Distributed State Estimation with Sporadic Information Exchange over Directed Switching Networks

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Abstract: This paper presents a distributed state estimation problem by a group of networked observer agents for a jointly observable linear time-invariant (LTI) system, where agents communicate sporadically at discrete instants over a communication network that is dynamically changing. In this setting, no single agent can independently estimate the system state using only its own measurements. A distributed hybrid estimation protocol is developed to achieve asymptotically converging state estimates under intermittently arriving measurements and asynchronous inter-agent interactions. By employing Lyapunov-based stability analysis for hybrid systems, sufficient conditions are derived for global exponential stability of the zero estimation error set.

1. INTRODUCTION

The distributed state estimation problem requires the design of observers that reconstruct the state of the system from available system measurements and information received from neighboring observer agents. The primary challenge of this problem setting is that none of the observer agents can independently estimate the system's state using only its own output measurements. The distributed state estimation problem considering continuous communication between observer agents is studied in Kim et al. (2020), and piecewise continuous communication in Liu and Huang (2024); Wang and Guay (2024); Zhang et al. (2024) and Basu and Sanfelice (2024). In practice, however, continuous communication may not be feasible due to limited communication bandwidth, as noted in Hu et al. (2018). Thus, there is a need to develop state estimation schemes relying only on intermittent information exchange between agents.

Another difficulty in distributed estimation is that the communication topology may be time varying. Communication links can change over time, for instance, due to network failures or the need to reconstruct communication pathways. Consequently, when agents communicate at discrete time instances, the interaction may not always happen between the same set of neighbors.

In this paper, we solve a distributed state estimation problem under intermittent information exchange between agents over directed, switching communication networks. Our proposed observer addresses the challenges posed by (i) the limited observability of the system state; (ii) sporadic measurement updates and asynchronous communication between agents; and (iii) directed, aperiodic, and switching networks.

To model and analyze the resulting closed-loop system with inherent continuous and discrete dynamics associated with the impulsive updates in interconnected dynamical systems, we employ the hybrid systems framework in Goebel et al. (2012). A distributed hybrid state estimation protocol is proposed to achieve asymptotically converging estimates under communication constraints. Our approach utilizes the joint observability condition to project each agent's estimated state vector onto its unique observable subspaces, and enables the decomposition of the distributed state estimation problem into multiple decoupled estimation subproblems, each associated with a distinct observable subspace. We demonstrate that if the union of directed, switched graphs over some frequent time intervals guarantees mutual connectivity among all agents, then the distributed state estimation can be achieved asymptotically. This approach also relaxes the need for the bidirectional communication in Zhang et al. (2024) and Liu and Huang (2024), and the graph periodicity requirement in Wang and Guay (2024).

2. GRAPH THEORY PRELIMINARIES

The time-varying (switching) network topology of the agents is described by a digraph $\mathcal{G}_\sigma := (\mathcal{V}, \mathcal{E}_\sigma)$, where $\mathcal{V}$ represents the set of agent nodes, $\mathcal{E}_\sigma \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges representing communication links between them, and $\sigma : [0, \infty) \to \mathcal{P} := \{1, 2, \ldots, \rho_{\max}\}$ is a switching signal that dictates the changes in the interconnection topology. We consider σ as piecewise constant and right-continuous function that switches at times $\{t_s^\sigma\}_{s=1}^\infty$, i.e., $\sigma(t) = \rho_s$ for all $t \in [t_s^\sigma, t_{s+1}^\sigma)$ with the consecutive switching intervals $[t_s^\sigma, t_{s+1}^\sigma)$ adhering to a minimum dwell-time bound $T > 0$, i.e., for each $s \in \mathbb{N}_{\geq 1} := \{1, 2, \cdots\}$, $t_{s+1}^\sigma - t_s^\sigma \geq T$. For any $k > 0$ and $\ell \in \mathbb{N}_{\geq 1}$, a $p-$union

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digraph $\bigcup_{r=\ell}^{\ell+p}\mathcal{G}_{\sigma(t_r^\sigma)}$ is defined as a simple digraph (no self-loops or multiple directed edges between same pair of vertices) with node set $\mathcal{V}$ and edge set given by the union of the edge sets of all the graphs in the collection $\{\mathcal{G}_{\sigma(t_\ell^\sigma)}, \mathcal{G}_{\sigma(t_{\ell+1}^\sigma)}, \cdots, \mathcal{G}_{\sigma(t_{\ell+p}^\sigma)}\}$. A graph is strongly connected if there exists a directed path between each pair of vertices in $\mathcal{V}$.

For each $\sigma \in \mathcal{P}$, let the edge weights for the communication links between agents in the digraph $\mathcal{G}_\sigma$ be denoted as $a_{ik,\sigma}$ with $a_{ii,\sigma} = 1$, i.e., a node is always assumed to be connected to itself (Li et al. (2015)), and $a_{ik,\sigma} = 1$ if $(k,i) \in \mathcal{E}_\sigma$, i.e., an agent i receives information from agent k during the switched configuration of agents $\mathcal{G}_\sigma$, and it is zero otherwise. The set of neighbors of agent i is denoted by $\mathcal{N}_{i,\sigma} \coloneqq \{k \in \mathcal{V} : (k,i) \in \mathcal{E}_\sigma\}$. A directed path from a node i_ℓ to $i_{\ell+h}$ in $\mathcal{G}_\sigma$ is a sequence of directed edges $(i_\ell, i_{\ell+1}), (i_{\ell+1}, i_{\ell+2}), \cdots, (i_{\ell+h-1}, i_{\ell+h})$ from i_ℓ to $i_{\ell+h}$ and we say that the node $i_{\ell+h}$ is reachable from the node i_ℓ. A directed graph is strongly connected if there exists a directed path from every node to every other node. The normalized Laplacian of a time-varying digraph $\mathcal{G}_\sigma$ is denoted as a row-stochastic (unity row sum) matrix $\mathcal{L}_\sigma = [\ell_{ik,\sigma}] \in \mathbb{R}^{N\times N}$ with $\ell_{ii,\sigma} = 1/d_{i,\sigma}$ and $\ell_{ik,\sigma} = a_{ik,\sigma}/d_{i,\sigma}$, where $d_{i,\sigma} \coloneqq 1 + \sum_{k\in\mathcal{N}_{i,\sigma}} a_{ik,\sigma}$ denotes the in-degree of node $i \in \mathcal{V}$.

3. PROBLEM STATEMENT

3.1 System Description

Consider the linear time-invariant (LTI) system

$$\dot{x} = Ax \tag{1}$$

where the state $x \in \mathbb{R}^n$ is to be estimated by a group of N observer agents connected through a network that only allows intermittent information exchange, both between the agents and with system (1). Let

$$y_i = H_i x \in \mathbb{R}^{p_i} \tag{2}$$

denote the measured output of system (1), available sporadically to agent $i \in \mathcal{V}$ at time instants $\{\bar{t}_m^i\}_{m=1}^\infty$ which are not known *a priori* and are not synchronized across all agents. The sequence $\{\bar{t}_m^i\}_{m=1}^\infty$ is strictly increasing, and for such a sequence there exist two positive scalars $\bar{T}_1^i$ and $\bar{T}_2^i$ such that, for all $m \in \mathbb{N}_{\geq 1}$,

$$0 < \bar{T}_1^i \leq \bar{t}_{m+1}^i - \bar{t}_m^i \leq \bar{T}_2^i, \qquad \bar{t}_1^i \leq \bar{T}_2^i. \tag{3}$$

We assume that none of the agents in $\mathcal{V}$ can independently estimate the state x from y_i, i.e., for each $i \in \mathcal{V}$, the pair (A, H_i) is not observable. Due to the limited observability by each individual agent, the goal of the N-agent system is to collectively generate an unbiased estimate of x by relying on local inter-agent communication along with the sporadic system measurements in (2).

Similar to the measurement updates, let $\{\hat{t}_a^i\}_{a=1}^\infty$ denote a sequence of communication time instants, not known *a priori*, satisfying

$$0 < \hat{T}_1^i \leq \hat{t}_{a+1}^i - \hat{t}_a^i \leq \hat{T}_2^i, \qquad \hat{t}_1^i \leq \hat{T}_2^i, \tag{4}$$

when agent i receives an estimated state update from its neighboring agents. The arrival of new information—either due to the measurement updates from the system (1) at each $\bar{t}_m^i$ with $m \in \mathbb{N}_{\geq 1}$, or the estimated state updates from the neighboring agents at $\hat{t}_a^i$ with $a \in \mathbb{N}_{\geq 1}$—trigger a jump in the estimation dynamics of agent i. Since each observer agent updates its state estimates asynchronously, the transmission interval bounds $\bar{T}_1^i, \bar{T}_2^i$ in (3), and $\hat{T}_1^i$ and $\hat{T}_2^i$ in (4), although assumed to be known, may vary significantly across all agents.

Let $\{t_\ell^i\}_{\ell=1}^\infty$ denote a nondecreasing sequence of update time instants when any measurement or communication events occur for agent i. Therefore, $\{t_\ell^i\}_{\ell=1}^\infty \coloneqq \text{Sort}\left(\{\bar{t}_m^i\}_{m=1}^\infty \cup \{\hat{t}_a^i\}_{a=1}^\infty\right)$ is obtained from the ordered union of two sequences $\{\bar{t}_m^i\}_{m=1}^\infty$ and $\{\hat{t}_a^i\}_{a=1}^\infty$. Let $\{\bar{t}_m\}_{m=1}^\infty \coloneqq \text{Sort}\left(\cup_{i\in\mathcal{V}}\{\bar{t}_m^i\}_{m=1}^\infty\right)$ and $\{\hat{t}_a\}_{a=1}^\infty \coloneqq \text{Sort}\left(\cup_{i\in\mathcal{V}}\{\hat{t}_a^i\}_{a=1}^\infty\right)$ denote the nondecreasing sequences of time instants at which measurement update events and state estimate exchange events occur for all agents, respectively. By $\{t_\ell\}_{\ell=1}^\infty \coloneqq \text{Sort}\left(\cup_{i\in\mathcal{V}}\{t_\ell^i\}_{\ell=1}^\infty\right)$, we denote the nondecreasing sequence of event times across all agents.

Additionally, due to network unreliability, the interaction topology of the agents is subject to dynamic changes. Thus, at times $t \in \{\hat{t}_a^i\}_{a=1}^\infty$, agent i may not always receive the state estimate updates from the same set of neighbors. The changing interaction topologies of the agents are dictated by the piecewise constant switching signal $t \mapsto \sigma(t)$, introduced in Section 2, with the switching instants given by the sequence $\{t_s^\sigma\}_{s=1}^\infty$ satisfying a minimum dwell-time bound $T > 0$. Due to the limited observability, and intermittent information exchange both between agents and between the system and agents, all occurring over dynamically changing interaction topologies dictated by σ, estimating the state x collectively among the agent group is a nontrivial task. Next, we formulate a distributed observer to solve this estimation problem.

3.2 Observer Design

Given the intermittent nature of communication events, we consider a hybrid observer model that, for each agent $i \in \mathcal{V}$, has flow dynamics of the form

$$\dot{\hat{x}}_i = A\hat{x}_i \qquad \text{if } t \notin \{t_\ell^i\}_{\ell=1}^\infty, \tag{5}$$

and jumps due to the measurements updates and communication events as[1]

$$\hat{x}_i^+ = \hat{x}_i + \left(\frac{1}{d_{i,\sigma(t)}}\right) L_i(y_i - H_i\hat{x}_i) \qquad \text{if } t \in \{\bar{t}_m^i\}_{m=1}^\infty, \tag{6}$$

$$\hat{x}_i^+ = \hat{x}_i + \sum_{k\in\mathcal{N}_{i,\sigma(t)}} \left(\frac{a_{ik,\sigma(t)}}{d_{i,\sigma(t)}}\right) M_i(\hat{x}_k - \hat{x}_i) \quad \text{if } t \in \{\hat{t}_a^i\}_{a=1}^\infty, \tag{7}$$

where $\hat{x}_i \in \mathbb{R}^n$ is the observer state of the agent $i \in \mathcal{V}$, $A \in \mathbb{R}^{n\times n}$ is the system matrix in (1), $L_i \in \mathbb{R}^{n\times p_i}$ is an observer gain and $M_i \in \mathbb{R}^{n\times n}$ is a weighting matrix, which will be designed. If $\hat{t}_a^i \in [t_s^\sigma, t_{s+1}^\sigma)$ for some $a, s \in \mathbb{N}_{\geq 1}$, then, $\sigma(\hat{t}_a^i) = \sigma(t_s^\sigma) \coloneqq \rho_s \in \mathcal{P}$.

For each agent $i \in \mathcal{V}$ and each $\sigma \in \mathcal{P}$, the estimation error dynamics with state vector $\tilde{x}_i \coloneqq \hat{x}_i - x$, based on (1) and (5)–(7), is given by

$$\dot{\tilde{x}}_i = A\tilde{x}_i \qquad \text{if } t \notin \{t_\ell^i\}_{\ell=1}^\infty, \tag{8a}$$

$$\tilde{x}_i^+ = \left(I - \frac{1}{d_{i,\sigma}} L_i H_i\right)\tilde{x}_i \qquad \text{if } t \in \{\bar{t}_m^i\}_{m=1}^\infty, \tag{8b}$$

[1] The proposed hybrid modeling approach introduced in Section 4 allows for simultaneous events.

$$\tilde{x}_i^+ = \tilde{x}_i + \frac{1}{d_{i,\sigma}} \sum_{k\in\mathcal{N}_{i,\sigma}} a_{ik,\sigma} M_i\left(\tilde{x}_k - \tilde{x}_i\right) \text{ if } t \in \{\bar{t}_m^i\}_{m=1}^{\infty}. \quad (8c)$$

Let $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_N) \in \mathbb{R}^{Nn}$ denote the augmented error vector with the dynamics of $\tilde{x}_i$ is given in (8a)–(8c). Next, we define the problem to solve in this paper.

Problem 1. Given the system (1), sporadically measured outputs (2) by each agent $i \in \mathcal{V}$ with known inter-transmission interval bounds $\bar{T}_1^i, \bar{T}_2^i$ in (3), and intermittent information exchange between them with known interval bounds $\hat{T}_1^i$, $\hat{T}_1^i$ in (4), design gain matrices L_i and M_i in the distributed state estimation protocol (5)–(7) for each $i \in \mathcal{V}$, such that, for any initial conditions $x_0 \in \mathbb{R}^n$ and $\hat{x}_{i0} \in \mathbb{R}^n$, and for any arbitrary switching signal $t \mapsto \sigma(t)$ dictating transitions between directed switching network topology $\mathcal{G}_{\sigma(t)}$ at switching instants t_s^σ with minimum dwell-time $T > 0$, the set

$$\mathcal{A} := \{\tilde{x} \in \mathbb{R}^{Nn} : \tilde{x}_i = 0 \ \forall i \in \mathcal{V}\} \quad (9)$$

is globally asymptotically stable for (8a)–(8c).

4. MAIN RESULTS

To solve Problem 1, we consider the following assumptions on observability, system dynamics, and the connectivity of observer agents. To this end, we first define the collective measurement vector and output matrix as

$$y := (y_1, y_2, \cdots, y_N) = \bar{H}x, \ \bar{H} := (H_1, H_2, \cdots, H_N), \quad (10)$$

respectively, with y_i, given in (2), which is measured sporadically at time instants $\{\bar{t}_m^i\}_{m=1}^{\infty}$.

Assumption 1. The pair $(A, \bar{H})$ is observable.

Remark 1. For each agent $i \in \mathcal{V}$, let $\nu_i \in \mathbb{N}_{\geq 1}$ be the dimension of the observable subspace $\mathcal{S}_i$ of the pair (A, H_i), i.e., $\text{rank}(\mathcal{O}_i) = \nu_i$, where $\mathcal{O}_i := \left[H_i^\top, (H_iA)^\top, \ldots, (H_iA^{n-1})^\top\right]^\top$ is the observability matrix associated with (A, H_i). The unobservable subspace $\mathcal{U}_i := \text{Ker}(\mathcal{O}_i)$ has dimension $n - \nu_i$. Since $\mathcal{S}_i := \text{Im}(\mathcal{O}_i^\top)$, then $\mathcal{U}_i = \mathcal{S}_i^\perp$. Furthermore, by Assumption 1, we have that $\bigcap_{i=1}^N \mathcal{U}_i = \{0\}$.

Assumption 2. The matrix A in (1) is neutrally stable. [2]

Remark 2. Under Assumption 2, there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix A is similar to a block diagonal matrix $\bar{A} := P^{-1}AP$ with each block being a real Jordan block of the form $J_k = \left[\begin{smallmatrix} 0 & -b_k \\ b_k & 0 \end{smallmatrix}\right]$ corresponding to a complex-conjugate eigenvalue pair $\pm ib_k$ or a scalar block $J_k = 0$ corresponding to the zero eigenvalue. As a result, $\bar{A}^\top = -\bar{A}$. For simplicity, and without loss of generality, we consider A to be skew-symmetric, similar to Zhang et al. (2024) and Liu and Huang (2024).

Assumption 3. There exists a subsequence $\{\ell_k\}_{k=1}^{\infty}$ of the index sequence $\{\ell\}_{\ell=1}^{\infty}$ associated with event times t_ℓ such that, for some $v > \bar{T}_2 := \max_{\forall i\in\mathcal{V}} \bar{T}_2^i$ with $\bar{T}_2^i$ defined in (3), $t_{\ell_{k+1}} - t_{\ell_k} < v$ for each $k \in \mathbb{N}_{\geq 1}$, and the union digraph of N observer agents $\bigcup_{c=\ell_k}^{\ell_{k+1}-1} \mathcal{G}_{\sigma(t_c)}$ is strongly connected.

Remark 3. By Assumption 3, in every interval $[t_{\ell_k}, t_{\ell_{k+1}})$ uniformly bounded by $v > 0$, the following hold: (a) all the agents are reachable from each other in the digraph $\bigcup_{c=\ell_k}^{\ell_{k+1}-1} \mathcal{G}_{\sigma(t_c)}$, and (b) each agent receives measurement updates from system (1) at least once.

[2] All eigenvalues of A are on the imaginary axis and semi-simple (algebraic multiplicity = geometric multiplicity).

4.1 Observability Decomposition

For each agent $i \in \mathcal{V}$, let $U_i \in \mathbb{R}^{n\times(n-\nu_i)}$ be a matrix whose columns form an orthonormal basis of $\mathcal{U}_i \in \mathbb{R}^{n\times(n-\nu_i)}$ such that $\text{Im}(U_i) = \mathcal{U}_i = \ker(\mathcal{O}_i)$. Let $S_i \in \mathbb{R}^{n\times\nu_i}$ be a matrix whose columns form an orthonormal basis for $\mathcal{S}_i = \mathcal{U}_i^\perp$. Then, by construction, it follows that $\text{Im}(S_i) = \mathcal{S}_i = \text{Im}(\mathcal{O}_i^\top)$. We now define an orthogonal matrix Σ_i as

$$\Sigma_i := [S_i \ U_i] \in \mathbb{R}^{n\times n}, \ \Sigma_i^\top \Sigma_i = I_n. \quad (11)$$

We note that Σ_i satisfies a variant of the Kalman observability decomposition lemma, given below, the proof of which is given in Lemma 3.2 of Zhang et al. (2024).

Lemma 1. Under Assumption 2, for each $i \in \mathcal{V}$,

$$\Sigma_i^\top A \Sigma_i = \begin{bmatrix} \bar{A}_i & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \ H_i\Sigma_i = \begin{bmatrix} \bar{H}_i & 0 \end{bmatrix}, \quad (12)$$

where $\bar{A}_i \in \mathbb{R}^{\nu_i\times\nu_i}$, $\hat{A}_i \in \mathbb{R}^{(n-\nu_i)\times(n-\nu_i)}$, $\bar{H}_i \in \mathbb{R}^{p_i\times\nu_i}$, and

- (I) $(\bar{A}_i, \bar{H}_i)$ is an observable pair,
- (II) $\bar{A}_i$ and $\hat{A}_i$ are skew-symmetric,
- (III) $S_i^\top A S_i = \bar{A}_i$ and $U_i^\top A U_i = \hat{A}_i$.

4.2 Distributed Observer Design

Following the decentralized design approach in Kim et al. (2020), for each $i \in \mathcal{V}$, we construct

$$L_i = \Sigma_i \begin{bmatrix} \bar{L}_i \\ 0 \end{bmatrix}, \ M_i = \Sigma_i \begin{bmatrix} 0_{\nu_i\times\nu_i} & 0 \\ 0 & I_{n-\nu_i} \end{bmatrix} \Sigma_i^\top, \quad (13)$$

where Σ_i is given in (11) and $n - \nu_i$ is the dimension of $\mathcal{U}_i$. By using the orthogonal transformation (11), and substituting L_i and M_i from (13) into (8a)–(8c), for each $\sigma \in \mathcal{P}$, the estimation error dynamics projected onto the observable subspace $\mathcal{S}_i$ becomes

$$S_i^\top \dot{\tilde{x}}_i = \bar{A}_i S_i^\top \tilde{x}_i \qquad \text{if } t \notin \{\bar{t}_m^i\}_{m=1}^{\infty}, \quad (14)$$

$$S_i^\top \tilde{x}_i^+ = \left(I - \mu_{i,\sigma}\bar{L}_i\bar{H}_i\right) S_i^\top \tilde{x}_i \qquad \text{if } t \in \{\bar{t}_m^i\}_{m=1}^{\infty}, \quad (15)$$

while for the projection onto the unobservable subspace $\mathcal{U}_i$, we have

$$U_i^\top \dot{\tilde{x}}_i = \hat{A}_i U_i^\top \tilde{x}_i \qquad \text{if } t \notin \{\hat{t}_a^i\}_{a=1}^{\infty}, \quad (16)$$

$$U_i^\top \tilde{x}_i^+ = \mu_{i,\sigma}\Big(U_i^\top \tilde{x}_i + \sum_{k\in\mathcal{N}_{i,\sigma}} a_{ik,\sigma} U_i^\top \tilde{x}_k\Big) \quad \text{if } t \in \{\hat{t}_a^i\}_{a=1}^{\infty}, \quad (17)$$

where $\mu_{i,\sigma} := 1/d_{i,\sigma}$ and $d_{i,\sigma} \in \{1, 2, \cdots, N\}$. The structure of L_i and M_i in (13) leads to the dynamics associated with the measurement events, through projection to the observable subspace, and the inter-agent communication events, through projection to the unobservable subspace.

The decentralized design of these observer gains was initially proposed in Kim et al. (2020) for static graphs, where σ is constant, and later extended to undirected switching graphs in Zhang et al. (2024) with $a_{ij,\sigma} = a_{ji,\sigma}$ for all $\sigma \in \mathcal{P}$. In Wang and Guay (2024), the switching signal characterizes periodicity of networks, i.e., $\sigma(t + T) = \sigma(t)$ for all $t \geq 0$ with period $T > 0$. In this paper, we consider $t \mapsto \sigma(t)$ to be discrete, aperiodic, and that allows for transitions between directed network topologies.

4.3 Projections Onto Unique Observable Subspaces

Let each agent $i \in \mathcal{V}$ have a unique observable subspace $\mathcal{S}_i$ that collectively spans $\mathbb{R}^n$, i.e., $\mathcal{S}_i \cap \mathcal{S}_k = \{0\}$ (or $S_i^\top S_k = 0$) and $\cup_{i\in\mathcal{V}} \mathcal{S}_i = \mathbb{R}^n$ (or equivalently $\sum_{i=1}^N \nu_i = n$). Then, from Assumption 1, for each $i \in \mathcal{V}$,

$$\mathcal{U}_i = \bigcup\nolimits_{k\in\mathcal{V}\setminus\{i\}} \mathcal{S}_k. \tag{18}$$

By appropriately exploiting the joint observability condition and mutual connectivity of agents in Assumption 3, the estimation protocol (5)–(7) can also be extended to the case where multiple agents may have identical observable subspaces, as well as nonzero intersections between the observable subspaces of different agents. In that case, we create a hypothetical cluster containing agents with identical observable subspaces. The unobservable subspace of an agent belonging to a cluster is the observable subspace of the rest of the clusters in the network. Due to space limitations, this approach will be published elsewhere.

Under Assumption 1, the unobservable subspace of each agent is the same as the observable subspace of the remaining $N-1$ agents. For each $i \in \mathcal{V}$, let the matrix $S_{-i} \in \mathbb{R}^{n\times(n-\nu_i)}$ be constructed by concatenating matrices S_k where $k \in \mathcal{V}\setminus\{i\}$. Therefore, the column spaces of S_{-i} form an orthonormal basis for the subspace $\mathcal{S}_{-i} := \bigcup_{k\in\mathcal{V}\setminus\{i\}} \mathcal{S}_k$ in (18). However, these basis vectors may not necessarily be identical to those spanning $\mathcal{U}_i$. Then, the change of orthonormal basis vectors between two identical subspaces $\mathcal{U}_i$ and $\mathcal{S}_{-i}$ yields that there exists an orthogonal transformation $\hat{\Gamma}_i \in \mathbb{R}^{(n-\nu_i)\times(n-\nu_i)}$ such that

$$U_i^\top = \hat{\Gamma}_i S_{-i}^\top. \tag{19}$$

By substituting (19) into (16)–(17), and using Lemma 1, we derive the flow dynamics (if $t \neq \hat{t}_a^i$) and jump dynamics (if $t = \hat{t}_a^i$) for each $\sigma \in \mathcal{P}$ as

$$S_{-i}^\top \dot{\tilde{x}}_i = (S_{-i}^\top A S_{-i}) S_{-i}^\top \tilde{x}_i \qquad \text{if } t \notin \{\hat{t}_a^i\}_{a=1}^\infty, \tag{20}$$

$$S_{-i}^\top \tilde{x}_i^+ = \mu_{i,\sigma}\Big(S_{-i}^\top \tilde{x}_i + \sum_{k\in\mathcal{N}_{i,\sigma}} a_{ik,\sigma} S_{-i}^\top \tilde{x}_k\Big) \quad \text{if } t = \{\hat{t}_a^i\}_{a=1}^\infty. \tag{21}$$

Since S_{-i} is a concatenation of the matrices S_r with $r \in \mathcal{V}\setminus\{i\}$, then by Item (III) of Lemma 1, for each $i \in \mathcal{V}$ and each $r \in \mathcal{V}\setminus\{i\}$, (20)–(21) yield

$$S_r^\top \dot{\tilde{x}}_i = \bar{A}_r\, S_r^\top \tilde{x}_i, \qquad \text{if } t \notin \{\hat{t}_a^i\}_{a=1}^\infty. \tag{22}$$

$$S_r^\top \tilde{x}_i^+ = \mu_{i,\sigma}\Big(S_r^\top \tilde{x}_i + \sum_{k\in\mathcal{N}_{i,\sigma}} a_{ik,\sigma} S_r^\top \tilde{x}_k\Big) \qquad \text{if } t \in \{\hat{t}_a^i\}_{a=1}^\infty. \tag{23}$$

By using (19), the flow and jump maps of the unobservable subspace of agent $i \in \mathcal{V}$ in (16) and (17), respectively, are decomposed into N independent projected components, with one defined by (22) and the remaining $N-1$ components by (23). Now, we reorganize (22) and (23) in accordance to the projections onto the same subspace $\mathcal{S}_i$ for each $i \in \mathcal{V}$. To this end, the dynamics associated with the projections onto $\mathcal{S}_i$, for each $i \in \mathcal{V}$ and each $r \in \mathcal{V}\setminus\{i\}$, are

$$S_i^\top \dot{\tilde{x}}_r = \bar{A}_i\, S_i^\top \tilde{x}_r \qquad \text{if } t \notin \{\hat{t}_a^r\}_{a=1}^\infty, \tag{24}$$

$$S_i^\top \tilde{x}_r^+ = \mu_{r,\sigma}\Big(S_i^\top \tilde{x}_r + \sum_{k\in\mathcal{N}_{r,\sigma}} a_{rk,\sigma} S_i^\top \tilde{x}_k\Big) \quad \text{if } t \in \{\hat{t}_a^r\}_{a=1}^\infty. \tag{25}$$

Based on (18), the estimation problem with state $\hat{x}_i \in \mathbb{R}^n$ in (5)–(7), or the error state $\tilde{x}_i \in \mathbb{R}^n$ in (8a)–(8c) thus reduces to N decoupled estimation problems associated with N observable subspaces $\mathcal{S}_i$ for each $i \in \mathcal{V}$ and corresponding projected error vector components (14)–(15) and (24)–(25).

4.4 Leader-Follower Architecture in Decoupled Subproblem

In each of the N decoupled estimation problems, there is one node with the dynamics (14)—(15) that does not have an incoming edge, thanks to the structure of the gain and weighting matrices defined in (13). We refer to such nodes as master nodes. Corresponding to the digraph $\mathcal{G}_\sigma$ of N observer agents (5)–(7), let $\mathcal{L}_\sigma^i \in \mathbb{R}^{N\times N}$ be the normalized Laplacian matrix representing the connectivity among N nodes for the i^{th} decoupled problem with the master node dynamics in (14)–(15), and the remaining $N-1$ follower node dynamics in (24)–(25). Let us denote the digraph for the i^{th} decoupled problem as $\mathcal{G}_\sigma^i$.

Given that the master node i in the i^{th} decoupled problem has only outgoing edges or no edges at all, it mirrors a classical "leader-follower" architecture, and thus

$$\mathcal{L}_\sigma^i = \left[\begin{array}{c|c} 1 & 0_{1\times(N-1)} \\ \hline \Delta_\sigma^i \mathbf{1}_{N-1} & \mathbb{H}_\sigma^i \end{array}\right] \qquad \forall \sigma \in \mathcal{P}, \tag{26}$$

where $\Delta_\sigma^i \in \mathbb{R}^{(N-1)\times(N-1)}$ is a diagonal matrix with the diagonal elements $a_{ki,\sigma}$ for all $k \in \mathcal{V}\setminus\{i\}$ representing the outgoing edges from node i, and $\mathbb{H}_\sigma^i$ is a nonnegative matrix of order $(N-1)$. Since the master node for each decoupled problem is unique and has no incoming edges, the digraph $\mathcal{G}_\sigma^i$ (equivalently, $\mathcal{L}_\sigma^i$) is distinct for all N decoupled problems.

Let $\tilde{x}_{iS} := S_i^\top \tilde{x}_i \in \mathbb{R}^{\nu_i}$, $\tilde{\boldsymbol{x}}_{-i} := (I_{N-1} \otimes S_i^\top)\tilde{x}_{-i} \in \mathbb{R}^{(N-1)\nu_i}$, and $\tilde{x}_{-i} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_N) \in \mathbb{R}^{(N-1)n}$ be an augmented state vector with $N-1$ components $\tilde{x}_k$ for all $k \in \mathcal{V}\setminus\{i\}$. Let $\{\hat{t}_a^{-i}\}_{a=1}^\infty$ be a sequence of nondecreasing time instants, composed of the time instants $\hat{t}_a^r$ for each $r \in \mathcal{V}\setminus\{i\}$, when all but agent i receives an estimated state update from its neighbors. Therefore, $\{\hat{t}_a^{-i}\}_{a=1}^\infty := \text{Sort}\Big(\bigcup_{r\in\mathcal{V}\setminus\{i\}} \{\hat{t}_a^r\}_{a=1}^\infty\Big)$. Then, from (24)–(25), the follower node dynamics at the i^{th} decoupled problem with $\tilde{\zeta}_{iS} := \big(\Delta_\sigma^i \mathbf{1}_{N-1} \otimes \tilde{x}_{iS}\big)$ becomes

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}}_{-i} &= (I_{N-1} \otimes \bar{A}_i)\tilde{\boldsymbol{x}}_{-i} && \text{if } t \notin \{\hat{t}_a^{-i}\}_{a=1}^\infty, \\ \tilde{\boldsymbol{x}}_{-i}^+ &= \big(\mathbb{H}_\sigma^i \otimes I_{\nu_i}\big)\tilde{\boldsymbol{x}}_{-i} + \tilde{\zeta}_{iS} && \text{if } t \in \{\hat{t}_a^{-i}\}_{a=1}^\infty, \end{aligned} \tag{27}$$

while the master node dynamics (14)–(15) rewritten as

$$\begin{aligned} \dot{\tilde{x}}_{iS} &= \bar{A}_i \tilde{x}_{iS} && \text{if } t \notin \{\bar{t}_m^i\}_{m=1}^\infty, \\ \tilde{x}_{iS}^+ &= \big(I - \mu_{i,\sigma}\bar{L}_i \bar{H}_i\big)\,\tilde{x}_{iS} && \text{if } t \in \{\bar{t}_m^i\}_{m=1}^\infty. \end{aligned} \tag{28}$$

The master node i in (28) and its $(N-1)$ follower nodes in (27) together form a cascaded N-node system where (28) is an autonomous subsystem that drives (27) through a serial interconnection. Since the evolution of the state of the master node in (28) is unaffected by those of (27), the stability analysis of (28) can be conducted independently.

4.5 Stability Analysis of Decoupled Subproblem

By introducing a timer variable $\bar{\tau}_i$, the dynamics in (28), can be analyzed with the hybrid systems framework in Goebel et al. (2012). This timer variable keeps track of the duration of flows and triggers a jump when new measurements from the system (1) arrive at $\{\bar{t}_m^i\}_{m=1}^\infty$ satisfying (3). After each jump, the observer starts to flow again for at least $\bar{T}_1^i$ time until the next jump occurs, at which point $\bar{\tau}_i$ is reset to a value in the interval $[\bar{T}_1^i, \bar{T}_2^i]$.

Let $\tilde{\xi}_i := (\tilde{x}_{iS}, \bar{\tau}_i)$. Then, the master node dynamics (28) are represented by the hybrid system

$$\mathcal{H}_i \begin{cases} \dot{\tilde{\xi}}_i = \bar{f}_i(\tilde{\xi}_i) & \tilde{\xi}_i \in C_i \\ \tilde{\xi}_i^+ \in \bar{g}_i(\tilde{\xi}_i) & \tilde{\xi}_i \in D_i, \end{cases} \tag{29}$$

where the flow, jump sets, and corresponding maps are

$$C_i := \{\tilde{\xi}_i \in \mathbb{R}^{\nu_i} \times \mathbb{R}_{\geq 0} : \bar{\tau}_i \in [0, \bar{T}_2^i]\}, \tag{30}$$

$$D_i := \{\tilde{\xi}_i \in \mathbb{R}^{\nu_i} \times \mathbb{R}_{\geq 0} : \bar{\tau}_i = 0\}, \tag{31}$$

$$\bar{f}_i(\tilde{\xi}_i) := \big(\bar{A}_i \tilde{x}_{iS},\ -1\big) \qquad \forall \tilde{\xi}_i \in C_i \tag{32}$$

$$\bar{g}_i(\tilde{\xi}_i) := \big(\big(I - \mu_{i,\sigma} \bar{L}_i \bar{H}_i\big)\, \tilde{x}_{iS},\ [\bar{T}_1^i, \bar{T}_2^i]\big) \quad \forall \tilde{\xi}_i \in D_i. \tag{33}$$

Let $\mathcal{A}_i := \{\tilde{\xi}_i \in \mathbb{R}^{\nu_i} \times \mathbb{R}_{\geq 0} : \tilde{x}_{iS} = 0, \bar{\tau}_i \in [0, \bar{T}_2^i]\}$ denote the set to be rendered globally exponentially stable for $\mathcal{H}_i$. We present the following result, the proof of which follows from the Theorem 1 of Ferrante et al. (2016).

Lemma 2. Let $\bar{T}_1^i > 0$ and $\bar{T}_2^i > 0$ be given. Suppose there exist $P_i \in \mathbb{R}^{\nu_i \times \nu_i} \succ 0$, a matrix $\bar{L}_i \in \mathbb{R}^{\nu_i \times p_i}$, and a scalar $\mu_i \in [1/N, 1]$ such that, for all $v_i \in [\bar{T}_1^i, \bar{T}_2^i]$,

$$(I - \mu_i \bar{L}_i \bar{H}_i)^\top e^{\bar{A}_i^\top v_i} P_i e^{\bar{A}_i v_i} (I - \mu_i \bar{L}_i \bar{H}_i) - P_i \prec 0. \tag{34}$$

Then the set $\mathcal{A}_i$ is globally exponentially stable for $\mathcal{H}_i$.

By introducing a timer variable $\hat{\tau}_{-i}$ to trigger jump events at $\hat{t}_a^{-i}$ for each $a \in \mathbb{N}_{\geq 1}$, the follower nodes with state vector $\tilde{\xi}_{-i} := (\tilde{\boldsymbol{x}}_{-i}, \hat{\tau}_{-i})$ with $\tilde{\boldsymbol{x}}_{-i}$ in (27), and input $\tilde{\zeta}_{iS} \in \mathbb{R}^{(N-1)\nu_i}$ defined above (27) can be formulated as

$$\mathcal{H}_{-i} \begin{cases} \dot{\tilde{\xi}}_{-i} = \bar{f}_{-i}(\tilde{\xi}_{-i}) & \tilde{\xi}_{-i} \in C_{-i} \\ \tilde{\xi}_{-i}^+ = \bar{g}_{-i}(\tilde{\xi}_{-i}) + \tilde{\zeta}_{iS} & \tilde{\xi}_{-i} \in D_{-i}, \end{cases} \tag{35}$$

where the flow, jump sets, and corresponding maps are

$$C_{-i} := \{\tilde{\xi}_{-i} \in \mathbb{R}^{(N-1)\nu_i} \times \mathbb{R}_{\geq 0} : \hat{\tau}_{-i} \in [0, \hat{T}_2^{-i}]\}, \tag{36}$$

$$D_{-i} := \{\tilde{\xi}_{-i} \in \mathbb{R}^{(N-1)\nu_i} \times \mathbb{R}_{\geq 0} : \hat{\tau}_{-i} = 0\}, \tag{37}$$

$$\bar{f}_{-i}(\tilde{\xi}_{-i}) := \big((I_{N-1} \otimes \bar{A}_i)\tilde{\boldsymbol{x}}_{-i},\ -1\big) \qquad \forall \tilde{\xi}_{-i} \in C_{-i} \tag{38}$$

$$\bar{g}_{-i}(\tilde{\xi}_{-i}) := \Big((\mathbb{H}_\sigma^i \otimes I_{\nu_i})\tilde{\boldsymbol{x}}_{-i},\ [\hat{T}_1^{-i}, \hat{T}_2^{-i}]\Big) \ \forall \tilde{\xi}_{-i} \in D_{-i} \tag{39}$$

where $\hat{T}_1^{-i} := \min_{r \in \mathcal{V} \setminus \{i\}} \{\hat{T}_1^r\}$, $\hat{T}_2^{-i} := \max_{r \in \mathcal{V} \setminus \{i\}} \{\hat{T}_2^r\}$, with $\hat{T}_1^i$ and $\hat{T}_2^i$ introduced in (4).

Due to the linearity of the flow map $\bar{f}_{-i}$ in (35) finite time escape through flow is not possible. Furthermore, $\tilde{\zeta}_{iS}$ affects the state component $\tilde{\boldsymbol{x}}_{-i}$ of $\tilde{\xi}_{-i}$ only, but not the timer variable $\hat{\tau}_{-i}$, reset of which triggers the jump events for $\mathcal{H}_{-i}$. As a result, $\tilde{\zeta}_{iS}$ has no influence on the flow duration and jump times of $\mathcal{H}_{-i}$. Additionally, for any $\tilde{\xi}_{-i} \in D_{-i}$, the jump map $\bar{g}_{-i}(\tilde{\xi}_{-i}) + \tilde{\zeta}_{iS} \in (C_{-i} \cup D_{-i}) = \mathbb{R}^{(N-1)\nu_i} \times \mathbb{R}_{\geq 0}$. Thus, given any solution $\tilde{\zeta}_{iS}$ to $\mathcal{H}_i$, and any initial condition $\tilde{\xi}_{-i}(0,0) \in C_{-i} \cup D_{-i}$, every maximal solution $\tilde{\xi}_{-i}$ to $\mathcal{H}_{-i}$ is complete.

For any such solution $\tilde{\xi}_{-i}$ to $\mathcal{H}_{-i}$, from direct integration of the flow dynamics, and the jump condition in (35), for any $(t,j) \in \operatorname{dom} \tilde{\boldsymbol{x}}_{-i}$ with $t \in [\hat{t}_j^{-i}, \hat{t}_{j+1}^{-i})$,

$$\begin{aligned} \tilde{\boldsymbol{x}}_{-i}(t,j) &= \prod_{k=1}^{j} \Big(\mathbb{H}^i_{\sigma(\hat{t}_k^{-i})} \otimes e^{\bar{A}_i t}\Big) \tilde{\boldsymbol{x}}_{-i}(0,0) \\ &+ \Big(I_{N-1} \otimes e^{\bar{A}_i (t - \hat{t}_j^{-i})}\Big) \tilde{\zeta}_{iS}(\hat{t}_j^{-i}, j-1) \qquad (40) \\ &+ \sum_{k=1}^{j-1} \left(\prod_{r=j-k+1}^{j} \mathbb{H}^i_{\sigma(\hat{t}_r^{-i})} \otimes\ e^{\bar{A}_i (t - \hat{t}_{j-k}^{-i})} \right) \tilde{\zeta}_{iS}(\hat{t}_{j-k}^{-i}, j-k-1). \end{aligned}$$

Theorem 1. Let Assumptions 1–3 hold. Given system (1) with measurements y_i in (2) that are sporadically available at instants $\{\bar{t}_m^i\}_{m=1}^{\infty}$ with inter-transmission interval bounds $\bar{T}_1^i$, $\bar{T}_2^i > 0$ satisfying (3), the distributed state estimation protocol (5)–(7) with L_i, M_i given in (13) and the gain matrix $\bar{L}_i$ in the construction of L_i satisfying (34) in Lemma 2 solves Problem 1.

Sketch of the Proof: By Lemma 2, at the i^{th} decoupled problem with $i \in \mathcal{V}$, the hybrid system $\mathcal{H}_{-i}$ given in (35) is fed by the exponentially decaying $(t,j) \mapsto \tilde{\zeta}_{iS}(t,j)$ generated by the autonomous system $\mathcal{H}_i$ in (29). By Assumption 2 and Remark 2, we have $|e^{\bar{A}_i t}| = 1$ for any $t \geq 0$. Therefore, $t \mapsto e^{\bar{A}_i (t)}$ in all three terms of (40) are bounded functions. In (40), the terms $\tilde{\zeta}_{iS}(\hat{t}_j^{-i}, j-1)$ denote the value of $\tilde{\zeta}_{iS}$ just before the j^{th} jump in $\mathcal{H}_{-i}$ in (35). Since the stability of $\mathcal{H}_i$ in (29) is unaffected by the jumps in $\mathcal{H}_{-i}$ at each time instant $\hat{t}_j^{-i}$, the exponential convergence of $\tilde{\zeta}_{iS}$ is preserved even upon events occurring in $\mathcal{H}_{-i}$. Thus, with increasing $t+j$ where $(t,j) \in \operatorname{dom} \tilde{\boldsymbol{x}}_{-i}$, as $\hat{t}_j^{-i}$ increases, the distance to the set $\tilde{\mathcal{A}}_i$ for $\mathcal{H}_i$ exponentially decreases, and as a result,

$$\lim_{t+j \to \infty} \tilde{\boldsymbol{x}}_{-i}(t,j) = \lim_{t+j \to \infty} \prod_{k=1}^{j} \Big(\mathbb{H}^i_{\sigma(\hat{t}_k^{-i})} \otimes e^{\bar{A}_i t}\Big) \tilde{\boldsymbol{x}}_{-i}(0,0). \tag{41}$$

By Assumption 3, across every interval $[t_{\ell_k}, t_{\ell_{k+1}})$, the union digraph $\bigcup_{c=\ell_k}^{\ell_{k+1}-1} \mathcal{G}_{\sigma(t_c)}$ formed by the inter-agent information exchange events is strongly connected. As the observable component with the state vector $\tilde{x}_{iS}$ in (28) only has outgoing edges, the union digraph $\bigcup_{c=\ell_k}^{\ell_{k+1}-1} \mathcal{G}^i_{\sigma(t_c)}$ ($\mathcal{G}^i_\sigma$ is the digraph of N nodes corresponding to $\mathcal{L}^i_\sigma$ in (26)) has a master node i with state vector $\tilde{x}_{iS}$, that does not have a parent node, but has a directed path to the remaining $N-1$ follower nodes, resembling a leader-follower architecture with a directed spanning tree rooted at node i. Then, by (Ren and Beard, 2005, Lemma 3.9, Corollary 3.5), the matrix $\prod_{c=\ell_k}^{\ell_{k+m}-1} \mathbb{H}^i_{\sigma(t_c)}$ can be shown to be Schur stable.

Since the switching signal σ takes only finitely many values in $\mathcal{P}$, every interval product of the Schur stable matrices $\prod_{c=\ell_{k+q-1}}^{\ell_{k+q}-1} \mathbb{H}^i_{\sigma(t_c)}$ across $[t_{\ell_{k+q-1}}, t_{\ell_{k+q}})$ for any $q \in \mathbb{N}_{\geq 1}$ come from a finite set. We denote this set as $\mathcal{M} := \{\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_\kappa\}$ for some $\kappa \in \mathbb{N}_{\geq 1}$. All products of matrix elements with arbitrary length from $\mathcal{M}$ have spectral radius strictly less than 1. Hence for any $\gamma \in [0,1)$ there exists an integer $k \geq 1$ such that for every sequence $\mathcal{M}_{r_1}, \ldots, \mathcal{M}_{r_k} \in \mathcal{M}$, we have $\rho\big(\mathcal{M}_{r_k} \cdots \mathcal{M}_{r_1}\big) < \gamma < 1$. Then, by (Morris, 2012, Theorem 1.1) and Berger and Wang (1992), all such infinite products of matrices from $\mathcal{M}$ converges to zero asymptotically. Thus, from (41), we have $\lim_{t+j \to \infty} \tilde{\boldsymbol{x}}_{-i}(t,j) = 0$ for all $i \in \mathcal{V}$. Consequently, the error vector $\tilde{x}_i$ in (8a)–(8c), or equivalently $\tilde{x}$, converges to the origin asymptotically. □

The inequality (34) is nonlinear in P_i and $\bar{L}_i$, and it needs to be verified for infinitely many values of v_i in $[\bar{T}_1^i, \bar{T}_2^i]$, making it computationally infeasible. A tractable design methodology, inspired by the convex polytopic embedding approach in Ferrante et al. (2016) can be adopted to design $\bar{L}_i$, which, however, is skipped here due to limited space.

5. SIMULATION RESULTS

For the numerical example, we consider a triple frequency harmonic oscillator system with A in (1) given by

$$A = \text{diag}(18.5,\ 14.1,\ 7.6) \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which is monitored by four sensor agents. Let the inter-transmission interval bounds for the measurement updates (3) to each agent be given as $\bar{T}_1^i = 0.05$, $\bar{T}_2^i := 0.1i$ for each $i \in \mathcal{V} := \{1,2,3,4\}$. The local measurements y_i to the sensor agents for each i arrive at the instants $\bar{t}_i^{m+1} = \bar{t}_i^m + \bar{T}_2^i$ with $\bar{t}_i^0 = 0$ and output matrices

$$H_1 = [1\ 0_{1\times 5}], \qquad H_2 = [0_{1\times 3}\ 1\ 0_{1\times 2}],$$
$$H_3 = [0_{1\times 4}\ 1\ 0], \qquad H_4 = [0\ 1\ 0_{1\times 4}].$$

None of the pairs (A, H_i), $i \in \mathcal{V}$, is observable, and agents 1 and 4 have the same observable subspaces. Based on three unique observable subspaces, we create three clusters $\mathcal{C}_1 = \{1,4\}$, $\mathcal{C}_2 = \{2\}$, $\mathcal{C}_3 = \{3\}$. The unobservable subspace of one cluster is the collective observable subspace of the remaining clusters, satisfying Assumption 1.

Let the inter-transmission interval bounds for state estimate exchange events (4) for the agents be given as $\hat{T}_1^i = 0.1$ for each $i \in \mathcal{V}$, $\hat{T}_2^2 = \hat{T}_2^4 = 0.6$, and $\hat{T}_2^1 = \hat{T}_2^3 = 1.2$. Therefore, agents 1 and 3 may potentially receive twice as many updates as agents 2 and 4. Let us assume that these four agents communicate over a switching digraph $\mathcal{G}_\sigma$ with

$$\sigma(t) = \text{mod}\left(\frac{s}{4}\right) + 1 \quad \forall t \in [sT/4, (s+1)T/4),\ \forall s \in \mathbb{N}_{\geq 1}$$

where $T = 0.8$. In every $[sT, (s+1)T)$ time-interval, the union digraph is strongly connected, and each agent $i \in \mathcal{V}$ receives $(8/i)$ many measurement updates, thus satisfying Assumption 3 with $v = T$. Next, we construct the distributed observers (5)–(7) with

$$L_1 = [1\ -0.1\ 0_{4\times 1}]^\top,\ L_2 = [0_{2\times 1}\ 0.48\ 1\ 0_{2\times 1}]^\top,$$
$$L_3 = [0_{4\times 1}\ 1\ -0.1]^\top,\ L_4 = [-0.33\ 1\ 0_{4\times 1}]^\top,$$
$$M_1 = \text{diag}(0_{2\times 2}, I_4),\ M_2 = \text{diag}(I_2, 0_{2\times 2}, I_2),$$
$$M_3 = \text{diag}(I_4, 0_{2\times 2}),\ M_4 = \text{diag}(0_{2\times 2}, I_4).$$

These choices allow the observer agents to successfully track the system state vector despite communication constraints, as observed from Figure 1.

6. CONCLUSION

In this paper, we studied distributed state estimation problem of a linear system that is jointly observable by a group of observer agents that receives sporadic measurement or estimated state update from the system or its neighbors, respectively, over directed switching network topologies. Building on the decentralized observer design in Kim et al. (2020), by utilizing the joint observability property, and by assuming a mild connectivity condition, our proposed hybrid estimation protocol reconstructs the system state asymptotically despite network constraints that allow only sporadic information exchange. Characterizing robustness of the proposed algorithm is part of future work.

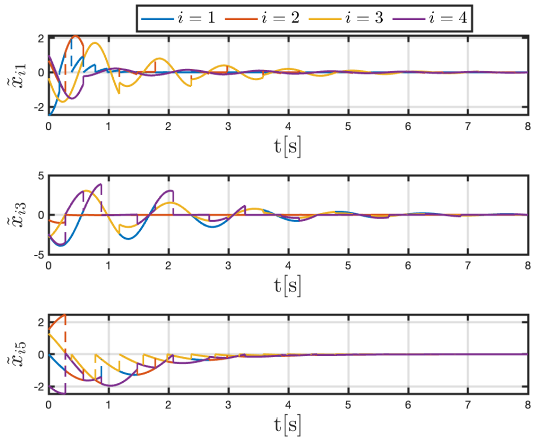

Fig. 1. The estimated error states by the observer agents.

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