Geometric Hybrid Dynamical Systems on Intrinsic Manifolds: Part I – Framework and Hybrid Lyapunov Theorem

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***Abstract*— This paper develops the hybrid Lyapunov theorem for geometric hybrid dynamical systems, namely hybrid inclusions evolving on intrinsic C^1-manifolds. We present various nonsmooth analysis notions, which aid in the formulation of sufficient conditions for existence of solutions. We also present topological definitions of uniform global stability, attractivity, and asymptotic stability of a nonempty, compact set $\mathcal{A}$. Finally, the Lyapunov theorem provides relaxed conditions for uniform global asymptotic stability of the set $\mathcal{A}$. An example illustrates the concepts and results.**

I. Introduction

A large class of systems, ranging from legged robots to spacecraft dynamics, evolve on non-Euclidean manifolds. Hybrid systems theory has proven useful to establish global asymptotic stability results for such systems when the underlying manifold is noncontractible [1], [2]. However, much of the literature on hybrid dynamical systems focuses on systems evolving on the Euclidean space $\mathbb{R}^n$ [3], [4]. Consequently, the characterization of uniform stability and attractivity of a desired nonempty set $\mathcal{A} \subset \mathbb{R}^n$ for a hybrid dynamical system in [4, Definition 3.1] assumes knowledge of the Riemannian metric, which, on $\mathbb{R}^n$, is assumed to be the standard metric.

However, Riemannian metrics on smooth manifolds, while they exist [5, Proposition 13.3], may not be known. Embedding the manifold into Euclidean space can induce a metric on the manifold. Although such an embedding approach has been used in the literature, even in the context of hybrid dynamical systems [2], we prefer to think of the manifold from an intrinsic geometric perspective – independent of the ambient Euclidean space – for the following reasons.

Firstly, while the Whitney Embedding theorem guarantees existence of an embedding, it does not characterize it, thus limiting its applicability in instances where the embedding is not apparent. Examples include quotient manifolds, which not only lack canonical embeddings, but also potentially lose intrinsic geometric structure when embedded in Euclidean space. For example, the isometry group of a 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the group of rotations, reflections, and translations in $\mathbb{R}^2$. However, embedding $\mathbb{T}^2$ in $\mathbb{R}^3$, thus resulting in the familiar $\mathbb{S}^1 \times \mathbb{S}^1$ 'doughnut' structure, results in loss of translations from the isometry group.

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Secondly, even on intrinsic manifolds without using embeddings, additional structure on the manifold may pose several challenges. Consider, for example, disconnected Riemannian manifolds which arise due to hybrid systems operating in multiple modes of operation; see [2], [6] for examples pertaining to attitude control and estimation. Even with the lack of a continuous curve connecting two points on the disconnected manifold, [5, Corollary 13.30] defines an appropriate distance between them. However, due to 'jumps' in the definition therein, there does not exist a global smooth Riemannian metric on $\mathcal{M}$ that yields the said distance definition. Furthermore, since this definition depends on arbitrarily chosen points on each connected component of the manifold $\mathcal{M}$, certifying stability of $\mathcal{A} \subset \mathcal{M}$ can be difficult as the Lyapunov function, defined using the Riemannian distance to $\mathcal{A}$, may increase along solutions in certain regions on $\mathcal{M}$. This issue can be circumvented by a topological approach to analysis of hybrid dynamical systems on manifolds, thus only using the intrinsic properties of the manifold.

Motivated by the above discussion, we observe that there is scarce literature on hybrid dynamical systems evolving on intrinsic smooth manifolds. Some results include [7], [8], where a class of hybrid systems operating in different modes are studied. A more general modeling approach is presented in [9], where topological hybrid systems are introduced. While these papers present novel and insightful modeling approaches, the dynamics therein enforce uniqueness of solutions as they are modeled using differential equations, and not differential inclusions, which are crucial to study properties of nonsmooth and perturbed dynamical systems [3, Chapter 4]. Such modeling for geometric hybrid systems is presented in [10] to study multistable inclusions. However, [10] considers a connected Riemannian manifold, thus limiting its applicability.

In light of a lack of tools for analysis of hybrid inclusions from a topological and geometric viewpoint, we study hybrid dynamical systems, modeled as hybrid inclusions, evolving on C^1-manifolds from a geometric point of view. We refer to these systems as *geometric hybrid dynamical systems*. First, we present various notions of nonsmooth analysis on manifolds that provide the foundation for the rest of this paper. Then, sufficient conditions for existence of nontrivial solutions are discussed. We also define topological notions of uniform global stability, (pre-)attractivity, and (pre-)asymptotic stability of a nonempty, compact set $\mathcal{A}$. Finally, we present the hybrid Lyapunov theorem that provides pointwise sufficient conditions for uniform global asymptotic stability of $\mathcal{A}$, analogous to the version on Euclidean space [4, Theorem 3.19], but without assuming additional structure on the manifold. Additional results on the invariance principle for geometric hybrid dynamical systems on C^1-manifolds are provided in the companion paper [11]. Due to space constraints, proofs will be presented elsewhere.

II. Preliminaries

A. Notation

We denote the set of real and nonnegative numbers by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, respectively. The set of natural numbers, including 0, is denoted by $\mathbb{N}$. The distance from a point $x \in \mathbb{R}^n$ to a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is defined as $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|$. Let $\mathcal{X}$ denote a topological space with some topology. The closure of a set $S \subset \mathcal{X}$ is denoted by $\overline{S}$, and its interior by $\operatorname{int} S$. The set S is compact if every open cover of S has a finite subcover, and it is precompact if $\overline{S}$ is compact. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$ function, denoted by $\alpha \in \mathcal{K}_\infty$, if α is zero at zero, continuous, strictly increasing, and unbounded. A function $\sigma : X \to \mathbb{R}_{\geq 0}$ is positive definite with respect to $\mathcal{A} \subset X$, denoted by $\sigma \in \mathcal{PD}(\mathcal{A})$, if $\sigma(x) = 0$ if and only if $x \in \mathcal{A}$. The composition of two maps $f : X \to Y$ and $g : Y \to Z$ is denoted by $g \circ f : X \to Z$. A set-valued map $F : X \rightrightarrows Y$ maps each point $x \in X$ to a subset $F(x) \subset Y$. Given a nonempty set $S \subset \mathbb{R}^n$, the tangent cone $\mathrm{T}_S(x)$ to S at $x \in S$ is the set of all vectors $w \subset \mathbb{R}^n$ for which there exist sequences $x_i \in S$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \searrow 0$, and $w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$.

B. Differential geometry

The notions of a topological manifold $\mathcal{M}$, coordinate charts, atlas, and maximal atlas of $\mathcal{M}$ are borrowed from [5]. A pair (U, φ) is a coordinate chart, or chart, on a topological manifold $\mathcal{M}$ if $U \subset \mathcal{M}$ is open and $\varphi : U \to \mathbb{R}^{\dim \mathcal{M}}$ is a homeomorphism. A chart (U, φ) of $\mathcal{M}$ is called a chart at $x \in \mathcal{M}$ if $x \in U$. Two charts (U, φ) and (W, ψ) of $\mathcal{M}$ are C^k-compatible if $U \cap W = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ onto its image is a C^k-diffeomorphism. A C^k-atlas of $\mathcal{M}$ is one whose charts are C^k-compatible. A C^k-manifold $\mathcal{M}$ is a topological manifold endowed with a maximal C^k-atlas. The tangent space to $\mathcal{M}$ at $x \in \mathcal{M}$ is denoted by $\mathrm{T}_x\mathcal{M}$, and the tangent bundle of $\mathcal{M}$ is denoted by $\mathrm{T}\mathcal{M}$.

A map $f : \mathcal{M} \to \mathcal{N}$ between two C^k-manifolds is C^k at $x \in \mathcal{M}$ if there exists a coordinate chart (U, φ) at $x \in \mathcal{M}$ and a chart (W, ψ) at $f(x) \in \mathcal{N}$ such that the coordinate representation $\psi \circ f \circ \varphi^{-1}$ is C^k in the usual Euclidean sense. The map f is said to be C^k if it is C^k at each $x \in \mathcal{M}$. The tangent map of f at $x \in \mathcal{M}$, denoted by $\mathrm{d}f_x : \mathrm{T}_x\mathcal{M} \to \mathrm{T}_{f(x)}\mathcal{N}$, is defined by $\mathrm{d}f_x(v) := \frac{df}{dt}(\gamma(t))\Big|_{t=0}$, where $\mathcal{I} \ni t \mapsto \gamma(t)$, with $\mathcal{I} \subset \mathbb{R}$ such that $0 \in \mathcal{I}$, is a smooth curve on $\mathcal{M}$ satisfying $\gamma(0) = x$ and $\gamma'(0) = v \in \mathrm{T}_x\mathcal{M}$.

A Riemannian manifold [12] is a pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is a C^k-manifold and g is a Riemannian metric on $\mathcal{M}$ whose value at each $x \in \mathcal{M}$ defines an inner-product on $\mathrm{T}_x\mathcal{M}$. The gradient of a C^1-function $f : \mathcal{M} \to \mathbb{R}$ at $x \in \mathcal{M}$, denoted by $\operatorname{grad} f(x)$, is such that $\mathrm{d}f_x(v) = g(\operatorname{grad} f(x), v)$ for each $v \in \mathrm{T}_x\mathcal{M}$. The set of critical points of f is denoted by $\operatorname{crit} f := \{x \in \mathcal{M} : \operatorname{grad} f(x) = 0\}$.

C. Regularity of set-valued maps

A set-valued map $F : \mathcal{M} \rightrightarrows \mathcal{N}$ between topological manifolds is outer semicontinuous at $x \in \mathcal{M}$ if for each sequence of points x_i converging[1] to x and each convergent sequence of points $y_i \in F(x_i)$, it follows that $y \in F(x)$, where $\lim_{i\to\infty} y_i = y$. The map F is outer semicontinuous if it is outer semicontinuous at each $x \in \mathcal{M}$. Given a nonempty set $S \subset \mathcal{M}$, F is outer semicontinuous relative to S the restriction of F to S is outer semicontinuous at each $x \in S$.

Furthermore, F is locally precompact at $x \in \mathcal{M}$ if there exist a neighborhood $\mathcal{U}$ of x such that $F(\mathcal{U})$ is precompact. The map F is said to be locally precompact if it is locally precompact at each $x \in \mathcal{M}$. Given a nonempty set $S \subset \mathcal{M}$, the map F is said to be locally precompact relative to S if F, restricted to S, is locally precompact at each $x \in S$.

Finally, a single-valued function $f : \mathcal{M} \to \mathbb{R}$ is lower semicontinuous if, at each $x \in \mathcal{M}$, $f(x) \leq \liminf_{y\to x} f(y)$.

III. Geometric Hybrid Dynamical Systems

In this paper, we define geometric hybrid dynamical systems as follows:

$$\mathcal{H} : \begin{cases} \dot{x} \;\; \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{1}$$

where $x \in \mathcal{M}$ is the state, $C \subset \mathcal{M}$ and $D \subset \mathcal{M}$ are the flow set and the jump set, respectively, $G : D \rightrightarrows \mathcal{M}$ is the jump map, and $\mathcal{M}$ is a finite dimensional C^1-manifold. The flow map is a set-valued map such that $F(x) \subset \mathrm{T}_x\mathcal{M}$ for each $x \in C$, and is denoted as $F : C \rightrightarrows \mathrm{T}\mathcal{M}$. This hybrid system is denoted by $\mathcal{H} = (C, F, D, G, \mathcal{M})$. We will assume $\mathcal{M}$ to be a finite-dimensional C^1-manifold.

Solutions to $\mathcal{H}$ are defined on hybrid time domains [4, Definition 2.26], parametrized by ordinary time $t \in \mathbb{R}_{\geq 0}$, that denotes the amount of time for which the solution has flowed, and a jump counter $j \in \mathbb{N}$ that denotes the number of jumps that have occurred. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if there exists $J \in \mathbb{N}$ such that

$$E = \bigcup_{j=0}^{J} [t_j, t_{j+1}] \times \{j\}, \tag{2}$$

for some finite sequence of times $\{t_j\}_{j=0}^{J+1}$ satisfying $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_J \leq t_{J+1}$. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if it is the union of compact hybrid time domains E_j such that $E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_j \ldots$.

Definition 1 (Solution to a hybrid system)**.** A function $\phi : \operatorname{dom}\phi \to \mathcal{M}$ is a solution to the geometric hybrid dynamical system $\mathcal{H} = (C, F, D, G, \mathcal{M})$ if $\operatorname{dom}\phi$ is a hybrid time domain, for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally

[1] Convergence is defined in the topological sense, without using a metric on the space. In particular, a sequence $\{x_i\}_{i=0}^{\infty} \subset \mathcal{M}$ converges to $x \in \mathcal{M}$, written as $\lim_{i\to\infty} x_i = x$, if for each neighborhood $U \subset \mathcal{M}$ of x, there exists $i^* \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq i^*$.

absolutely continuous on the interval $I^j := \{t : (t,j) \in \operatorname{dom}\phi\}$, $\phi(0,0) \in \overline{C} \cup D$, and

- for each $j \in \mathbb{N}$ such that $\operatorname{int} I^j \neq \varnothing$, ϕ satisfies

$$\begin{aligned} \phi(t,j) &\in C && \text{for all} && t \in \operatorname{int} I^j, \\ \dot{\phi}(t,j) &\in F(\phi(t,j)) && \text{for almost all} && t \in I^j; \end{aligned}$$

- for each $(t,j) \in \operatorname{dom}\phi$ such that $(t,j+1) \in \operatorname{dom}\phi$,

$$\begin{aligned} \phi(t,j) &\in D, \\ \phi(t,j+1) &\in G(\phi(t,j)). \end{aligned}$$

Note that the notion of local absolute continuity used in the definition above is borrowed from [13, Section A.2.1]. We reproduce it here for completeness.

Definition 2 (Local absolute continuity)**.** A function $\gamma : \mathbb{R} \supset I \to \mathcal{M}$ is said to be locally absolutely continuous if, for each C^1-function $\psi : \mathcal{M} \to \mathbb{R}$, the composition $\psi \circ \gamma : I \to \mathbb{R}$ is locally absolutely continuous.

Given a solution ϕ to $\mathcal{H}$, its total time of flow is given by $\sup_t \operatorname{dom}\phi := \sup\{t : \exists j \text{ s.t. } (t,j) \in \operatorname{dom}\phi\}$. Similarly, the number of times ϕ jumps is given by $\sup_j \operatorname{dom}\phi := \sup\{j : \exists t \text{ s.t. } (t,j) \in \operatorname{dom}\phi\}$. Then, we define $\sup \operatorname{dom}\phi := (\sup_t \operatorname{dom}\phi, \sup_j \operatorname{dom}\phi)$.

A solution ϕ is *nontrivial* if $\operatorname{dom}\phi$ contains at least two points, it is *maximal* if it cannot be extended, and it is *complete* if its domain is unbounded. Given a nonempty set $X_o \subset \overline{C} \cup D$, we denote by $\hat{\mathcal{S}}_{\mathcal{H}}(X_o)$ (resp., $\mathcal{S}_{\mathcal{H}}(X_o)$) the set of solutions (resp., maximal solutions) to $\mathcal{H}$ from X_o.

Next, we provide some regularity conditions that impart desirable properties to $\mathcal{H}$, some of which are explored in the companion paper [11].

Definition 3 (Geometric hybrid basic conditions)**.** The geometric hybrid dynamical system $\mathcal{H} = (C, F, D, G, \mathcal{M})$ is said to satisfy the *geometric hybrid basic conditions* if

1) $\mathcal{M}$ is a C^1-manifold;
2) C and D are subsets of $\mathcal{M}$ that are closed relative to $\mathcal{M}$;
3) $F : C \rightrightarrows \mathrm{T}\mathcal{M}$ is outer semicontinuous and locally precompact relative to C, and $F(x)$ is convex for all $x \in C$;
4) $G : D \rightrightarrows \mathcal{M}$ is outer semicontinuous and locally precompact relative to D.

If $\mathcal{M} = \mathbb{R}^n$ and the hybrid system $\mathcal{H}$ satisfies Definition 3, it is shown in [3, Chapter 6] that $\mathcal{H}$ is well-posed; in particular, the set of solutions to $\mathcal{H}$ with compact time domains is sequentially compact, and dynamical properties of $\mathcal{H}$ are robust to arbitrarily small perturbations. This notion of well-posedness requires knowledge of a distance metric on the manifold. Since we assume $\mathcal{M}$ to be a C^1-manifold, a distance metric on $\mathcal{M}$ may be unknown. Consequently, a similar notion is not yet defined for the class of systems in (1). A weaker notion only requiring sequential compactness of the set of compact solutions to $\mathcal{H}$, namely, nominal well-posedness, is defined in the companion paper [11].

In the following example, we look at a geometric hybrid dynamical system on a Möbius band, which we will revisit in the following sections to illustrate the results therein.

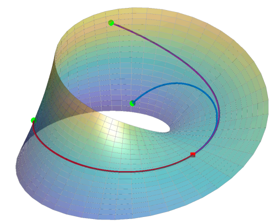

Fig. 1: Illustration of the open Möbius band embedded in $\mathbb{R}^3$, with sample trajectories converging to $\widetilde{\mathcal{A}}$.

Example 1 (The Möbius band). Consider the open Möbius band $\mathscr{B} := ([0,1] \times (-1,1))/\sim$ with the equivalence relation given by $(0, z_2) \sim (1, -z_2)$ for each $z_2 \in (-1,1)$. See Fig. 1 for an illustration of the Möbius band. Note that, due to the quotient operation, $\mathscr{B}$ is a quotient space. With some abuse of notation, an element of $\mathscr{B}$ is denoted by $[z]_{\mathscr{B}} := [z_1, z_2]_{\mathscr{B}}$, where the notation $[\cdot]_{\mathscr{B}}$ denotes the corresponding equivalence class. Equipped with smooth coordinate charts according to [5, Ex. 10.3], $\mathscr{B}$ is a smooth manifold, i.e., a C^∞-manifold. We also endow $\mathscr{B}$ with a Riemannian metric $g_{\mathscr{B}}$ according to [12, Ex. 2.35].

Let $Q := \{-1, 1\}$ and define the smooth manifold $\mathcal{M} := \mathscr{B} \times Q$. For each $q \in Q$, define a C^1-function $V_q \in \mathcal{PD}(\widetilde{\mathcal{A}})$ for some nonempty, compact set $\widetilde{\mathcal{A}} \subset \mathscr{B}$. Suppose that

$$\mu := \min_{\substack{q \in Q \\ [z]_{\mathscr{B}} \in \operatorname{crit} V_q \setminus \widetilde{\mathcal{A}}}} \left\{ V_q([z]_{\mathscr{B}}) - \min_{p \in Q} V_p([z]_{\mathscr{B}}) \right\} > 0. \quad (3)$$

With this construction, we pick $\delta \in (0, \mu)$ and define a geometric hybrid dynamical system $\mathcal{H} = (C, F, D, G, \mathcal{M})$ with state $x := ([z]_{\mathscr{B}}, q) \in \mathcal{M}$ as follows:

$$\begin{aligned} C &:= \left\{ ([z]_{\mathscr{B}}, q) \in \mathcal{M} : V_q([z]_{\mathscr{B}}) - \min_{q \in Q} V_q([z]_{\mathscr{B}}) \leq \delta \right\}, \\ F(x) &:= \begin{pmatrix} -\operatorname{grad} V_q([z]_{\mathscr{B}}) \\ 0 \end{pmatrix} \qquad \forall x \in C, \\ D &:= \left\{ ([z]_{\mathscr{B}}, q) \in \mathcal{M} : V_q([z]_{\mathscr{B}}) - \min_{q \in Q} V_q([z]_{\mathscr{B}}) \geq \delta \right\}, \\ G(x) &:= \begin{pmatrix} [z]_{\mathscr{B}} \\ \arg\min_{q \in Q} V_q([z]_{\mathscr{B}}) \end{pmatrix} \qquad \forall x \in D. \end{aligned}$$

Since $\mu > 0$ and $\operatorname{card}(Q) = 2$, G is single valued. ■

IV. Nonsmooth Analysis on Manifolds

In this section, we define various notions of nonsmooth analysis on manifolds that will be used throughout the paper. First, we present the notion of tangent cone to a subset of a C^1-manifold. This notion is closely related to the notion of tangent cones to subsets of the Euclidean space.

Definition 4 (Tangent cone to subsets of manifolds [14, Definition 3.2])**.** Let $\mathcal{M}$ be an n-dimensional C^1-manifold and let S be a nonempty subset of $\mathcal{M}$. The tangent cone $\mathrm{T}_S^{\mathcal{M}}(x)$ at $x \in S$ is defined as

$$\mathrm{T}_S^{\mathcal{M}}(x) := (\mathrm{d}\varphi_x)^{-1}(\mathrm{T}_{\varphi(S \cap U)}(\varphi(x))),$$

where (U, φ) is any coordinate chart of $\mathcal{M}$ at x, and $\mathrm{T}_{\varphi(S \cap U)}(\varphi(x))$ denotes the tangent cone to the set $\varphi(S \cap U)$ at the point $\varphi(x)$.

In the above definition, the inverse of the map $\mathrm{d}\varphi_x : \mathrm{T}_x\mathcal{M} \to \mathbb{R}^n$ exists as it is an isomorphism [5, Prop. 3.6(d)]. Next, we define the notion of locally Lipschitz function between manifolds as follows. This notion will be instrumental in relaxing the requirement of a Lyapunov function candidate to be continuously differentiable, as will be seen in the forthcoming Section VII.

Definition 5 (Locally Lipschitz function between manifolds)**.** Given C^1-manifolds $\mathcal{M}$ and $\mathcal{N}$, a function $f : \mathcal{M} \to \mathcal{N}$ is locally Lipschitz on $\mathcal{M}$ if for each pair of coordinate charts (U, φ) and (W, ψ) on $\mathcal{M}$ and $\mathcal{N}$, respectively, such that $V \cap f(U) \neq \varnothing$, the function $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(W)) \to \psi(W)$ is locally Lipschitz, where $f^{-1}(W) := \{x \in \mathcal{M} : f(x) \in W\}$.

Remark 1. For Definition 5 to be invariant under the choice of coordinate charts, the transition map [5, Ch.1] between coordinate charts on $\mathcal{M}$ (resp., $\mathcal{N}$) should be locally Lipschitz. This holds since each transition map is a C^1-diffeomorphism.

Remark 2. Local Lipschitz continuity of a function between metric spaces is often defined using distance between points. This definition solely depends on the choice of the metrics on the spaces. Definition 5, however, implicitly assumes the standard metric on $\mathbb{R}^n$ to comment about local Lipschitz continuity of the coordinate representation of the map f therein. Therefore, if $\mathbb{R}^n$ is endowed with a metric that is not equivalent [15, p.22] to the standard metric, then Definition 5 differs from the metric-based definition [5, p.609].

Next, we provide a version of the Rademacher's theorem for locally Lipschitz functions on manifolds. This result will be instrumental in bounding the rate of change of a locally Lipschitz Lyapunov function candidate.

Proposition 1 (Rademacher's theorem on manifolds)**.** *Consider a finite-dimensional C^1-manifold $\mathcal{M}$ and a locally Lipschitz function $f : \mathcal{M} \to \mathbb{R}$. Then, f is differentiable almost everywhere on $\mathcal{M}$.*

Following Proposition 1, since a locally Lipschitz function is not differentiable everywhere, we define the following notion of generalized directional derivative that determines the rate of change of a function at a given point along a specified direction, even at its points of nondifferentiability.

Definition 6 (Generalized directional derivative [14])**.** Suppose $f : \mathcal{M} \to \mathbb{R}$ is locally Lipschitz on a C^1-manifold $\mathcal{M}$. Then, the generalized directional derivative of f at $x \in \mathcal{M}$ in the direction $v \in \mathrm{T}_x\mathcal{M}$, denoted by $f^\circ(x, v)$, is defined as

$$f^\circ(x, v) := \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{\widetilde{f}(\varphi(y) + t\mathrm{d}\varphi_x(v)) - \widetilde{f}(\varphi(y))}{t} \tag{4}$$

where (U, φ) is a chart at x, $\widetilde{f} := f \circ \varphi^{-1}$, and recall that $\mathrm{d}\varphi_x : \mathrm{T}_x\mathcal{M} \to \mathbb{R}^{\dim \mathcal{M}}$ denotes the differential of φ at x.

It is shown in [14, Lemma 3.3] that Definition 6 is independent of the choice of coordinate chart.

V. Existence of Nontrivial Solutions to $\mathcal{H}$

Using the nonsmooth analysis tools presented in the previous section, we provide viability conditions for existence of nontrivial solutions to the hybrid system $\mathcal{H}$ in (1). The result follows similarly to the existence result presented in [3, Prop. 6.10] for hybrid systems evolving on Euclidean space.

Proposition 2 (Existence of solutions)**.** *Consider the geometric hybrid dynamical system $\mathcal{H} = (C, F, D, G, \mathcal{M})$ that satisfies the geometric hybrid basic conditions in Definition 3. Pick $\xi \in C \cup D$. If $\xi \in D$ or*

(VC^g) there exists an open neighborhood U of ξ such that, for each $x \in U \cap C$,

$$F(x) \cap \mathrm{T}_C^{\mathcal{M}}(x) \neq \varnothing$$

then there exists a nontrivial solution $\phi \in \hat{\mathcal{S}}_{\mathcal{H}}(\xi)$. If ($VC^g$) holds for every $\xi \in C \setminus D$, then there exists a nontrivial solution to $\mathcal{H}$ from each point of $C \cup D$, and each maximal solution ϕ to $\mathcal{H}$ satisfies exactly one of the following conditions:

a) ϕ is complete;

b) $\operatorname{dom}\phi$ is bounded, the interval $I^J = \{t : (t, J) \in \operatorname{dom}\phi\}$, where $J = \sup_j \operatorname{dom}\phi$, has a nonempty interior, and $t \mapsto \phi(t, J)$ is a maximal solution to $\dot{x} \in F(x)$ $x \in C$ such that there does not exist a compact set $K \subset \mathcal{M}$ satisfying $\lim_{t \nearrow T} \phi(t, J) \in \operatorname{int} K$, where $T = \sup_t \operatorname{dom}\phi$;

c) $\phi(T, J) \notin C \cup D$, where $(T, J) := \sup \operatorname{dom}\phi$.

Furthermore, the following hold:

1) If $G(D) \subset C \cup D$, then item c above does not hold.

2) If C is compact, then item b above does not hold.

Example 2 (The Möbius band, revisited). In Example 1, the sets C and D are closed, and the maps F and G are single valued and continuous. Therefore, $\mathcal{H} = (C, F, D, G, \mathcal{M})$ therein satisfies the geometric hybrid basic conditions. Note that $C \setminus D = \operatorname{int} C$. Then, $\mathrm{T}_{C \setminus D}^{\mathcal{M}}(x) = \mathrm{T}_x\mathcal{M}$ for each $x \in C \setminus D$, and (VC^g) holds trivially for each $x \in C \setminus D$. Therefore, due to Proposition 2, there exists a nontrivial solution to $\mathcal{H}$ from each point in $C \setminus D$. ■

VI. Stability, Attractivity, and Asymptotic Stability on C^1-Manifolds

In this section, we first define several notions for the system in (1), and later provide Lyapunov characterizations for said notions. Recall that as $\mathcal{M}$ is a C^1-manifold, we may not know the Riemannian metric on it. Hence, the distance between points on this manifold may not be defined. The following definitions are formulated considering this fact. In the presence of a distance metric, these definitions reduce to what is present in the literature; see, e.g., [3].

Definition 7 (Uniform global stability)**.** A nonempty, compact set $\mathcal{A} \subset \mathcal{M}$ is uniformly globally stable for $\mathcal{H}$ if, for each compact neighborhood W of $\mathcal{A}$, there exists a compact neighborhood U of $\mathcal{A}$ such that $U \subset W$ and a compact neighborhood X of W such that each solution ϕ to $\mathcal{H}$ satisfies

$$\left.\begin{array}{lcl} \phi(0,0) \in U & \Longrightarrow & \phi(t,j) \in W \\ \phi(0,0) \in W & \Longrightarrow & \phi(t,j) \in X \end{array}\right\} \quad \forall (t,j) \in \operatorname{dom} x.$$

The first implication in the above definition implies that $\mathcal{A}$ is stable for $\mathcal{H}$, while the second implication implies that $\mathcal{A}$ is Lagrange stable for $\mathcal{H}$. This definition when $\mathcal{A}$ is a compact set and the hybrid system evolves on a Riemannian manifold simplifies to the existence of a class-$\mathcal{K}_\infty$ function that upper bounds the distance from a solution of the hybrid system at each hybrid time to the set $\mathcal{A}$. The following result proves this statement when said Riemannian manifold is $\mathbb{R}^n$, hence reconciling Definition 7 with [4, Definition 3.7].

Proposition 3. *A nonempty, compact set $\mathcal{A} \subset \mathbb{R}^n$ is uniformly globally stable for $\mathcal{H}$ on $\mathbb{R}^n$ if and only if there exists $\alpha \in \mathcal{K}_\infty$ such that each solution ϕ to $\mathcal{H}$ satisfies $|\phi(t,j)|_\mathcal{A} \le \alpha(|\phi(0,0)|_\mathcal{A})$ for each $(t,j) \in \operatorname{dom}\phi$.*

Definition 8 (Uniform global pre-attractivity)**.** A nonempty, compact set $\mathcal{A}$ is uniformly globally pre-attractive for $\mathcal{H}$ on $\mathcal{M}$ if, for each compact neighborhoods $U, W \subset \mathcal{M}$ of $\mathcal{A}$, there exists $T \ge 0$ such that for each solution $\phi \in \hat{\mathcal{S}}_\mathcal{H}(U)$, $(t,j) \in \operatorname{dom}\phi$ and $t + j \ge T$ implies $\phi(t,j) \in W$.

Definition 9 (Uniform global pre-asymptotic stability)**.** A nonempty, compact set $\mathcal{A} \subset \mathcal{M}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}$ on $\mathcal{M}$ if it is uniformly globally stable and uniformly globally pre-attractive.

In the definitions above, the prefix "pre-" enables maximal solutions to not be complete. If each maximal solution to $\mathcal{H}$ is complete, then the prefix is dropped.

VII. Lyapunov Theorem for Geometric Hybrid Dynamical Systems

We provide sufficient conditions for uniform global pre-asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$ using locally Lipschitz Lyapunov functions. Then, using the results in Section IV, we obtain upper bounds on the rate of change during flows and at jumps of a Lyapunov function along solutions to $\mathcal{H}$, which will be used in the forthcoming Hybrid Lyapunov theorem.

Definition 10 (Lyapunov function candidate)**.** Let $\mathcal{H} = (C, F, D, G, \mathcal{M})$ be a geometric hybrid dynamical system. Given nonempty sets $\mathcal{U}, \mathcal{A} \subset \mathcal{M}$, a function $V : \operatorname{dom} V \to \mathbb{R}$ defines a Lyapunov function candidate on $\mathcal{U}$ with respect to $\mathcal{A}$ for $\mathcal{H}$ if the following conditions hold:

1) $(\overline{C} \cup D \cup G(D)) \cap \mathcal{U} \subset \operatorname{dom} V$;
2) $\mathcal{U}$ contains an open neighborhood of $\mathcal{A} \cap (C \cup D \cup G(D))$;
3) V is continuous on $\mathcal{U}$ and locally Lipschitz on an open set containing $\overline{C} \cup \mathcal{U}$;
4) V, restricted to $C \cup D \cup G(D)$, satisfies $V \in \mathcal{PD}(\mathcal{A})$.

Due to item 3 in the definition above, V is differentiable almost everywhere on $\overline{C} \cup \mathcal{U}$. Then, the rate of change of V along flows of the solutions to $\mathcal{H}$ is defined using the generalized directional derivative.

Definition 11 ($\dot{V}$ and ΔV)**.** Given a geometric hybrid dynamical system $\mathcal{H} = (C, F, D, G, \mathcal{M})$, sets $\mathcal{U}, \mathcal{A} \subset \mathcal{M}$, and a function $V : \operatorname{dom} V \to \mathbb{R}$ that defines a Lyapunov function candidate on $\mathcal{U}$ with respect to $\mathcal{A}$ for $\mathcal{H}$,

- the change of V along flows is given by
$$\dot{V}(x) := \sup_{v \in F(x) \cap \mathrm{T}_C^\mathcal{M}(x)} V^\circ(x, v) \quad \forall x \in C \cap \mathcal{U};$$
- the change of V at jumps is given by
$$\Delta V(x) := \sup_{g \in G(x)} V(g) - V(x) \quad \forall x \in D \cap \mathcal{U}.$$

The following result proves that $\dot{V}$ and ΔV, as defined above, upper bound the rate of change of the Lyapunov function candidate V along flows and during jumps, respectively.

Lemma 1. *Let $\mathcal{H} = (C, F, D, G, \mathcal{M})$ be a geometric hybrid dynamical system. For each solution $(t,j) \mapsto \phi(t,j)$ to $\mathcal{H}$ and each $(T, J) \in \operatorname{dom}\phi$, let $0 \le t_0 \le t_1 \le \ldots \le t_{J+1} = T$ satisfy (2) with $E = \operatorname{dom}\phi \cap ([0,T] \times \{0, 1, \ldots, J\})$. Then,*

i) for each $j \in \{0, 1, \ldots, J\}$,
$$\frac{dV}{dt}(\phi(t,j)) \le \dot{V}(\phi(t,j)) \quad \text{for almost all } t \in [t_j, t_{j+1}];$$
ii) for each $j \in \{0, 1, \ldots J\}$ with $(t_{j+1}, j+1) \in \operatorname{dom}\phi$,
$$V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \le \Delta V(\phi(t_{j+1}, j)).$$

Following [3, Theorem 3.18] and using the results in [16], we obtain the following theorem. Note that, in the following theorem, the hybrid basic conditions are not needed.

Theorem 1 (Hybrid Lyapunov Theorem)**.** *Consider a nonempty, compact set $\mathcal{A} \subset \mathcal{M}$ and a function $V : \mathcal{M} \to \mathbb{R}_{\ge 0}$ that defines a Lyapunov function candidate with respect to $\mathcal{A}$ for $\mathcal{H} = (C, F, D, G, \mathcal{M})$. The set $\mathcal{A}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}$ if the Lyapunov function candidate $V : \operatorname{dom} V \to \mathbb{R}_{\ge 0}$ is proper[2], and one of the following conditions hold:*

(a) Strict decrease during flows and jumps: there exist lower semicontinuous functions $\rho_\mathrm{C}, \rho_\mathrm{D} \in \mathcal{PD}(\mathcal{A})$ such that
$$\dot{V}(x) \le -\rho_\mathrm{C}(x) \qquad \forall x \in C, \tag{5}$$
$$\Delta V(x) \le -\rho_\mathrm{D}(x) \qquad \forall x \in D. \tag{6}$$

(b) Strict decrease during flows and no increase at jumps: there exists a lower semicontinuous function $\rho_\mathrm{C} \in \mathcal{PD}(\mathcal{A})$ such that (5) holds, (6) holds with $\rho_\mathrm{D} \equiv 0$ and, for each compact neighborhood R of $\mathcal{A}$, there exist $\gamma \in \mathcal{K}_\infty$ and $N \ge 0$ such that for each solution $\phi \in \hat{\mathcal{S}}_\mathcal{H}(R \backslash \mathcal{A})$, $(t,j) \in \operatorname{dom}\phi$ implies $t \ge \gamma(t+j) - N$.

(c) Strict decrease at jumps and no increase during flows: there exists a lower semicontinuous function $\rho_\mathrm{D} \in \mathcal{PD}(\mathcal{A})$ such that (6) holds, (5) holds with $\rho_\mathrm{C} \equiv 0$ and, for each compact neighborhood R of $\mathcal{A}$, there exists $\gamma \in \mathcal{K}_\infty$ and $N \ge 0$ such that for each solution $\phi \in \hat{\mathcal{S}}_\mathcal{H}(R \backslash \mathcal{A})$, $(t,j) \in \operatorname{dom}\phi$ implies $j \ge \gamma(t+j) - N$.

(d) Increase balanced by decrease: there exist constants $\lambda_c \in \mathbb{R}$ and $\lambda_d \in \mathbb{R}$ such that
$$\dot{V}(x) \le \lambda_c V(x) \qquad \forall x \in C,$$
$$\Delta V(x) \le (\exp(\lambda_d) - 1)V(x) \qquad \forall x \in D,$$

[2] A map $f : X \to Y$ between topological spaces is proper if the preimage of each compact set is compact.

and there exist $M, \gamma > 0$ such that, for each solution ϕ to $\mathcal{H}$ and each $(t,j) \in \operatorname{dom}\phi$, $\lambda_c t + \lambda_d j \le M - \gamma(t+j)$.

(e) Strict decrease during flows and finite number of jumps: there exist a lower semicontinuous function $\rho_{\mathrm{C}} \in \mathcal{PD}(\mathcal{A})$ and $\lambda \in \mathcal{K}_\infty$ such that (5) holds, $V(\chi) \le \lambda(V(x))$ for each $x \in D$ and each $\chi \in G(x)$, and for each compact neighborhood R of $\mathcal{A}$, there exists $J > 0$ such that each solution $\phi \in \hat{\mathcal{S}}_{\mathcal{H}}(R \setminus \mathcal{A})$ satisfies $\sup_j \operatorname{dom} x \le J$.

(f) Strict decrease at jumps and bounded time of flow: there exists a lower semicontinuous function $\rho_D \in \mathcal{PD}(\mathcal{A})$ and $\lambda \in \mathbb{R}$ such that (6) holds, $\dot{V}(x) \le \lambda V(x)$ for each $x \in C$, and, for each compact neighborhood R of $\mathcal{A}$, there exists $T \ge 0$ such that each solution $\phi \in \hat{\mathcal{S}}_{\mathcal{H}}(R \setminus \mathcal{A})$ satisfies $\sup_t \operatorname{dom}\phi \le T$.

The above theorem assumes that the Lyapunov function candidate V is proper. In the case when $\mathcal{M}$ is the Euclidean space, the following result establishes a class-$\mathcal{K}_\infty$ lower bound on the value of V at every point in its domain.

Proposition 4. *Given a nonempty, compact set $\mathcal{A} \subset \mathbb{R}^n$, a continuous function $V \in \mathcal{PD}(\mathcal{A})$ is proper if and only if there exists $\alpha \in \mathcal{K}_\infty$ such that $\alpha(|x|_{\mathcal{A}}) \le V(x)$ for each $x \in \mathbb{R}^n$.*

The result in Theorem 1 can be specialized to hybrid systems evolving on $\mathbb{R}^n$ using Proposition 4; see [16, Thm.3].

Example 3 (The Möbius band, revisited). We revisit Example 1. Let $\widetilde{\mathcal{A}} := \{[z_1, z_2]_{\mathscr{B}} \in \mathscr{B} : z_1 \in \{0,1\}, z_2 = 0\}$ and, for each $z_1 \in [0,1]$, define the equivalence class $[\cdot]_{\mathbb{S}^1}$ such that $[z_1]_{\mathbb{S}^1} = \{z_1\}$ for each $z_1 \in (0,1)$, and $[0]_{\mathbb{S}^1} = [1]_{\mathbb{S}^1} = \{0,1\}$. Then, for each $q \in Q$, let $V_q([z]_{\mathscr{B}}) := U(\Gamma([z_1]_{\mathbb{S}^1}, q)) + z_2^2/2$ for each $(z_1, z_2, q) \in [0,1] \times (0,1) \times Q$, where $U([z_1]_{\mathbb{S}^1}) := (1 - \cos(2\pi z_1))/2$ for each $z_1 \in [0,1]$, $\Gamma([z_1]_{\mathbb{S}^1}, q) := [(z_1 + q \arcsin(U([z_1])/2)/\pi) \operatorname{mod}(1)]_{\mathbb{S}^1}$ for each $(z_1, q) \in [0,1] \times Q$, and mod denotes the modulo operator. Note that $V_q \in \mathcal{PD}(\widetilde{\mathcal{A}})$ is proper and continuously differentiable for each $q \in Q$. This construction of the map Γ is motivated by ideas from synergistic control [17].

Let $\mathcal{A} := \widetilde{\mathcal{A}} \times Q$, and $V(x) := V_q([z]_{\mathscr{B}})$ for each $x = ([z]_{\mathscr{B}}, q) \in \mathcal{M}$. Note that $V \in \mathcal{PD}(\mathcal{A})$, and V is proper and C^1 on $\mathcal{M}$. For each $x \in C$, we have $\dot{V}(x) = \mathrm{d}(V_q)_{[z]_{\mathscr{B}}}(-\operatorname{grad} V_q([z]_{\mathscr{B}})) = -\rho_{\mathrm{C}}(x)$, where $\rho_{\mathrm{C}}(x) := g_{\mathscr{B}}(\operatorname{grad} V_q([z]_{\mathscr{B}}), \operatorname{grad} V_q([z]_{\mathscr{B}}))$ for all $x \in \mathcal{M}$. Note that $\rho_{\mathrm{C}} \in \mathcal{PD}(\mathcal{A})$ and is continuous. Then, (5) holds.

Similarly, from the definition of D and G in Example 1, we obtain $\Delta V(x) \le -\delta$ for each $x \in D$. Noting that $D \cap \mathcal{A} = \varnothing$, the condition in (6) holds with $\rho_{\mathrm{D}} \in \mathcal{PD}(\mathcal{A})$ defined as $\rho_{\mathrm{D}}(x) = 0$ if $x \in \mathcal{A}$, and $\rho_{\mathrm{D}}(x) = \delta$ if $x \notin \mathcal{A}$. Note that ρ_{D} is lower semicontinuous as $\mathcal{A}$ is closed. For each $\xi \in \mathcal{M}$, let $r := V(\xi)$. Following (5) and (6), each $\phi \in \hat{\mathcal{S}}_{\mathcal{H}}(\xi)$ satisfies $\phi(t,j) \in \{x \in \mathcal{M} : V(x) \le V(r)\}$ for each $(t,j) \in \operatorname{dom}\phi$. As V is proper, we use Prop. 2 and the fact that $C \cup D = \mathcal{M}$ to conclude each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{M})$ is complete. Then, $\mathcal{A}$ is uniformly globally asymptotically stable for $\mathcal{H}$. Since $\mathscr{B}$ is noncontractible, $\widetilde{\mathcal{A}}$ cannot be globally asymptotically stabilized by gradient descent using either function V_q. Introducing hybrid dynamics overcomes this topological obstruction – sample solutions to $\mathcal{H}$ in Fig. 1 illustrate this fact. ■

VIII. Conclusion

We develop a relaxed hybrid Lyapunov theorem for geometric hybrid inclusions on intrinsic C^1-manifolds. By leveraging tools from nonsmooth analysis and topology, we provide sufficient conditions for the existence of nontrivial solutions to such hybrid systems, and introduce topological definitions of uniform global stability, (pre-)attractivity, and (pre-)asymptotic stability. The presented results extend existing hybrid systems theory beyond Euclidean spaces, enabling the analysis of hybrid systems on manifolds where a Riemannian metric may not be known. Extensions of this work, which dive into the invariance principle for geometric hybrid inclusions on intrinsic manifolds, are presented in the companion paper [11].

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References

[1] C. G. Mayhew and A. R. Teel, "Synergistic potential functions for hybrid control of rigid-body attitude," in *Proc. of the 2011 Amer. Control Conf.*, pp. 875–880, 2011.

[2] P. Casau, R. Cunha, R. G. Sanfelice, and C. Silvestre, "Hybrid Control for Robust and Global Tracking on Smooth Manifolds," *IEEE Trans. Autom. Control*, vol. 65, no. 5, pp. 1870–1885, 2020.

[3] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[4] R. G. Sanfelice, *Hybrid Feedback Control*. Princeton University Press, 2021.

[5] J. M. Lee, *Introduction to Smooth Manifolds*. Springer, 2013.

[6] P. P. Jirwankar and R. G. Sanfelice, "Robust Global Hybrid Passive Complementary Filter on SO(2)," in *2024 IEEE 63rd Conference on Decision and Control (CDC)*, pp. 7946–7951, 2024.

[7] S. N. Simić, K. H. Johansson, S. Sastry, and J. Lygeros, "Towards a geometric theory of hybrid systems," in *Hybrid Systems: Computation and Control* (N. Lynch and B. H. Krogh, eds.), vol. 1790 of *Lecture Notes in Computer Science*, (Berlin, Heidelberg), pp. 421–436, Springer, 2000.

[8] M. Barbero Liñán, J. Cortés, D. Martín de Diego, S. Martínez, and M. Muñoz Lecanda, "Global controllability tests for geometric hybrid control systems," *Nonlinear Analysis: Hybrid Systems*, vol. 38, p. 100935, 2020.

[9] M. D. Kvalheim, P. Gustafson, and D. E. Koditschek, "Conley's fundamental theorem for a class of hybrid systems," *SIAM Journal on Applied Dynamical Systems*, vol. 20, no. 2, pp. 784–825, 2021.

[10] P. Forni and D. Angeli, "Smooth lyapunov functions for multistable hybrid systems on manifolds," in *Proc. 56th IEEE Conf. Decision and Control (CDC)*, pp. 5481–5486, 2017.

[11] P. P. Jirwankar, D. E. Ochoa, and R. G. Sanfelice, "Geometric Hybrid Dynamical Systems on Intrinsic Manifolds: Part II – Hybrid Invariance Principle," in *Proc. 64th IEEE Conf. Decision and Control*, 2025.

[12] J. M. Lee, *Introduction to Riemannian Manifolds*. Graduate Texts in Mathematics, Springer Cham, 2 ed., 2019.

[13] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*, vol. 49 of *Texts in Applied Mathematics*. New York-Heidelberg-Berlin: Springer Verlag, 2004.

[14] D. Motreanu and N. Pavel, "Quasi-tangent vectors in flow-invariance and optimization problems on Banach manifolds," *Journal of Mathematical Analysis and Applications*, vol. 88, no. 1, pp. 116–132, 1982.

[15] S. Willard, *General Topology*. Addison-Wesley Pub. Co., 1970.

[16] P. K. Wintz and R. G. Sanfelice, "Relaxed Lyapunov Conditions for Compact Sets in Dynamical Systems," in *Proc. of the 2025 American Control Conference*, 2025.

[17] S. Berkane and A. Tayebi, "Construction of Synergistic Potential Functions on SO(3) With Application to Velocity-Free Hybrid Attitude Stabilization," *IEEE Trans. Autom. Control*, vol. 62, no. 1, pp. 495–501, 2017.