Two-Player Zero-Sum Hybrid Games

Santiago J. Leudo and Ricardo G. Sanfelice

Electrical and Computer Engineering Department, University of California, Santa Cruz, CA 95064, USA.

Abstract

In this paper, we formulate a two-player zero-sum game under dynamic constraints defined by hybrid dynamical equations. The game consists of a min-max problem involving a cost functional that depends on the actions and resulting solutions to the hybrid system, defined as functions of hybrid time and, hence, can flow or jump. A conveniently-defined terminal set allows to recast both finite and infinite horizon problems. We present sufficient conditions given in terms of Hamilton–Jacobi–Bellman-Isaacs equations to guarantee that a solution to the game is attained. It is shown that when the players select the optimal strategy, the value function can be evaluated without computing solutions to the hybrid system. Under additional conditions, we show that the optimal state-feedback laws render a set asymptotically stable for the resulting hybrid closed-loop system. Applications of these games, presented here as robust control problems, include disturbance rejection and security problems.

Key words: Game Theory; Optimal Control; Hybrid Systems; Robust Control.

1 Introduction

1.1 Background

A game is an optimization problem with multiple decision makers (players), a set of constraints (potentially dynamic) that enforces the "rules" of the game, and a set of payoff functions to be optimized by selecting decision variables. Constraints on the state and decision variables formulated as dynamic relationships lead to dynamic games; see [4] and the references therein. In this setting, an interesting scenario arises when the players have different incentives, e.g., to minimize or maximize their own cost function. Dynamic noncooperative games focus on the case in which the players select their actions with no coalition formation, such that an individual benefit does not necessarily imply a benefit to the other players [20,21,36]. This type of dynamic game has been thoroughly studied in the literature when the dynamic constraints are given in terms of difference equations or differential equations.

Challenges arise when the players' dynamics exhibit both continuous and discrete behaviors, for instance, due to intermittent information availability, resets, timers in the control algorithms that expire, or non-smooth mechanical behaviors exhibiting impacts, among others. Hybrid dynamical systems conveniently capture this kind of behavior [17,41]. Under certain assumptions, differential algebraic equation (DAEs) [28] – also known as descriptor systems – can be recast as hybrid equations, see [35, Lemma 2]. Specifically, when the initial condition to a DAE is *consistent* and the data pair of the system is *regular* (for each subsystem, in the case of switched DAEs [30]), a solution to the DAE is also a solution to the equivalent hybrid system defined as in [17]. However, when designing algorithms that make optimal choices of the decision variables under constraints given by hybrid dynamics, relying only on continuous-time or discrete-time approaches potentially results in suboptimal solutions. Unfortunately, tools for the design of algorithms for games with such hybrid dynamic constraints, which we refer to as hybrid games, are not fully developed.

Particular classes of dynamic games involving hybrid dynamic constraints have been recently studied in the literature. A game-theoretic control design approach is presented for timed automata in [2,3], for hybrid finite-state automata in [19,44], and for o-minimal hybrid systems in [5]. In these articles, the specifications to be guaranteed by the system are defined in terms of temporal logic formulae. When the payoff is defined in terms of a terminal cost, such approach allows designing reachability-based controllers through the satisfaction of Hamilton-Jacobi conditions to certify safety of hybrid finite-state automata [9]. Following an approach that allows for richer dynamics, [45] studies a class of reachability games between a controller and the environment, under constraints defined by hybrid automata (STORMED games) for which at each decision step, the players can choose either to have their variables evolve

Email addresses: `sjimen28@ucsc.edu` (Santiago J. Leudo), `ricardo@ucsc.edu` (Ricardo G. Sanfelice).

continuously or discretely, following predefined rules. For continuous-time systems with state resets, tools for the computation of the region of attraction for hybrid limit cycles under the presence of disturbances are provided in [7], with inputs defined only in continuous time.

Efforts pertaining to differential games with impulsive elements include [38], where the interaction between the players is modeled similarly to switched systems, [8], which establishes continuity of bounds on value functions and viscosity solutions, [18], which formulates necessary and sufficient conditions for optimality in bimodal linear-quadratic differential games, and [6], which studies a class of stochastic two-player differential games in match race problems.

1.2 Contribution and Applications

Motivated by the lack of tools for the design of algorithms for general hybrid games, we formulate a framework for the study of two-player zero-sum games with hybrid dynamic constraints. Specifically, we optimize a cost functional, which includes

- a stage cost that penalizes the evolution of the state and the input during flows,
- a stage cost that penalizes the evolution of the state and the input at jumps, and
- a terminal cost to penalize the final value of the solutions.

Following the framework in [17,41], we model the hybrid dynamic constraints as a hybrid dynamical system, denoted $\mathcal{H}$ and given in terms of the hybrid equation

$$\mathcal{H}\begin{cases} \dot{x} \;=\; F(x,(u_{C1},u_{C2})) & (x,(u_{C1},u_{C2}))\in C \\ x^{+} \;=\; G(x,(u_{D1},u_{D2})) & (x,(u_{D1},u_{D2}))\in D \end{cases} \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $(u_{C1},u_{D1})\in\mathbb{R}^{m_{C1}}\times\mathbb{R}^{m_{D1}}$ is the input chosen by player P_1, $(u_{C2},u_{D2})\in\mathbb{R}^{m_{C2}}\times\mathbb{R}^{m_{D2}}$ is the input chosen by player P_2, and the data of (C,F,D,G) is given as follows [1] :

- The *flow map* $F:\mathbb{R}^n\times\mathbb{R}^{m_C}\to\mathbb{R}^n$ captures the continuous evolution of the system when the state and the input are in the *flow set* $C\subset\mathbb{R}^n\times\mathbb{R}^{m_{C1}+m_{C2}}$.
- The *jump map* $G:\mathbb{R}^n\times\mathbb{R}^{m_D}\to\mathbb{R}^n$ describes the discrete evolution of the system when the state and the input are in the *jump set* $D\subset\mathbb{R}^n\times\mathbb{R}^{m_{D1}+m_{D2}}$.

For such broad class of systems, we consider a Bolza-form cost functional $\mathcal{J}$ associated to the solution to $\mathcal{H}$ from ξ and study a zero-sum two-player hybrid game that, informally, is given as

$$\min_{(u_{C1},u_{D1})}\ \max_{(u_{C2},u_{D2})}\ \mathcal{J}(\xi,(u_{C1},u_{C2}),(u_{D1},u_{D2})) \tag{2}$$

This game captures the dynamics of systems operating in contested scenarios with hybrid dynamics, such as continuous-time dynamics with logical modes, multiple modes of operation, and dynamics or control signals that change abruptly or impulsively. Such dynamics can be represented by switching systems, hybrid automata, or impulsive differential equations, all of which can be modeled as in (1); see [17,15].

Several applications lead to the game in (2). The following robust control scenario is of particular interest and concrete instances are considered in this paper:

($\star$) Given the system $\mathcal{H}$ as in (1) with state x, the robust control problem consists of establishing conditions such that player P_1 selects a control input (u_{C1},u_{D1}) that minimizes a cost functional $\mathcal{J}$ until the game ends, which occurs when the state enters a set X, in the presence of a disturbance (u_{C2},u_{D2}) chosen by P_2 (modeled as maximizing the cost).

The solution of the game formulated in this paper, known as a *saddle-point equilibrium*, is given in terms of the actions of the players. Informally, when a player unilaterally deviates from the equilibrium action, it does not improve its individual outcome. Thus, by formulating the applications above as two-player zero-sum hybrid games, we can synthesize the saddle-point equilibrium and determine the control action that minimizes the cost $\mathcal{J}$ for the maximizing adversarial action. The main contributions of this paper are summarized as follows.

- In Section III, we present a framework for the study of two-player zero-sum games with hybrid dynamic constraints.
- We present in Theorem 4.1 sufficient conditions based on Hamilton–Jacobi–Bellman-Isaacs-like equations to design a saddle-point equilibrium and evaluate the game value function without computing solutions to the hybrid system.
- Connections between optimality and asymptotic stability of a set are proposed in Section V and framed in the game theoretical approach employed.
- We present in Section VI applications to robust control scenarios by formulating and solving them as two-player zero-sum hybrid dynamic games.

This work extends our preliminary conference paper [25] since a more general problem is considered. By including a terminal set, we formulate problems with variable terminal time and with infinite horizon. Sufficient conditions to solve the general problem are provided in Theorem 4.1. with the corresponding proof, that uses as a preliminary step Proposition 4.2. Remarks 4.5 and 4.6 provide a discussion on the existence and computation of a value function. Theorem 5.4 provides connections between optimality and stability for the general problem plus considering special cases that allow to guarantee pre-asymptotic stability. Proposition 6.1 allows to

[1] Here, $m_C=m_{C1}+m_{C2}$ and $m_D=m_{D1}+m_{D2}$.

solve a robust LQR game with aperiodic jumps. Complete proofs of the results are provided. We provide examples of robust control applications, namely, a disturbance rejection problem with periodic jumps and a security problem for a bouncing ball system under attacks.

Related to (2) are the zero-sum games for DAEs studied in the literature [14]. A min-max principle built upon Pontryagin's Maximum Principle is provided in [48]. Linear dynamics and quadratic costs result in coupled Riccati differential equations, and conditions for their solvability are provided in [10] and [49]. In [42], noncooperative games for Markov switching DAEs are studied and Hamilton–Jacobi–Bellman-Isaacs equations are derived. When the initial condition to a switching DAE is consistent and the data pair of the system is regular for each subsystem, a deterministic version of the problem solved in [42] can be recast as (2).

In recent works, optimality for hybrid systems modeled as in (1) is certified via Lyapunov-like conditions [12], providing cost evaluation results for the case in which the data is given in terms of set-valued maps. The work in [16] provides sufficient conditions to guarantee the existence of optimal solutions. A receding-horizon algorithm to implement these ideas is presented in [1]. Cost evaluation results and conditions to guarantee asymptotic stability of a set are established for a discrete-time system under adversarial scenarios in [22]. A fixed finite-horizon case of the hybrid game formulated in this paper is studied in [26]. The conditions on the optimization problem formulated therein are similar to their counterparts in the differential/dynamic game theory literature. Nevertheless, in contrast to this work, the end of the game therein is attained when the time of solutions to $\mathcal{H}$ reach a terminal set $\mathcal{T}$. An inverse optimal approach is formualted in [33]. To account for hybrid time domains, which are introduced in Section 2, a hybrid time domain-like geometry is assumed for $\mathcal{T}$ as in [1]. This results in optimality conditions in terms of PDEs and the optimal feedback laws are not stationary.

Due to space constrains, a technical report is provided in [23], with proofs and details omitted here.

1.3 Notation

Given two vectors x, y, we use the equivalent notation $(x, y) = [x^\top y^\top]^\top$ and $\langle x, y\rangle$ denotes the Euclidean inner product. The symbol $\mathbb{N}$ denotes the set of natural numbers including zero. The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative reals. Given a nonempty set $\mathcal{A}$, the distance from the vector x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{y\in\mathcal{A}} |x-y|$. We denote with $\mathbb{S}^n_+$ the set of real positive definite matrices of dimension n, and with $\mathbb{S}^n_{0+}$ the set of real positive semidefinite matrices of dimension n. Given a nonempty set C, we denote by $\operatorname{int} C$ its interior and by $\overline{C}$ its closure. We denote by $\mathbb{1}_n$ the n-dimensional vector on ones.

2 Preliminaries

2.1 Hybrid Systems with Inputs

Since solutions to the dynamical system $\mathcal{H}$ as in (1) can exhibit both continuous and discrete behavior, we use ordinary time t to determine the amount of flow, and a counter $j \in \mathbb{N}$ that counts the number of jumps. Based on this parametrization, the concept of a hybrid time domain, in which solutions are defined, of a hybrid arc, and of a hybrid input are presented in [17, Section 2.2]. A hybrid signal is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I^j_\phi := \{t : (t, j) \in \operatorname{dom}\phi\}$. Let $\mathcal{X}$ be the set of hybrid arcs $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ and $\mathcal{U} := \mathcal{U}_C \times \mathcal{U}_D$ the set of hybrid inputs $u = (u_C, u_D) : \operatorname{dom} u \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, where $u_C = (u_{C1}, u_{C2})$, $m_{C1} + m_{C2} = m_C$, $u_D = (u_{D1}, u_{D2})$, and $m_{D1} + m_{D2} = m_D$. A solution to the hybrid system $\mathcal{H}$ with input is defined as follows.

Definition 2.1 (Solution to $\mathcal{H}$) *A hybrid signal (ϕ, u) defines a solution pair to (1) if $\phi \in \mathcal{X}$, $u = (u_C, u_D) \in \mathcal{U}$, $\operatorname{dom}\phi = \operatorname{dom}u$, and*

- $(\phi(0, 0), u_C(0, 0)) \in \overline{C}$ *or* $(\phi(0, 0), u_D(0, 0)) \in D$,
- *For each $j \in \mathbb{N}$ such that I^j_ϕ has a nonempty $\operatorname{int} I^j_\phi$, we have, for all $t \in \operatorname{int} I^j_\phi$, $(\phi(t, j), u_C(t, j)) \in C$ and, for almost all $t \in I^j_\phi$, $\frac{d\phi}{dt}(t, j) = F(\phi(t, j), u_C(t, j))$,*
- *For all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$, $(\phi(t, j), u_D(t, j)) \in D$, $\phi(t, j+1) = G(\phi(t, j), u_D(t, j))$.*

A solution pair (ϕ, u) is a compact solution pair if ϕ is a compact hybrid arc [17, Definition 2.5].

Given a solution pair (ϕ, u), the component ϕ is referred to as the state trajectory. In this article, the same symbols are used to denote input actions and their values. The context clarifies the meaning of u, as follows: "the function u," "the signal u," or "the hybrid signal u" that appears in "the solution pair (ϕ, u)" refer to the input action, whereas "u" refers to the input value as a point in $\mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ in any other case.

A solution pair (ϕ, u) to $\mathcal{H}$ from $\xi \in \mathbb{R}^n$ is complete if $\operatorname{dom}(\phi, u)$ is unbounded. It is maximal if there is no solution (ψ, w) from ξ such that $\phi(t, j) = \psi(t, j)$ and $u(t, j) = w(t, j)$ for all $(t, j) \in \operatorname{dom}(\phi, u)$ and $\operatorname{dom}(\phi, u)$ is a proper subset of $\operatorname{dom}(\psi, w)$. We denote by $\hat{\mathcal{S}}_{\mathcal{H}}(M)$ the set of solution pairs (ϕ, u) to $\mathcal{H}$ as in (1) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_{\mathcal{H}}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}}(M)$ denotes all maximal solution pairs from M. Given $\xi \in \mathbb{R}^n$, we define the set of input actions that yield maximal solutions to $\mathcal{H}$ from ξ as $\mathcal{U}_{\mathcal{H}}(\xi) := \{u : \exists(\phi, u) \in \hat{\mathcal{S}}_{\mathcal{H}}(\xi)\}$. For a given $u \in \mathcal{U}$, we denote the set of maximal state trajectories to $\mathcal{H}$ from ξ for u by $\mathcal{R}(\xi, u) = \{\phi : (\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)\}$. We say u renders a maximal trajectory ϕ to $\mathcal{H}$ from ξ if $\phi \in \mathcal{R}(\xi, u)$. A complete solution (ϕ, u) is discrete if $\operatorname{dom}(\phi, u) \subset \{0\} \times \mathbb{N}$ and continuous if $\operatorname{dom}(\phi, u) \subset \mathbb{R}_{\geq 0} \times \{0\}$.

For each $\star \in \{C, D\}$, we define the projection of $\star \subset \mathbb{R}^n \times \mathbb{R}^{m_\star}$ as $\Pi(\star) := \{\xi \in \mathbb{R}^n : \exists u_\star \in \mathbb{R}^{m_\star} \text{ s.t. } (\xi, u_\star) \in \star\}$ and the set-valued maps that output the allowed input values at a given state x as $\Pi_u^\star(x) = \{u_\star \in \mathbb{R}^{m_\star} : (x, u_\star) \in \star\}$. Moreover, $\sup_t \operatorname{dom} \phi := \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$, $\sup_j \operatorname{dom} \phi := \sup\{j \in \mathbb{N} : \exists t \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$, and $\sup \operatorname{dom} \phi := (\sup_t \operatorname{dom} \phi, \sup_j \operatorname{dom} \phi)$. Whenever $\operatorname{dom} \phi$ is compact, $\operatorname{dom} \phi \ni \max \operatorname{dom} \phi := \sup \operatorname{dom} \phi$.

2.2 Hybrid Closed-Loop Systems

Given a hybrid system $\mathcal{H}$ as in (1) and a function $\kappa := (\kappa_C, \kappa_D)$ with $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, the autonomous hybrid system resulting from assigning $u = \kappa(x)$, namely, the hybrid closed-loop system, is given by

$$\mathcal{H}_\kappa \begin{cases} \dot{x} \ = \ F(x, \kappa_C(x)) & x \in C_\kappa \\ x^+ \ = \ G(x, \kappa_D(x)) & x \in D_\kappa \end{cases} \tag{3}$$

where, for $\star \in \{C, D\}$, $\star_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_\star(x)) \in \star\}$.

A solution to the hybrid closed-loop system $\mathcal{H}_\kappa$ is defined similar to Definition 2.1. We denote by $\hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ the set of solutions ϕ to $\mathcal{H}_\kappa$ as in (3) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_{\mathcal{H}_\kappa}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ denotes all maximal solutions from M.

3 Two-Player Zero-Sum Hybrid Games

3.1 Game Formulation

Following the formulation in [4], for each $i \in \{1, 2\}$, consider the i-th player P_i with dynamics described by $\mathcal{H}_i$ as in (1) with data (C_i, F_i, D_i, G_i), state $x_i \in \mathbb{R}^{n_i}$, and input $u_i = (u_{Ci}, u_{Di}) \in \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$, where $C_i \subset \mathbb{R}^n \times \mathbb{R}^{m_C}$, $F_i : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}^{n_i}$, $D_i \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$ and $G_i : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}^{n_i}$, with $n_1 + n_2 = n$. We denote by $\mathcal{U}_i = \mathcal{U}_{Ci} \times \mathcal{U}_{Di}$ the set of hybrid inputs for $\mathcal{H}_i$; see Definition 2.3.

Notice that each player's dynamics are described in terms of maps and sets defined in the entire state and input space rather than the individual spaces ($\mathbb{R}^n$ and $\mathbb{R}^m$ rather than $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$, respectively). This allows to model the ability of each player's state to evolve according to the state variables and input of the other players.

Definition 3.1 (Elements of a two-player zero-sum hybrid game) *A two-player zero-sum hybrid game is composed by*

1) *The state $x = (x_1, x_2) \in \mathbb{R}^n$, where, for each $i \in \{1, 2\}$, $x_i \in \mathbb{R}^{n_i}$ is the state of player P_i.*
2) *The set of joint input actions $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ with elements $u = (u_1, u_2)$, where, for each $i \in \{1, 2\}$, $u_i = (u_{Ci}, u_{Di})$ is a hybrid input. For each $i \in \{1, 2\}$, P_i selects u_i independently of P_{-i}, who selects u_{-i}, namely, the joint input action u has components u_i that are independently chosen by each player* [2].
3) *The dynamics of the game, described as in* (1) *and denoted by $\mathcal{H}$, with data*

$$\begin{aligned} C &:= C_1 \cap C_2 \\ F(x, u_C) &:= (F_1(x, u_C), F_2(x, u_C)) \\ D &:= D_1 \cup D_2 \\ G(x, u_D) &:= \{\hat{G}_i(x, u_D) : (x, u_D) \in D_i, i \in \{1, 2\}\} \end{aligned}$$

   where $\hat{G}_1(x, u_D) = (G_1(x, u_D), \mathbb{1}_{n_2})$, $\hat{G}_2(x, u_D) = (\mathbb{1}_{n_1}, G_2(x, u_D))$, $u_C = (u_{C1}, u_{C2})$, and $u_D = (u_{D1}, u_{D2})$.
4) *For each $i \in \{1, 2\}$, a strategy space $\mathcal{K}_i$ of P_i defined as a collection of mappings $\kappa_i : \mathbb{R}^n \to \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$. The strategy space of the game, namely $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$, is the collection of mappings with elements $\kappa = (\kappa_1, \kappa_2)$, where $\kappa_i \in \mathcal{K}_i$ for each $i \in \{1, 2\}$, such that every maximal solution (ϕ, u) to $\mathcal{H}$ with input assigned as $\operatorname{dom} \phi \ni (t, j) \mapsto u_i(t, j) = \kappa_i(\phi(t, j))$ for each $i \in \{1, 2\}$ is complete. Each $\kappa_i \in \mathcal{K}_i$ is said to be a permissible pure* [3] *feedback strategy for P_i.*
5) *A scalar-valued functional $(\xi, u) \mapsto \mathcal{J}_i(\xi, u)$ defined for each $i \in \{1, 2\}$ and called the cost associated to P_i. For each $u \in \mathcal{U}$, we refer to a single cost functional $\mathcal{J} := \mathcal{J}_1 = -\mathcal{J}_2$ as the cost associated to the unique solution to $\mathcal{H}$ from ξ for u, and its structure is defined for each type of game.*

Remark 3.2 (Players' state) *In scenarios where each player has its own dynamics, as in pursue-evasion [47] or target defense [37] games, it is common to have a state associated to each player, namely x_1 for P_1 and x_2 for P_2, justifying the partition of the state x in x_1 and x_2. When the players do not have their own dynamics but can independently select an input, e.g., P_1 selects u_1 and P_2 selects u_2 to control a common state x, such state can be associated, without loss of generality, to either of the players, e.g., $x = x_1$ with $n = n_1$ and $n_2 = 0$.*

Notice that Definition 3.1 is general enough to cover games with a finite horizon, for which additional conditions specify the end of the game, e.g., a terminal set in the state space or fixed duration specifications [26].

From a given initial condition ξ, a given strategy $\kappa \in \mathcal{K}$ potentially leads to nonunique solutions [4] $(\phi^1, u^1), (\phi^2, u^2), \ldots, (\phi^k, u^k)$ to $\mathcal{H}$, where $u^l = \kappa \circ \phi^l$ and $\phi^l(0, 0) = \xi$ for each $l \in \{1, 2, \ldots k\}$. Thus, for the formulation in Definition 3.1 to be in Kuhn's extensive form [4], so it admits a solution, an appropriate cost definition is required so each strategy $\kappa \in \mathcal{K}$ has a unique cost correspondence, namely, every solution (ϕ^l, u^l) with $u^l = \kappa \circ \phi^l$, $l \in \{1, 2, \ldots, k\}$ is assigned the same cost.

[2] The subindex $-i$ refers to the player P_{3-i}.

[3] This, in contrast to when $\mathcal{K}_i$ is defined as a probability distribution, namely, when $\mathcal{K}_i$ is the space of mixed strategies.

[4] A given strategy κ can lead to multiple input actions due to a nonempty $\Pi(C) \cap \Pi(D)$.

3.2 Equilibrium Solution Concept

Given the formulation of the elements of a hybrid game in Definition 3.1, its solution is defined as follows.

Definition 3.3 (Feedback saddle-point equilibrium) *Consider a two-player zero-sum game, with dynamics $\mathcal{H}$ as in (1) with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$, for a given cost functional $\mathcal{J} : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$. We say that a strategy $\kappa = (\kappa_1, \kappa_2) \in \mathcal{K}$ is a feedback saddle-point equilibrium if for each $\xi \in \overline{\Pi(C)} \cup \Pi(D)$, every hybrid input $u^* = (u_1^*, u_2^*)$ such that there exists $\phi^* \in \mathcal{R}(\xi, u^*)$, with components defined as $\operatorname{dom} \phi^* \ni (t,j) \mapsto u_i^*(t,j) = \kappa_i(\phi^*(t,j))$, $i \in \{1,2\}$, satisfies*

$$\mathcal{J}(\xi, (u_1^*, u_2)) \leq \mathcal{J}(\xi, u^*) \leq \mathcal{J}(\xi, (u_1, u_2^*)) \tag{4}$$

for all hybrid inputs u_1 and u_2 such that $\mathcal{R}(\xi, (u_1^*, u_2))$ and $\mathcal{R}(\xi, (u_1, u_2^*))$ are nonempty, and every such hybrid input u^* is strongly time consistent – see Remark 3.4.

Definition 3.3 is a generalization of the classical pure strategy Nash equilibrium [4, (6.3)] to the case where the players exhibit hybrid dynamics and opposite optimization goals. In words, we refer to the strategy $\kappa = (\kappa_1, \kappa_2)$ as a feedback saddle-point when a player P_i cannot improve the cost $\mathcal{J}_i$ by playing any strategy different from κ_i when the player P_{-i} is playing κ_{-i}. Condition (4) is verified over the set of inputs that define joint input actions (u_1^*, u_2) and (u_1, u_2^*), yielding at least one nontrivial solution to $\mathcal{H}$ from ξ.

Remark 3.4 (Time consistency and subgame perfection) *The permissible strategies considered in this work have a feedback information structure, in the sense that they depend only on the current value of the state, and not on any past history of the values of the state or hybrid time. Given $\xi \in \mathbb{R}^n$, we say that an input action u^* is* strongly time consistent *if even when the past history of input values that led $\mathcal{H}$ as in* (1) *to ξ were not optimal, the action u^* is still a solution for the remaining of the game (subgame), which is defined in the forthcoming Problem ($\diamond$), starting from ξ. When this property holds for every state $\xi \in \Pi(C) \cup \Pi(D)$, we say that u^* is* subgame perfect, *see [13]. Then, under a strategy space that does not impose structural restrictions on the permissible strategies, (e.g., a linear dependence on the state) the saddle-point equilibrium strategy, when it exists, is said to be strongly time consistent if its components κ_C and κ_D lead to input actions that are strongly time consistent for each $\xi \in \Pi(C) \cup \Pi(D)$. Notice that given the hybrid time horizon structure of the input actions considered in this work, the saddle-point equilibrium is time independent. This results in truncations of input actions not keeping track of previous hybrid time values, i.e., if there exists any past history of strategies that led to the current state, this is hidden for the evaluation of the saddle-point equilibrium at the current state, which results in preservation of optimality in the subgame, property known as* permanent optimality *[4, Section 5.6].*

3.3 Problem Statement

We formulate an optimization problem to solve a two-player zero-sum hybrid game with variable terminal time and a terminal set, and provide sufficient conditions to characterize its solution.

Following the formulation in Definition 3.1, consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ described by (1) with data (C, F, D, G). Let the closed set $X \subset \Pi(C) \cup \Pi(D)$ be the terminal constraint set. We say that a solution (ϕ, u) to $\mathcal{H}$ is *feasible* if there exists $(T, J) \in \operatorname{dom}(\phi, u)$ such that $\phi(T, J) \in X$. In addition, we make (T, J) to be both the terminal time of (ϕ, u) and the first time at which ϕ reaches X, i.e., there does not exist $(t, j) \in \operatorname{dom} \phi$ with $t + j < T + J$ such that $\phi(t, j) \in X$ and $(T, J) = \max \operatorname{dom}(\phi, u)$; hence $\operatorname{dom} \phi$ is compact [5]. Uniqueness of solutions for a given input implies a unique correspondence from cost to control action, which allows this type of games to be *well-defined*, so that an equilibrium solution is defined [4, Remark 5.3]. This justifies the following assumption.

Assumption 3.5 *The flow map F and the flow set C are such that solutions to $\dot{x} = F(x, u_C)$ $(x, u_C) \in C$ are unique for each input u_C. The jump map G is single valued.*

Sufficient conditions to guarantee that Assumption 3.5 holds include Lipschitz continuity of the flow map F, provided it is a single-valued function. Under Assumption 3.5, solutions to $\mathcal{H}$ are unique [6] for each $u \in \mathcal{U}$.

Given $\xi \in \Pi(C) \cup \Pi(D)$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ, under Assumption 3.5, as

$$\begin{aligned} \mathcal{J}(\xi, u) := & \sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt \\ & + \sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) \\ & + \limsup_{\substack{t+j \to \sup_t \operatorname{dom} \phi + \sup_j \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t,j)) \end{aligned} \tag{5}$$

where $t_{\sup_j \operatorname{dom} \phi + 1} = \sup_t \operatorname{dom} \phi$ defines the upper limit of the last integral, and $\{t_j\}_{j=0}^{\sup_j \operatorname{dom} \phi}$ is a nondecreasing sequence associated to the definition of the hybrid time

[5] When $X = \emptyset$, the requirement that ϕ belongs to X is not enforced, hence, there is no terminal constraint and the two-player zero-sum hybrid game evolves over an infinite (hybrid) horizon when $\operatorname{dom} \phi$ is unbounded, i.e., ϕ is complete.

[6] Under Assumption 3.5, the domain of the input u specifies whether from points in $\Pi(C) \cap \Pi(D)$ a jump or flow occur.

domain of (ϕ, u) [17, Definition 2.3].

When X is nonempty, the set $\mathcal{S}_{\mathcal{H}}^X(\xi) \subset \mathcal{S}_{\mathcal{H}}(\xi)$ denotes all maximal solutions from ξ that reach X at their terminal time. When X is empty, $\mathcal{S}_{\mathcal{H}}^X(\xi)$ is the set of complete solutions from ξ. We define the set of input actions that yield maximal solutions to $\mathcal{H}$ from ξ entering X as $\mathcal{U}_{\mathcal{H}}^X(\xi) := \{u : \exists(\phi, u) \in \mathcal{S}_{\mathcal{H}}^X(\xi)\}$. The feasible set $\mathcal{M} \subset \Pi(C) \cup \Pi(D)$ is the set of states ξ such that there exists $(\phi, u) \in \hat{\mathcal{S}}_{\mathcal{H}}^X(\xi)$ with $\phi(T, J) \in X$, where (T, J) is the terminal time of $\mathrm{dom}(\phi, u)$.

We now formulate the two-player zero-sum game.

Problem $(\diamond)$: Given the terminal set X, the feasible set $\mathcal{M}$, and $\xi \in \mathcal{M}$, under Assumption 3.5, solve

$$\underset{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}}{\text{minimize}} \; \underset{u_2}{\text{maximize}} \; \mathcal{J}(\xi, u) \tag{6}$$

over the space of feedback strategies.

Remark 3.6 (Infinite horizon games) *When the terminal set X is empty and maximal solutions are complete, Problem $(\diamond)$ reduces to an infinite horizon hybrid game as in [25], as stated in footnote 5. In this case, the feasible set satisfies $\mathcal{M} = \Pi(C) \cup \Pi(D)$ and, for each $\xi \in \mathcal{M}$, the set of complete solutions $\mathcal{S}_{\mathcal{H}}^X(\xi)$ is nonempty. For such games, the set $\mathcal{U}_{\mathcal{H}}^X$ in Problem $(\diamond)$ denotes the inputs that yield maximal complete solutions to $\mathcal{H}$.*

Remark 3.7 (Feedback saddle-point equilibrium and min-max input action) *A solution to Problem $(\diamond)$, when it exists, can be expressed in terms of the feedback saddle-point equilibrium $\kappa = (\kappa_1, \kappa_2)$ for the two-player zero-sum game. Each $u^* = (u_1^*, u_2^*)$ that renders a state trajectory $\phi^* \in \mathcal{R}(\xi, u^*)$, with components defined as $\mathrm{dom}\,\phi^* \ni (t,j) \mapsto u_i^*(t,j) = \kappa_i(\phi^*(t,j))$ for each $i \in \{1, 2\}$, satisfies*

$$u^* = \arg \min_{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \max_{u_2} \mathcal{J}(\xi, u) = \arg \max_{\substack{u_2 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \min_{u_1} \mathcal{J}(\xi, u)$$

and it is referred to as a min-max input action at ξ.

Definition 3.8 (Value function) *Given the terminal set X, the feasible set $\mathcal{M} \subset \Pi(\overline{C}) \cup \Pi(D)$, and $\xi \in \mathcal{M}$, under Assumption 3.5, the value function at ξ is given by*

$$\mathcal{J}^*(\xi) := \min_{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \max_{u_2} \mathcal{J}(\xi, u) = \max_{\substack{u_2 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \min_{u_1} \mathcal{J}(\xi, u) \tag{7}$$

4 Hamilton–Jacobi–Bellman-Isaacs Equations for Two-Player Zero-Sum Hybrid Games

The following result provides sufficient conditions to characterize the value function and the feedback law that attains it. It addresses the solution to Problem $(\diamond)$ showing that the optimizer is the saddle-point equilibrium. It involves the feasible set $\mathcal{M}$ to reduce the set over which the sufficient conditions need to be checked. When $\mathcal{M}$ is not known, it could just be replaced by $\mathbb{R}^n$.

Theorem 4.1 (Hamilton–Jacobi–Bellman-Isaacs (HJBI) for Problem $(\diamond)$) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (1) with data (C, F, D, G) satisfying Assumption 3.5, stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, (potentially empty) terminal set X, and feasible set $\mathcal{M}$, suppose the following hold:*

1) *There exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\Pi(C)$ and that satisfies the Hamilton–Jacobi–Bellman-Isaacs (HJBI) hybrid equations given as*

$$\begin{aligned} 0 &= \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \max_{u_{C2}} \; \mathcal{L}_C(x, u_C) \\ &= \max_{\substack{u_{C2} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \min_{u_{C1}} \; \mathcal{L}_C(x, u_C) \quad \forall x \in \Pi(C) \cap \mathcal{M}, \end{aligned} \tag{8}$$

where $\mathcal{L}_C(x, u_C) := L_C(x, u_C) + \langle \nabla V(x), F(x, u_C) \rangle$,

$$\begin{aligned} V(x) &= \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \max_{u_{D2}} \; \mathcal{L}_D(x, u_D) \\ &= \max_{\substack{u_{D2} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \min_{u_{D1}} \; \mathcal{L}_D(x, u_D) \quad \forall x \in \Pi(D) \cap \mathcal{M}, \end{aligned} \tag{9}$$

where $\mathcal{L}_D(x, u_D) := L_D(x, u_D) + V(G(x, u_D))$.

2) *For each $\xi \in \mathcal{M}$, each $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^X(\xi)$ satisfies*[7]

$$\limsup_{\substack{t+j \to \sup_t \mathrm{dom}\,\phi + \sup_j \mathrm{dom}\,\phi \\ (t,j)\in\mathrm{dom}\phi}} V(\phi(t,j)) = \limsup_{\substack{t+j \to \sup_t \mathrm{dom}\,\phi + \sup_j \mathrm{dom}\,\phi \\ (t,j)\in\mathrm{dom}\phi}} q(\phi(t,j)) \tag{10}$$

Then

$$\mathcal{J}^*(\xi) = V(\xi) \qquad \forall \xi \in \Pi(\overline{C}) \cup \Pi(D), \tag{11}$$

and any feedback law $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ with values

$$\kappa_C(x) \in \arg \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \max_{u_{C2}} \mathcal{L}_C(x, u_C) \qquad \forall x \in \Pi(C) \cap \mathcal{M} \tag{12}$$

$$\kappa_D(x) \in \arg \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \max_{u_{D2}} \mathcal{L}_D(x, u_D) \qquad \forall x \in \Pi(D) \cap \mathcal{M} \tag{13}$$

is a pure strategy saddle-point equilibrium for Problem $(\diamond)$ with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$, where $\mathcal{J}$ is as in (5).

When the players select the optimal strategy, the value function equals the function V. The result does not

[7] The boundary condition (10) matches the value of V to the terminal cost q at the final value of ϕ.

require computing solutions to $\mathcal{H}$, at the price of finding the function V satisfying the conditions therein.

The terminal set X determines the size of the compact hybrid time domain of the solutions considered in Theorem 4.1. Based on reachability tools, given X, the feasible set $\mathcal{M}$ can be computed for certain class of systems. When $\mathcal{M}$ is known a priori, the set of states for which equations (8) and (9) need to be enforced could be smaller than in the infinite horizon counterpart.

4.1 Proof of Theorem 4.1

Before we present the proof of Theorem 4.1, we present the following results providing sufficient conditions to bound and exactly evaluate the cost of the game. These results are instrumental on guaranteeing that the saddle-point equilibrium is attained and in evaluating the value function of the game.

Proposition 4.2 (Time-dependent conditions for upper bound) *Consider $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^X(\xi)$ with $u = (u_C, u_D)$, such that*

1) *for each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior*[8] *$\text{int} I_\phi^j$,*

$$L_C(\phi(t,j), u_C(t,j)) + \frac{dV}{dt}(\phi(t,j)) \le 0 \quad \forall t \in \text{int} I_\phi^j \tag{14}$$

2) *for every $(t_{j+1}, j) \in \operatorname{dom}\phi$ such that $(t_{j+1}, j+1) \in \operatorname{dom}\phi$,*

$$\begin{aligned} &L_D(\phi(t_{j+1},j), u_D(t_{j+1},j)) \\ &+V(\phi(t_{j+1},j+1)) - V(\phi(t_{j+1},j)) \le 0. \end{aligned} \tag{15}$$

Then

$$\begin{aligned} &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt \\ &+ \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} L_D(\phi(t_{j+1},j), u_D(t_{j+1},j)) \\ &+ \limsup_{\substack{t+j \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} V(\phi(t,j)) \le V(\xi). \end{aligned} \tag{16}$$

Proof. Given a $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)$, where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated with the hybrid time domain of (ϕ, u) as in [17, Definition 2.3], for each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior $\text{int} I_\phi^j$, by integrating (14) over I_ϕ^j, we obtain

$$0 \ge \int_{t_j}^{t_{j+1}} \left(L_C(\phi(t,j), u_C(t,j)) + \frac{dV}{dt}(\phi(t,j)) \right) dt$$

from where we have

$$\begin{aligned} 0 \ge \int_{t_j}^{t_{j+1}} & L_C(\phi(t,j), u_C(t,j))dt \\ &+V(\phi(t_{j+1},j)) - V(\phi(t_j,j)) \end{aligned}$$

Pick any $(t^*, j^*) \in \operatorname{dom}(\phi, u)$. Summing from $j = 0$ to $j = j^*$ we obtain

$$\begin{aligned} 0 \ge &\sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt \\ &+ \sum_{j=0}^{j^*} (V(\phi(t_{j+1},j)) - V(\phi(t_j,j))) \end{aligned}$$

Then, solving for V at the initial condition $\phi(0,0)$, we obtain

$$\begin{aligned} V(\phi(0,0)) \ge &\sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt \\ &+V(\phi(t_1,0)) + \sum_{j=1}^{j^*} (V(\phi(t_{j+1},j)) - V(\phi(t_j,j))) \end{aligned} \tag{17}$$

In addition, if $j^* > 0$, adding (15) from $j = 0$ to $j = j^* - 1$, we obtain

$$\begin{aligned} \sum_{j=0}^{j^*-1} V(\phi(t_{j+1},j)) \ge &\sum_{j=0}^{j^*-1} L_D(\phi(t_{j+1},j), u_D(t_{j+1},j)) \\ &+ \sum_{j=0}^{j^*-1} V(\phi(t_{j+1},j+1)) \end{aligned}$$

Then, solving for V at the first jump time, we obtain

$$\begin{aligned} V(\phi(t_1,0)) \ge &V(\phi(t_1,1)) + \sum_{j=0}^{j^*-1} L_D(\phi(t_{j+1},j), u_D(t_{j+1},j)) \\ &+ \sum_{j=1}^{j^*-1} (V(\phi(t_{j+1},j+1)) - V(\phi(t_{j+1},j))) \end{aligned} \tag{18}$$

In addition, given that $\phi(0,0) = \xi$, lower bounding $V(\phi(t_1,0))$ in (17) by the right-hand side of (18), we obtain

$$\begin{aligned} V(\xi) \ge &\sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + V(\phi(t_1,0)) \\ &+ \sum_{j=1}^{j^*} (V(\phi(t_{j+1},j)) - V(\phi(t_j,j))) \end{aligned}$$

[8] When $j = \sup_j \operatorname{dom}\phi \in \mathbb{N}$ and $\sup_t \operatorname{dom}\phi = \infty$, we define $t_{j+1} := \infty$.

Since

$$V(\phi(t_1,1)) + \sum_{j=1}^{j^*-1} (V(\phi(t_{j+1},j+1)) - V(\phi(t_{j+1},j))) + \sum_{j=1}^{j^*} (V(\phi(t_{j+1},j)) - V(\phi(t_j,j))) = V(\phi(t_{j^*+1},j^*))$$

then we have

$$V(\xi) \geq V(\phi(t_{j^*+1},j^*)) + \sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j),u_C(t,j))dt + \sum_{j=0}^{j^*-1} L_D(\phi(t_{j+1},j),u_D(t_{j+1},j))$$

By taking the limit when $t_{j^*+1} + j^* \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi$, we establish (16). Notice that if $j^* = 0$, the solution (ϕ, u) is continuous and (16) reduces to

$$V(\xi) \geq \limsup_{t^* \to \sup_t \operatorname{dom}\phi} \int_{t_0}^{t^*} L_C(\phi(t,0),u_C(t,0))dt + V(\phi(t^*,0)).$$

On the other hand, if $t_{j^*+1} = 0$ for all j^*, the solution (ϕ, u) is discrete and (16) reduces to

$$V(\xi) \geq \limsup_{j^* \to \sup_j \operatorname{dom}\phi} \sum_{j=0}^{j^*-1} L_D(\phi(0,j),u_D(0,j)) + V(\phi(0,j^*)).$$

□

The following corollary is immediate from the proof of Proposition 4.2.

Corollary 4.3 (Change of Signs) *If the inequalities in the conditions in Proposition 4.2 are inverted, namely, if "$\leq$" in* (14) *and* (15) *is replaced with "$\geq$", then* (16) *holds with the inequality inverted. Likewise, if the conditions in Proposition 4.2 hold with equalities, then* (16) *holds with equality.*

We are ready to prove Theorem 4.1. To show it, we proceed as follows:

a) Pick an initial condition $\xi \in \mathcal{M}$ and evaluate the cost associated to any solution from ξ yielded by $\kappa = (\kappa_C, \kappa_D)$, with values as in (12) and (13). Show that this cost coincides with the value of the function V at ξ.

b) Lower bound the cost associated to any solution from ξ when P_2 plays $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$ by the value of the function V evaluated at ξ.

c) Upper bound the cost associated to any solution from ξ when P_1 plays $\kappa_1 := (\kappa_{C1}, \kappa_{D1})$ by the value of the function V evaluated at ξ.

d) By showing that the cost of any solution from ξ when P_1 plays κ_1 is not greater than the cost of any solution yield by κ from ξ, and by showing that the cost of any solution from ξ when P_2 plays κ_2 is not less than the cost of any solution yield by κ from ξ, we show optimality of κ in Problem ($\diamond$) in the min-max sense.

Proceeding as in item a above, pick any $\xi \in \mathcal{M}$ and any $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^X(\xi)$ with $\operatorname{dom}\phi^* \ni (t,j) \mapsto u^*(t,j) = \kappa(\phi^*(t,j))$. We show that (ϕ^*, u^*) is optimal in the min-max sense. Given that V satisfies (8), and κ_C is as in (12), for each $j \in \mathbb{N}$ such that $I_{\phi^*}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi^*}^j$, we have, for all $t \in \operatorname{int} I_{\phi^*}^j$,

$$0 = L_C(\phi^*(t,j), \kappa_C(\phi^*(t,j))) + \langle \nabla V(\phi^*(t,j)), F(\phi^*(t,j), \kappa_C(\phi^*(t,j))) \rangle$$

and $\phi^*(t,j) \in C_\kappa$, as in (3). Given that V is continuously differentiable on a neighborhood of $\Pi(C)$, we can express its total derivative along ϕ^* as

$$\frac{dV}{dt}(\phi^*(t,j)) = \langle \nabla V(\phi^*(t,j)), F(\phi^*(t,j), \kappa_C(\phi^*(t,j))) \rangle \tag{19}$$

for every $(t,j) \in \operatorname{int}(I_{\phi^*}^j) \times \{j\}$ with $\operatorname{int}(I_{\phi^*}^j)$ nonempty. Given that V satisfies (9) and κ_D is as in (13), for every $(t_{j+1}, j) \in \operatorname{dom}\phi^*$ such that $(t_{j+1}, j+1) \in \operatorname{dom}\phi^*$, we have that

$$\begin{aligned} V(\phi^*(t_{j+1},j)) &= L_D(\phi^*(t_{j+1},j), \kappa_D(\phi^*(t_{j+1},j))) \\ &\quad + V(G(\phi^*(t_{j+1},j), \kappa_D(\phi^*(t_{j+1},j)))) \\ &= L_D(\phi^*(t_{j+1},j), \kappa_D(\phi^*(t_{j+1},j))) \\ &\quad + V(\phi^*(t_{j+1},j+1)) \end{aligned} \tag{20}$$

where $\phi^*(t_{j+1},j) \in D_\kappa$ is defined in (3). Now, given that (ϕ^*, u^*) is maximal with $\operatorname{dom}\phi^* \ni (t,j) \mapsto u^*(t,j) = \kappa(\phi^*(t,j))$, thanks to (19) and (20), from Corollary 4.3 and (10), we have that

$$V(\xi) = \mathcal{J}(\xi, u^*). \tag{21}$$

Continuing with item b as above, pick any $(\phi_s, u^s) \in \mathcal{S}_{\mathcal{H}}^s(\xi)$, where $\mathcal{S}_{\mathcal{H}}^s(\xi) \subset \mathcal{S}_{\mathcal{H}}^X(\xi)$ is the set of solutions (ϕ, u) with $u = (u_1, u_2)$, $\operatorname{dom}\phi \ni (t,j) \mapsto u_1(t,j) = \bar{\kappa}_1(\phi(t,j))$ for some $\bar{\kappa}_1 \in \mathcal{K}_1$, $\operatorname{dom}\phi \ni (t,j) \mapsto u_2(t,j) = \kappa_2(t,j,\phi(t,j))$ for $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$ as in (12) and (13). Since $\bar{\kappa}_1$ does not necessarily attain the minimum in (8), then, for each $j \in \mathbb{N}$ such that $I_{\phi_s}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi_s}^j$, we have for every $t \in \operatorname{int} I_{\phi_s}^j$,

$$0 \leq L_C(\phi_s(t,j), u_C^s(t,j)) + \langle \nabla V(\phi_s(t,j)), F(\phi_s(t,j), u_C^s(t,j)) \rangle .$$

Similarly to (19), we have

$$\frac{dV}{dt}(\phi_s(t,j)) := \langle \nabla V(\phi_s(t,j)), F(\phi_s(t,j), u^s_C(t,j))\rangle \tag{22}$$

for every $(t,j) \in \operatorname{int}(I^j_{\phi_s}) \times \{j\}$ with $\operatorname{int}(I^j_{\phi_s})$ nonempty. In addition, since $\bar{\kappa}_1$ does not necessarily attain the minimum in (9), then for every $(t_{j+1}, j) \in \operatorname{dom}\phi_s$ such that $(t_{j+1}, j+1) \in \operatorname{dom}\phi_s$, we have

$$\begin{aligned} V(\phi_s(t_{j+1}, j)) \leq L_D(\phi_s(t_{j+1}, j), u^s_D(t_{j+1}, j)) \\ + V(\phi_s(t_{j+1}, j+1)). \end{aligned} \tag{23}$$

Now, given that (ϕ_s, u^s) is maximal, with $u^s = (u^s_1, u^s_2)$, u^s_1 defined by any $\bar{\kappa}_1 \in \mathcal{K}_1$, and u^s_2 defined by κ_2 as in (12) and (13), thanks to (22) and (23), from Proposition 4.2 and (10), we have

$$V(\xi) \leq \mathcal{J}(\xi, u^s). \tag{24}$$

Proceeding with item c as above, pick any $(\phi_w, u^w) \in \mathcal{S}^w_{\mathcal{H}}(\xi)$, where $\mathcal{S}^w_{\mathcal{H}}(\xi) \subset \mathcal{S}^X_{\mathcal{H}}(\xi)$ is the set of solutions (ϕ, u) with $u = (u_1, u_2)$, $\operatorname{dom}\phi \ni (t,j) \mapsto u_1(t,j) = \kappa_1(\phi(t,j))$ for $\kappa_1 := (\kappa_{C1}, \kappa_{D1})$ as in (12) and (13), $\operatorname{dom}\phi \ni (t,j) \mapsto u_2(t,j) = \bar{\kappa}_2(\phi(t,j))$ for some $\bar{\kappa}_2 \in \mathcal{K}_2$. Since $\bar{\kappa}_2$ does not necessarily attain the maximum in (8), then, for each $j \in \mathbb{N}$ such that $I^j_{\phi_w} = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I^j_{\phi_w}$, we have for every $t \in \operatorname{int} I^j_{\phi_w}$,

$$\begin{aligned} 0 \geq L_C(\phi_w(t,j), u^w_C(t,j)) \\ + \langle \nabla V(\phi_w(t,j)), F(\phi_w(t,j), u^w_C(t,j))\rangle. \end{aligned}$$

Similarly to (19), we have

$$\frac{dV}{dt}(\phi_w(t,j)) := \langle \nabla V(\phi_w(t,j)), F(\phi_w(t,j), u^w_C(t,j))\rangle \tag{25}$$

for every $(t,j) \in \operatorname{int}(I^j_{\phi_w}) \times \{j\}$ with $\operatorname{int}(I^j_{\phi_w})$ nonempty. In addition, since $\bar{\kappa}_2$ does not necessarily attain the maximum in (9), then for every $(t_{j+1}, j) \in \operatorname{dom}\phi_w$ such that $(t_{j+1}, j+1) \in \operatorname{dom}\phi_w$, we have

$$\begin{aligned} V(\phi_w(t_{j+1}, j)) \geq L_D(\phi_w(t_{j+1}, j), u^w_D(t_{j+1}, j)) \\ + V(\phi_w(t_{j+1}, j+1)). \end{aligned} \tag{26}$$

Now, given that (ϕ_w, u^w) is maximal, with $u^w = (u^w_1, u^w_2)$, u^w_1 defined by κ_1 as in (12) and (13), and u^w_2 defined by any $\bar{\kappa}_2 \in \mathcal{K}_2$, thanks to (25) and (26), from Corollary 4.3 and (10), we have

$$V(\xi) \geq \mathcal{J}(\xi, u^w). \tag{27}$$

Finally, by proceeding as in item d above, by applying the infimum on (24) over the set $\mathcal{S}^s_{\mathcal{H}}(\xi)$, we obtain

$$V(\xi) \leq \inf_{(\phi_s, u^s) \in \mathcal{S}^s_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, u^s) =: \overline{V}(\xi) \tag{28}$$

Notice that the infimum in (28) is attained in $\mathcal{S}^s_{\mathcal{H}}(\xi)$ if there exists $(\phi_s, u^s) \in \mathcal{S}^s_{\mathcal{H}}(\xi)$ such that $\mathcal{J}(\xi, u^s) = \overline{V}(\xi)$. By applying the supremum on (27) over the set $\mathcal{S}^w_{\mathcal{H}}(\xi)$,

$$V(\xi) \geq \sup_{(\phi_w, u^w) \in \mathcal{S}^w_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, u^w) =: \underline{V}(\xi). \tag{29}$$

Notice that the supremum in (29) is attained in $\mathcal{S}^w_{\mathcal{H}}(\xi)$ if there exists $(\phi_w, u^w) \in \mathcal{S}^w_{\mathcal{H}}(\xi)$ such that $\mathcal{J}(\xi, u^w) = V(\xi)$, at which $\underline{V}(\xi) = V(\xi)$. Given that $V(\xi) = \mathcal{J}(\xi, u^*)$ from (21), we have that for any $\xi \in \mathcal{M}$, each $(\phi^*, u^*) \in \mathcal{S}^X_{\mathcal{H}}(\xi)$ with $u^* = (\kappa_1(\phi^*), \kappa_2(\phi^*))$ satisfies

$$\underline{V}(\xi) \leq \mathcal{J}(\xi, u^*) \leq \overline{V}(\xi) \tag{30}$$

Thanks to $(\phi^*, u^*) \in \mathcal{S}^s_{\mathcal{H}}(\xi) \cap \mathcal{S}^w_{\mathcal{H}}(\xi) \subset \mathcal{S}^X_{\mathcal{H}}(\xi)$, we have

$$\overline{V}(\xi) = \inf_{(\phi_s, u^s) \in \mathcal{S}^s_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, u^s) = \mathcal{J}(\xi, u^*) = V(\xi) \tag{31}$$

$$\underline{V}(\xi) = \sup_{(\phi_w, u^w) \in \mathcal{S}^w_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, u^w) = \mathcal{J}(\xi, u^*) = V(\xi) \tag{32}$$

Since the infimum and supremum are attained in (31) and (32), respectively, by (ϕ^*, u^*), (30) leads to

$$\mathcal{J}(\xi, u^*) = \min_{\substack{u_1 \\ (u_1,u_2) \in \mathcal{U}^X_{\mathcal{H}}(\xi)}} \max_{u_2} \mathcal{J}(\xi, (u_1, u_2)) \tag{33}$$

Thus, from (21) and (33), $V(\xi)$ is the value function for $\mathcal{H}$, as in Definition 3.8 and from (30), κ is the saddle-point equilibrium as in Definition 3.3. □

Remark 4.4 (Connections between Theorem 4.1 and Problem ($\diamond$)) *Given $\xi \in (\Pi(\overline{C}) \cup \Pi(D)) \cap \mathcal{M}$, if there exist a function V satisfying the conditions in Theorem 4.1, then a solution to Problem ($\diamond$) exists, namely there is an optimizer input action $u^* = (u^*_C, u^*_D) = ((u^*_{C1}, u^*_{C2}), (u^*_{D1}, u^*_{D2})) \in \mathcal{U}^X_{\mathcal{H}}(\xi)$ that satisfies (4), and V is the value function as in Definition 3.8. In addition, notice that the strategy $\kappa = (\kappa_C, \kappa_D) \in \mathcal{K}$, with elements as in (12) and (13), is such that every maximal solution to the closed-loop system $\mathcal{H}_\kappa$ from ξ has a cost equal to the min-max in (6), which equals the max-min.*

Remark 4.5 (Existence of a value function) *Theorem 4.1 does not explicitly rely on regularity conditions over the stage costs, flow and jump maps, convexity of $\mathcal{J}$, or compactness of the set of inputs $\mathcal{U}^X_{\mathcal{H}}$. Sufficient conditions to guarantee the existence of a solution to Problem ($\diamond$) are not currently available in the literature. One could expect that, as in any converse results, guaranteeing the existence of a value function satisfying* (8) *and* (9) *would require the data of the system and the game to satisfy certain regularity properties. Such is the case in optimal control problems [16].*

Remark 4.6 (Computation of the function V) *In Mayer-type games with dynamics defined by hybrid finite-state automata as in [44,9,31], reachability-based approaches allow to synthesize safety controllers and compute the value function through the satisfaction of HJB conditions. In several other cases, computing the saddle-point equilibrium strategy and the function V satisfying the HJBI hybrid equations is nontrivial. This is a challenge already present in the certification of asymptotic stability. However, the complexity associated to the computation of a Lyapunov function does not diminish the contribution that the sufficient conditions for stability have had in the field. In the same spirit, a contribution of Theorem 4.1, as an important step in games with dynamics defined as in [17], is in providing sufficient conditions that characterize value functions and saddle-point equilibria for such systems, similar to the results for continuous-time and discrete-time systems already available in the literature; e.g., [4].*

Remark 4.7 (Time invariance of saddle-point equilibrium) *In general, games with fixed horizon define time-varying strategies [26], with some exceptions in the case in which the system is autonomous and the stage and terminal costs are stationary – see [4, Remark 5.5]. Problems with variable terminal time are in general non-stationary, due to the end time being optimized. This leads to optimal strategies and value function depending explicitly on time. However, when a terminal set is fixed, though there is a variable (potentially unbounded) terminal time, the optimality conditions are stationary, (when the dynamics, the stage costs, and the terminal cost are time invariant). Notice that for such a case, the terminal time becomes implicit rather than explicit because the game ends when the state enters the terminal set. Hence, the time at which the game ends is not fixed a priori but rather determined by the evolution of the state. In other words, the time dependency is replaced by a state-dependent stopping time. Thus, the value function and optimal strategies depend only on the state.*

5 Asymptotic Stability for Hybrid Games

We present a result that connects optimality and asymptotic stability for hybrid games. First, we introduce positive definite functions.

Definition 5.1 (Positive definite functions) *A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite, also written as $\rho \in \mathcal{PD}$, if $\rho(s) > 0$ for all $s > 0$ and $\rho(0) = 0$. A function $\rho : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is positive definite with respect to a set $\mathcal{A} \subset \mathbb{R}^n$, in composition with $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, also written as $\rho \in \mathcal{PD}_\kappa(\mathcal{A})$, if $\rho(x, \kappa(x)) > 0$ for all $x \in \mathbb{R}^n \setminus \mathcal{A}$ and $\rho(\mathcal{A}, \kappa(\mathcal{A})) = \{0\}$.*

In the next result, we provide alternative conditions to those in Theorem 4.1 for the solution to Problem ($\diamond$).

Lemma 5.2 (Equivalent conditions) *Given $\mathcal{H}$ as in (1) with data (C, F, D, G), the terminal set X, the feasible set $\mathcal{M} \subset \Pi(\overline{C}) \cup \Pi(D)$, and feedback $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ that satisfies (12) and (13), if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\Pi(C)$ such that the data of $\mathcal{H}_\kappa$ as in (3) satisfies*[9] *$C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$, then (8), (9), (12), and (13) are satisfied if and only if*

$$\mathcal{L}_C(x, \kappa_C(x)) = 0 \quad \forall x \in C_\kappa \cap \mathcal{M}, \tag{34}$$

$$\begin{aligned} &\mathcal{L}_C(x, (u_{C1}, \kappa_{C2}(x))) \geq 0 \\ &\qquad \forall (x, u_{C1}) : \ (x, (u_{C1}, \kappa_{C2}(x))) \in C \cap \mathcal{M}, \end{aligned} \tag{35}$$

$$\begin{aligned} &\mathcal{L}_C(x, (\kappa_{C1}(x), u_{C2})) \leq 0 \\ &\qquad \forall (x, u_{C2}) : \ (x, (\kappa_{C1}(x), u_{C2})) \in C \cap \mathcal{M}, \end{aligned} \tag{36}$$

$$\mathcal{L}_D(x, \kappa_D(x)) = V(x) \quad \forall x \in D_\kappa \cap \mathcal{M}, \tag{37}$$

$$\begin{aligned} &\mathcal{L}_D(x, (u_{D1}, \kappa_{D2}(x))) \geq V(x) \\ &\qquad \forall (x, u_{D1}) : \ (x, (u_{D1}, \kappa_{D2}(x))) \in D \cap \mathcal{M}, \end{aligned} \tag{38}$$

$$\begin{aligned} &\mathcal{L}_D(x, (\kappa_{D1}(x), u_{D2})) \leq V(x) \\ &\qquad \forall (x, u_{D2}) : \ (x, (\kappa_{D1}(x), u_{D2})) \in D \cap \mathcal{M}. \end{aligned} \tag{39}$$

Under additional properties, the conditions above establish pre-asymptotic stability (pAS) of a set $\mathcal{A}$. We define pAS for hybrid systems as in [41, Definition 3.1]

Theorem 5.3 (Saddle-point equilibrium under the existence of a Lyapunov function) *Consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (1) with data (C, F, D, G) satisfying Assumption 3.5, and $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ defining the hybrid closed-loop system $\mathcal{H}_\kappa$ as in (3) such that $C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$. Given the terminal set X, the feasible set $\mathcal{M} \subset \Pi(\overline{C}) \cup \Pi(D)$, and a closed set $\mathcal{A} \subset \Pi(C) \cup \Pi(D)$, continuous functions $L_C : C \to \mathbb{R}_{\geq 0}$ and $L_D : D \to \mathbb{R}_{\geq 0}$ defining the stage costs for flows and jumps, respectively, and $q : \mathbb{R}^n \to \mathbb{R}$ defining the terminal cost, suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{C_\kappa}$, satisfying (34)-(39), and such that for each $\xi \in (\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M}$, each $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$ satisfies (10). If one of the following conditions holds*[10]

1) *$L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ and $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$;*
2) *$L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$ and there exists a continuous function $\eta \in \mathcal{PD}$ such that $L_C(x, \kappa_D(x)) \geq \eta(|x|_\mathcal{A})$ for all $x \in C_\kappa \cap \mathcal{M}$;*
3) *$L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ and there exists a continuous function $\eta \in \mathcal{PD}$ such that $L_D(x, \kappa_D(x)) \geq \eta(|x|_\mathcal{A})$ for all $x \in D_\kappa \cap \mathcal{M}$;*
4) *$L_C \equiv 0$, $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$, and for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for every*

[9] Notice that $C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$ when $\kappa_C(x) \in \Pi^C_u(x)$ for all $x \in \Pi(C)$ and $\kappa_D(x) \in \Pi^D_u(x)$ for all $x \in \Pi(D)$.
[10] Class-$\mathcal{K}_\infty$ functions are defined as usual, see, e.g., [17].

solution $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$, $|\phi(0,0)|_{\mathcal{A}} \in (0,r]$, $(t,j) \in \operatorname{dom}\phi$, $t+j \geq T$ *imply* $j \geq \gamma_r(T) - N_r$*;*

5) $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$, $L_D \equiv 0$, *and for each* $r > 0$, *there exist* $\gamma_r \in \mathcal{K}_\infty$ *and* $N_r \geq 0$ *such that for every solution* $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$, $|\phi(0,0)|_{\mathcal{A}} \in (0,r]$, $(t,j) \in \operatorname{dom}\phi$, $t+j \geq T$ *imply* $t \geq \gamma_r(T) - N_r$*;*

6) $L_C(x,\kappa_C(x)) \geq -\lambda_C V(x)$ *for all* $x \in C_\kappa$, $L_D(x,\kappa_D(x)) \geq (1-e^{\lambda_D})V(x)$ *for all* $x \in D_\kappa$, *and there exist* $\gamma, M > 0$ *such that, for each* $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$, $(t,j) \in \operatorname{dom}\phi$ *implies* $\lambda_C t + \lambda_D j \leq M - \gamma(t+j)$*;*

then $\mathcal{J}^*(\xi) = V(\xi)$ *for all* $\xi \in (\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M}$*. Furthermore, the feedback law* κ *is the saddle-point equilibrium (see Definition 3.3) and it renders* $\mathcal{A}$ *pre-asymptotically stable for* $\mathcal{H}_\kappa$ *with basin of attraction containing the largest sublevel set of* V *contained in* $\mathcal{M}$*.*

Proof. Since, by assumption, we have that $C_\kappa = \Pi(C)$, $D_\kappa = \Pi(D)$, and $V, \kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2}))$ are such that (34)-(39) hold, then, thanks to Lemma 5.2, V and κ satisfy (8), (9), (12), and (13). Since for each $\xi \in (\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M}$, each $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$ satisfies (10), we have from Theorem 4.1 that V is the value function as in (7) for $\mathcal{H}_\kappa$ at $(\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M}$ and the feedback law κ with values (12), (13) is the saddle-point equilibrium for this game. Then, V is a Lyapunov candidate for $\mathcal{H}_\kappa$ [17, Def. 3.16] since $\overline{C_\kappa} \cup D_\kappa \subset \operatorname{dom} V = \mathbb{R}^n$ and V is continuously differentiable on an open set containing $\overline{C_\kappa}$. From (34) and (37), we have that

$$\langle \nabla V(x), F(x,\kappa_C(x))\rangle \leq -L_C(x,\kappa_C(x)) \quad \forall x \in C_\kappa \cap \mathcal{M}, \tag{40}$$

$$V(G(x,\kappa_D(x))) - V(x) \leq -L_D(x,\kappa_D(x)) \quad \forall x \in D_\kappa \cap \mathcal{M}. \tag{41}$$

Moreover, if

a) Item 1, item 4, or item 5 above hold, define

$$\rho(x,\kappa(x)) := \begin{cases} L_C(x,\kappa_C(x)) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{L_C(x,\kappa_C(x)), L_D(x,\kappa_D(x))\} & \text{if } x \in C_\kappa \cap D_\kappa \\ L_D(x,\kappa_D(x)) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

b) Item 2 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} \eta(|x|_{\mathcal{A}}) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{\eta(|x|_{\mathcal{A}}), L_D(x,\kappa_D(x))\} & \text{if } x \in C_\kappa \cap D_\kappa \\ L_D(x,\kappa_D(x)) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

c) Item 3 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} L_C(x,\kappa_C(x)) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{L_C(x,\kappa_C(x)), \eta(|x|_{\mathcal{A}})\} & \text{if } x \in C_\kappa \cap D_\kappa \\ \eta(|x|_{\mathcal{A}}) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

Thus, given that from (40) and (41), for each case above the function ρ satisfies

$$\langle \nabla V(x), F(x,\kappa_C(x))\rangle \leq -\rho(x,\kappa(x)) \quad \forall x \in C_\kappa \cap \mathcal{M}, \tag{42}$$

$$V(G(x,\kappa_D(x))) - V(x) \leq -\rho(x,\kappa(x)) \quad \forall x \in D_\kappa \cap \mathcal{M}. \tag{43}$$

Thanks to [41, Theorem 3.19], the set $\mathcal{A}$ is pAS for $\mathcal{H}_\kappa$. The case when item 6 holds follows directly from item e in [41, Theorem 3.19]. □

6 Applications

We illustrate Theorem 4.1 in a disturbance rejection and a security problem by recasting them as zero-sum hybrid games.

6.1 Robust hybrid LQR with aperiodic jumps

In this section, we study a special case that emerges in applications featuring sample-and-hold control implementations and intermittent information scenarios – see, e.g., [11,12,29,39,40]. We introduce a state variable τ that plays the role of a timer. Once τ reaches an element in a threshold set $\{T_1, T_2\}$ with $0 \leq T_1 \leq T_2$, it potentially [11] triggers a jump in the state and resets τ to zero. More precisely, we consider a hybrid system with state $x = (x_p, \tau) = ((x_{p1}, x_{p2}), \tau) \in \mathbb{R}^n \times [0, T_2]$, input $u = (u_C, u_D) = ((u_{C1}, u_{C2}), (u_{D1}, u_{D2})) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (1), defined by

$$\begin{aligned} C &:= \mathbb{R}^n \times [0, T_2] \times \mathbb{R}^{m_C} \\ F(x,u_C) &:= (A_C x_p + B_C u_C, 1) \quad \forall (x,u_C) \in C \\ D &:= \mathbb{R}^n \times \{T_1, T_2\} \times \mathbb{R}^{m_D} \\ G(x,u_D) &:= (A_D x_p + B_D u_D, 0) \quad \forall (x,u_D) \in D \end{aligned} \tag{44}$$

with $A_C = \begin{bmatrix} A_{C1} & 0 \\ 0 & A_{C2} \end{bmatrix}$, $B_C = \begin{bmatrix} B_{C1} & B_{C2} \end{bmatrix}$, $A_D = \begin{bmatrix} A_{D1} & 0 \\ 0 & A_{D2} \end{bmatrix}$, and $B_D = \begin{bmatrix} B_{D1} & B_{D2} \end{bmatrix}$. Here, the input $u_1 := (u_{C1}, u_{D1})$ plays the role of the control and is assigned by player P_1, and $u_2 := (u_{C2}, u_{D2})$ is the disturbance input, which is assigned by player P_2. The problem of minimizing the effect of the worst-case disturbance u_2 in the cost of complete solutions to $\mathcal{H}$ is formulated as a two-player zero-sum game as in Section 3.1. Thus, by solving Problem $(\diamond)$ for every $\xi \in \Pi(C) \cup \Pi(D)$, the control objective is achieved.

The following result presents a tool for the solution of optimal control problems for hybrid systems with linear maps and aperiodic jumps under an adversarial action.

Proposition 6.1 (Hybrid Riccati equation for disturbance rejection with aperiodic jumps) *Given a hybrid system* $\mathcal{H}$ *as in* (1) *defined by* (C,F,D,G) *as in* (44) *with state* $x = (x_p, \tau) \in \mathbb{R}^n \times [0, T_2]$, *let* $0 \leq T_1 \leq T_2$, *and, with the aim of pursuing minimum energy and*

[11] When $T_1 < T_2$, solutions can either evolve via flow or jump when $\tau = T_1$. A sequence $\{T_i\}_{i=1}^N$ can be handled similarly.

distance to the origin, consider the cost functions $L_C(x,u_C) := x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2}$, $L_D(x,u_D) := x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, *and terminal cost* $q(x) := x_p^\top P(\tau) x_p$ *defining* $\mathcal{J}$ *as in (5), with* $Q_C, Q_D \in \mathbb{S}_+^n$, $R_{C1} \in \mathbb{S}_+^{m_{C_1}}$, $-R_{C2} \in \mathbb{S}_+^{m_{C_2}}$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, *and* $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$. *Suppose there exists a matrix function* $P : [0, T_2] \to \mathbb{S}_+^n$ *that is continuously differentiable and such that*

$$-\frac{dP}{d\tau}(\tau) = -P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau) + Q_C + P(\tau)A_C + A_C^\top P(\tau) \quad \forall \tau \in (0, T_2), \tag{45}$$

$$-R_{D2} - B_{D2}^\top P(0) B_{D2},\ R_{D1} + B_{D1}^\top P(0) B_{D1} \in \mathbb{S}_{0+}^{m_D}, \tag{46}$$

the matrix $R_v := R_D + [B_{D1}\, B_{D2}]^\top P(0)[B_{D1}\, B_{D2}]$, *with* $R_D := \mathrm{diag}(R_{D1}, R_{D2})$, *is invertible, and*

$$\begin{aligned} P(\bar{T}) = Q_D &+ A_D^\top P(0) A_D \\ &- A_D^\top P(0)[B_{D1}\ B_{D2}] R_v^{-1} [B_{D1}\ B_{D2}]^\top P(0) A_D \end{aligned} \tag{47}$$

at each $\bar{T} \in \{T_1, T_2\}$, *where* $A_C, B_{C1}, B_{C2}, A_D, B_{D1}$, *and* B_{D2} *are defined below (44). Then, the feedback law* $\kappa := (\kappa_C, \kappa_D)$, *with values*

$$\kappa_C(x) = (-R_{C1}^{-1}B_{C1}^\top P(\tau)x_p, -R_{C2}^{-1}B_{C2}^\top P(\tau)x_p) \quad \forall x \in \Pi(C), \tag{48}$$

$$\kappa_D(x) = -R_v^{-1}[B_{D1}\ B_{D2}]^\top P(0) A_D x_p \quad \forall x \in \Pi(D) \tag{49}$$

is a pure strategy saddle-point equilibrium for the two-player zero-sum hybrid game with periodic jumps. In addition, for each $x = (x_p, \tau) \in \Pi(\overline{C}) \cup \Pi(D)$, *the value function is equal to* $V(x) := x_p^\top P(\tau) x_p$.

Proof. We show that when (45)-(47) hold, by using Theorem 4.1, the value function is equal to V and the feedback law $\kappa := (\kappa_C, \kappa_D)$ with values in (48) and (49), such a cost is attained. We can write (8) in Theorem 4.1 as

$$\begin{aligned} 0 &= \min_{\substack{u_{C1} \\ u_C = (u_{C1}, u_{C2}) \in \Pi_u^C(x)}} \max_{u_{C2}} \mathcal{L}_C(x, u_C), \\ \mathcal{L}_C(x, u_C) &= x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} \\ &\quad + 2x_p^\top P(\tau)(A_C x_p + B_C u_C) + x_p^\top \frac{dP}{d\tau}(\tau) x_p \end{aligned} \tag{50}$$

First, thanks to (45) and $x_p^\top(P(\tau)A_C + A_C^\top P(\tau))x_p = 2x_p^\top P(\tau) A_C x_p$, for every $x \in \Pi(C)$, one has

$$\begin{aligned} \mathcal{L}_C(x, u_C) = \\ & x_p^\top P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau)x_p \\ & + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2x_p^\top P(\tau) B_C u_C \end{aligned}$$

The first-order necessary conditions for optimality $\frac{\partial \mathcal{L}_C}{\partial u_{C1}}(x, u_C)\big|_{u_C^*} = 0$, $\frac{\partial \mathcal{L}_C}{\partial u_{C2}}(x, u_C)\big|_{u_C^*} = 0$ for all $(x, u_C) \in C$ are satisfied by the point $u_C^* = (u_{C1}^*, u_{C2}^*)$, with values

$$u_{C1}^* = -R_{C1}^{-1} B_{C1}^\top P(\tau) x_p, \quad u_{C2}^* = -R_{C2}^{-1} B_{C2}^\top P(\tau) x_p \tag{51}$$

for each $x = (x_p, \tau) \in \Pi(C)$. Since $R_{C1}, -R_{C2} \in \mathbb{S}_+^{m_D}$, the second-order sufficient conditions for optimality $\frac{\partial^2 \mathcal{L}_C}{\partial u_{C1}^2}(x, u_C)\big|_{u_C^*} \succeq 0$, $\frac{\partial^2 \mathcal{L}_C}{\partial u_{C2}^2}(x, u_C)\big|_{u_C^*} \preceq 0$, hold for all $(x, u_C) \in C$, rendering u_C^* as in (51) as an optimizer of the min-max problem in (50). In addition, it satisfies $\mathcal{L}_C(x, u_C^*) = 0$, making $V(x) = x_p^\top P(\tau) x_p$ a solution to (8) in Theorem 4.1.

On the other hand, we can write (9) as

$$\begin{aligned} x_p^\top P(\bar{T}) x_p &= \min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \Pi_u^D(x)}} \max_{u_{D2}} \mathcal{L}_D(x, u_D), \\ \mathcal{L}_D(x, u_D) &= x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2} \\ &\quad + (A_D x_p + B_D u_D)^\top P(0)(A_D x_p + B_D u_D) \end{aligned} \tag{52}$$

Similar to the case along flows, the first-order necessary conditions for optimality are satisfied by the point $u_D^* = (u_{D1}^*, u_{D2}^*)$, such that, for each $(x_p, \tau) \in \Pi(D)$,

$$u_D^* = -R_v^{-1}[B_{D1}\ B_{D2}]^\top P(0) A_D x_p, \tag{53}$$

Thanks to (46), the second-order sufficient conditions for optimality are satisfied, rendering u_D^* as in (53) as an optimizer of the problem in (52). In addition, u_D^* satisfies $\mathcal{L}_D(x, u_D^*) = x_p^\top P(\bar{T}) x_p$ with $\bar{T} \in \{T_1, T_2\}$ and $P(\bar{T})$ as in (47), making $V(x) = x_p^\top P(\tau) x_p$ a solution of (9).

Then, given that V is continuously differentiable on a neighborhood of $\Pi(C)$ and that Assumption 3.5 holds, by applying Theorem 4.1, in particular from (11), for every $\xi = (\xi_p, \xi_\tau) \in \Pi(\overline{C}) \cup \Pi(D)$ the value function is $\mathcal{J}^*(\xi) = \xi_p^\top P(\xi_\tau) \xi_p$. From (12) and (13), when P_1 plays u_1^* defined by $\kappa_1 = (\kappa_{C1}, \kappa_{D1})$ with values as in (48) and (49), and P_2 plays any disturbance u_2 such that solutions to $\mathcal{H}$ with data as in (44) are complete, then the cost is upper bounded by $\mathcal{J}(\xi, u^*)$, satisfying (4). □

Notice that when $T_1 < T_2$ are finite, the jumps are not necessarily periodic, since they can occur when $\tau = T_1$ or when $\tau = T_2$. When $T_1 = T_2 = 0$ we recover the discrete-time LQR robust problem, when $T_1 = T_2 = \infty$ we recover the continuous-time LQR robust problem, and when $T_1 = T_2$ are finite, we have a hybrid game with periodic jumps as in in Figure 1.

6.2 *Security jumps-actuated hybrid game*

Consider a hybrid system with state $x \in \mathbb{R}^n$, input $u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (1),

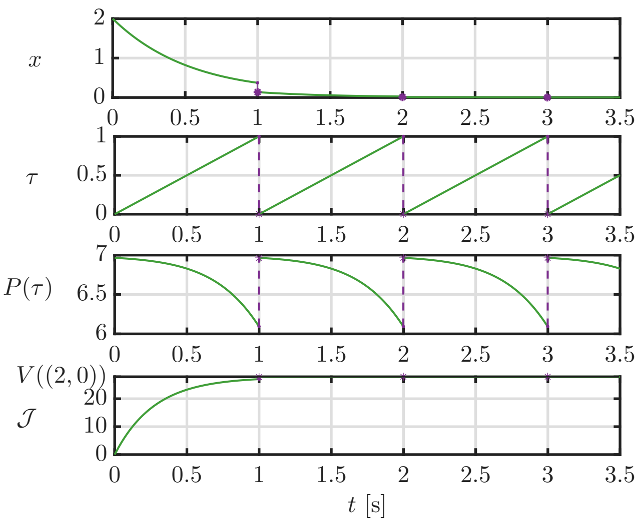

Fig. 1. 1D Robust hybrid LQR with periodic jumps. Dynamics as in (44) with $A_C = 1.8, B_C = [1,1], A_D = 2, B_D = [1,1], Q_C = 0.1, R_C = \mathrm{diag}(1.304, -4), Q_D = 1, R_D = \mathrm{diag}(1.304, -8), P(0) = 6.9653, T_1 = T_2 = 1.$

described by

$$\begin{aligned} \dot{x} &= F(x) & x \in C \\ x^+ &= A_D x + \begin{bmatrix} B_{D1} \; B_{D2} \end{bmatrix} u_D & (x, u_D) \in D \end{aligned} \tag{54}$$

with $F : \mathbb{R}^n \to \mathbb{R}^n$, $A_D \in \mathbb{R}^{n\times n}$, and $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$, such that $C \cup \Pi(D)$ is nonempty. The input u_{D1} plays the role of the control and u_{D2} the disturbance input [12]. Here, the problem of minimizing a cost $\mathcal{J}$ in the presence of the maximizing attack u_{D2} is formulated as a two-player zero-sum game as in Problem ($\diamond$).

The following result presents a tool for the solution of optimal control problems for jumps-actuated hybrid systems and state-affine flow maps under a malicious input attack designed to cause as much damage as possible.

Corollary 6.2 (Hybrid Riccati equation for security) *Given a hybrid system $\mathcal{H}$ as in (1) defined by (C, F, D, G) as in (54), and, with the aim of pursuing minimum energy and distance to the origin, consider the cost functions $L_C(x, \cdot) := 0$, $L_D(x, u_D) := x^\top Q_D x + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, and terminal cost $q(x) := x^\top P x$, defining $\mathcal{J}$ as in (5), with $Q_D \in \mathbb{S}_+^n$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$ and $P \in \mathbb{S}_+^n$. Suppose there exists a matrix $P \in \mathbb{S}_+^n$ such that*

$$0 = 2x^\top P F(x) \qquad \forall x \in C, \tag{55}$$

$$-R_{D2} - B_{D2}^\top P B_{D2}, R_{D1} + B_{D1}^\top P B_{D1} \in \mathbb{S}_{0+}^{m_D}, \tag{56}$$

the matrix $R_v := R_D + [B_{D1}\; B_{D2}]^\top P [B_{D1}\; B_{D2}]$, with $R_D := \mathrm{diag}(R_{D1}, R_{D2})$, is invertible, and

$$\begin{aligned} 0 = &-P + Q_D + A_D^\top P A_D \\ &- A_D^\top P [B_{D1}\; B_{D2}] R_v^{-1} [B_{D1}\; B_{D2}]^\top P A_D \end{aligned} \tag{57}$$

Then, the feedback law

$$\kappa_{D1}(x) = -[R_{v_{11}}^{-1}\; R_{v_{12}}^{-1}][B_{D1}\; B_{D2}]^\top P A_D x \qquad \forall x \in \Pi(D) \tag{58}$$

minimizes the cost functional $\mathcal{J}$ in the presence of the maximizing attack u_2, given by

$$\kappa_{D2}(x) = -[R_{v_{21}}^{-1}\; R_{v_{22}}^{-1}][B_{D1}\; B_{D2}]^\top P A_D x \qquad \forall x \in \Pi(D) \tag{59}$$

In addition, for each $x \in \overline{C} \cup \Pi(D)$, the value function is equal to $V(x) := x^\top P x$.

Proof. We show that when conditions (55)-(57) hold, by using the result in Theorem 4.1 with $X = \emptyset$,, the value function is equal to the function V and under the feedback law as in (58) such a cost is attained in the presence of the maximizing attack given by (59). We can write (8) as $0 = 2x^\top P F(x)$ for all $x \in C$, which is satisfied thanks to (55). Likewise, we can write (9) as

$$\begin{aligned} x^\top P x &= \min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \Pi_u^D(x)}} \max_{u_{D2}} \mathcal{L}_D(x, u_D), \\ \mathcal{L}_D(x, u_D) &= x^\top Q_D x + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2} \\ &\quad + (A_D x + B_D u_D)^\top P (A_D x + B_D u_D). \end{aligned} \tag{60}$$

The first order necessary conditions for optimality are satisfied by $u_D^* = (u_{D1}^*, u_{D2}^*)$, defined for each $x \in \Pi(D)$ as

$$u_D^* = -R_v^{-1} [B_{D1}\; B_{D2}]^\top P A_D x \tag{61}$$

Given that (56) holds, the second-order sufficient conditions for optimality are satisfied, rendering u_D^* as in (61) as an optimizer of the problem in (60). In addition, u_D^* satisfies $\mathcal{L}_D(x, u_D^*) = x^\top P x$, with P as in (57), leading $V(x) = x^\top P x$ as a solution of (9).

Thus, given that V is continuously differentiable in $\mathbb{R}^n$ and Assumption (3.5) holds, by applying Theorem 4.1, in particular from (11), for every $\xi \in \overline{C} \cup \Pi(D)$ the value function is $\mathcal{J}^*(\xi) = \xi^\top P \xi$. From (12) and (13), when P_2 plays u_2^* defined by κ_{D2} as in (59), P_1 minimizes the cost of complete solutions to $\mathcal{H}$ by playing u_1^* defined by κ_{D1} as in (58), attaining $\mathcal{J}(\xi, u^*)$, and satisfying (4). □

Notice that the saddle-point equilibrium $\kappa_D := (\kappa_{D1}, \kappa_{D2})$ is composed by P_1 playing the minimizer strategy κ_{D1} as in (58), and P_2 playing the maximizing

[12] "Jumps-actuated" makes reference to the lack of inputs during flows. Since this example is general, any condition involving x and u_D can be specified in D to trigger jumps.

attack κ_{D2} as in (59). Given that the flow map F does not have inputs, as long as it satisfies (55), the two-player discrete time Riccati algebraic equation (57) allows to charazterize the optimal at-jumps-only strategy.

Example 6.3 (Bouncing ball with terminal set) Inspired by the problem in [43], consider a simplified model of a juggling system as in [34], with state $x = (x_p, x_v) \in \mathbb{R}^2$, input $u_D := (u_{D1}, u_{D2}) \in \mathbb{R}^2$, and dynamics $\mathcal{H}$ as in (1), with data

$$\begin{aligned} C &= \mathbb{R}_{\geq 0} \times \mathbb{R}, \qquad F(x) = (x_v, -1) \quad \forall x \in C \\ D &= \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2, \\ G(x, u_D) &= (0, -\lambda x_v + u_{D1} + u_{D2}) \quad \forall (x, u_D) \in D \end{aligned} \tag{62}$$

where u_{D1} is the control input, u_{D2} is the action of an attacker, and $\lambda \in (0,1)$ is the coefficient of restitution of the ball. The scenario in which u_{D1} is designed to minimize a cost functional $\mathcal{J}$ until the game ends, which occurs when the state enters a set X, under the presence of the worst-case disturbance u_{D2} is formulated as a two-player zero-sum game. With the aim of pursuing minimum velocity and control effort at jumps, consider the cost functions $L_C(x, u_C) := 0$, $L_D(x, u_D) := {x_v}^2 Q_D + u_D^\top R_D u_D$, and terminal cost $q(x) := \frac{1}{2}{x_v}^2 + x_p$ defining $\mathcal{J}$ as in (5), with $R_D := \begin{bmatrix} R_{D1} & 0 \\ 0 & R_{D2} \end{bmatrix}$ and Q_D, R_{D1}, $-R_{D2} > 0$. Here, u_{D1} is designed by player P_1, which aims to minimize $\mathcal{J}$, while player P_2 seeks to maximize it by choosing u_{D2}. A game of kind [4, Section 5.2] arises and its solution characterizes a division of the state space into two dominance regions, $\mathcal{M}, \Psi \subset \Pi(C) \cup \Pi(D)$, in which, under optimal play, it can be determined whether the terminal set X is reached or not as a function of the initial condition. If the initial state satisfies $\xi \in \mathcal{M}$ (the feasible set), then, under optimal play, the ball reaches the terminal set X at some time (T, J) and the game ends. On the other hand, if $\xi \in \Psi$, under optimal play, we have an infinite horizon game (if maximal solutions are complete).

The function $x \mapsto V(x) := x_p + \frac{1}{2}{x_v}^2$ is such that $\langle \nabla V(x), F(x) \rangle = 0$ for all $x \in C$, making V a solution to (8). In addition, notice that V satisfies

$$\min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^2}} \max_{u_{D2}} \mathcal{L}_D(x, u_D) = \frac{1}{2}{x_v}^2 \tag{63}$$

for all $(x, u_D) \in D$. Equality (63) is attained by $\kappa_D(x) := (\kappa_{D1}(x), \kappa_{D2}(x))$ with $\kappa_{D1}(x) := \frac{R_{D2}\lambda}{R_{D1} + R_{D2} + 2R_{D1}R_{D2}} x_v$ and $\kappa_{D2}(x) := \frac{R_{D1}\lambda}{R_{D1} + R_{D2} + 2R_{D1}R_{D2}} x_v$ when

$$Q_D = \frac{-2R_{D1}R_{D2}\lambda^2 + R_{D1} + R_{D2} + 2R_{D1}R_{D2}}{2R_{D1} + 2R_{D2} + 4R_{D1}R_{D2}}, \tag{64}$$

which makes V a solution to (9) with saddle-point equilibrium κ_D. Thus, given that V is continuously differentiable on $\mathbb{R}^2$, and that (8) and (9) hold thanks to (63) and (64), from Theorem 4.1, the value function is $\mathcal{J}^*(\xi) = {\xi_v}^2/2 + \xi_p$, where $\xi = (\xi_p, \xi_v) \in \mathbb{R}^2$. Figure 2 displays this behavior with $\xi \in \mathcal{M}$ and both players playing the saddle point equilibrium. The terminal set X is reached at $t = 8$s and the cost of the solution is $V(\xi)$.

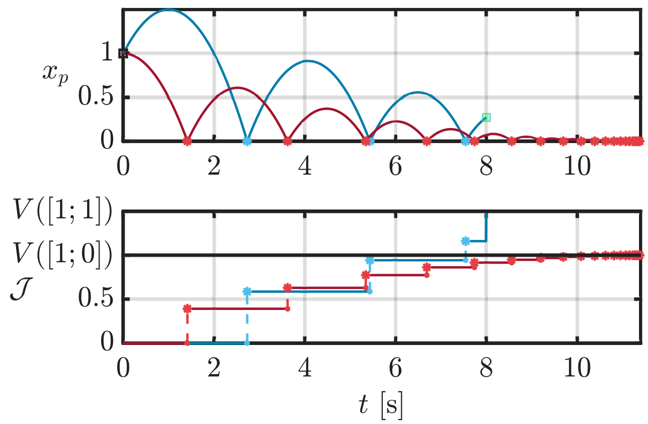

Fig. 2. Bouncing ball solutions attaining minimum cost under worst-case u_2, with $X = \{x \in \mathbb{R}^2 : 0 \leq x_p \leq 0.3, -0.37 \leq x_v \leq 0.37\}$, $\lambda = 0.8$, $R_{D1} = 10, R_{D2} = -20$, and $Q_D = 0.189$. Solution in reachable set (blue). Complete solution (red). Value function (black). Initial conditions (squares). Terminal set (green).

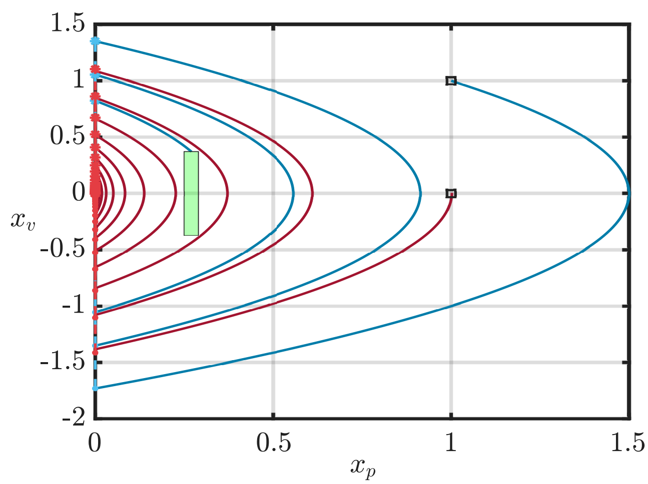

Fig. 3. Bouncing ball phase portrait. Terminal set (green) and initial condition (square).

As an alternative version of this game, consider the case in which $X = \emptyset$ and let $\mathcal{A} = \{0\}$, encoding the goal of stabilizing the ball to rest under the effect of an attacker. This implies that $\mathcal{M} = \emptyset$ and $\Psi = C \cup \Pi(D)$. This is formulated as a two-player zero-sum infinite horizon hybrid game via solving Problem ($\diamond$) over the set of complete input actions. The function V is a solution to (9) with saddle-point equilibrium κ_D. Similarly, given that V is continuously differentiable on $\mathbb{R}^2$, and that (8) and (9) hold thanks to (63) and (64), from Theorem 4.1, the value function is $\mathcal{J}^*(\xi) = {\xi_v}^2/2 + \xi_p$, where $\xi := (\xi_p, \xi_v) \in \mathbb{R}^2$. Figure 2 displays this behavior.

Furthermore, given that $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$, and (34)-(39) hold, by setting $\alpha_1(s) = \min\left\{\frac{1}{2}\left(\frac{s}{\sqrt{2}}\right)^2, \frac{s}{\sqrt{2}}\right\}$ and $\alpha_2(s) = \frac{1}{2}s^2 + s$ for all $s \geq 0$, from Theorem 5.3, we have that κ_D is the saddle-point equilibrium and renders $\mathcal{A} = \{0\}$ pre-asymptotically stable for $\mathcal{H}$.

□

A special hybrid game emerges under a *capture-the-flag* setting, in which two teams compete to grab the opponents flag and return it to their base without getting tagged. This is formulated as in Problem ($\diamond$) in [24].

7 Further Connections with the Literature

Some results provided in this paper have direct counterparts in the continuous-time and discrete-time game theory literature. The definition of a game in terms of its elements can betraced back to [4], as explained below.

Given a discrete-time two-player zero-sum game with final time [13] "J", f_k and X defining the single-valued jump map and jump set, respectively, as in [4], setting the data of $\mathcal{H}$ as $C = \emptyset$, $G = f_k$ for $k \in \mathbb{N}_{\leq J}$, and $D = X$ reduces Definition 3.1 to [4, Def. 5.1] for the case in which the output of each player is equal to its state and there is a feedback information structure as in [4, Def. 5.2]. Thus, items $(vi) - (ix)$ in [4, Def. 5.1] are omitted in the formulation herein and items $(i) - (v)$ and $(x) - (xi)$ are covered by Definition 3.1, the definition of the hybrid time domain with final time $(0, J)$, and the set $\mathcal{S}_\mathcal{H}$.

Given a continuous-time two-player zero-sum game with final time [14] "T", f and $\mathcal{S}^0$ defining the single-valued flow map and flow set, respectively, as in [4], setting the data of $\mathcal{H}$ as $D = \emptyset$, $F = f$, and $C = \mathcal{S}^0$ reduces Definition 3.1 to [4, Def. 5.5] for the case in which the output of each player is equal to its state and there is a feedback information structure as in [4, Def. 5.6]. Thus, items $(vi) - (vii)$ in [4, Def. 5.5] are omitted in the formulation herein and items $(i) - (v)$ and $(viii), (ix)$ are covered by Definition 3.1, the definition of the hybrid time domain with final time $(0, T)$, and the set $\mathcal{S}_\mathcal{H}$.

Remark 7.1 (Equivalent costs) *Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and a strategy $\kappa^* = (\kappa_1^* \kappa_2^*) \in \mathcal{K}$, denote by $\mathcal{U}^*(\xi, \kappa^*)$ the set of joint actions $u = (u_1, u_2)$ rendering a maximal trajectory ϕ to $\mathcal{H}$ from ξ with components defined as $\operatorname{dom}\phi \ni (t,j) \mapsto u_i(t,j) = \kappa_i^*(\phi(t,j))$ for each $i \in \{1,2\}$. By expressing the largest cost associated to the solutions to $\mathcal{H}$ from ξ under the strategy κ^* as $\hat{\mathcal{J}}(\xi, \kappa^*) := \sup_{u \in \mathcal{U}^*(\xi,\kappa^*)} \mathcal{J}(\xi, u)$, an equivalent condition to (4) for when $\mathcal{J}(\xi, u) = \hat{\mathcal{J}}(\xi, \kappa^*)$ for every $u \in \mathcal{U}^*(\xi, \kappa^*)$ is $\hat{\mathcal{J}}(\xi, (\kappa_1^*, \kappa_2)) \leq \hat{\mathcal{J}}(\xi, \kappa^*) \leq \hat{\mathcal{J}}_1(\xi, (\kappa_1, \kappa_2^*))$ for all $\kappa_i \in \mathcal{K}_i$, $i \in \{1, 2\}$.*

Conditions for computing value functions for linear quadratic problems have been widely studied, concerning solving differential and algebraic Riccati equations. The computation of value functions for systems with nonlinear dynamics is an open research problem and has seen interesting learning-based contributions in the last years, e.g., [32]. The computation of value functions for DAEs is discussed in [27], [46], for the case of linear differential games under algebraic constraints. Such value functions have a similar structure to the ones provided herein for hybrid systems with linear jump and flow maps and algebraic constraints encoded by the flow set C.

The design of value functions for switched DAEs imposes additional challenges that follow the discussion in [30] on the existence of Lyapunov functions and asymptotic stability. In some cases, a common Lyapunov function for all the subsystems of a switched DAE does not exist and even when it exists, it is not enough to guarantee asymptotic stability due to arbitrary switching. To solve this, conditions over switching are provided in [30, Theorem 4.1], and for the optimality of hybrid systems, such conditions are resembled by the point-wise conditions on the change of V along jumps. In [42], there are coupled value functions associated to each subsystem of a switched DAE in a zero-sum game, which result in coupled Riccati differential equations with optimal feedback strategies described by linear-time-varying functions of the state. Note that both scenarios are accounted for in the design of a value function for hybrid games based on optimality pointwise conditions provided in this work.

8 Conclusion and Future Work

In this paper, we formulate a two-player zero-sum game under dynamic constraints given in terms of hybrid dynamical systems, as in [17]. Scenarios in which the control action is selected by a player P_1 to accomplish an objective and counteract the damage caused by an adversarial player P_2 are studied. By encoding the objectives of the players in the optimization of a cost functional, sufficient conditions in Hamilton–Jacobi–Bellman-Isaacs form are provided to upper bound the cost for any disturbance. The main result allows the optimal strategy of P_1 to minimize the cost under the maximizing adversarial action. Additional conditions are proposed to allow the saddle-point strategy to render a set of interest asymptotically stable by using the value function as a Lyapunov function.

Future work includes generalizing results to mixed strategies. Structural conditions on the system that do not involve V and guarantee the existence of a solution to Problem ($\diamond$) based on the smoothness and regularity of the data of the system, as in [16], will be studied.

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Santiago J. Leudo received the B.S. degrees in Electronic Engineering and Mechanical Engineering in 2014, and the M.S. degree in Electronic and Computer Engineering in 2017, from University of Los Andes, Colombia. Santiago was a fellow of the Control Research Group GIAP and an Instructor Professor at University of Los Andes in 2017-2018. In 2018, Santiago received the Fulbright Ph.D. Scholarship for high-level skills in Information Technologies - Robotics. He received the M.S. and Ph.D degrees in Electrical and Computer Engineering - Robotics and Control from the University of California, Santa Cruz, in 2021 and 2024, respectively. His research interests are in connections between control theory and game theory for nonlinear, hybrid, and multi-agent systems, and their applications to robotic networks, UAVs, security, and aerospace systems.

Ricardo G. Sanfelice received the B.S. degree in Electronics Engineering from the Universidad de Mar del Plata, Buenos Aires, Argentina, in 2001, and the M.S. and Ph.D. degrees in Electrical and Computer Engineering from the University of California, Santa Barbara, in 2004 and 2007, respectively. In 2007 and 2008, he held postdoctoral positions at the Laboratory for Information and Decision Systems at MIT and at the Centre Automatique et Systèmes at the École de Mines de Paris. In 2009, he joined the faculty of the Department of Aerospace and Mechanical Engineering at the University of Arizona, Tucson, where he was an Assistant Professor. In 2014, he joined the University of California, Santa Cruz, where he is currently Professor and Chair in the Department of Electrical and Computer Engineering. Prof. Sanfelice is the recipient of the 2013 SIAM Control and Systems Theory Prize, the National Science Foundation CAREER award, the Air Force Young Investigator Research Award, the 2010 IEEE Control Systems Magazine Outstanding Paper Award, and the 2020 Test-of-Time Award from the Hybrid Systems: Computation and Control Conference. He is Associate Editor for Automatica, Communicating Editor for the Journal of Nonlinear Science, and a Fellow of the IEEE. His research interests are in modeling, stability, robust control, observer design, and simulation of nonlinear and hybrid systems with applications to power systems, aerospace, and biology.