mathbb{R}^s_{\ge 0};\beta,\lambda\ge 0} \lambda \quad \text{s.t. } P > \lambda I, (8), (9) \tag{10}$$

as design procedure and get the upper bound on the norm of the residual $r := y - C\,\hat{x}$ given by

$$\begin{aligned} |r| = |y - C\,\hat{x}| &= |C\,e + E\,w| \le |C\,e| + |\overline{E}\,\bar{w}| \\ &\le \theta_{\mathrm{th}} := \frac{|C|}{\sqrt{\lambda_{\min}(P)}} + \sqrt{k}|\overline{E}| \end{aligned}$$

at steady state, where $\overline{E} \in \mathbb{R}^{m\times k}$ is the matrix made by the non-null columns of E and $\bar{w} \in [-1, 1]^k$ is the vector of the corresponding components of w, with the integer $k \le s$. In [6], hybrid switched-gain observers triggered by the residual norm were developed with a discrete state q that remains constant during flows and takes values 1 or 2 at a jump time t, according to a hysteresis mechanism based on $|r|$. In more detail, $q = 1$ is associated with "large" residuals (i.e., $|r| \ge \delta_1\theta_{\mathrm{th}}$), while $q = 2$ corresponds to "small" residuals (i.e., $|r| \le \delta_2\theta_{\mathrm{th}}$), where $\delta_1, \delta_2 \in (0, 1)$, $\delta_1 < \delta_2$:

$$\left.\begin{aligned} \dot{\hat{x}} &= A\,\hat{x} + B\,u + L_q\,(y - C\,\hat{x}) \\ \dot{q} &= 0 \end{aligned}\right\} (\hat{x}, q, y, u, w) \in \mathcal{C} \tag{11a}$$

$$\left.\begin{aligned} \hat{x}^+ &= \hat{x} \\ q^+ &= 3 - q \end{aligned}\right\} (\hat{x}, q, y, u, w) \in \mathcal{D} \tag{11b}$$

where $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$,

$$\begin{aligned} \mathcal{C}_1 &:= \{(\hat{x}, q, y, u, w) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \times \mathbb{R}^p \\ &\qquad \times [-1, 1]^s : |y - C\hat{x}| \ge \delta_1\theta_{\mathrm{th}}, q = 1\}\,, \\ \mathcal{D}_1 &:= \{(\hat{x}, q, y, u, w) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \\ &\qquad \times\mathbb{R}^p \times [-1, 1]^s : |y - C\hat{x}| \le \delta_1\theta_{\mathrm{th}}, q = 1\}\,, \\ \mathcal{C}_2 &:= \{(\hat{x}, q, y, u, w) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \\ &\qquad \times\mathbb{R}^p \times [-1, 1]^s : |y - C\hat{x}| \le \delta_2\theta_{\mathrm{th}}, q = 2\}\,, \\ \mathcal{D}_2 &:= \{(\hat{x}, q, y, u, w) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \\ &\qquad \times\mathbb{R}^p \times [-1, 1]^s : |y - C\hat{x}| \ge \delta_2\theta_{\mathrm{th}}, q = 2\}\,. \end{aligned}$$

The interconnection between the plant in (1) and the hybrid observer in (11) results in a hybrid system with state $(x, \hat{x}, q)$, control input u, and disturbance input w, which is well-posed in the sense of [7, Definition 6.29, p. 133] (for the proof, see [6, Proposition 1]).

III. Design of Switched-Gain Observers Based on Max/Min Lyapunov Functions

The use of a common Lyapunov function may be too restrictive for letting the gain L_1 and L_2 take on "high" and "low" values, respectively (see, for example, the gains of the switched-gain and hold-time switched-gain observers in Section IV). Thus, max/min Lyapunov functions are proposed in the following to construct (11). Next theorem establishes sufficient conditions to ensure the existence of an invariant set for the estimation error given by the switched-gain observer (11) by using the sublevel set of a max Lyapunov function as invariant set.

Theorem 2: If there exist two symmetric matrices $P_1, P_2 > 0$, $Y_1, Y_2 \in \mathbb{R}^{n\times m}$, $\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22} \in \mathbb{R}^s_{\geq 0}$, and scalars $\beta_{11}, \beta_{21}, \beta_{12}, \beta_{22}, \gamma_{11}, \gamma_{21}, \gamma_{12}, \gamma_{22}, \tau_{11}, \tau_{21}, \tau_{12}, \tau_{22} \geq 0$, $\delta_1, \delta_2 \in (0,1)$ such that

$$\begin{pmatrix} A^\top P_1 - C^\top Y_1 + P_1 A - Y_1 C \\ +\beta_{11} P_1 - \tau_{11}(P_2 - P_1) + \gamma_{11} C^\top C & \begin{matrix} P_1 D - Y_1 E \\ +\gamma_{11} C^\top E \end{matrix} \\ \star & \begin{matrix} \gamma_{11} E^\top E \\ -\mathrm{diag}(\alpha_{11}) \end{matrix} \end{pmatrix} < 0 \tag{12}$$

$$\begin{pmatrix} A^\top P_2 - C^\top Y_1 + P_2 A - Y_1 C \\ +\beta_{21} P_2 - \tau_{21}(P_1 - P_2) + \gamma_{21} C^\top C & \begin{matrix} P_2 D - Y_1 E \\ +\gamma_{21} C^\top E \end{matrix} \\ \star & \begin{matrix} \gamma_{21} E^\top E \\ -\mathrm{diag}(\alpha_{21}) \end{matrix} \end{pmatrix} < 0 \tag{13}$$

$$\begin{pmatrix} A^\top P_1 - C^\top Y_2 + P_1 A - Y_2 C \\ +\beta_{12} P_1 - \tau_{12}(P_2 - P_1) - \gamma_{12} C^\top C & \begin{matrix} P_1 D - Y_2 E \\ -\gamma_{12} C^\top E \end{matrix} \\ \star & \begin{matrix} -\gamma_{12} E^\top E \\ -\mathrm{diag}(\alpha_{12}) \end{matrix} \end{pmatrix} < 0 \tag{14}$$

$$\begin{pmatrix} A^\top P_2 - C^\top Y_2 + P_2 A - Y_2 C \\ +\beta_{22} P_2 - \tau_{22}(P_1 - P_2) - \gamma_{22} C^\top C & \begin{matrix} P_2 D - Y_2 E \\ -\gamma_{22} C^\top E \end{matrix} \\ \star & \begin{matrix} -\gamma_{22} E^\top E \\ -\mathrm{diag}(\alpha_{22}) \end{matrix} \end{pmatrix} < 0 \tag{15}$$

$$1^\top \alpha_{11} - \beta_{11} - \gamma_{11}\delta_1^2\theta_{\mathrm{th}}^2 \leq 0 \tag{16}$$
$$1^\top \alpha_{21} - \beta_{21} - \gamma_{21}\delta_1^2\theta_{\mathrm{th}}^2 \leq 0 \tag{17}$$
$$1^\top \alpha_{12} - \beta_{12} + \gamma_{12}\delta_2^2\theta_{\mathrm{th}}^2 \leq 0 \tag{18}$$
$$1^\top \alpha_{22} - \beta_{22} + \gamma_{22}\delta_2^2\theta_{\mathrm{th}}^2 \leq 0 \tag{19}$$
$$\delta_1 < \delta_2 \tag{20}$$

hold, the estimation error given by the observer (11) with gains $L_1 = P_1^{-1} Y_1$ and $L_2 = P_2^{-1} Y_2$ is invariant with $V(e) = \max\{e^\top P_1 e, e^\top P_2 e\}$.

Proof. Inspired by the proof of [6, Theorem 2], from (1) and (11) it follows that

$$\left.\begin{aligned} \dot{e} &= (A - L_q C)\,e + (D - L_q E)\,w \\ \dot{q} &= 0 \end{aligned}\right\}(e,q,w) \in \tilde{\mathcal{C}} \tag{21a}$$

$$\left.\begin{aligned} e^+ &= e \\ q^+ &= 3 - q \end{aligned}\right\}(e,q,w) \in \tilde{\mathcal{D}} \tag{21b}$$

where $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_1 \cup \tilde{\mathcal{D}}_2$ with

$$\begin{aligned} \tilde{\mathcal{C}}_1 &:= \{(e,q,w) \in \mathbb{R}^n \times \{1,2\} \times [-1,1]^m : |Ce + Ew| \\ &\qquad \geq \delta_1\theta_{\mathrm{th}}, q = 1\}, \\ \tilde{\mathcal{D}}_1 &:= \{(e,q,w) \in \mathbb{R}^n \times \{1,2\} \times [-1,1]^m : |Ce + Ew| \\ &\qquad \leq \delta_1\theta_{\mathrm{th}}, q = 1\}, \\ \tilde{\mathcal{C}}_2 &:= \{(e,q,w) \in \mathbb{R}^n \times \{1,2\} \times [-1,1]^m : |Ce + Ew| \\ &\qquad \leq \delta_2\theta_{\mathrm{th}}, q = 2\}, \\ \tilde{\mathcal{D}}_2 &:= \{(e,q,w) \in \mathbb{R}^n \times \{1,2\} \times [-1,1]^m : |Ce + Ew| \\ &\qquad \geq \delta_2\theta_{\mathrm{th}}, q = 2\} \end{aligned}$$

where $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ correspond to "large" residuals and "small" residuals, respectively.

To ensure invariance, we have to impose the decrease of $V(e) = \max\{e^\top P_1 e, e^\top P_2 e\}$ along the trajectories of (21) with

$$V(e) > 1 \tag{22}$$

and

$$w_i^2 - 1 \leq 0\,,\; i = 1, 2, \ldots, s. \tag{23}$$

If $(e,q,w) \in \tilde{\mathcal{C}}_1$, we need to account for the condition $|Ce + Ew| \geq \delta_1\theta_{\mathrm{th}}$, which leads to

$$\begin{aligned} &- e^\top C^\top C e - e^\top C^\top E w - w^\top E^\top C e - w^\top E^\top E w \\ &+ \delta_1^2\theta_{\mathrm{th}}^2 \leq 0\,. \end{aligned} \tag{24}$$

If $(e,q,w) \in \tilde{\mathcal{C}}_2$, from $|Ce + Ew| \leq \delta_2\theta_{\mathrm{th}}$ it follows that

$$\begin{aligned} &e^\top C^\top C e + e^\top C^\top E w + w^\top E^\top C e + w^\top E^\top E w \\ &- \delta_2^2\theta_{\mathrm{th}}^2 \leq 0 \end{aligned} \tag{25}$$

by squaring both sides.

The function

$$\begin{aligned} e \mapsto V(e) &= \max\{e^\top P_1 e, e^\top P_2 e\} \\ &= \begin{cases} e^\top P_1 e & \text{if } e^\top P_1 e \geq e^\top P_2 e \\ e^\top P_2 e & \text{if } e^\top P_1 e \leq e^\top P_2 e \end{cases} \end{aligned} \tag{26}$$

is continuously differentiable almost everywhere and thus $\nabla V(\cdot)$ exists almost everywhere. The Clarke generalized directional derivative at $e \in \mathbb{R}^n$ in the direction of $v \in \mathbb{R}^n$ is denoted by

$$V^\circ(e,v) := \max_{\zeta \in \partial V(e)} \langle \zeta, v \rangle$$

(see [18, Prop. 2.1.2, p. 27]). Moreover, using [18, Prop. 2.2.1, p. 31], there exists $\zeta \in \partial V(e)$ such that $\dot{V}(e,q,w) = \langle \zeta, \dot{e} \rangle$

and thus $\dot{V}(e,q,w) \le V^\circ(e,\dot{e}) = V^\circ(e,q,w)$ by using (21a), where is given by

$$V^\circ(e,q,w)=\begin{cases}\Phi_{e,w}(P_1,Y_1) & \text{if } e^\top(P_2-P_1)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_1 \qquad (27a)\\ \Phi_{e,w}(P_2,Y_1) & \text{if } e^\top(P_1-P_2)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_1 \qquad (27b)\\ \Phi_{e,w}(P_1,Y_2) & \text{if } e^\top(P_2-P_1)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_2 \qquad (27c)\\ \Phi_{e,w}(P_2,Y_2) & \text{if } e^\top(P_1-P_2)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_2 \qquad (27d)\end{cases}$$

where

$$\Phi_{e,w}(P,Y) := e^\top(A^\top P - C^\top Y^\top + PA - YC)\,e + w^\top(D^\top P - E^\top Y^\top)e + e^\top(PD - YE)w\,.$$

Now we will establish conditions for which $V^\circ(e,q,w)<0$ and thus $\dot{V}(e,q,w)<0$ by using the S-procedure (see, e.g., [13, p. 23]).

First, consider (27a): a sufficient condition for $\dot{V}(e,q,w) \le V^\circ(e,q,w) = \Phi_{e,w}(P_1,Y_1) < 0$ for all $e\in\mathbb{R}^n$ and $w\in[-1,1]^m$ s.t. (22) (i.e., $1-e^\top P_1 e<0$), (23), $e^\top(P_2-P_1)e<0$, and (24) to hold is the existence of $P_1>0$, Y_1, $\beta_{11}\ge 0$, $\alpha_{11}\in\mathbb{R}^s_{\ge 0}$, $\tau_{11}\ge 0$, and $\gamma_{11}\ge 0$ s.t. (12) and (16) are satisfied. Thus, there exists $\varepsilon_{11}>0$ such that $\dot{V}(e,q,w)\le -\varepsilon_{11}|e|^2-\varepsilon_{11}|w|^2$, which ensures the decrease of $V(e)$ along the trajectories of (21) while satisfying (22) and (23).

In a similar way, we proceed from (27b) to get (13) and (17), the satisfaction of which ensures that (4) holds and the existence of ε_{21} such that $\dot{V}(e,q,w)\le -\varepsilon_{21}|e|^2-\varepsilon_{21}|w|^2$; from (27c), to get (14) and (18) and the existence of $\varepsilon_{12}>0$ such that $\dot{V}(e,q,w)\le -\varepsilon_{12}|e|^2-\varepsilon_{12}|w|^2$; from (27d), to get (15) and (19) and the existence of $\varepsilon_{22}>0$ such that $\dot{V}(e,q,w)\le -\varepsilon_{22}|e|^2-\varepsilon_{22}|w|^2$. Putting all together, we get $\dot{V}(e,q,w)\le -\min\{\varepsilon_{11},\varepsilon_{21},\varepsilon_{12},\varepsilon_{22}\}(|e|^2+|w|^2)<0$ along the trajectories of (21) while satisfying (22) and (23), which ensures the decrease of $V(e)$. □

To keep a small asymptotic error at regime with "small" residuals, the design of the observer is accomplished by maximizing the minimum eigenvalue of P_2, as follows:

$$\begin{aligned}&\max_{\substack{P_1,\,P_2\,>\,0;\,Y_1,\,Y_2;\,\alpha\,\in\,\mathbb{R}^{4s}_{\ge 0};\,\beta\,\in\,\mathbb{R}^4_{\ge 0};\\ \gamma\,\in\,\mathbb{R}^4_{\ge 0};\,\tau\,\in\,\mathbb{R}^4_{\ge 0};\,\lambda\,\ge\,0;\,\delta_1,\,\delta_2\,\in\,(0,1)}} \lambda\\ &\text{s.t. } P_2>\lambda I,\,(12)-(20)\end{aligned} \qquad (28)$$

where, for the sake of brevity, $\alpha := (\alpha_{11},\alpha_{21},\alpha_{12},\alpha_{22})$, $\beta := (\beta_{11},\beta_{21},\beta_{12},\beta_{22})$, $\gamma := (\gamma_{11},\gamma_{21},\gamma_{12},\gamma_{22})$, and $\tau := (\tau_{11},\tau_{21},\tau_{12},\tau_{22})$.

A similar result holds by using a min Lyapunov function.

Theorem 3: If there exist two symmetric matrices $P_1,P_2>0$, $Y_1,Y_2\in\mathbb{R}^{n\times m}$, $\alpha_{11},\alpha_{21},\alpha_{12},\alpha_{22}\in\mathbb{R}^s_{\ge 0}$, and scalars $\beta_{11},\beta_{21},\beta_{12},\beta_{22},\gamma_{11},\gamma_{21},\gamma_{12},\gamma_{22},\tau_{11},\tau_{21},\ \tau_{12},\tau_{22}\ \ge\ 0$, $\delta_1,\delta_2\in(0,1)$ such that

$$\begin{pmatrix} A^\top P_1 - C^\top Y_1 + P_1 A - Y_1 C & P_1 D - Y_1 E\\ +\beta_{11}P_1 - \tau_{11}(P_1-P_2) + \gamma_{11}C^\top C & +\gamma_{11}C^\top E\\ \star & \gamma_{11}E^\top E\\ & -\mathrm{diag}(\alpha_{11})\end{pmatrix}<0 \qquad (29)$$

$$\begin{pmatrix} A^\top P_2 - C^\top Y_1 + P_2 A - Y_1 C & P_2 D - Y_1 E\\ +\beta_{21}P_2 - \tau_{21}(P_2-P_1) + \gamma_{21}C^\top C & +\gamma_{21}C^\top E\\ \star & \gamma_{21}E^\top E\\ & -\mathrm{diag}(\alpha_{21})\end{pmatrix}<0 \qquad (30)$$

$$\begin{pmatrix} A^\top P_1 - C^\top Y_2 + P_1 A - Y_2 C & P_1 D - Y_2 E\\ +\beta_{12}P_1 - \tau_{12}(P_1-P_2) - \gamma_{12}C^\top C & -\gamma_{12}C^\top E\\ \star & -\gamma_{12}E^\top E\\ & -\mathrm{diag}(\alpha_{12})\end{pmatrix}<0 \qquad (31)$$

$$\begin{pmatrix} A^\top P_2 - C^\top Y_2 + P_2 A - Y_2 C & P_2 D - Y_2 E\\ +\beta_{22}P_2 - \tau_{22}(P_2-P_1) - \gamma_{22}C^\top C & -\gamma_{22}C^\top E\\ \star & -\gamma_{22}E^\top E\\ & -\mathrm{diag}(\alpha_{22})\end{pmatrix}<0 \qquad (32)$$

$$1^\top\alpha_{11} - \beta_{11} - \gamma_{11}\delta_1^2\theta_{\rm th}^2 \le 0 \qquad (33)$$
$$1^\top\alpha_{21} - \beta_{21} - \gamma_{21}\delta_1^2\theta_{\rm th}^2 \le 0 \qquad (34)$$
$$1^\top\alpha_{12} - \beta_{12} + \gamma_{12}\delta_2^2\theta_{\rm th}^2 \le 0 \qquad (35)$$
$$1^\top\alpha_{22} - \beta_{22} + \gamma_{22}\delta_2^2\theta_{\rm th}^2 \le 0 \qquad (36)$$
$$\delta_1 < \delta_2 \qquad (37)$$

hold, the estimation error given by the observer (11) with gains $L_1 = P_1^{-1}Y_1$ and $L_2 = P_2^{-1}Y_2$ is invariant with $V(e) = \min\{e^\top P_1\, e, e^\top P_2\, e\}$.

Proof. It is omitted for the sake of brevity since it the same of Theorem 2 with

$$V(e) = \begin{cases} e^\top P_1 e & \text{if } e^\top P_1 e \le e^\top P_2 e\\ e^\top P_2 e & \text{if } e^\top P_1 e \ge e^\top P_2 e\end{cases}$$

and

$$V^\circ(e,q,w)=\begin{cases}\Phi_{e,w}(P_1,Y_1) & \text{if } e^\top(P_1-P_2)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_1\\ \Phi_{e,w}(P_2,Y_1) & \text{if } e^\top(P_2-P_1)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_1\\ \Phi_{e,w}(P_1,Y_2) & \text{if } e^\top(P_1-P_2)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_2\\ \Phi_{e,w}(P_2,Y_2) & \text{if } e^\top(P_2-P_1)e<0\,,\\ & (e,q,w)\in\tilde{\mathcal{C}}_2\,.\end{cases}$$

□

Likewise in (28), the design problem is addressed as follows:

$$\begin{aligned}&\max_{\substack{P_1,\,P_2\,>\,0;\,Y_1,\,Y_2;\,\alpha\,\in\,\mathbb{R}^{4s}_{\ge 0};\,\beta\,\in\,\mathbb{R}^4_{\ge 0};\\ \gamma\,\in\,\mathbb{R}^4_{\ge 0};\,\tau\,\in\,\mathbb{R}^4_{\ge 0};\,\lambda\,\ge\,0;\,\delta_1,\,\delta_2\,\in\,(0,1)}} \lambda\\ &\text{s.t. } P_1>\lambda I, P_2>\lambda I,\,(29)-(37)\,.\end{aligned} \qquad (38)$$

In the next section, we will show numerical results concerning what proposed so far.

IV. Numerical Results

For the purpose of comparison, as a case study we dealt with the oscillator with a measurement bias and subject to disturbances considered in [6], [17]:

$$A=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\quad \begin{matrix}C=(1\quad 0\quad 1)\\ \\ E=(0.1\quad 0)\end{matrix}\quad D=\begin{pmatrix}0&0\\0&0.1\\0&0\end{pmatrix}.$$

All the observers for this system were constructed by using YALMIP [19], which was adopted also to solve BMI conditions of Section III by means of grid search methods [15].

First of all, we designed the observer (2) according to the approach proposed in [17] by solving (10) with (i) a bisection method to find the maximum $\beta > 0$ and (ii) a fixed-step reduction of β to maximize the minimum eigenvalue of P. We got $L = (1.0542 \;\; 0.9364 \;\; 0.4742)^\top$, $\theta_{\text{th}} = 0.6822$ and denote this observer as CGO (constant-gain observer). Moreover, we constructed an H_∞ filter and obtained $L = (1.0544 \;\; 0.937 \;\; 0.4745)^\top$ by minimizing the $\mathcal{L}_2$ gain between error and disturbance using a standard LMI program.

We solved the design problem to construct the observer (11) by using a quadratic Lyapunov function (see [6, (28), p. 5]) and found $L_1 = (0.8520 \;\; 1.1343 \;\; 0.4742)^\top$ and $L_2 = (1.1151 \;\; 1.2468 \;\; 0.6771)^\top$. This observer is called switched-gain observer (SGO).

By using a quadratic Lyapunov function, we designed the hold-time switched-gain observer (HTSGO) according to [6, (35), p. 6]. We obtained $L_1 = (1.1151 \;\; 1.2468 \;\; 0.6771)^\top$ and $L_2 = (1.1151 \;\; 1.2468 \;\; 0.6771)^\top$, $\delta_2 = 0.9999$, and chose $T_{\text{h}} = 3$ (threshold of the hold-time timer τ_{h}), $\delta_1 = \delta_2/2$.

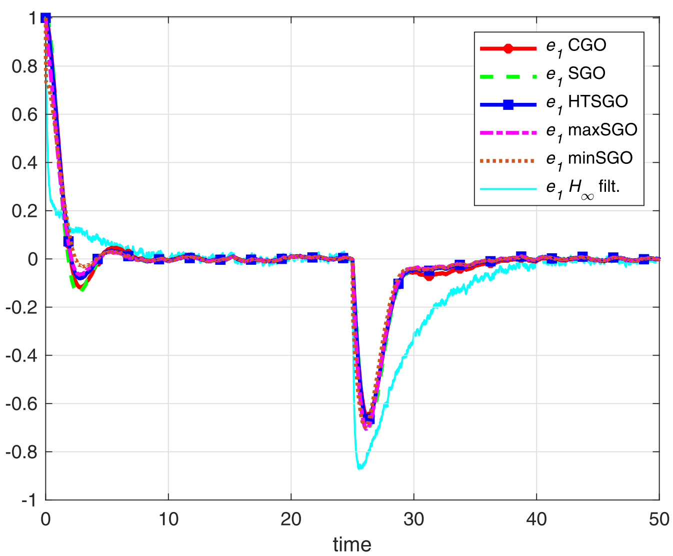

Fig. 2. Evolution of the estimation errors in the first state variable.

Finally, we designed the observers based on max/min Lyapunov functions by solving (28) and (38): we obtained $L_1 = (1.6813 \;\; 2.3899 \;\; 1.0845)^\top$, $L_2 = (1.1371 \;\; 1.2505 \;\; 0.6965)^\top$, $\delta_1 = 0.7500$, $\delta_2 = 0.7750$ and $L_1 = (80.9607 \;\; 75.3800 \;\; 35.4935)^\top$, $L_2 = (1.0113 \;\; 1.0675 \;\; 0.6965)^\top$, $\delta_1 = 0.7542$, $\delta_2 = 0.7639$, respectively. We will denote such observers as maxSGO and minSGO, respectively.

It is worth noting that for the SGO the gain L_2 "larger" than L_1 ($|L_2| > |L_1|$), thus missing the goal to assign a small gain at steady state with 'small" residuals. The HTSGO design provided the same values of L_1 and L_2 but turns out really effective in the transient due to the hold time setup. Instead, the Lyapunov functions adopted to deal with the set invariance of the estimation error for the maxSGO and minSGO depend

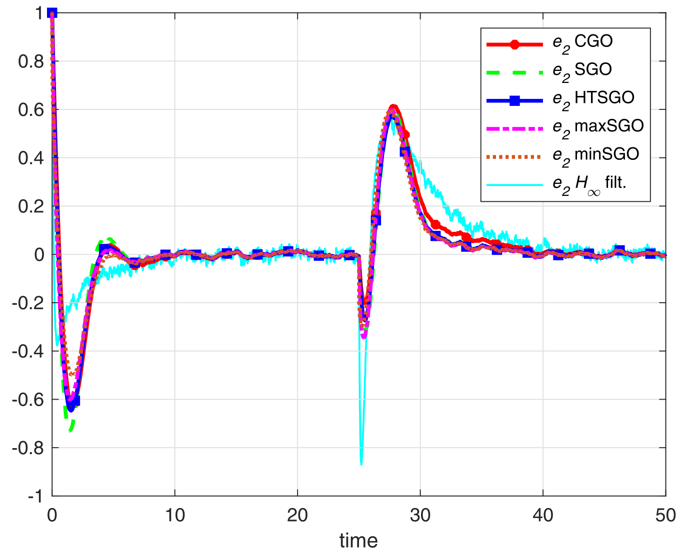

Fig. 3. Evolution of the estimation errors in the second state variable.

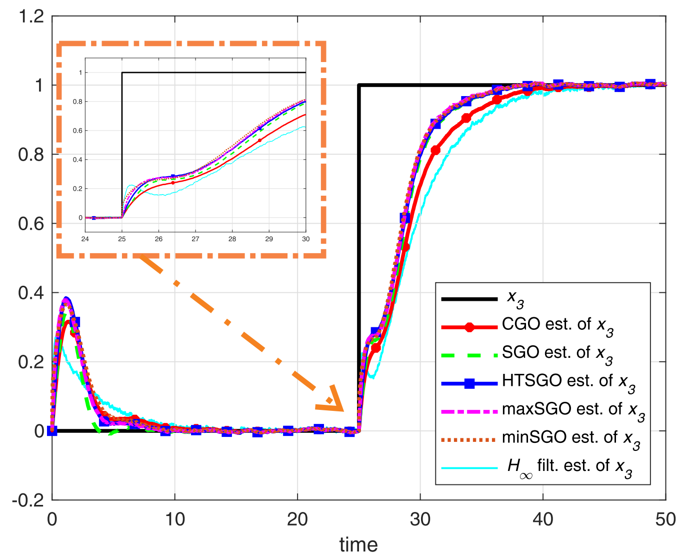

Fig. 4. Evolution of x_3 and its estimates with a zoom around $t = 25$ s.

on a larger number of parameters that can be differently tuned in the transient and at steady state. In other words, the richer structure of the max/min Lyapunov functions may allow for a successful design of the maxSGO and minSGO, as compared with SGO and HTSGO. Figs. 2-5 show the results of a simulation run with initial state $(0.5\,,\,0.5\,,\,0)$, estimated initial states all equal to $(-0.5\,,\,-0.5\,,\,0)$, disturbances chosen according to a uniform distribution in $[-1, 1]$, and a unitary bias on the output occurring at time $t = 25\,\text{s}$ as a step function

TABLE I
COMPUTATIONAL TIMES FOR DESIGN (IN s) AND DECAY RATES (IN $1/\text{s}$)

	CGO	SGO	HTSGO	maxSGO	minSGO	H_∞ filt.
comp. time	1.87	76.16	34.89	773.18	264.86	1.28
decay rate	0.3025	0.2483	0.3977	0.4195	0.3535	0.2669
UAB	0.4117	0.4020	0.4020	0.4017	0.4024	0.5011

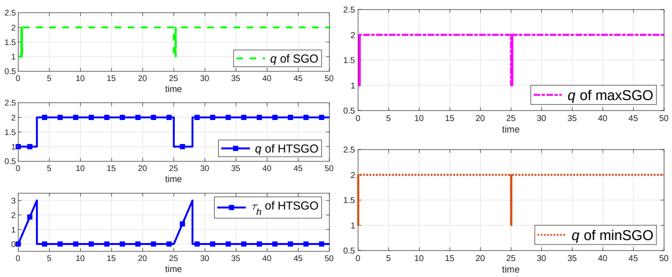

Fig. 5. Evolution of q given by the SGO; q, τ_{h} given by the HTSGO; q given by the maxSGO; q given by minSGO.

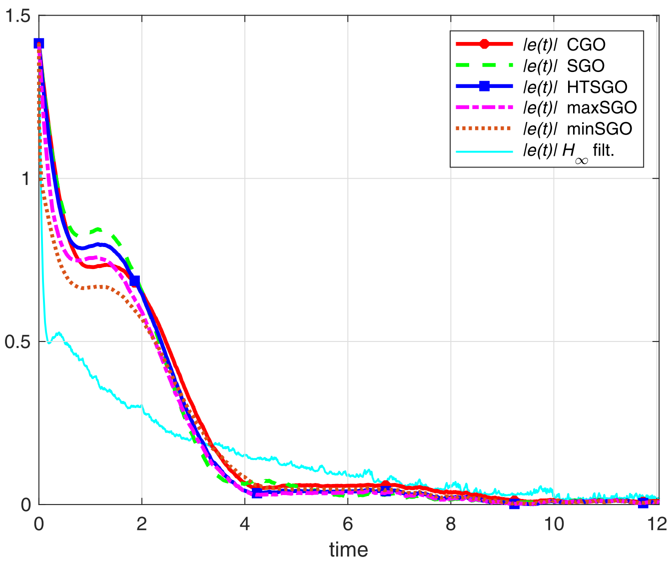

Fig. 6. Evolution of the norms of the all estimation errors during the transient in the same simulation run of Figs. 2-5.

are presented.

Table I presents the computational times required to design the various estimators. The maxSGO and minSGO design methods require significantly more computational resources due to the necessity of employing search grid techniques. However, these methods ensure superior performance in terms of both transient and static precision. Specifically, they achieve the desired goals, such as correct gain assignment, with maxSGO exhibiting a smaller UAB compared to SGO, HTSGO, and minSGO, and the best decay rate for maxSGO. Conversely, the H_∞ approach yields the poorest results (see also Fig. 6), likely because it does not account for the boundedness of the noises.

V. Conclusion

We have investigated switched-gain observers for linear systems affected by bounded disturbances and new design conditions based on the construction of invariant sets described by sublevel sets of max/min Lyapunov functions have been devised. We have demonstrated that the proposed design utilizing Lyapunov functions outperforms the H_∞ filter and the estimators based on the design approaches presented in [6]. This is evidenced by an increased decay rate achieved without compromising performance at regime. Nevertheless, further enhancements will concern the investigation of Lyapunov functions that may depend on time [20], [21] or on the discrete state. Multiple Lyapunovs functions defined over different regions of the state space [22], [23] will be considered as well. The extension to deal with nonlinear systems by using max/min Lyapunov functions seems to be challenging in general, but hopefully with some chance of success under the assumption of Lipschitz nonlinearities [4].

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