LYAPUNOV-LIKE DESCRIPTIONS OF STRONG FORWARD INVARIANCE FOR DIFFERENTIAL INCLUSIONS

RAFAL K. GOEBEL, RICARDO G. SANFELICE AND ANDREW R. TEEL*

Abstract. The strong forward invariance property of a set is studied in the setting of differential inclusions. The main results are necessary / converse Lyapunov-like characterizations of this property, in terms of Lyapunov-like functions. These functions are smooth and not increasing too fast outside of the invariant set, and in some cases smooth everywhere and decreasing. Applications to safety and barrier functions, robust invariance, and to invariance of interconnections, are highlighted.

Key words. Lyapunov functions, differential inclusions, strong invariance, converse Lyapunov results, barriers and safety

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1. Introduction.

For a classical differential equation, or a perhaps generalized dynamical system with unique solutions, a set is forward invariant if every solution starting in it at a time t_0 remains in the set for all relevant $t \geq t_0$. For a differential equation with an irregular right-hand side, or when generalized solutions to such a differential equation are considered, or for a differential inclusion, or for other kinds of generalized dynamics, solutions need not be unique. Then, one may distinguish between weak and strong forward invariance, and weak and strong versions of other properties, for example stability. The weak version requires that, for each relevant initial condition, there exist a solution from it satisfying the property in question. The strong version requires that all solutions from all relevant initial condition satisfy the property. Early appearance of weak stability is in [50], with strong and weak properties featured in the later book by the same author [51]. Early appearance of strongly invariant sets is [56]. [1] Weak invariance and invariance, without the "strong" adjective but in that sense, for differential equations are treated by [59]. The same concepts, called viability and invariance, in a more general setting of differential inclusions, appear in the seminal books [4] and [3].

The main object of study in this paper is strong forward invariance of sets for an autonomous differential inclusion

$$\dot{x} \in F(x),$$

without the expectation of uniqueness of solutions to it. Reasons for considering differential inclusions include: the concepts of generalized solutions to irregular differential equations, dating back to [30] and [17], with a more recent survey in [15]; robustness considerations for a differential equation $\dot{x} = f(x, d)$ with disturbances $d \in D$, leading to $\dot{x} \in F(x) := \{f(x, d) \,|\, d \in D\}$, see, for example [55]; optimality conditions in optimal control, where Euler-Lagrange or Hamiltonian equations feature nonsmooth functions and become inclusions, see the early [11] or the recent [12]; extensions of the gradient flow / steepest descent to nonsmooth functions; and more.

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[1] Even earlier appearance of weak and strong invariance in dynamics, but in a different meaning, is [52].

The main results of this paper provide necessary conditions for strong forward invariance in terms of (smooth-enough) Lyapunov-like functions. Lyapunov functions, as sufficient conditions for asymptotic stability, date back to [34], and are an often-used and textbook tool [27]. Converse Lyapunov results, stating that the existence of a Lyapunov function, with different degrees of regularity, is a necessary condition for asymptotic stability in a differential equation, have a rich history too; see the survey [26]. For differential inclusions, the existence of smooth Lyapunov functions as a sufficient and, more importantly, necessary condition for asymptotic stability of an equilibrium, with relation to robustness, was observed in [13], with generalization to asymptotically stable sets in [57], and to more general hybrid dynamics in [7], [8].

Lyapunov-like conditions for properties related to asymptotic stability, for example input-to-state stability, also abound [54]. Lyapunov-like conditions for forward completeness — which amounts to forward maximal solutions being forward complete and the open set of initial conditions being strongly forward invariant — date back to [25] and [6], where irregular Lyapunov-like functions were featured. Smooth Lyapunov-like, or barrier, functions for that purpose appear in [2], for differential equations and, essentially, Lipschitz continuous differential inclusions, and in [7] for hybrid inclusions. A special case of the result of [7], that generalizes [2], is stated below as Theorem 2.9 and used in the proof of one of the main results. Sufficient conditions for strong forward invariance, involving Lyapunov-like functions, were used by [22], [21], [9], [10], [16]. However, Lyapunov-like characterizations of strong forward invariance of a general set, in the setting of differential inclusions, and — especially — necessary conditions featuring smooth functions, do not appear to have been formulated before. [2] This is the purpose of Section 3 and Section 4.

Sufficient conditions for strong forward invariance in terms of smooth Lyapunov-like functions, including those stated in Subsection 2.1, can of course be generalized to the use of nonsmooth functions and generalized differentiation techniques. This is not a goal of this work.

Necessary and sufficient conditions for strong forward invariance, of a closed set, in terms of "subtangential" conditions on the dynamics date back to [39], [44], were revisited by [59], and recast in the language of tangent cones to the (viable or invariant) set in [4], [3], and are reasonably well-known. Some of these results are recalled in Subsection 2.3 below. Closely related to the subtangential conditions are the necessary and sufficient conditions for strong forward invariance of the set $K = \{x \in \mathbb{R}^n \mid h(x) \leq 0\}$ for a differential equation with unique solutions, in terms of the (smooth) function h used as a Lyapunov-like function, in [28].

The notion of safety, in a dynamical or control system, and the barrier functions used for that purpose, has recently gained attention in the control literature: see [46] and the subsequent [1], [58], [48], [28], [33], [41], [36], and [38]. Safety and strong forward invariance of certain sets are closely related. The relationship is clarified in Subsection 5.1 and some applications of the main results on invariance to safety are provided there. A different application of the main results is to establishing invariance of sets for an interconnection of several differential equations with inputs. This is done, and further discussed, in Subsection 5.2.

[2] The main results of this paper have been announced, without proofs, in [20]. Further differences from [20] include the application to interconnections in Subsection 5.2; a more detailed treatment of the Lipschitz case in Proposition 4.3, Lemma 4.4, and Corollary 4.5; the discussion of sets with regular boundary in Subsection 3.4; the discussion of (sub)tangential conditions in Subsection 2.3; and the inclusion of several examples.

2. Setting and background.

Throughout this paper, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, i.e., for every $x \in \mathbb{R}^n$, $F(x) \subset \mathbb{R}^n$ is a set. A *solution* to the differential inclusion

$$\dot{x} \in F(x) \tag{2.1}$$

is a function $\phi : I \to \mathbb{R}^n$ such that $I \subset \mathbb{R}$ is an interval (consisting of more than one point) and, for every $[a, b] \subset I$, ϕ is absolutely continuous on $[a, b]$, and $\dot{\phi}(t) \in F(\phi(t))$ for almost all $t \in I$. A solution is *forward maximal* if it cannot be extended forward in time, and *forward complete* if I is unbounded from above.

2.1. Strong invariance and sufficient conditions.

A set $K \subset \mathbb{R}^n$ is *strongly forward invariant* for (2.1) if every solution $\phi : [0, T] \to \mathbb{R}^n$ with $\phi(0) \in S$ satisfies $\phi(t) \in S$ for all $t \in [0, T]$. Equivalently, K is strongly invariant if there exists no solution $\phi : [0, T] \to \mathbb{R}^n$ to (2.1) with $\phi(0) \in S$ and $\phi(T) \notin S$. [3]

Example 2.1. (reachable sets) Let $S \subset \mathbb{R}^n$ be any set, and define the (infinite-horizon) reachable set $\mathcal{R}_S \subset \mathbb{R}^n$ as

$$\mathcal{R}_S := S \cup \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists\, T > 0 \text{ and a solution } \phi : [0, T] \to \mathbb{R}^n \\ \text{such that } \phi(0) \in S,\ \phi(T) = x \end{array} \right\}.$$

Then, $\mathcal{R}_S$ is strongly forward invariant, essentially by its definition. △

Example 2.1 requires no regularity of F. Consequently, the reachable set need not be closed, or otherwise regular, even when S is. The next example involves commonly encountered open and closed invariant sets.

Example 2.2. (nonincreasing V) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. For $r \in \mathbb{R}$, consider the level set $V_{=r}$ and the closed sublevel set $V_{\leq r}$, respectively, defined by

$$V_{=r} := \{x \in \mathbb{R}^n \,|\, V(x) = r\}, \quad V_{\leq r} := \{x \in \mathbb{R}^n \,|\, V(x) \leq r\}.$$

If there exists a neighborhood $U \subset \mathbb{R}$ of $V_{=r}$ such that

$$\nabla V(x) \cdot f \leq 0 \quad \forall x \in U,\ \forall f \in F(x), \tag{2.2}$$

then $V_{\leq r}$ is strongly forward invariant. Indeed, standard arguments show that, for any solution $\phi : [a, b] \to U$ to (2.1), $t \mapsto V(\phi(t))$ is absolutely continuous on $[a, b]$,

$$\frac{d}{dt} V(\phi(t)) = \nabla V(\phi(t)) \cdot \dot{\phi}(t) \leq 0$$

for almost all $t \in [a, b]$, and thus V is nonincreasing along solutions to (2.1). For the open sublevel set $V_{<r}$, defined by

$$V_{<r} := \{x \in \mathbb{R}^n \,|\, V(x) < r\},$$

to be strongly forward invariant, it suffices that there exists a neighborhood $U \subset \mathbb{R}$ of $V_{=r}$ such that (2.2) holds not at all $x \in U$ but all $x \in U \cap V_{<r}$. Furthermore, if (2.2) holds with $U = \mathbb{R}^n$, then for every $r \in \mathbb{R}$, the closed sublevel set $V_{\leq r}$ and the open sublevel set $V_{<r}$ are strongly forward invariant. △

[3] The definition of strong forward invariance is intentionally stated without any assumptions on F, and thus without any guarantees of the existence of solutions to (2.1). In an extreme case, this renders every set $K \subset \mathbb{R}^n$ invariant for (2.1) where $F(x) = \emptyset$ for all $x \in \mathbb{R}^n$ and where there exist no solutions to (2.1) at all. In [18, Definition 6.25], to highlight the issues of existence of nontrivial solutions and of maximal solutions being not complete, the name "strong forward pre-invariance" was proposed. This work drops the "pre" term.

Example 2.3. Consider the differential equation $\dot{x} = f(x)$, where the continuous but not locally Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \sqrt{x} & \text{if} \quad x \geq 0. \end{cases}$$

The set $K := (-\infty, 0]$ is not strongly forward invariant, despite the fact that $f(x) = 0$ for every $x \in K$. Indeed, $\phi(t) := t^2/4$ is a solution on $[0, \infty)$, with $\phi(0) = 0 \in K$. The non-uniqueness of solutions, from 0, is to blame for this. Also, note that the function $V(x) = x$ satisfies (2.2) at every $x \in K$ (but not at $x \notin K$) and $V_{\leq 0} = K$ is not strongly forward invariant. △

The issues highlighted in Example 2.3 suggest that conditions like (2.2) enforced only on a closed set need not imply its strong forward invariance. This is reflected in the sufficient condition below, which applies at points in a neighborhood of the closed set in question, and outside that set. On the other hand, the Lyapunov-like function V is allowed to grow, but not too fast.

PROPOSITION 2.4. *(sufficient condition for a closed set) Let $K \subset \mathbb{R}^n$ be a non-empty closed set. Suppose that there exist an open neighborhood $U \subset \mathbb{R}^n$ of the boundary ∂K of K, a continuous function $V : U \to \mathbb{R}$ which is continuously differentiable on $U \setminus K$ and such that $V(x) = 0$ if $x \in \partial K$ and $V(x) > 0$ if $x \in U \setminus K$, and $\lambda \in \mathbb{R}$ so that*

$$\nabla V(x) \cdot f \leq \lambda V(x) \quad \forall x \in U \setminus K, \ \forall f \in F(x). \tag{2.3}$$

Then, K is strongly forward invariant for (2.1).

Proof. Suppose, for the sake of contradiction, that K is not strongly forward invariant. Then, there exists a solution $\phi : [0, T] \to \mathbb{R}^n$ of (2.1) such that $\phi(0) \in K$ and $\phi(T) \notin K$. Subject to reducing T, one can assume that $\phi : [0, T] \to U$. Let τ be the maximum of $t \in [0, T)$ such that $\phi(t) \in K$. Then, for every $\tau' \in (\tau, T)$, the function $t \mapsto V(\phi(t))$ is absolutely continuous on $[\tau', T]$ and, for almost all $t \in [\tau', T]$,

$$\frac{d}{dt} V(\phi(t)) = \nabla V(\phi(t)) \cdot \dot{\phi}(t) \leq \lambda V(\phi(t))$$

and thus

$$V(\phi(T)) \leq e^{\lambda(T-\tau')} V(\phi(\tau')).$$

Continuity of V and letting $\tau' \searrow \tau$ yields $0 < V(\phi(T)) \leq e^{\lambda(T-\tau)} V(\phi(\tau)) = e^{\lambda(T-\tau)} 0 = 0$ which is absurd. Thus, K is strongly forward invariant. □

Example 2.5. For the simple case of $f(x) := x \in \mathbb{R}$ and $K := \{0\}$, the function $V(x) := x^2$ satisfies $V'(x) f(x) = 2V(x)$ for all $x \in U := \mathbb{R}$, where V' is the derivative of the (scalar) function V. This confirms invariance of K for $\dot{x} = f(x)$. In fact, considering U to be any open neighbourhood of 0 sufficies. Of course, for the locally Lipschitz function f, K consisting of an equilibrium guarantees invariance. △

For the f and K in Example 2.5, there is no continuous V that is positive definite with respect to K and nonincreasing along solutions. This suggests that increasing, but not too fast, Lyapunov functions need to be considered in necessary conditions.

The right-hand side of the Lyapunov inequality (2.3) in Proposition 2.4 involves the function $v \mapsto \lambda v$. More general functions that guarantee the same outcome can be used; see, for example, the "minimal functions" in [28, Definition 1], the "uniqueness functions" in [37, Definition 4], and the earlier references therein. Note that the condition in [37, Theorem 1] is the special case of (2.3) with λ equal to zero.

Let $O \subset \mathbb{R}^n$ be a nonempty and open set. A function $V : O \to \mathbb{R}$ is a *barrier for* O if $\lim_{i\to\infty} V(x_i) = \infty$ for every sequence of points $x_i \in \mathbb{R}^n$ such that $\lim_{i\to\infty} x_i \notin O$ or $\lim_{i\to\infty} \|x_i\| = \infty$. Here and in what follows, $\|\cdot\|$ is the Euclidean norm.

Proposition 2.6. *(sufficient condition for an open set) Let $O \subset \mathbb{R}^n$ be nonempty and open. Suppose that there exist a continuously differentiable function $V : O \to \mathbb{R}$, which is a barrier for O, and $\lambda \in \mathbb{R}$ so that*

$$\nabla V(x) \cdot f \leq \lambda V(x) \quad \forall x \in O,\ \forall f \in F(x). \tag{2.4}$$

Then O is strongly forward invariant.

Proof. Suppose, for the sake of contradiction, that O isn't strongly forward invariant. Then there exists a solution $\phi : [0, T] \to \mathbb{R}^n$ to (2.1) such that $\phi(t) \in O$ for all $t \in [0, T)$ and $\phi(T) \notin O$. Then $V(\phi(t_i)) \leq e^{\lambda t_i} V(\phi(0))$ for any sequence $t_i \nearrow T$, while $\lim_{i\to\infty} V(\phi(t_i)) = \infty$ because V is a barrier for O and $\phi(T) \notin O$. This is a contradiction. □

The sufficient condition in Proposition 2.6 is not necessary. Indeed, condition (2.4) does not allow for a finite-time blow-up of a solution, i.e., for the existence of a solution $\phi : [0, T) \to \mathbb{R}^n$ such that $\lim_{t\nearrow T} \|\phi(t)\| = \infty$. Thus, no continuously differentiable barrier V satisfying (2.4) exists for, for example, the differential equation $\dot{x} = x^2$ and the obviously forward invariant set $O = \mathbb{R}$: every forward maximal solution with a positive initial condition blows up in finite time.

Example 2.7. For the differential equation $\dot{x} = f(x) := -x$, on $\mathbb{R}$, the open set $O := (0, \infty)$ is forward invariant. The function $V : O \to \mathbb{R}$ given by $V(x) := x + 1/x$ is a barrier for O. For every $x \in O$,

$$V'(x) \cdot f(x) = -x + \frac{1}{x} \leq x + \frac{1}{x} = V(x).$$

Similarly, this V confirms that O is forward invariant for $\dot{x} = x$. △

2.2. Basic Assumptions and consequences.

The differential inclusion (2.1) or just the mapping F satisfies the *Basic Assumptions* if $F(x)$ is nonempty and convex for every $x \in \mathbb{R}^n$, and F is outer semicontinuous (equivalently, it has a closed graph) and locally bounded. Equivalently, one may say that $F(x)$ is nonempty, convex, and compact for every $x \in \mathbb{R}^n$ and F is upper semicontinuous. Another assumption used in what follows is of local Lipschitz continuity. The mapping F is *locally Lipschitz continuous* on $\mathbb{R}^n$ if for every bounded set $S \subset \mathbb{R}^n$ there exists $k_S > 0$ such that

$$F(x') \subset F(x) + k_S \|x' - x\| \mathbb{B} \quad \forall x, x' \in S. \tag{2.5}$$

The mapping F is *Lipschitz continuous* if there exists $k_S > 0$ for $S = \mathbb{R}^n$. When the values of F are closed, local Lipschitz continuity implies outer semicontinuity. The terminology and notation, used above and below, follows [49]; see also [19].

Under the Basic Assumptions, for every $x_0 \in \mathbb{R}^n$ there exists $T > 0$ and a solution $\phi : [0, T] \to \mathbb{R}^n$ to (2.1) with $\phi(0) = x_0$, and every such solution can be extended to a forward maximal solution ψ on $[0, \tau)$, either with $\tau = \infty$, which means that ψ is forward complete, or with ψ blowing up in finite time τ. Furthermore, under the Basic Assumptions, and subject to excluding finite-time blow-up, the sets of solutions on $[0, T]$ from x_0 depend outer semicontinuously on x_0. If, additionally, F is locally Lipschitz continuous, the sets of solutions depend continuously on x_0. See [4] or [19].

Example 2.8. (asymptotic stability and invariance) Suppose F satisfies the Basic Assumptions and that a nonempty and compact set $\mathcal{A} \subset \mathbb{R}^n$ is asymptotically stable for (2.1). Then, the basin of attraction O of $\mathcal{A}$ is open, and there exists a continuously differentiable $V : O \to [0, \infty)$, positive definite with respect to $\mathcal{A}$ and a barrier for O, such that (2.4) holds with $\lambda = -1$. See [57], or the earlier [13] for the case of $\mathcal{A} = \{0\}$ and $O = \mathbb{R}^n$. The open and closed sublevel sets of any such V are strongly forward invariant, as suggested in Example 2.2. The basin O is strongly forward invariant itself, essentially by its definition or by Proposition 2.6. △

The differential inclusion (2.1) is *forward complete on a set S* if S is strongly forward invariant and every forward maximal solution from $x_0 \in S$ is forward complete. For example, both simple differential equations in Example 2.7 are forward complete on the set O from that example. (This is clear from inspection, but can also be deduced using the barrier function in Example 2.7 which ensures forward invariance and no finite-time blow-up which, in turn, ensures forward completeness.

Forward completeness facilitates a kind of a converse result to Proposition 2.6.

THEOREM 2.9. *(Lyapunov characterization of forward completeness) Let F satisfy the Basic Assumptions and $O \subset \mathbb{R}^n$ be open. Then, the following are equivalent:*

(a) (2.1) *is forward complete on O;*

(b) For every barrier function $\omega : O \to (0, \infty)$ for O, there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 and a smooth function $V : O \to (0, \infty)$ such that, for all $x \in O$,

$$\alpha_1(\omega(x)) \le V(x) \le \alpha_2(\omega(x)), \tag{2.6}$$

$$\nabla V(x) \cdot f \le V(x) \quad \forall f \in F(x).$$

The equivalence follows from a generalization [7, Theorem 8.1] to hybrid dynamics of the result [2, Theorem 2] given for differential equations with perturbations and so, essentially, for a Lipschitz continuous differential inclusion. A direct approach can be outlined for the setting of this paper. Basic Assumptions and forward completeness ensure that finite-horizon reachable sets on $[0, T]$ from every initial condition are compact subsets of O. Then, given a barrier ω, one can find a function $\beta : [0, \infty)^2 \to [0, \infty)$ that is nondecreasing in each of its arguments, such that

$$\omega(\phi(t)) \le \beta(\omega(\phi(0)), t) \quad \forall t \in I,$$

and then, two class-$\mathcal{K}_\infty$ functions κ_i so that

$$\kappa_1(\phi(t)) \le \kappa_2(\phi(0)) e^{t/2} \quad \forall t \in I,$$

for every solution $\phi : I \to \mathbb{R}^n$ to (2.1). From here, the construction of the desired V is similar to what is done to construct, from a similar bound that involves exponential decay, a smooth Lyapunov function in [57].

2.3. (Sub)tangential conditions. Necessary and sufficient conditions for viability and invariance date back to Nagumo [44], were revisited, among others, by [59], and have been reformulated in terms of tangent cones to a set.

Let $S \subset \mathbb{R}^n$ be nonempty and closed, and suppose that F satisfies the Basic Assumptions. Let $T_S(x)$ be the tangent cone (also called the Bouligand tangent cone, or the contingent cone) to S at x. Then, the following equivalences hold:

(i) for every $x_0 \in S$ there exists $T > 0$ and a solution $\phi : [0, T] \to S$ of (2.1) if and only if

(ii) $F(x) \cap T_S(x)$ for every $x \in \partial S$.

If, furthermore, F is locally Lipschitz continuous on a neighborhood of S, then

(iii) S is strongly forward invariant

if and only if

(iv) $F(x) \subset T_S(x)$ for every $x \in \partial S$.

For formal statements and proofs, see [4, Chapter 4, Section 2, Theorem 1 and Theorem 3] or [3, Theorem 3.3.2 and Theorem 5.3.4]. Conditions (ii) and (iv) can be equivalently posed for every $x \in S$, because they trivially hold at points $x \in \operatorname{int} S$ at which $T_S(x) = \mathbb{R}^n$. Note that, while the condition *(iv)* is posed on S only, local Lipschitz continuity of F is assumed on a neighborhood of S. Also note that (i) does not ensure the existence of a forward complete solution from x_0. [4]

Example 2.10. Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and set $S = h_{\leq 0}$. For any x such that $h(x) = 0$ and $\nabla h(x) \neq 0$, which ensures that $x \in \partial S$, one has

$$T_S(x) = \{v \in \mathbb{R}^n \mid \nabla h(x) \cdot v \leq 0\}. \tag{2.7}$$

Suppose that $\nabla h(x) \neq 0$ for every $x \in \partial S = \{y \in \mathbb{R}^n \mid h(y) = 0\}$. If F satisfies the Basic Assumptions and is locally Lipschitz continuous, then S is strongly forward invariant if and only if

$$\nabla h(x) \cdot f \leq 0 \quad \forall x \in \partial S, \ \forall f \in F(x). \tag{2.8}$$

If F is, in fact, given by locally Lipschitz continuous function f, then S is strongly forward invariant if and only if

$$\nabla h(x) \cdot f(x) \leq 0 \quad \forall x \in \partial S. \tag{2.9}$$

△

In the setting of the example above, when $\nabla h(x) = 0$ occurs at a boundary point of S, (2.7) may fail and conditions (2.8) or (2.9) need not ensure invariance, if assumed only for $x \in \partial S$. When assumed on a neighborhood of S and not just on ∂S, the necessity and sufficiency of conditions similar to (2.9), under some further assumptions on f and h, is treated in [28].

The result below translates some of the implications above to the setting of differential equations

$$\dot{x} = f(x) \tag{2.10}$$

with a potentially discontinuous f. Such f can arise when applying a discontinuous feedback k to a control system $\dot{x} = f_c(x, u)$ with a continuous f_c, i.e., when considering $\dot{x} = f(x) := f_c(x, k(x))$. Given $f : \mathbb{R}^n \to \mathbb{R}^n$, a *Krasovskii solution* to (2.10) is a solution to (2.1), where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the Krasovskii regularization of f, i.e.,

$$F(x) := \bigcap_{\delta > 0} \overline{\operatorname{con}}\, f(x + \delta \mathbb{B}). \tag{2.11}$$

In casual words, F is the "smallest" mapping that satisfies the Basic Assumptions and has $f(x) \in F(x)$ for all $x \in \mathbb{R}^n$.

[4] In the control literature, weak invariance of a set S sometimes requests that there exist a forward complete solution $\phi : [0, \infty) \to S$ from every initial condition in S. This is in fact expected, under Basic Assumptions, from the ω-limits of solutions and used in the applications of the Invariance Principle. See, for example, [32].

Proposition 2.11. *(tangent cone condition for discontinuous f) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a locally bounded function and $S \subset \mathbb{R}^n$ be a nonempty and closed set. If $f(x) \in T_S(x)$ for all $x \in \partial S$ and forward maximal Krasovskii solutions to* (2.10) *are unique, then S is strongly forward invariant for Krasovskii solutions to* (2.10).

Proof. For every $x \in \mathbb{R}^n$, one has $f(x) \in F(x)$ for F given by (2.11). Thus $f(x) \in T_S(x)$ implies $F(x) \cap T_S(x) \neq \emptyset$, and the stated assumptions imply (ii). If S failed to be strongly forward invariant for Krasovskii solutions, then for some Krasovskii solution $\phi : [0, T] \to \mathbb{R}^n$ one has $\phi(0) \in S$ and $\phi(t) \notin S$ for all $t \in (0, T]$. But (ii) implies the existence of $\tau > 0$ and a Krasovskii solution $\psi : [0, \tau] \to S$ with $\psi(0) = \phi(0)$. This violates the uniqueness of Krasovskii solutions. □

Example 2.3 confirms that the conclusion of the proposition fails without uniqueness, even if f is continuous. The example below confirms that $f(x) \in T_S(x)$ is not necessary for strong forward invariance, even if Krasovskii solutions are unique. (Obviously, strong forward invariance does not imply uniqueness of Krasovskii solutions.)

Example 2.12. Let $f : \mathbb{R} \to \mathbb{R}$ and its Krasovskii regularization $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 0, \\ -1 & \text{if} \quad x \geq 0; \end{cases} \qquad F(x) = \begin{cases} 1 & \text{if} \quad x < 0, \\ [-1, 1] & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x > 0. \end{cases}$$

Krasovskii solutions to $\dot{x} = f(x)$, i.e., solutions to $\dot{x} \in F(x)$, are unique. The set $S := \{0\}$ is strongly forward invariant. But $f(0) = -1 \notin \{0\} = T_S(0)$. A slightly more interesting variant of this example, where invariance of S is verified by non-constant solutions, comes from $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = \begin{cases} (1, v) & \text{if} \quad x < 0, \\ (-1, v) & \text{if} \quad x \geq 0, \end{cases}$$

with any chosen $v \neq 0$. For the strongly forward invariant set S being the y-axis, $f(0, y) \in T_S(0, y)$ fails. △

If the word "Krasovskii" in Proposition 2.11 is replaced, both times, by "piecewise smooth", or "Caratheodory", or "Filippov", then the result becomes false. Indeed:

Example 2.13. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1$ if $x \neq 0$ and $f(0) = -7$. Then, piecewise smooth, or Caratheodory, or Filippov solutions to (2.10) are just classical solutions to $\dot{x} = 1$. (For the case of Filipov, the construction of the set-valued map defining Filippov solutions ignores values of the dynamics on sets of 0 measure, and so $f(0)$ is irrelevant.) The set $S := (-\infty, 0]$ is not forward invariant, while $f(0) = -7 \in (-\infty, 0] = T_S(0)$. △

The (sub)tangential property and invariance have been characterized in terms of other, related, notions of nonsmooth analysis. For example, [14, Theorem 3.8] says that for a locally Lipschitz F, S is strongly forward invariant if and only if

$$H(x, N_S^P(x)) \leq 0 \quad \forall\, x \in S. \tag{2.12}$$

Here, $N_S^P(x)$ is the proximal normal cone to S at x,

$$H(x, p) = \sup\{p \cdot v \mid v \in F(x)\}$$

is the upper Hamiltonian associated to F, and (2.12) is understood to say that $H(x, p) \leq 0$ for all $p \in N_S^P(x)$. The connection to (iv) above is that, for a convex set S, $N_S^P(x)$ equals the normal cone in the sense of convex analysis, and then (2.12) is equivalent to (iv).

3. Converse results under Basic Assumptions.

3. Converse results under Basic Assumptions. Throughout this section, it is assumed that

- F satisfies the Basic Assumptions.

3.1. Time-varying Lyapunov-like function. The result below first gives a sufficient condition for strong forward invariance of a closed set K in terms of a smooth time-varying Lyapunov-like function that does not increase faster than exponentially. The condition turns out necessary for a compact K and then, an exponentially decreasing Lyapunov-like function exists. The conditions are inspired by the use of time-varying barrier functions in the context of safety, for example [38, Theorem 2].

THEOREM 3.1. *(time-varying certificate of invariance) Let $K \subset \mathbb{R}^n$ be closed. If*

(a) there exists an open neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of $K \times [0,1]$, $\lambda \in \mathbb{R}$, and a continuous function $V : \mathcal{N} \to \mathbb{R}$ that is continuously differentiable on $\mathcal{N} \setminus (K \times [0,1])$ such that

(i) $V(x,\tau) = 0$ for all $(x,\tau) \in \partial K \times [0,1]$,

(ii) $V(x,\tau) > 0$ for all $(x,\tau) \in \mathcal{N} \setminus (K \times [0,1])$,

(iii) $\nabla V(x,\tau) \cdot (f,1) \leq \lambda V(x,\tau)$ for all $(x,\tau) \in \mathcal{N} \setminus (K \times [0,1])$ and all $f \in F(x)$;

then

(b) K is strongly forward invariant for (2.1).

If K is nonempty and compact, and (b) holds, then there exist an open neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of $K \times [0,1]$ and a smooth function $V : \mathcal{N} \to [0,\infty)$ such that $V(x,\tau) = 0$ for all $(x,\tau) \in K \times [0,1]$, (ii) above holds, and

(iv) $\nabla V(x,\tau) \cdot (f,1) \leq -V(x,\tau)$ for all $(x,\tau) \in \mathcal{N}$ and all $f \in F(x)$.

If, additonally, (2.1) has no finite-time blow-up, one can take $\mathcal{N} = \mathbb{R}^{n+1}$.

The proof of (a) implying (b) is essentially identical to that of Proposition 2.4, and is not provided. The proof that (b) implies (a) relies on augmenting the differential inclusion (2.1) to a hybrid inclusion, in the framework of [18]. Consult that reference, or [19], for the definition of a solution, asymptotic stability, etc. A symbolic representation of a hybrid inclusion is

$$\begin{aligned} \dot{x} &\in F(x) \qquad x \in C \\ x^+ &\in G(x) \qquad x \in D \end{aligned} \tag{3.1}$$

The result that facilitates the proof of Theorem 3.1 is the converse Lyapunov result for a hybrid inclusion recalled below; see [8, Theorem 3.13] or [18, Corollary 7.32].

THEOREM 3.2. *(hybrid converse Lyapunov) Suppose that the data (F, C, G, D) of (3.1) satisfies the Hybrid Basic Assumptions. Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty compact set that is pre-asymptotically stable for (3.1). Then, its basin of attraction $O \subset \mathbb{R}^n$ is an open neighborhood of $\mathcal{A}$, and for every $\omega : O \to [0,\infty)$ that is a barrier for O and positive definite with respect to $\mathcal{A}$, there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 and a smooth function $V : \mathbb{R}^n \to [0,\infty)$ such that the bounds (2.6) hold for all $x \in O$, and*

$$\nabla V(x) \cdot f \leq -V(x) \quad \forall x \in C,\ \forall f \in F(x);$$

$$V(g) \leq \frac{1}{e} V(x) \quad \forall x \in D,\ \forall g \in G(x).$$

Proof of (b) implying (a) in Theorem 3.1. Suppose that K is nonempty, compact, and strongly forward invariant for (2.1). Consider the hybrid inclusion

$$
(3.2) \qquad \begin{array}{ll} (x,\tau) \in \mathbb{R}^n \times [0,1] & \begin{bmatrix} \dot{x} \\ \dot{\tau} \end{bmatrix} \in \begin{bmatrix} F(x) \\ 1 \end{bmatrix}, \\ (x,\tau) \in \mathbb{R}^n \times [0,1] & \begin{bmatrix} x^+ \\ \tau^+ \end{bmatrix} \in \begin{bmatrix} K \\ 0 \end{bmatrix}. \end{array}
$$

It satisfies the hybrid basic assumptions. The compact set

$$\mathcal{A} := K \times [0,1]$$

is strongly forward invariant for (3.2). Also, $\mathcal{A}$ is uniformly pre-attractive from a neighborhood of itself; in particular, if (t,j) belongs to the domain of a solution and $t+j>1$ then $(x(t,j),\tau(t,j)) \in K \times [0,1]$. Thus, according to [18, Proposition 7.5], the set $\mathcal{A}$ is locally pre-asymptotically stable for (3.2). Then, Theorem 3.2 implies that there exist a smooth Lyapunov function $V : \mathcal{N} \to [0,\infty)$ verifying this pre-asymptotic stability, where $\mathcal{N} \subset \mathbb{R}^{n+1}$ is the open basin of attraction. In particular, this Lyapunov function satisfies (i), (ii), and (iv). If, furthermore, (2.1) has no finite-time blow-up the every maximal solution to (3.2) flows until $\tau = 1$ and then jumps to K. Then, the basin of attraction of $\mathcal{A}$ is $\mathbb{R}^{n+1}$ and a $V : \mathbb{R}^{n+1} \to [0,\infty)$ with the described properties exist. ◻

Pre-asymptotic stability of a compact set, under hybrid basic assumptions, is robust, in the sense that it is preserved under small-enough perturbations to F outside of the compact set. See [18, Theorem 7.21]. Such understanding of robustness, in the setting of differential inclusions, goes back to [13]. A stronger concept of robustness is discussed in Section 3.3. From the proof of Theorem 3.1, one can then deduce:

Corollary 3.3. *(robustness of invariance) Let $K \subset \mathbb{R}^n$ be a nonempty, compact set that is strongly forward invariant for* (2.1)*. Then, the strong forward invariance of K for* (2.1) *is robust, in the following sense: there exists a continuous $\rho : \mathbb{R}^n \to [0,\infty)$ such that $\rho(x) > 0$ if $x \notin K$ such that K is strongly forward invariant for*

$$
(3.3) \qquad \dot{x} \in F_\rho(x),
$$

where the inflation *$F_\rho : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of F is the mapping given at each $x \in \mathbb{R}^n$ by*

$$
(3.4) \qquad F_\rho(x) := \operatorname{con} F(x + \rho(x)\mathbb{B}) + \rho(x)\mathbb{B}.
$$

Example 3.4. Consider the differential equation $\dot{x} = f(x)$ for the continuous, but not Lipschitz continuous, function $f : \mathbb{R} \to \mathbb{R}$ given by

$$
f(x) = \begin{cases} \sqrt{x}\sin(\pi/x) & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \le 0. \end{cases}
$$

Strong forward invariance of the set $K := (-\infty, 0]$ can be verified directly, or from the existence of a continuously differentiable function $V : \mathbb{R} \to [0,\infty)$ that is 0 on $(-\infty,0]$, bounded below and above on $(0,\infty)$ by, respectively, $x \mapsto x^2$ and $x \mapsto 2x^2$, and is increasing on the intervals where f is negative and nonincreasing on the intervals where f is positive. Such a V satisfies $V'(x) \cdot f(x) \le 0 \le V(x)$ for all x, which ensures that K is stable and strongly forward invariant. To observe that the invariance of K is not robust to perturbations of a size proportional to the distance to K, take $\rho(x) := \max\{0, ax\}$ with an arbitrary $a > 0$ and consider the inflation (3.4) with $F(x) = f(x)$. It is an exercise to verify that $\sqrt{x} \in F_\rho(x)$ for all $x \in [0, a/(2(a+1))]$ and thus for all positive x close enough to 0. Consequently, K is not strongly forward invariant for (3.3). Corollary 3.3 ensures robustness to smaller perturbations. △

3.2. Autonomous Lyapunov-like function. The next result is a necessary and sufficient condition for strong forward invariance of a set, in terms of a continuous Lyapunov-like function that is smooth outside the set and does not increase faster than exponentially along solutions. It features an assumption on the reversed dynamics

$$\dot{x} \in -F(x). \tag{3.5}$$

Observe that, for any set $K \subset \mathbb{R}^n$, K is strongly forward invariant for (2.1) if and only if $\mathbb{R}^n \setminus K$ is strongly forward invariant for (3.5).

THEOREM 3.5. *(autonomous certificate of invariance) Suppose that* (3.5) *has no finite-time blow-up and $K \subset \mathbb{R}^n$ is nonempty and closed. The following are equivalent:*

(a) K is strongly forward invariant for (2.1)*.*

(b) There exists a continuous function $V : \mathbb{R}^n \to [0,\infty)$ which is smooth on $\mathbb{R}^n \setminus K$ such that $V(x) = 0$ if and only if $x \in K$ and

$$\nabla V(x) \cdot f \leq V(x) \quad \forall x \in \mathbb{R}^n \setminus K, \ \forall f \in F(x).$$

For (2.1), one condition that excludes finite-time blow up is the existence of a globally asymptotically stable compact set. Another is linear growth of F: the existence of $c, d > 0$ such that

$$\|v\| \leq c\|x\| + d \quad \forall x \in \mathbb{R}^n, \ \forall v \in F(x).$$

If F has linear growth then so does $-F$, and thus linear growth of F excludes finite-time blow-up for (3.5). (Mappings that satisfy the Basic Assumptions and have linear growth are sometimes called *Marchaud maps* [3, Definition 2.2.4].)

The implication (b) $\implies$ (a) in Theorem 3.5 is a special case of Proposition 2.4. A key step in the proof of the other implication is noting that invariance of K is equivalent to invariance of its open complement for (3.5), which, when combined with no finite-time blow up, admits a Lyapunov-like characterization by Theorem 5.4.

Proof of (a) implies (b) in Theorem 3.5. Suppose (a). Then, the open set

$$O := \mathbb{R}^n \setminus K$$

is strongly forward invariant for (3.5). Because (3.5) has no finite-time blow-up, O is forward complete for (3.5). Theorem 2.9 provides a smooth function $W : O \to (0,\infty)$ that is a barrier for O and such that

$$\nabla W(x) \cdot (-f) \leq W(x) \quad \forall x \in O, \ \forall f \in F(x).$$

For $x \in O$, define

$$V(x) := \frac{1}{W(x)}$$

and for $x \in K$, set $V(x) := 0$. Because W is a barrier for O, $V : \mathbb{R}^n \to [0,\infty)$ is continuous, and 0 only on K. At $x \in O$, for every $f \in F(x)$,

$$\begin{aligned} \nabla V(x) \cdot f &= -\frac{1}{W^2(x)} \nabla W(x) \cdot f = \frac{1}{W^2(x)} \nabla W(x) \cdot (-f) \\ &\leq \frac{1}{W^2(x)} W(x) = \frac{1}{W(x)} = V(x). \end{aligned}$$

This verifies (b). □

Example 3.6. For the differential equation $\dot{x} = x$, the set $K := (-\infty, 0]$ is strongly forward invariant. Example 2.7 suggested that $W(x) = x + 1/x$ confirms invariance of $O := \mathbb{R} \setminus K = (0, \infty)$ for the reverse dynamics $\dot{x} = -x$. The function $V : \mathbb{R} \to [0, \infty)$, constructed from W following the proof of Theorem 3.5, is

$$V(x) = \max\left\{0, \frac{x}{1 + x^2}\right\}.$$

It is a simple exercise to verify that $V'(x)x \leq V(x)$ for all $x > 0$. △

Example 3.7. It is a pleasant but pointless calculus exercise to verify directly that a function V as in (b) of Theorem 3.5 does not exist for the differential equation in Example 2.3 and the not strongly forward invariant set $K = (-\infty, 0]$. Suppose that there exists a continuous $V : [0, c) \to [0, \infty)$, continuously differentiable and positive on $(0, c)$, with $V(0) = 0$, and such that

$$V'(x)\sqrt{x} \leq V(x) \quad \forall x \in (0, c).$$

Then, for every $0 < a < b < c$,

$$\ln V(b) - \ln V(a) = \int_a^b \frac{V'(x)}{V(x)}\, dx \leq \int_a^b \frac{dx}{\sqrt{x}} = 2\sqrt{b} - 2\sqrt{a}.$$

Letting $a \searrow 0$, so that $V(a) \searrow 0$, produces a contradiction. △

For a compact invariant set K, one can dispose of the extra assumption of no blow-up, with the price to pay being that the Lyapunov-like function is constructed not on $\mathbb{R}^n$ but on a neighborhood of K. This is done next.

COROLLARY 3.8. *(local autonomous certificate of invariance) Let $K \subset \mathbb{R}^n$ be a nonempty compact set. The following are equivalent:*

(a) K is strongly forward invariant for (2.1).

(b) For any bounded and open neighborhood $U \subset \mathbb{R}^n$ of K there exists a continuous function $V : U \to [0, \infty)$ that is smooth on $U \setminus K$ such that $V(x) = 0$ if and only if $x \in K$ and

$$\nabla V(x) \cdot f \leq V(x) \quad \forall x \in U \setminus K,\ \forall f \in F(x).$$

Proof. The implication from (b) to (a) is proved almost identically to what occurred in Proposition 2.4; given a solution ϕ that leaves K one just needs to pick U large enough to contain the range of ϕ.

For the opposite implication, pick any bounded and open neighborhood U of K. Let $h : \mathbb{R}^n \to [0, 1]$ be any continuous function such that $h(x) = 1$ for $x \in U$ and $h(x) = 0$ for $x \notin U + \mathbb{B}$. For example, let $h(x) = \max\{1, d_{\mathbb{R}^n \setminus (U + \mathbb{B})}(x)\}$. [5] Define $F_h : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$F_h(x) := h(x)F(x).$$

Then $F_h(x) = \{0\}$ for all $x \notin U + \mathbb{B}$, so that the augmented differential inclusion $\dot{x} \in F_h(x)$ has no finite-time blow-up to ∞, and $F_h(x) = F(x)$ for all $x \in U$, so that K is strongly forward invariant for this augmented inclusion. Because F_h satisfies the Basic Asssumptions, Theorem 3.5 yields a function W on $\mathbb{R}^n$ with the listed properties for F_h. When restricted to U, because $F_h(x) = F(x)$ for all $x \in U$, this function is what is described in (b). □

[5] Here, and in what follows, $d_S : \mathbb{R}^n \to [0, \infty)$ is the distance to a nonempty set $S \subset \mathbb{R}^n$, i.e., $d_S(x) := \inf\{\|y - x\| \mid y \in S\}$.

Remark 3.9. *If a smooth, where needed, function $V : U \to \mathbb{R}$ satisfies*

$$\nabla V(x) \cdot f \leq \lambda V(x) \quad \forall x \in U \setminus K,\ \forall f \in F(x),$$

then the function $W : U \times \mathbb{R} \to \mathbb{R}$ given by $W(x,t) = e^{-\lambda' t} V(x)$ with $\lambda' > \lambda$ satisfies

$$\begin{aligned} \nabla W(x,t) \cdot (f,1) &= e^{-\lambda' t} \nabla V(x) \cdot f - \lambda' e^{-\lambda' t} V(x) \\ &\leq e^{-\lambda' t} \lambda V(x) - \lambda' W(x,t) = (\lambda - \lambda') W(x,t). \end{aligned}$$

Thus, the autonomous Lyapunov-like functions from Theorem 3.5 and Corollary 3.8, which don't increase faster than exponentially along solutions, can be used to produce time-varying functions that satisfy the Lyapunov inequality of Theorem 3.1, and decrease exponentially.

3.3. Strongly robust invariance. Corollary 3.3 noted that strong forward invariance of a compact set is robust, i.e., it persists for the inflated dynamics (3.3) given by the inflation (3.4), where the inflation size ρ vanishes on the compact set. This subsection addresses strong robustness, where ρ is positive everywhere.

THEOREM 3.10. *(necessary condition for strongly robust invariance) Let $K \subset \mathbb{R}^n$ be a nonempty and compact set. If*

(a) K is strongly robustly strongly forward invariant, i.e., there exists a continuous $\rho : \mathbb{R}^n \to (0,\infty)$ such that K is strongly forward invariant for (3.3),

then

(b) there exists a smooth function $V : \mathbb{R}^n \to [0,\infty)$, positive definite with respect to K and proper, and a neighborhood $U \subset \mathbb{R}^n$ of K, such that

$$\nabla V(x) \cdot f \leq -V(x) \quad \forall x \in U,\ \forall f \in F(x).$$

What facilitates this result is the nice observation in [33, Theorem 19], simplified and generalized to hybrid dynamics by [42, Proposition 3.8], and summarized in the theorem below, that strong robustness of invariance relates to asymptotic stability. The result below deals with the differential inclusions

$$\dot{x} \in L(x) + \rho_1(x)\mathbb{B}, \tag{3.6}$$

$$\dot{x} \in L(x) + \rho_2(x)\mathbb{B}. \tag{3.7}$$

THEOREM 3.11. *(from invariance to stability) Let $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy the Basic Assumptions and be locally Lipschitz continuous. Let $\rho_1 : \mathbb{R}^n \to (0,\infty)$ be continuous and $\rho_2 : \mathbb{R}^n \to (0,\infty)$ be locally Lipschitz continuous and such that $\rho_2(x) < \rho_1(x)$ for all $x \in \mathbb{R}^n$. If*

- *the set $X \subset \mathbb{R}^n$ is strongly forward invariant for* (3.6)

then

- *its closure $\overline{X}$ is (locally) asymptotically stable for* (3.7).

Another ingredient in the proof of Theorem 3.10 is the approximation of F and its inflations with a locally Lipschitz set-valued mapping. The next result can be deduced from [57, Lemma 8], which extracted the essential conclusions from [13, Proposition 3.5], and from [18, Lemma 7.36].

LEMMA 3.12. *(Lipschitz inflation) For any continuous function $\rho : \mathbb{R}^n \to (0,\infty)$ there exist*

(i) a set-valued mapping $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ that satisfies the Basic Assumptions and is locally Lipschitz continuous;

(ii) continuous functions $\rho_0, \rho_1 : \mathbb{R}^n \to (0,\infty)$
such that, for every $x \in \mathbb{R}^n$,

$$F(x) \subset F_{\rho_0}(x) \subset L(x) \subset L(x) + \rho_1(x)\mathbb{B} \subset F_\rho(x). \tag{3.8}$$

Proof of Theorem 3.10. Suppose that K is robustly strongly invariant, and let L and ρ_1 come from Lemma 3.12. Let $\rho_2 : \mathbb{R}^n \to (0,\infty)$ be as described Theorem 3.11. Then K is (locally) asymptotically stable for (3.7), with an open basin of attraction. Then, there exists a compact neighborhood U of K, with U being a subset of the mentioned basin of attraction, such that the constrained differential inclusion given by (3.7) and $x \in N$ is (globally) pre-asymptotically stable, in the terminology of [18], [8]. Then, through Theorem 3.2 one obtains a smooth, positive definite with respect to K, and proper $V : \mathbb{R}^n \to [0,\infty)$ such that

$$\nabla V(x) \cdot (L(x) + \rho_2(x)\mathbb{B}) \le -V(x) \tag{3.9}$$

for all $x \in U$. Because $F(x) \subset L(x)$, the inequality in (b) follows. □

The necessary condition in Theorem 3.10 can be altered to a necessary and sufficient, and slightly more technical, condition. The implication from (a) to (b) in the result below follows from (3.8) and (3.9). For the opposite implication, one can use ρ' in the definition of strong robustness of invariance.

COROLLARY 3.13. *(certificate of strongly robust invariance) Let $K \subset \mathbb{R}^n$ be a nonempty and compact set. The following are equivalent:*

(a) K is strongly robustly strongly forward invariant.

(b) There exist a continuous function $\rho' : \mathbb{R}^n \to (0,\infty)$ and a smooth function $V : \mathbb{R}^n \to [0,\infty)$, positive definite with respect to K and proper, and a neighborhood $N \subset \mathbb{R}^n$ of K, such that

$$\nabla V(x) \cdot f \le -V(x) \quad \forall x \in N,\ \forall f \in F_{\rho'}(x).$$

Remark 3.14. *If F, besides satisfying the Basic Assumptions, is continuous, then the robustness of invariance, as described in (a) of Theorem 3.10, is equivalent to the existence of a continuous function $\rho' : \mathbb{R}^n \to (0,\infty)$ such that K is strongly forward invariant for $\dot{x} \in F(x) + \rho'(x)\mathbb{B}$. Indeed, for any continuous $\rho : \mathbb{R}^n \to (0,\infty)$ there exists a continuous $\rho' : \mathbb{R}^n \to (0,\infty)$ such that*

$$F(x + \rho'(x)\mathbb{B}) + \rho'(x)\mathbb{B} \subset F(x) + \rho(x)\mathbb{B} \qquad \forall x \in \mathbb{R}^n.$$

Remark 3.15. *If a nonempty and compact set $K \subset \mathbb{R}^n$ is strongly robustly strongly forward invariant, then there exists a smooth function $V : O \to [0,\infty)$, where O is an open neighborhood of K, and $\varepsilon > 0$ such that*

$$\nabla V(x) \cdot f \le -\varepsilon \quad \forall x \in \partial K,\ \forall f \in F(x).$$

Indeed, the ω-limit from K for (2.1), $A := \Omega(K)$, is contained in the interior of K, and thus is asymptotically stable and its basin of attraction contains K. To see that $A \subset \operatorname{int} K$, note that $A \subset K$ thanks to strong forward invariance, and suppose that $a \in A \cap \partial K$. Because A is backward invariant, there exists a solution $\phi : [0,T] \to \mathbb{R}^n$ to (2.1) with $\phi(0) =: x_0$ for some $x_0 \in A \subset K$, and with $\phi(T) = a$. Then, there exists $\delta > 0$ such that, for every $a' \in a + \delta\mathbb{B}$, there exists a solution $\psi : [0,T] \to \mathbb{R}^n$ to (3.3), where ρ comes from the definition of strong robustness of strong forward invariance of

K for (2.1)*, such that* $\psi(0) = x_0$ *and* $\psi(T) = a'$*. This contradicts the strong forward invariance of* K *for* (3.3)*, and thus* $A \subset \operatorname{int} K$*. The* V *claimed at the beginning in the remark is any smooth Lyapunov function, defined on the basin of attraction of* A*, confirming asymptotic stability of* A *for* (2.1)*.*

The necessary condition stated and justified above is, unfortunately, not a sufficient condition for strong forward invariance of K*. Indeed,* K *is not guaranteed to be a sublevel set of* V*, and thus the strict decrease of* V *at the boundary of* K *does not ensure invariance.*

3.4. Sets with regular boundary. In this subsection, $M \subset \mathbb{R}^n$ is a compact C^k submanifold of dimension $n-1$ which is a boundary of a bounded open set $O \subset \mathbb{R}^n$. Here, $k \in \{1, 2, \dots\}$. Let $D_M : \mathbb{R}^n \to \mathbb{R}$ be the *signed distance from* M:

$$D_M(x) := \begin{cases} d_M(x) & \text{if} \quad x \notin O \\ -d_M(x) & \text{if} \quad x \in O \end{cases}$$

(In [29], the sign choice was opposite.) Also, let $P_M : \mathbb{R}^n \rightrightarrows M$ be the projection onto M, or set of closest points in M to x, i.e., $P_M(x) := \{y \in M \mid \|x - y\| = d_M(x)\}$. Then, following [29],

(i) If $k = 1$ and $P_M : \mathbb{R}^n \rightrightarrows M$ is single-valued on some open neighborhood U of M then D_M is continuously differentiable on U.

(ii) If $k \geq 2$ then D_M is C^k on some open neighborhood U of M.

THEOREM 3.16. *Let* $K := O \cup M$*, which is a compact set in* $\mathbb{R}^n$*. Suppose that* $k \geq 2$*, or* $k = 1$ *and* P_M *is single-valued on some open neighborhood* U *of* M*. The following are equivalent:*

(a) K *is strongly robustly strongly forward invariant for* (2.1)*.*

(b) There exists $\varepsilon > 0$ *such that* $\nabla D_M(x) \cdot f \leq -\varepsilon$ *for all* $x \in M$*,* $f \in F(x)$*.*

(c) There exist an open neighborhood U *of* M*, a continuously differentiable function* $V : U \to \mathbb{R}$ *such that* $V(x) = 0$ *if* $x \in M$ *and* $V(x) > 0$ *if* $x \notin K$*, and* $\varepsilon > 0$ *such that* $\nabla V(x) \cdot f \leq -\varepsilon$ *for all* $x \in U$*,* $f \in F(x)$*. One such* V *is* D_M*.*

Proof. The implications from (b) to (c) and from (c) to (a) are straightforward. To prove that (a) implies (b), assume (a) and suppose that (b) fails. Then, thanks to the continuity of ∇D_M and to outer semicontinuity and local boundedness of F, there exists $x \in M$ and $f \in F(x)$ such that $\nabla D_M(x) \cdot f \geq 0$. Let $\rho : \mathbb{R} \to (0, \infty)$ come from the definition of strong robustness, so that K is strongly forward invariant for $\dot{x} \in F_\rho(x)$. Because ρ is positive and continuous, there exists a neighborhood N of x such that $f \in F(x' + \rho(x')\mathbb{B})$ for all $x' \in N$. Furthermore, there exists $a > 0$ and a neighborhood $N' \subset N$ of x such that

$$v := f + a\nabla D_M(x) \in F(x' + \rho(x')\mathbb{B}) + \rho(x')\mathbb{B} = F_\rho(x')$$

for all $x' \in N'$. Then, $\phi(t) := x + tv$ defines a solution to $\dot{x} \in F_\rho(x)$ on $[0, T]$ with small enough $T > 0$. For this solution

$$\begin{aligned} \frac{d}{dt} D_M(\phi(t)) &= \nabla D_M(\phi(t)) \cdot (f + a\nabla D_M(x)) \\ &= \nabla D_M(x + tv) \cdot f + a\nabla D_M(x + tv) \cdot \nabla D_M(x) \\ &\geq \frac{1}{2} a \|\nabla D_M(x)\|^2 > 0 \end{aligned}$$

for all small enough $t \geq 0$, because $\nabla D_M(x) \cdot f \geq 0$, ∇D_M is continuous, and $\nabla D_M(x) \neq 0$. Then, for small enough $t > 0$, $D_M(\phi(t)) > 0$, because $D_M(\phi(0)) = 0$. This contradicts strong forward invariance. ◻

Note that the set $K := O \cup M$ can be represented as $\{x \in \mathbb{R}^n \mid h(x) \le 0\}$, where $h = D_M$. This relates the developments above to Example 2.10, where sets defined by more general functions h were considered.

Not the signed distance, but the distance to the invariant set is used in the next section as a natural candidate for a Lyapunov-like function verifying invariance.

4. Converse results under Lipschitz continuity.

Throughout this section, it is assumed that

- F is outer semicontinuous, locally bounded and $F(x) \neq \emptyset$ for every $x \in \mathbb{R}^n$.

In contrast to Basic Assumptions, convexity of $F(x)$ is not assumed. Further, Lipschitz continuity assumptions on F are stated as needed. The main result is as follows:

THEOREM 4.1. *(autonomous certificate of invariance) Suppose that F is Lipschitz continuous on $\mathbb{R}^n$, with a Lipschitz constant L. Let $K \subset \mathbb{R}^n$ be a nonempty compact set. The following are equivalent:*

(a) K is strongly forward invariant for (2.1).

(b) For every $0 < a_1 < 1 < a_2$, $\lambda > 2L$, and $\zeta > 2$ there exists a continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$, smooth on $\mathbb{R}^n \setminus K$, such that, for every $x \in \mathbb{R}^n$, one has

$$a_1 d_K^2(x) \le V(x) \le a_2 d_K^2(x), \tag{4.1}$$

$$\nabla V(x) \cdot f \le \lambda V(x) \qquad \forall f \in F(x), \tag{4.2}$$

$$\|\nabla V(x)\| \le \zeta d_K(x). \tag{4.3}$$

Previous converse results, Theorem 3.1 and Theorem 3.5, rely on known results guaranteeing the existence of smooth Lyapunov or Lyapunov-like functions certifying certain properties. In the proof of Theorem 4.1, a natural candidate for a Lyapunov-like function, namely the distance from the invariant set K, requires a smoothing procedure. The following technical preliminary result [57, Lemma 16] is inspired by an earlier result by [13] and has roots dating back to [31].

LEMMA 4.2. *(smoothing tool) Let $O \subset \mathbb{R}^n$ be open and the three functions $\alpha : O \to \mathbb{R}$ and $\mu, \nu : O \to (0, \infty)$ be continuous. Suppose $V_0 : O \to \mathbb{R}$ is locally Lipschitz on O, F is convex-valued and locally Lipschitz on O, and for almost all $x \in O$ and all $f \in F(x)$,*

$$\nabla V_0(x) \cdot f \le \alpha(x).$$

Then, there exists a smooth function $V : O \to \mathbb{R}$ such that, for all $x \in O$,

$$|V(x) - V_0(x)| \le \mu(x),$$

and, for all $x \in O$ and all $f \in F(x)$,

$$\nabla V(x) \cdot f \le \alpha(x) + \nu(x).$$

Proof of Theorem 4.1. Only the implication from (a) to (b) deserves a proof. The function $d_K : \mathbb{R}^n \to [0, \infty)$, where $d_K(x) := \min_{y \in K} \|x - y\|$ is the distance from x to K, is Lipschitz continuous with constant 1. Let $L > 0$ be a Lipschitz constant for F. It is claimed that, for every $x \in \mathbb{R}^n$ and $f \in F(x)$,

$$Dd_K(x; f) := \liminf_{\substack{v \to f \\ t \searrow 0}} \frac{d_K(x + tv) - d_K(x)}{t} \le L d_K(x).$$

The subderivative $Dd_K(x; f)$ of d_K at x for f, as called in [49], and also referred to as the contingent derivative [3], [4], [5], reduces to what is often called the lower Dini derivative because d_K is Lipschitz continuous, and to $\nabla d_K(x) \cdot f$ at each x at which d_K is differentiable [49, Theorem 9.18]. The complement of the set of all such points x is of measure 0, by the Rademacher's theorem [49, Theorem 9.60]. Thus, for almost all $x \in \mathbb{R}^n$ and all $f \in F(x)$, one has

$$\nabla d_K(x) \cdot f \leq L d_K(x). \tag{4.4}$$

To see that the claim is true, pick $x \in \mathbb{R}^n$ and $f \in F(x)$. By [4, Theorem 2, Section 3, Chapter 2] there exists $T > 0$ and a continuously differentiable $\phi : [0, T] \to \mathbb{R}^n$ solving (2.1) and such that $\phi(0) = x$ and $\dot{\phi}(0) = f$. Let $y \in K$ be such that $\|x - y\| = d_K(x)$. By [4, Theorem 1, Section 4, Chapter 2] there exists $\psi : [0, T] \to \mathbb{R}^n$ solving (2.1) such that $\psi(0) = y$ and, for all $t \in [0, T]$,

$$\|\phi(t) - \psi(t)\| \leq e^{Lt} \|x - y\|.$$

Since K is strongly forward invariant, $\psi(t) \in K$ for all $t \in [0, T]$, and then

$$d_K(\phi(t)) \leq \|\phi(t) - \psi(t)\| \leq e^{Lt} \|x - y\| = e^{Lt} d_K(x).$$

Because $\phi(t) = x + t\frac{\phi(t)-x}{t}$ and $\frac{\phi(t)-x}{t} = \frac{\phi(t)-\phi(0)}{t} \to \dot{\phi}(0) = f$ when $t \searrow 0$, one has

$$Dd_K(x; f) \leq \liminf_{t \searrow 0} \frac{d_K(\phi(t)) - d_K(x)}{t} \leq d_K(x) \liminf_{t \searrow 0} \frac{e^{Lt} - 1}{t} = L d_K(x).$$

This establishes the claim.

It is clear that (4.4) holds at almost all $x \in \mathbb{R}^n$, not just for all $f \in F(x)$ but for all $f \in \text{con}F(x)$. Let $O := \mathbb{R}^n \setminus K$ and define $\mu, \nu : O \to (0, \infty)$ by

$$\mu(x) = \nu(x) := \kappa d_K(x)$$

for small $\kappa > 0$. Lemma 4.2, applied with O, μ, ν as above and with $V_0 = d_K$ and $\alpha = L d_K$, yields a smooth function $V_1 : O \to \mathbb{R}$ such that, for all $x \in O$,

$$(1 - \kappa) d_K(x) \leq V_1(x) \leq (1 + \kappa) d_K(x), \tag{4.5}$$

and, for all $x \in O$ and $f \in \text{con}F(x)$,

$$\nabla V_1(x) \cdot f \leq L d_K(x) + \kappa d_K(x) \leq \left(\frac{L}{1 - \kappa} + \kappa \right) V_1(x).$$

Extend V_1 from O to $\mathbb{R}^n$ by setting $V_1(x) = 0$ if $x \in K$, and define $V : \mathbb{R}^n \to [0, \infty)$ by $V(x) := (V_1(x))^2$. It is straightforward to check that this V satisfies (4.1) with $\alpha_1 = (1 - \kappa)^2 < 1$, $\alpha_2 = (1 + \kappa)^2 > 1$, thanks to (4.5), and (4.2) with

$$\lambda = 2 \left(\frac{L}{1 - \kappa} + \kappa \right)^2 > 2L.$$

To verify (4.3), recall that d_K is Lipschitz continuous with constant 1, so $\nabla d_K(x)$, wherever it exists, satisfies $\|\nabla d_K(x)\| \leq 1$. Then, the function V_σ defined in the proof of Lemma 16 in [57], recalled here as Lemma 4.2, satisfies, thanks to (366) in [57], the following bound:

$$
\begin{aligned}
|\nabla V_\sigma(x)\cdot v| &= \left|\int \nabla d_K(x+\sigma\xi)\cdot v\,\psi(\xi)\,d\xi\right| \\
&\leq \int \|\nabla d_K(x+\sigma\xi)\|\,\|v\|\,\psi(\xi)\,d\xi \leq \|v\|\int \psi(\xi)\,d\xi = \|v\|
\end{aligned}
$$

for every $x, v \in \mathbb{R}$. From this, $\|\nabla V_\sigma(x)\| \leq 1$. Then, the definition given in [57] under (372), when applied to V_1 here, ensures that $\|\nabla V_1(x)\| \leq 1$. Then, $\nabla V(x) = 2V_1(x)\nabla V_1(x)$ and the bound (4.3), with $\zeta = 2(1+\kappa)$, follows. Now, $\kappa > 0$ can be chosen so that α_1, α_2 are arbitrarily close to 1, λ is arbitrarily close to $2L$, and ζ is arbitrarily close to 2. □

In the proof of Theorem 4.1, one passed from (4.4) holding for all $f \in F(x)$ to (2.1) holding for all $f \in \mathrm{con}F(x)$, where $\mathrm{con}F(x)$, the convex hull of $F(x)$, is the smallest convex set containing $F(x)$. A different approach to handling the lack of convexity of F, so that Lemma 4.2 can be invoked, is the following simple observation:

PROPOSITION 4.3. *Suppose that F is Lipschitz continuous and $K \subset \mathbb{R}^n$ is a nonempty closed set. The following are equivalent:*

(a) K is strongly forward invariant for (2.1).

(b) K is strongly forward invariant for

$$\dot{x} \in \mathrm{con}F(x). \tag{4.6}$$

Proof. Because $F(x) \subset \mathrm{con}F(x)$, (b) implies (a). Now suppose that (b) fails, so that there exists a solution $\phi : [0,T] \to \mathbb{R}^n$ to (4.6) with $\phi(0) \in K$ and $\phi(T) \notin K$. By the Filippov-Ważewski relaxation theorem and its immediate consequences (see, for example, [5, Theorem 10.4.4, Corollary 10.4.5]) solutions to (4.6) can be approximated by solutions to (2.1) with the same initial condition, and in particular there exists a solution $\psi : [0,T] \to \mathbb{R}^n$ to (2.1) with $\psi(0) = \phi(0) \in K$ and $\psi(T) \notin K$. This violates strong forward invariance of K for (2.1). □

The application of invariance-based arguments in Subsection 5.2 requires a local version of Theorem 4.1. Such a version is obtained in the corollary below.

LEMMA 4.4. *Let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping $K \subset \mathbb{R}^n$ be a nonempty compact set, and $\delta > 0$ be such that $\mathrm{dom}\, G = \mathbb{R}^n$ and G is Lipschitz continuous on $K_\delta := K + \delta\mathbb{B}$. Then, for every $\delta' \in (0,\delta)$ there exists a set-valued mapping $G' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $G'(x) = G(x)$ for all $x \in K_{\delta'} := K + \delta'\mathbb{B}$ and G' is Lipschitz continuous on $\mathbb{R}^n$.*

Proof. Pick $\delta' \in (0,\delta)$ and a smooth function $\alpha : \mathbb{R}^n \to [0,1]$ such that $\alpha(x) = 1$ for all $x \in K_{\delta'}$ and $\alpha(x) = 0$ for all $x \notin K_\delta$. Let $L > 0$ be a Lipschitz constant for G on K_δ, $M > 0$ be a Lipschitz constant for α, and $N := \sup\{|y| \,|\, y \in F(x),\, x \in K_\delta\}$. Define $G' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$G'(x) := \alpha(x)G(x) \quad \forall x \in \mathbb{R}^n.$$

Then, for any $x, y \in \mathbb{R}^n$,

$$
\begin{aligned}
d_H(G'(x), G'(y)) &= d_H(\alpha(x)G(x), \alpha(y)G(y)) \\
&\leq d_H(\alpha(x)G(x), \alpha(x)G(y)) + d_H(\alpha(x)G(y), \alpha(y)G(y)) \\
&\leq d_H(G(x), G(y)) + |\alpha(x) - \alpha(y)|N \\
&\leq L\|x-y\| + M\|x-y\|N = (L+MN)\|x-y\|.
\end{aligned}
$$

□

In the setting of Lemma 4.4, if K is strongly forward invariant for $\dot{x} \in G(x)$, then it is strongly forward invariant for $\dot{x} \in G'(x)$, because G and G' are the same on a neighborhood of K. Also, if G is outer semicontinuous and locally bounded, then so is G'. Thus, the next result can be deduced from Theorem 4.1 and Lemma 4.4. An application of it, to invariance in interconnections, is in Subsection 5.2.

COROLLARY 4.5. *(local autonomous certificate of invariance) Suppose that F is Lipschitz continuous on a neighborhood of a nonempty compact set $K \subset \mathbb{R}^n$. The following are equivalent:*

(a) K is strongly forward invariant for (2.1)*.*

(b) For every $0 < a_1 < 1 < a_2$, $\lambda > 2L$, and $\zeta > 2$ there exists a continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$, smooth on $\mathbb{R}^n \setminus K$, such that, for every $x \in \mathbb{R}^n$, (4.1) *and* (4.3) *hold, and for every x in a neighborhood of K,* (4.2) *holds.*

5. Connections and applications. The last section underlines connections and highlights potential applications of the Lyapunov-like characterizations of strong forward invariance to two topics: barrier functions for safety, and invariance in interconnections.

5.1. Barriers and safety. Throughout this section

- $X_g, X_b \subset \mathbb{R}^n$ are nonempty sets with $X_g \cap X_b = \emptyset$.

The differential inclusion (2.1) is *safe with respect to* (X_g, X_b) if there exists no solution from X_g that reaches X_b. Some immediate connections of the concept of safety to the concept of strong forward invariance are as follows: Obviously, for any set $K \subset \mathbb{R}^N$, (2.1) is safe with respect to $(K, \mathbb{R}^n \setminus K)$ if and only if K is strongly forward invariant for (2.1). More broadly, this is known:

PROPOSITION 5.1. *The following are equivalent:*

(a) The differential inclusion (2.1) *is safe with respect to (X_g, X_b).*

(b) There exists a strongly forward invariant for (2.1) *set $K \subset \mathbb{R}^n$ such that*

$$X_g \subset K \subset \mathbb{R}^n \setminus X_b. \tag{5.1}$$

Indeed, if (2.1) is safe with respect to (X_g, X_b), then the strongly forward invariant infinite horizon reachable set (recall Example 2.1) from X_g plays the role of a set K such that (5.1) holds. This reachable set need not, in general, be closed or open even if X_g is. The implication from (b) to (a) is obvious.

A natural sufficient condition for safety, dating back to [45] and given originally for a differential equation, involves a continuously differentiable barrier function $B : \mathbb{R}^n \to \mathbb{R}$ such that $B(x) \leq 0$ for $x \in X_g$, $B(x) > 0$ for $x \in X_b$, and

$$\nabla B(x) \cdot f \leq 0 \quad \forall x \in \mathbb{R}^n, \forall f \in F(x). \tag{5.2}$$

Naturally, (5.2) renders the set

$$K := \{x \in \mathbb{R}^n \, : \, B(x) \leq 0\} \tag{5.3}$$

strongly forward invariant. This, and thus also safety, remains true if (5.2) is altered to hold as a strict inequality but only at $x \in \partial K$.

Already in [45], (5.2) was shown to be necessary, for a differential equation, and subject to compactness of X_g, X_b and of the whole state space, and a further somewhat restrictive assumption. Versions and extensions of this sufficient condition, and converse statements, appear in [47], [58], [1], [48], [38] [33], and [41]. For example, in the current setting of a differential inclusion subject to the Basic Assumptions, [38] provides a necessary and sufficient condition for safety with time varying, lower semicontinuous, and nonincreasing barrier function. Characterizations of strongly robust safety are discussed below, after Proposition 5.3.

The converse results producing barrier functions yield functions that are not increasing. Most of these results can't be immediately applied to a strongly forward invariant closed set $K =: X_g$ and its complement $X_b := \mathbb{R}^n \setminus K$, because such X_b is not closed and because often, for example in [47], the state space itself is compact and not $\mathbb{R}^n$, and further assumptions are made. In fact, by the simple Example 2.5, there may be no positive definite function that is nonincreasing along all solutions. Accordingly, the converse results in Section 3 allow for a not-too-fast increase.

One path to applying the converse results in Section 3 in the context of safety is as follows. Following [3], given a set $S \subset \mathbb{R}^n$, the (strong forward) *invariance kernel* of S is the largest closed subset of S that is strongly forward invariant for (2.1). From [3, Theorem 5.4.2], it follows that:

THEOREM 5.2. *(invariance kernel) Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be Lipschitz continuous. Then for any closed $S \subset \mathbb{R}^n$ there exists the (possibly empty) invariance kernel. It consists of all initial conditions in S from which all solutions stay in S.*

With this terminology and result, if X_b is open and if the invariance kernel of $\mathbb{R}^n \setminus X_b$ contains X_g, then (2.1) is safe with respect to (X_g, X_b).

PROPOSITION 5.3. *(invariance kernel for safety) Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be Lipschitz continuous. Suppose that* (2.1) *is safe with respect to $(X_g, \widetilde{X_b})$, where $\widetilde{X_b}$ is an open set containing X_b. Then, $K \subset \mathbb{R}^n$ defined as the invariance kernel of $\mathbb{R}^n \setminus \widetilde{X_b}$ is nonempty and a strongly forward invariant for* (2.1) *closed set such that* (5.1) *holds.*

Now, one can apply the results in Section 3 and Section 4 to K. The additional margin of safety assumed in Proposition 5.3, through the open set $\widetilde{X_b}$, resembles what is sometimes done in the barrier/safety literature and what can be deduced from robustness assumptions, like in [33].

In the context of strongly robust safety for (2.1), made precise in the result below, [36] states a converse result that yields a continuously differentiable barrier B that is nonpositive on X_g, positive on X_b, and $\nabla B(x) \cdot f < 0$ for all x in the boundary of K in (5.3). This generalizes earlier results, [48, Theorem 1] and [33, Theorem 16], from differential equations or Lipschitz differential inclusions, to more general dynamics. A different generalization, to hybrid dynamics with Lipschitz maps, is in [42]. The proof of [36, Theorem 2] relies on a converse result [35] with a technical proof. Under stronger assumptions, a version of [36, Theorem 2] and a generalization of [33, Theorem 16] can be deduced from Theorem 3.10. This is done below, where the construction of V from [33] is applied to a Lipschitz inflation of F. Note that strict decrease of a barrier, represented by $\nabla B(x) \cdot f < 0$ above or (ii) below, cannot be guaranteed without robustness, as assumed in Theorem 5.4, or some other extra assumptions beyond safety. Indeed, the trivial dynamics $\dot{x} = 0$ is safe with respect to any two disjoint sets.

THEOREM 5.4. *(certificate of safety) Suppose that F satisfies the Basic Assump-*

tions and that (2.1) *is strongly robustly safe with respect to* (X_g, X_b)*, in the sense that there exists a continuous function* $\rho : \mathbb{R}^n \to (0, \infty)$ *such that* (3.3) *is safe with respect to* (X_g, X_b)*. Suppose that* $\overline{\mathcal{R}^{\rho}_{X_q}}$ *is bounded and such that* $\overline{\mathcal{R}^{\rho}_{X_q}} \cap \overline{X_b} = \emptyset$*, where* $\mathcal{R}^{\rho}_{X_q}$ *is the (infinite horizon) reachable set from* X_g *for* (3.3)*. Then, there exists a smooth function* $B : \mathbb{R}^n \to [0, \infty)$ *and* $c > 0$ *such that*

(i) $B(x) < 0$ *for all* $x \in X_g$*, and* $B(x) > 0$ *for all* $x \in X_b$*.*

(ii) $\nabla B(x) \cdot f \leq -c$ *for all* $f \in F(x)$ *and* x *such that* $B(x) = 0$*.*

Proof. Let L and ρ_0 come from Lemma 3.12, so that (3.8) holds for all $x \in \mathbb{R}^n$. Let $\mathcal{R}^{L}_{X_g}$ be the (infinite horizon) reachable set from X_g for $\dot{x} \in L(x)$. Because L is locally Lipschitz continuous, $K := \overline{\mathcal{R}^{L}_{X_g}}$ is strongly forward invariant. By (3.8), K is strongly robustly strongly forward invariant for (2.1), in the sense of Theorem 3.10. Let the function V and the neighborhood N of K come from Theorem 3.10. Because $K \cap \overline{X_b} \subset \overline{\mathcal{R}^{\rho}_{X_q}} \cap \overline{X_b} = \emptyset$, there exists an arbitrarily small $c > 0$ such that $V(x) \leq c$ implies that $x \in N$ and $x \notin \overline{X_b}$. One can then consider $B(x) := V(x) - c$. ☐

5.2. Interconnections. The literature contains a rich history of stability analysis tools for the interconnection of nonlinear systems. Contributions include the ideas of loop gain, conicity, and positivity in [60], stability for large-scale systems [43], and nonlinear small-gain theorems [40], [24], [23]. Less attention has been given to conditions that guarantee strong forward invariance of a compact set for an interconnection of nonlinear systems. A notable exception is [53].

This section considers strong forward invariance of a compact set for an interconnection of differential inclusions. The compact set is the Cartesian product of compact sets, one for each subsystem in the interconnection. This problem has been studied in [53, §6] in the context of assume-guarantee contracts. Theorem 5.8 below, which pertains to Lipschitz differential inclusions, is similar to [53, Theorem 3]. However, the proof technique used here is different. Rather than invoking tangent cone conditions for strong forward invariance of Lipschitz differential inclusions, the local version of the converse theorem for strong forward invariance, given in Corollary 4.5, is used. Its consequences for the task at hand are summarized in the "Property P.". Theorem 5.11 below extends the conclusion of Theorem 5.8 to not necessarily Lipschitz differential inclusions.

Let $N \geq 2$ and consider the interconnection of N continuous-time systems, where the state of the i-th system is $x_i \in \mathbb{R}^{n_i}$, its input is $w_i \in \mathbb{R}^{m_i}$, and, for each $i \in \{1, \ldots, N\}$, the state and input satisfy

$$\dot{x}_i \in F_i(x_i, w_i) \tag{5.4}$$

Let $n := \sum_{i=1}^{N} n_i$, $m := \sum_{i=1}^{M} m_i$, $x := (x_1, \ldots, x_N) \in \mathbb{R}^n$, $w := (w_1, \ldots, w_N) \in \mathbb{R}^m$, and

$$\widehat{F}(x, w) := F_1(x_1, w_1) \times F_2(x_2, w_2) \times \cdots \times F_N(x_N, w_N)$$

for all $(x, w) \in \mathbb{R}^{n+m}$, so that the state/input pair (x, w) satisfies

$$\dot{x} \in \widehat{F}(x, w). \tag{5.5}$$

The systems are interconnected through the constraint

$$(x, w) \in H \subset \mathbb{R}^{n+m}. \tag{5.6}$$

The question addressed below is that of strong forward invariance, for (5.5), of a set $K \subset \mathbb{R}^n$ of the form

$$K = K_1 \times \cdots \times K_N, \tag{5.7}$$

where, for $i = 1, \ldots, N$, the set $K_i \subset \mathbb{R}^{n_i}$ is compact The following is assumed.

Assumption 5.5. *There exist compact sets $W_i \subset \mathbb{R}^{m_i}$, $i = 1, \ldots, N$, and constants $\delta > 0$ and $\mu > 0$ such that, with the definition $W := W_1 \times \cdots \times W_N$,*

1. *the following implication holds:*

$$\left.\begin{array}{rcl} x & \in & K + \delta\mathbb{B} \\ (x, w) & \in & H \end{array}\right\} \quad \Longrightarrow \quad d_W(w) \leq \mu d_K(x),$$

2. *for $i = 1, \ldots, N$,*
   (a) *the mapping $F_i : \mathbb{R}^{n_i + m_i} \rightrightarrows \mathbb{R}^{n_i}$ is Lipschitz on $(K_i \times W_i) + \delta\mathbb{B}$ and*
   (b) *K_i is strongly forward invariant for $\dot{x}_i \in F_i(x_i, W_i)$.*

Remark 5.6. *Note that the first item of Assumption 5.5 implies that if $x \in K$ and $(x, w) \in H$ then $w \in W$.*

The following is a consequence of Corollary 4.5.

Corollary 5.7. *Conditions (a) and (b) in the second item of Assumption 5.5 imply the following property:*

(P) For each $i = 1, \ldots, N$ there exist a continuously differentiable function $V_i : \mathbb{R}^{n_i} \to [0, \infty)$ and positive real numbers $\underline{\alpha}_i$, $\overline{\alpha}_i$, c_i, λ_i, and ε_i such that, for all $x_i \in K_i + \varepsilon_i\mathbb{B}$ and all $f_{i,\circ} \in F_i(x_i, W_i)$,

$$\underline{\alpha}_i d_{K_i}^2(x_i) \leq V_i(x_i) \leq \overline{\alpha}_i d_{K_i}^2(x_i)$$

$$|\nabla V_i(x_i)| \leq c_i |x_i|_{K_i}$$

$$\langle \nabla V_i(x_i), f_{i,\circ} \rangle \leq \lambda_i V_i(x_i).$$

Theorem 5.8. *If Assumption 5.5 holds then the set K defined in (5.7) is strongly forward invariant for the system (5.4)-(5.6).*

Proof. Let V_i, $i = 1, \ldots, N$, satisfying the property (P) come from Corollary 5.7. Consider the function

$$V(x) := \sum_{i=1}^{N} V_i(x_i).$$

One can verify that there exist strictly positive real numbers $\underline{\alpha}$, $\overline{\alpha}$, and ε such that

$$\underline{\alpha} d_K^2(x) \leq V(x) \leq \overline{\alpha} d_K^2(x) \qquad \forall x \in K + \varepsilon\mathbb{B}. \tag{5.8}$$

Fix $(x, w) \in H$ such that $x \in K + \delta \min\left\{1, \mu^{-1}\right\}\mathbb{B}$, where $\delta > 0$ and $\mu > 0$ come from Assumption 5.5, and such that $x_i \in K_i + \varepsilon_i\mathbb{B}$, where $\varepsilon_i > 0$ comes from (P). It follows from the first item of Assumption 5.5 that $d_W(w) \leq \mu d_K(x) \leq \delta$. Let L_i denote the Lipschitz constant of F_i on $(K_i \times W_i) + \delta\mathbb{B}$, which exists due item 2b) of Assumption 5.5. Let $f_i \in F_i(x_i, w_i)$. Let $v_i \in W_i$ satisfy $|v_i - w_i| = d_{W_i}(w_i)$ and, using the Lipschitz property of F_i in Assumption 5.5, let $f_{i,\circ} \in F_i(x_i, v_i)$ be such that $|f_i - f_{i,\circ}| \leq L_i |w_i - v_i|$. Then

$$\begin{aligned} \langle \nabla V_i(x_i), f_i \rangle &= \langle \nabla V_i(x_i), f_{i,\circ} + f_i - f_{i,\circ} \rangle \leq \lambda_i V_i(x) + |\nabla V_i(x_i)| L_i d_{W_i}(w_i) \\ &\leq \lambda_i V_i(x) + |\nabla V_i(x_i)| L_i \mu d_K(x) \leq \lambda_i V_i(x) + c_i |x_i|_{K_i} L_i \mu d_K(x) \\ &\leq \lambda_i V_i(x) + M_i d_K^2(x) \end{aligned}$$

where $M_i := c_i L_i \mu$. Define

$$\overline{\lambda} := \max_{i\in\{1,\ldots,N\}} \lambda_i, \qquad \widetilde{M} := \sum_{i=1}^{N} M_i.$$

Then, for $\widehat{f} = (f_1^T, \ldots, f_N^T)^T$,

$$\text{(5.9)} \qquad \langle \nabla V(x), \widehat{f}\rangle \le \sum_{i=1}^{N} \lambda_i V_i(x) + \sum_{i=1}^{N} M_i d_K^2(x) \le \left(\overline{\lambda} + \widetilde{M}\underline{\alpha}^{-1}\right) V(x).$$

The result now follows from (5.8) and (5.9). □

The proof technique for Theorem 5.8 indicates that the local Lipschitz assumption can be relaxed as long as Property P in Corollary 5.7 holds. Thus, one can consider the following relaxed assumption:

Assumption 5.9. *There exist compact sets $W_i \subset \mathbb{R}^{m_i}$, $i \in \{1,\ldots,N\}$, $\delta > 0$, and $\mu > 0$ such that, with the definition $W := W_1 \times \cdots \times W_N$,*

1. *item 1) of Assumption 5.5 holds and*
2. (a) *for each $i \in \{1,\ldots,N\}$, there exists $L_i > 0$ such that, for each $(x_i, w_i) \in (K_i \times W_i) + \delta\mathbb{B}$, $f_i \in F_i(x_i, w_i)$ and $v_i \in W_i$ such that $|v_i - w_i| = d_{W_i}(w_i)$, there exists $f_{i,\circ} \in F_i(x_i, v_i)$ such that $|f_i - f_{i,\circ}| \le L_i |w_i - v_i|$.*
   (b) *Property P in Corollary 5.7 holds.*

Remark 5.10. *A sufficient condition for item 2a) is that F_i is locally Lipschitz in its second argument uniformly in its first argument. However, it is emphasized here that F_i does not need to be Lipschitz in x_i.*

The proof of the following extension of Theorem 5.8 follows the same lines as the proof of Theorem 5.8 and thus is omitted.

THEOREM 5.11. *If Assumptions 5.9 holds then the set K defined in (5.7) is strongly forward invariant for the system (5.4)-(5.6).*

The following example generalizes [53, Example 11].

Example 5.12. Let $\delta > 0$ and $b \ge 0$. For $i \in \{1,2\}$, define $K_i := [0,b]$, let $\lambda_i \ge 0$, and let $\Gamma_i, \Psi_i : \mathbb{R} \rightrightarrows \mathbb{R}$ be nonempty on $K_i + \delta\mathbb{B}$ and such that, for $\alpha \in K_i + \delta\mathbb{B}$ with $\alpha > \beta = b$ or $\alpha < \beta = 0$, $\gamma_i \in \Gamma_i(\alpha)$ and $\psi_i \in \Psi_i([0,b])$, we have

$$2(\alpha - \beta)(\gamma_i - \psi_i) \ge -\lambda_i(\alpha - \beta)^2.$$

When, for each $i \in \{1,2\}$, $\Gamma_i(s) = \Lambda_i(s) = a_i s$ for some $a_i \ge 0$, as in [53, Example 11], the bound above holds with $\lambda_i = 0$.

It is supposed that there exists $\delta > 0$ and, for each $i \in \{1,2\}$ there exists $L_i > 0$ such that, for each $w_i \in [0,b] + \delta\mathbb{B}$, $\psi_i \in \Psi_i(w_i)$, and $v_i \in [0,b]$ such that $|v_i - w_i| = d_{[0,b]}(w_i)$ there exists $\psi_{i,\circ} \in \Psi_i(v_i)$ such that $|\psi_i - \psi_{i,\circ}| \le L_i |w_i - v_i|$. This amounts to a growth condition on Ψ_i as its argument moves away from the set $[0,b]$. A sufficient condition for this property is that Ψ_i is locally Lipschitz.

To study forward invariance of $K := K_1 \times K_2$ for the interconnection of the systems

$$\dot{x}_i \in -\Gamma_i(x_i) + \Psi_i(w_i) =: F_i(x_i, w_i) \quad i \in \{1,2\}$$

with the constraint

$$(x,w) \in H := \{(x,w) : w_1 = x_2, w_2 = x_1\},$$

define $W_i := [0, b]$ for $i \in \{1, 2\}$ and $W := W_1 \times W_2$. It is then immediate from the definitions of H, W and K that $(x, w) \in H$ implies that $|w|_W = |x|_K$ so that the first item of Assumption 5.9 holds with $\mu = 1$. Item 2a) of Assumption 5.9 also holds, due to the growth assumption on Ψ_i imposed above. It remains to establish Property P. Consider $V_i(x_i) := d^2_{K_i}(x_i)$. It is then easy to see that Property P holds with $\underline{\alpha}_i = \overline{\alpha}_i = 1$, $c_i = 2$, λ_i given above, and $\varepsilon_i = \delta$. This follows from the fact that $d_{K_i}(x_i) = x_i - \beta$ when $x_i > \beta = b$ or when $x_i < \beta = 0$. Thus, the set K is strongly forward invariant due to Theorem 5.11. △

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