Asymptotic stability in hybrid systems via nested Matrosov functions

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***Abstract*— A theorem on nested Matrosov functions is extended to time-varying hybrid systems. It provides sufficient conditions for uniform global asymptotic stability of a compact set. An application to parameter identification with state resets is made and illustrated on an example.**

I. Introduction

Matrosov's theorem provides sufficient conditions for uniform global asymptotic stability of the origin in time-varying differential equations. The sufficient condition reported by Matrosov [13] is that, given a continuously differentiable (C^1) function V that establishes uniform global stability of the origin, there exists an auxiliary C^1 function with derivative that is "definitely nonzero" in the set where the derivative of V vanishes. Several alternative versions of Matrosov's theorem have appeared in the literature; see [8] and the references therein. Matrosov's theorem has been applied to nonlinear control problems, including tracking control [17], output feedback [16], and adaptive control [12], among others.

The most recent versions of Matrosov's theorem have provided extra flexibility by using multiple auxiliary functions rather than only one as in the original work of Matrosov. For continuous-time systems see [8], where five auxiliary functions are used in stability analysis for nonholonomic vehicles, and [20], where $3n-2$ auxiliary functions are used for the interconnection of n subsystems; for discrete-time systems see [15] and [11]. These auxiliary functions need to satisfy *nested conditions* specifying the points where they are negative. A Matrosov theorem with one auxiliary function but a weaken negativity condition, expressed in terms of persistency of excitation, has been proposed for a class of single-valued, time-varying hybrid systems in [11] (see also [10]). In [11], and also in [14], these conditions have been shown to facilitate the construction of strictly decreasing Lyapunov functions.

To the best of our knowledge, all instances of Matrosov's theorem in the literature have focused on time-varying systems. In this note, we emphasize that it can also be used for time-invariant systems to assist in applying invariance-principle-based stability analysis tools; see [9], [4], and [19]. In fact, to apply Matrosov's theorem, neither notions of invariance nor specific conditions guaranteeing sequential compactness of solutions are needed for its application.

Building from the ideas in [18], we develop a nested Matrosov theorem for hybrid systems allowing for set-valued dynamics, nonuniqueness of solutions, and Zeno solutions. After a brief introduction to hybrid systems and stability in Section II, in Section III we present the result for time-invariant systems and apply it to the bouncing ball system. In Section IV, we give the main result, which is for time-varying systems, and apply it to parameter identification with state resets. Its proof is given in Section V.

$\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}$ the real numbers, $\mathbb{R}_{\geq 0} := [0,\infty)$, $\mathbb{Z}$ the integers, and $\mathbb{Z}_{\geq k}$ the integers greater than or equal to the integer k. $\mathbb{B}$ is the open unit ball in an Euclidean space. Given a set S, $\overline{S}$ is its closure. Given $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y\in S} |x-y|$, where $|\cdot|$ is Euclidean norm. Given $S \subset \mathbb{R}^n$ and constants $\delta, \Delta, 0 \leq \delta \leq \Delta$, $\Omega_S(\delta,\Delta) := \{x \in \mathbb{R}^n \mid \delta \leq |x|_S \leq \Delta\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

II. Hybrid Systems

A. Modeling framework and solutions

We follow the presentation in [6] and [7]. Cf. [2], [21], [9]. Hybrid systems are dynamical systems with a state $x \in \mathbb{R}^n$ that can change continuously during flows, and discontinuously at jumps. The state may include physical variables, like positions and velocities, as well as logic variables taking values like "on" and "off", which are typically identified with integers embedded in Euclidean space. A hybrid system $\mathcal{H}$ is defined by four objects comprising its *data*:

- *Flow map:* A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the flows (or continuous evolution).
- *Flow set:* A set $C \subset \mathbb{R}^n$ specifying the set of points where flows are possible.
- *Jump map:* A set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the jumps (or discrete evolution).
- *Jump set:* A set $D \subset \mathbb{R}^n$ specifying the set of points where jumps are possible.

A hybrid system $\mathcal{H} := (F, C, G, D)$ can be written in the form

$$\mathcal{H}: \qquad \left\{ \begin{array}{rcll} \dot{x} & \in & F(x) & x \in C \\ x^+ & \in & G(x) & x \in D \end{array} \right.$$

the solutions for which are now made precise.

Definition 2.1: (*hybrid time domain*) A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \,...\leq t_J$. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0,T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Definition 2.2: (*hybrid arc*) A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and if for

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each $j \in \mathbb{Z}_{\geq 0}$, the function $t \mapsto x(t,j)$ is locally absolutely continuous.

A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}$ if $x(0,0) \in \overline{C} \cup D$ and:
(S1) For all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t,j) \in \operatorname{dom} x$,[1] $x(t,j) \in C$ and $\dot{x}(t,j) \in F(x(t,j))$.
(S2) For all $(t,j) \in \operatorname{dom} x$ such that $(t,j+1) \in \operatorname{dom} x$, $x(t,j) \in D$ and $x(t,j+1) \in G(x(t,j))$.

A solution x is called *maximal* if there does not exist a solution x' such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$.

The results in [7] give mild conditions on the data (F,C,G,D) to guarantee certain regularity properties for the set of solutions to a hybrid system. These conditions are critical for sequential compactness of solutions and inherent robustness of asymptotic stability [7], invariance principles [19], and converse Lyapunov theorems [3]. However, these conditions are not critical in sufficient conditions for nominal asymptotic stability, like those in this paper.

B. Globally asymptotically stable sets

To establish sufficient conditions for uniform global asymptotic stability for hybrid systems, we consider closed, not necessarily compact, sets.

Definition 2.3: (UGS, UGA & UGAS) The closed set $\mathcal{A} \subset \mathbb{R}^n$ for the system $\mathcal{H}$ is said to be

- *uniformly globally stable* (UGS) if there exists $\alpha \in \mathcal{K}_\infty$ such that any solution x satisfies $|x(t,j)|_{\mathcal{A}} \leq \alpha(|x(0,0)|_{\mathcal{A}})$ for all $(t,j) \in \operatorname{dom} x$;
- *uniformly globally attractive* (UGA) if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution x, $|x(0,0)|_{\mathcal{A}} \leq r$, $(t,j) \in \operatorname{dom} x$, and $t + j \geq T$ imply $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$;
- *uniformly globally asymptotically stable* (UGAS) if it is both UGS and UGA.

UGAS does not imply that solutions exist from every point in $\mathbb{R}^n$ nor that maximal solutions have unbounded time domains. A sufficient condition for UGS of a closed set $\mathcal{A}$ is given next.

Theorem 2.4: The closed set $\mathcal{A} \subset \mathbb{R}^n$ is UGS for the hybrid system $\mathcal{H} = (F,C,G,D)$ if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, C^1 on an open set containing $\overline{C}$, and class-$\mathcal{K}_\infty$ functions α_1, α_2 such that $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$ for all $x \in \overline{C} \cup D \cup G(D)$, $\sup_{f \in F(x)} \langle \nabla V(x), f \rangle \leq 0$ for all $x \in C$, and $\sup_{g \in G(x)} V(g) - V(x) \leq 0$ for all $x \in D$.

C. Time-varying systems

Time-varying hybrid systems with state $x \in \mathbb{R}^n$ have the form

$$\mathcal{H}_{tv}: \quad \begin{cases} \dot{x} \in F(x,\tau,k) & (x,\tau,k) \in C, \\ x^+ \in G(x,\tau,k) & (x,\tau,k) \in D, \end{cases} \tag{1}$$

where τ increments with ordinary time and k increments with jumps, and $C, D \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Time-varying systems can be converted to time-invariant systems by considering the augmented system with state (x,τ,k)

$$\mathcal{H}_{aug}: \quad \begin{cases} \left.\begin{array}{l} \dot{x} \in F(x,\tau,k) \\ \dot{\tau} = 1, \quad \dot{k} = 0 \end{array}\right\} & (x,\tau,k) \in C, \\ \left.\begin{array}{l} x^+ \in G(x,\tau,k) \\ \tau^+ = \tau, \quad k^+ = k+1 \end{array}\right\} & (x,\tau,k) \in D. \end{cases}$$

Then UGAS of a compact set $\mathcal{A}$ for the state x in $\mathcal{H}_{tv}$ is identified with UGAS of the closed (not compact) set $\mathcal{A} \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ for the augmented system $\mathcal{H}_{aug}$. This requires giving special attention to the possibly unbounded states τ and k. See Section IV.

When $\mathcal{H}_{tv}$ is periodic, with period $T > 0$ with respect to τ and period $N \in \mathbb{Z}_{\geq 1}$ with respect to k, the augmented system can be written as

$$\mathcal{H}_{tv,p}: \begin{cases} \left.\begin{array}{l} \dot{x} \in F(x,\angle z,k) \\ \dot{z} = \dfrac{1}{2\pi T}\begin{bmatrix} -z_2 \\ z_1 \end{bmatrix} \\ \dot{k} = 0 \end{array}\right\} & \begin{array}{c} (x,\angle z,k) \in C, \\ z \in S^1, \\ k \in \{1,\ldots,N\}, \end{array} \\ \left.\begin{array}{l} x^+ \in G(x,\angle z,k) \\ z^+ = z \\ k^+ = (k \bmod N) + 1 \end{array}\right\} & \begin{array}{c} (x,\angle z,k) \in D, \\ z \in S^1, \\ k \in \{1,\ldots,N\}, \end{array} \end{cases}$$

where S^1 denotes the unit circle and $\angle : S^1 \to [0, 2\pi)$ is such that $\angle z$ denotes the angle, positive in the counterclockwise direction, between z and the positive horizontal axis. In this case, UGAS of a compact set $\mathcal{A}$ for the state x in $\mathcal{H}_{tv}$ is identified with UGAS of the compact set $\mathcal{A} \times S^1 \times \{1,\ldots,N\}$ for $\mathcal{H}_{tv,p}$. Thus, the time-invariant result in Section III can be applied directly to periodic time-varying hybrid systems.

III. Nested Matrosov functions: the time-invariant case

We first state a Matrosov theorem for time-invariant hybrid systems. It provides sufficient conditions for UGAS of compact sets that relax classical Lyapunov conditions. It is a convenient alternative to LaSalle's invariance principle for establishing attractivity of a stable compact set. In contrast to invariance principles, no knowledge about solutions is required. Like Lyapunov theorems, only bounds on derivatives and differences must be established.

Theorem 3.1: (Time-invariant nested Matrosov) *Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact, UGS set for the hybrid system $\mathcal{H} = (F,C,G,D)$. $\mathcal{A}$ is UGAS if there exist $m \in \mathbb{Z}_{\geq 1}$ and, for each $0 < \delta < \Delta$,*

- *a number $\mu > 0$,*
- *continuous functions $u_{c,i} : \overline{C} \cap \Omega_{\mathcal{A}}(\delta,\Delta) \to \mathbb{R}$, $u_{d,i} : \overline{D} \cap \Omega_{\mathcal{A}}(\delta,\Delta) \to \mathbb{R}$, $i \in \{1,2,\ldots,m\}$,*
- *functions $V_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1,2,\ldots,m\}$, C^1 on an open set containing $\overline{C} \cap \Omega_{\mathcal{A}}(\delta,\Delta)$,*

such that, for each $i \in \{1,2,\ldots,m\}$,

$$|V_i(x)| \leq \mu \qquad \forall x \in (\overline{C} \cup D) \cap \Omega_{\mathcal{A}}(\delta,\Delta), \tag{2}$$

$$\sup_{f \in F(x)} \langle \nabla V_i(x), f \rangle \leq u_{c,i}(x) \quad \forall x \in C \cap \Omega_{\mathcal{A}}(\delta,\Delta), \tag{3}$$

$$\sup_{g \in G(x) \cap (\overline{C} \cup D) \cap \Omega_{\mathcal{A}}(\delta,\Delta)} V_i(g) - V_i(x) \leq u_{d,i}(x) \qquad \forall x \in D \cap \Omega_{\mathcal{A}}(\delta,\Delta), \tag{4}$$

[1] When UGAS holds for the solution concept that uses (S1), it also holds for the more restrictive solution concept where $x(t,j) \in C$ for all t except possibly at the beginning and end of intervals of non-zero length. When C is closed, these two solution concepts are equivalent.

and, with the constant functions $u_{c,0}, u_{d,0} : \mathbb{R}^n \to \{0\}$ *and* $u_{c,m+1}, u_{d,m+1} : \mathbb{R}^n \to \{1\}$, *for each* $j \in \{0, 1, \dots, m\}$,

1) *if* $x \in \overline{C} \cap \Omega_{\mathcal{A}}(\delta, \Delta)$ *and* $u_{c,i}(x) = 0$ *for all* $i \in \{0, 1, \dots, j\}$ *then* $u_{c,j+1}(x) \le 0$,
2) *if* $x \in \overline{D} \cap \Omega_{\mathcal{A}}(\delta, \Delta)$ *and* $u_{d,i}(x) = 0$ *for all* $i \in \{0, 1, \dots, j\}$ *then* $u_{d,j+1}(x) \le 0$.

The theorem imposes a nested negative semi-definite condition on the functions $u_{c,i}$ and $u_{d,i}$, which bound the change in V_i along flows and jumps, respectively. Through the definition of $u_{c,0}$ and $u_{d,0}$, the nested condition requires that $u_{c,1}$ and $u_{d,1}$ are never positive. The function $u_{c,2}$ (respectively, $u_{d,2}$) can be positive only where $u_{c,1}$ (respectively, $u_{d,1}$) is negative, and so on. Finally, through the definitions of $u_{c,m+1}$ and $u_{d,m+1}$, there are no points in $\Omega_{\mathcal{A}}(\delta, \Delta)$ where all of the $u_{c,i}$ (respectively, $u_{d,i}$) are zero.

The existence of μ satisfying (2) is guaranteed when V_i is continuous on $\Omega_{\mathcal{A}}(\delta, \Delta)$. However, continuity is not required in general. The theorem is stated for functions V_i that are continuously differentiable on an open set containing $\overline{C} \cap \Omega_{\mathcal{A}}(\delta, \Delta)$, but a similar result holds for functions locally Lipschitz on this set. Such a result requires working with a generalized notion of derivative, like the Clarke generalized gradient [5].

When the first function in Matrosov's theorem satisfies the conditions in Theorem 2.4, it can be used to establish the required UGS property. This fact and Theorem 3.1 are illustrated next.

Example 3.2: (Bouncing ball) Consider a ball bouncing on the ground with vertical position x_1 and vertical velocity x_2. In between bounces, the equations of motion are given by $\dot{x}_1 = x_2$, $\dot{x}_2 = -\gamma$, where $\gamma > 0$ is the gravitational constant and the state $x := (x_1, x_2)$ is in the set $C := \{x \in \mathbb{R}^2 \mid x_1 > 0\}$. Bounces occur when the state x is in the set $D := \{x \in \mathbb{R}^2 \mid x_1 = 0 \text{ and } x_2 < 0\}$ with the update rule $x_1^+ = 0$, $x_2^+ = -\varrho x_2$, where $\varrho \in [0, 1)$ is the restitution coefficient. There are various ways to show that $\mathcal{A} := (0, 0)$ (or even the system with C and D replaced by their closures) is UGAS. In [3] a strictly decreasing Lyapunov function was presented. Invariance principles, like those in [19], can also be applied using the energy function $V_1(x) := \frac{1}{2}x_2^2 + \gamma x_1$. To use Matrosov's theorem, we start with this same function V_1 and find that the conditions of Theorem 2.4 hold, so that the origin is UGS, and conditions (3) and (4) in Theorem 3.1 hold for $i = 1$ with

$$u_{c,1}(x) := 0 \ \forall x \in \overline{C}\ , \ u_{d,1}(x) := -\frac{1}{2}(1 - \varrho^2)x_2^2 \ \forall x \in \overline{D}.$$

Since these functions are never positive, items 1) and 2) in Theorem 3.1 hold for $j = 0$. In fact, since there are no points outside of $\mathcal{A}$ where $u_{d,1}$ is zero (points in $\overline{D}$ have $x_1 = 0$), item 2 in Theorem 3.1 will hold for all j no matter what $u_{d,i}$ is for $i > 1$. Next, we pick $V_2 : \mathbb{R}^2 \to \mathbb{R}$ to be given by $V_2(x) := \gamma x_2$. Conditions (3) and (4) in Theorem 3.1 hold for $i = 2$ with

$$u_{c,2}(x) := -\gamma^2 \ \forall x \in \overline{C}\ , \ u_{d,2}(x) := -\gamma(\varrho + 1)x_2 \ \forall x \in \overline{D}.$$

Since $u_{c,2}$ is always negative, item 1) of Theorem 3.1 holds for all j. The origin is UGAS. △

To show UGAS of the origin of the system in Example 3.2 via invariance principles, it is required to define a notion of invariance, verify certain regularity conditions under which the invariance principle is applicable, as well as have rudimentary knowledge of the system solutions. The application of Theorem 3.1 only requires the ability to construct the Matrosov functions as Example 3.2 demonstrates. Another application of Theorem 3.1 is given in [18, Example 4.2].

IV. Nested Matrosov functions: the time-varying case

A. Main result

Like in [15], [8], [10], [11], the time-varying version is given in the spirit of the result in [17], which is less general than Matrosov's original result but is easier to state and check.

Given $\mathcal{A} \subset \mathbb{R}^n$, define $\Upsilon_{\mathcal{A}}(\delta, \Delta) := \Omega_{\mathcal{A}}(\delta, \Delta) \times \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0}$. Given a set $S \subset \mathbb{R}^n \times \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0}$, define $\Pi(S) := \{x \in \mathbb{R}^n \mid (x, \tau, k) \in S \text{ for some } (\tau, k) \in \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0}\}$.

*Theorem 4.1: (*Time-varying nested Matrosov*) Let* $\mathcal{A} \subset \mathbb{R}^n$ *be a compact, UGS set for the hybrid system* $\mathcal{H}_{tv} = (F, C, G, D)$. $\mathcal{A}$ *is UGAS if there exist* $m, s \in \mathbb{Z}_{\ge 1}$ *and, for each* $0 < \delta < \Delta$,

- *a number* $\mu > 0$,
- *a function* $\phi : \mathbb{R}^n \times \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0} \to \mathbb{R}^s$,
- *continuous functions* $u_{c,i} : \left(\overline{\Pi(C)} \cap \Omega_{\mathcal{A}}(\delta, \Delta)\right) \times \mathbb{R}^s \to \mathbb{R}$, $u_{d,i} : \left(\overline{\Pi(D)} \cap \Omega_{\mathcal{A}}(\delta, \Delta)\right) \times \mathbb{R}^s \to \mathbb{R}$, $i \in \{1, 2, \dots, m\}$,
- *functions* $V_i : \mathbb{R}^n \times \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0} \to \mathbb{R}$, $i \in \{1, 2, \dots, m\}$, C^1 *on an open set containing* $\overline{C} \cap \Upsilon_{\mathcal{A}}(\delta, \Delta)$,

such that, for each $i \in \{1, 2, \dots, m\}$,

$$\max\{|V_i(x, \tau, k)|, |\phi(x, \tau, k)|\} \le \mu \qquad \forall (x, \tau, k) \in (\overline{C} \cup D) \cap \Upsilon_{\mathcal{A}}(\delta, \Delta), \tag{5}$$

$$\sup_{f \in F(x,\tau,k)} \langle \nabla_x V_i(x, \tau, k), f \rangle + \nabla_\tau V_i(x, \tau, k) \le u_{c,i}(x, \phi(x, \tau, k)) \qquad \forall (x, \tau, k) \in C \cap \Upsilon_{\mathcal{A}}(\delta, \Delta), \tag{6}$$

$$\sup_{\substack{g \in G(x,\tau,k) \cap \Omega_{\mathcal{A}}(\delta, \Delta) \\ (g, \tau, k+1) \in (\overline{C} \cup D)}} V_i(g, \tau, k+1) - V_i(x, \tau, k) \le u_{d,i}(x, \phi(x, \tau, k)) \qquad \forall (x, \tau, k) \in D \cap \Upsilon_{\mathcal{A}}(\delta, \Delta), \tag{7}$$

and, with the constant functions $u_{c,0}, u_{c,m+1} : \mathbb{R}^{n+s} \to \{0\}$ *and* $u_{d,0}, u_{d,m+1} : \mathbb{R}^{n+s} \to \{1\}$, *for each* $j \in \{0, 1, \dots, m\}$,

1) *if* $x \in \overline{\Pi(C)} \cap \Omega_{\mathcal{A}}(\delta, \Delta)$, $|\psi| \le \mu$, *and* $u_{c,i}(x, \psi) = 0$ *for all* $i \in \{0, 1, \dots, j\}$, *then* $u_{c,j+1}(x, \psi) \le 0$,
2) *if* $x \in \overline{\Pi(D)} \cap \Omega_{\mathcal{A}}(\delta, \Delta)$, $|\psi| \le \mu$, *and* $u_{d,i}(x, \psi) = 0$ *for all* $i \in \{0, 1, \dots, j\}$, *then* $u_{d,j+1}(x, \psi) \le 0$.

It can be verified that the conditions for uniform asymptotic stability in [10], [11] can be cast as those in Theorem 4.1.

The utility of Theorem 4.1 for continuous-time and discrete-time systems has been illustrated in [15], [8], [20]. Next, we illustrate its usefulness for time-varying hybrid systems $\mathcal{H}_{tv}$.

B. Application: Parameter identification with state resetting

Consider the class of systems with state $(\xi,\zeta,\rho,\tau)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}\times\mathbb{R}$, flow and jump sets $C:=\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times[0,T_2]\times\mathbb{R}_{\geq 0}$, $D:=\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times[T_1,T_2]\times\mathbb{R}_{\geq 0}$, respectively, where $0<T_1\leq T_2<\infty$, and dynamics

$$\dot{\xi}=A\xi+B(\tau,\xi)\zeta,\ \dot{\zeta}=-\exp(-\beta\rho)\Gamma B^\top(\tau,\xi)P\xi,\ \dot{\rho}=1 \tag{8}$$

when $(\xi,\zeta,\rho,\tau)\in C$, and

$$\xi^+\in\varphi(\xi),\ \zeta^+=\zeta,\ \rho^+=0, \tag{9}$$

when $(\xi,\zeta,\rho,\tau)\in D$, where $A,P\in\mathbb{R}^{n_1\times n_1}$, $B:\mathbb{R}\times\mathbb{R}^{n_1}\to\mathbb{R}^{n_1\times n_2}$, $\beta\in\mathbb{R}$, $\Gamma\in\mathbb{R}^{n_2\times n_2}$, and $\varphi:\mathbb{R}^{n_1}\rightrightarrows\mathbb{R}^{n_1}$. In the case where $\beta=0$ and $T_1=T_2=\infty$, the system (8)-(9) corresponds to the dynamics of a classical parameter identification algorithm. In particular, for the system $\dot{\eta}=\chi(t,\eta)+\vartheta(t,\eta)\theta$ where $\theta\in\mathbb{R}^{n_2}$ is an unknown constant vector, consider the parameter identification algorithm

$$\begin{aligned}\dot{\hat{\eta}}&=A(\hat{\eta}-\eta)+\chi(t,\eta)+\vartheta(t,\eta)\hat{\theta},\\ \dot{\hat{\theta}}&=-\Gamma\vartheta^\top(t,\eta)P(\hat{\eta}-\eta).\end{aligned}$$

Defining $\xi:=\hat{\eta}-\eta$, $\zeta:=\hat{\theta}-\theta$, and $B(\tau,\xi):=\vartheta(\tau,\eta(\tau))$ results in the system (8)-(9) with $\beta=0$, and $T_1=T_2=\infty$, so that D is empty. See also the systems considered in [8].

For the system (8)-(9), we make the following assumptions:

Assumption 4.2: The matrices Γ and P are symmetric, positive definite,

$$v\in\varphi(\xi)\Longrightarrow v^\top Pv\leq\min\{\exp(-\beta T_1),\exp(-\beta T_2)\}\,\xi^\top P\xi,$$

and there exist $\lambda\in(-\infty,\beta)$ such that $A^\top P+PA\leq\lambda P$.

Assumption 4.3: The function $\tau\mapsto B(\tau,0)$ is continuously differentiable. Moreover, there exist strictly positive real numbers σ, ς, ε, and T, and $\alpha\in\mathcal{K}_\infty$, such that, for all $\tau\geq 0$,

1) $|B(\tau,0)|\leq\sigma$,
2) $\left|\dfrac{d}{d\tau}B(\tau,0)\right|\leq\varsigma$,
3) $|B(\tau,\xi)-B(\tau,0)|\leq\alpha(|\xi|)$, and
4) $\varepsilon I\leq\displaystyle\int_\tau^{\tau+T}B^\top(s,0)B(s,0)ds.$

The next result for the system (8)-(9) follows from Theorem 4.1.

Corollary 4.4: For the system (8)-(9) *and under Assumptions 4.2 and 4.3, the compact set $\mathcal{A}:=\{0\}\times\{0\}\times[0,T_2]$ is UGAS.*

Proof. Let $x:=(\xi,\zeta,\rho)$, $F(x,\tau):=(A\xi+B(\tau,\xi)\zeta,-\exp(-\beta\rho)\Gamma B^\top(\tau,\xi)P\xi,1)$ for all $(x,\tau)\in C$ and $G(x,\tau):=(\varphi(\xi),\zeta,0)$ for all $(x,\tau)\in D$. We establish UGS of $\mathcal{A}$. Consider the Lyapunov function candidate $V(x):=\exp(-\beta\rho)\xi^\top P\xi+\zeta^\top\Gamma^{-1}\zeta$. There exist strictly positive real numbers $\underline{\alpha}$ and $\overline{\alpha}$ such that $\underline{\alpha}|(x,\tau)|^2_{\mathcal{A}}\leq V(x)\leq\overline{\alpha}|(x,\tau)|^2_{\mathcal{A}}$ for all $(x,\tau)\in C\cup D$. Using Assumption 4.2, for all $(x,\tau)\in C$,

$$\begin{aligned}\langle\nabla V(x),F(x,\tau)\rangle&=-\beta\exp(-\beta\rho)\xi^\top P\xi\\&\quad+\exp(-\beta\rho)\xi^\top(A^\top P+PA)\xi\\&\quad+\exp(-\beta\rho)2\xi^\top PB(\tau,\xi)\zeta\\&\quad-2\zeta^\top\exp(-\beta\rho)B^\top(\tau,\xi)P\xi\\&\leq-(\beta-\lambda)\exp(-\beta\rho)\xi^\top P\xi\leq 0,\end{aligned}$$

and, for all $(x,\tau)\in D$ and $g\in G(x,\tau)$,

$$V(g)\leq\widetilde{\gamma}\xi^\top P\xi+\zeta^\top\Gamma^{-1}\zeta\leq V(x),$$

where $\widetilde{\gamma}:=\min\{\exp(-\beta T_1),\exp(-\beta T_2)\}$. It follows from Theorem 2.4 that $\mathcal{A}$ is UGS.

Now we use Theorem 4.1 to establish UGAS of $\mathcal{A}$. Using Assumption 4.2, let $\kappa_1>0$ be such that, for all $v\in\varphi(\xi)$, $v^\top v\leq(\exp(\kappa_1T_1)-1)\xi^\top\xi$. Also, let $c>0$. Then define

$$\begin{aligned}&V_1(x,\tau):=V(x),\qquad V_2(x,\tau):=\exp(\kappa_1\rho)\xi^\top\xi,\\&V_3(x,\tau):=-\zeta^\top B^\top(\tau,0)\xi,\\&V_4(x,\tau):=-\int_\tau^\infty\exp(\tau-s)|B(s,0)\zeta|^2ds,\\&V_5(x,\tau):=\exp(c\rho)|\zeta|^2,\quad\phi(x,\tau):=B(\tau,0)\zeta.\end{aligned}$$

Using Assumption 4.3, it follows for each $\Delta>0$ there exists $\mu>0$ such that $|(\xi,\zeta)|\leq\Delta$ implies $|\phi(x,\tau)|\leq\mu$ and $|V_i(x,\tau)|\leq\mu$ for $i\in\{1,2,\ldots,5\}$. Moreover, $V_4(x,\tau)\leq-\varepsilon\exp(-T)|\zeta|^2$ for any $T>0$. Define

$$\begin{aligned}u_{c,1}(x,\psi)&:=-(\beta-\lambda)\exp(-\beta\rho)\xi^\top P\xi,\\u_{c,2}(x,\psi)&:=\exp(\kappa_1T_2)(\kappa_1|\xi|^2+2|A||\xi|^2\\&\qquad+2(\sigma+\alpha(|\xi|))|\xi||\zeta|),\\u_{c,3}(x,\psi)&:=-|\psi|^2+\sigma|\zeta|^2\alpha(|\xi|)+\sigma|A||\xi||\zeta|+\varsigma|\xi||\zeta|\\&\qquad+\sigma\left(\sigma+\alpha(|\xi|)\right)\exp(-\beta\rho)|\xi|^2|P||\Gamma|,\\u_{c,4}(x,\psi)&:=-\varepsilon\exp(-T)|\zeta|^2+|\psi|^2\\&\qquad+2\sigma^2\exp(-\beta\rho)|\Gamma||P|\left(\sigma+\alpha(|\xi|)\right)|\xi||\zeta|,\\u_{c,5}(x,\psi)&:=c\exp(c\rho)|\zeta|^2\\&\qquad+\exp((c-\beta)\rho)2(\sigma+\alpha(|\xi|))|\Gamma||P||\zeta||\xi|,\end{aligned}$$

where by the norm of a matrix we mean its matrix 2-norm. Using Assumption 4.3, routine calculations establish the bound (6) for $i\in\{1,2,\ldots,5\}$.

Using Assumption 4.2, let $\kappa_2>0$ be such that, for all $v\in\varphi(\xi)$, $|v|\leq\kappa_2|\xi|$. Then define

$$\begin{aligned}&u_{d,1}(x,\psi):=0,\ u_{d,2}(x,\psi):=-\xi^\top\xi,\\&u_{d,3}(x,\psi):=(\kappa_2+1)\sigma|\xi||\zeta|,\ u_{d,4}(x,\psi):=0,\\&u_{d,5}(x,\psi):=\left(1-\exp(cT_1)\right)|\zeta|^2.\end{aligned}$$

Using Assumption 4.3, routine calculations establish the bound (7) for $i\in\{1,2,\ldots,5\}$.

Finally, it is also straightforward to verify conditions 1) and 2) of Theorem 4.1. ■

Example 4.5: Consider the system (8)-(9) with $B(\tau,\xi)=\left[\begin{array}{cc}\max\{0,\sin(\tau)\} & \min\{0,\sin(\tau)\}\end{array}\right]$, $A=-1$, $P=1$, $\Gamma=I$, $\beta=0$, $T_1=T_2=\pi$, and $\varphi(\xi)=0$. Assumptions 4.2 and 4.3 are satisfied except for the condition that B is continuously differentiable. In this case, the function V_3 in the proof of Corollary 4.4 would be only locally Lipschitz. However, the arguments for UGAS go through in the same way in this case but using a more general notion of derivative, like the Clarke generalized gradient [5].

Figure 1 shows solutions, projected onto the ordinary time axis, for this system compared to solutions when using $\varphi(\xi)=\xi$, i.e., the system without resets. Because of the structure of B in this example, state resetting is able to decouple the identification of the two unknown parameters in the system. In particular, when the initial estimate of one parameter is correct, it remains correct throughout the process of identifying the second parameter.

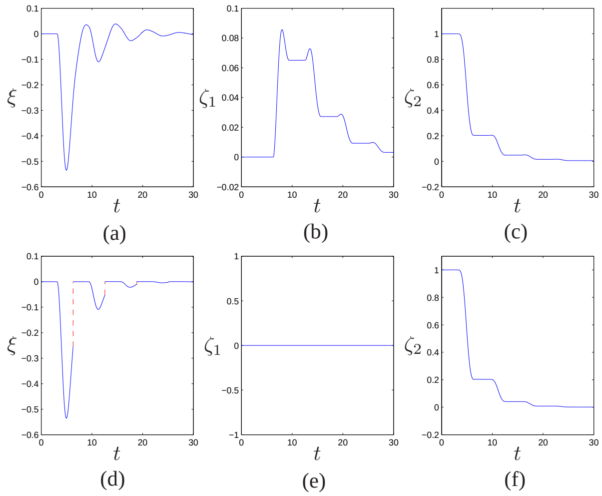

Fig. 1. Solutions to system (8)-(9) projected onto the t axis. (a)-(c) depict ξ, ζ_1, and ζ_2 for the case without resets while (d)-(f) show the case with resets. In both cases, the initial conditions are $\xi(0,0)=0, \zeta_1(0,0)=0$, and $\zeta_2(0,0)=1$. For the case without resets, ζ_1 leaves zero at $t=2\pi$ indicating that the estimate of θ_1 becomes incorrect. On the other hand, for the case with resets $\hat{\theta}_1 \equiv \theta_1$.

V. Proofs

Theorem 3.1 follows from Theorem 4.1. Our proof of Theorem 4.1 uses ideas in [15], [8].

A. Behavior of solutions

UGS is assumed; we establish UGA. Consider the hybrid system $\widetilde{\mathcal{H}}_{tv} = (F, C, \widetilde{G}, D)$, where $\widetilde{G}(x,\tau,k) := \left\{ g \in G(x,\tau,k) \;\middle|\; (g,\tau,k+1) \in \overline{C} \cup D \right\}$ for each $(x,\tau,k) \in D$. By assumption, $\mathcal{A}$ is UGS for $\widetilde{\mathcal{H}}_{tv}$. Let $\varepsilon > 0$ and $r > 0$. Let $\alpha \in \mathcal{K}_\infty$ come from the UGS property of $\widetilde{\mathcal{H}}_{tv}$ and define $\Delta := \alpha(r)$ and $\delta = \alpha^{-1}(\varepsilon)$. UGA is established if there exists $T > 0$ such that, for each solution x to $\widetilde{\mathcal{H}}_{tv}$, the set

$$\Theta_T(x) := \{(t,j) \in \operatorname{dom} x \mid T \le t+j \;,\; x(s,i) \in \Omega_{\mathcal{A}}(\delta,\Delta) \quad \forall (s,i) \in \operatorname{dom} x,\; s+i \le t+j\}$$

is empty. To establish this fact, we will use the following lemma which will be proved later.

Lemma 5.1: Under the conditions of Theorem 4.1 for $\widetilde{\mathcal{H}}_{tv}$, there exist a function $V : \mathbb{R}^n \times \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0} \to \mathbb{R}$, C^1 on an open set containing $\overline{C} \cap \Upsilon_{\mathcal{A}}(\delta,\Delta)$, and numbers $\eta, \rho > 0$ such that

$$\max\{|V(x,\tau,k)|, |\phi(x,\tau,k)|\} \le \eta \quad \forall (x,\tau,k) \in (\overline{C} \cup D) \cap \Upsilon_{\mathcal{A}}(\delta,\Delta), \tag{10}$$

$$\sup_{f \in F(x,\tau,k)} \langle \nabla_x V(x,\tau,k), f\rangle + \nabla_\tau V(x,\tau,k) \le -\rho \quad \forall (x,\tau,k) \in C \cap \Upsilon_{\mathcal{A}}(\delta,\Delta), \tag{11}$$

$$\sup_{g \in \widetilde{G}(x,\tau,k) \cap \Omega_{\mathcal{A}}(\delta,\Delta)} V(g,\tau,k+1) - V(x,\tau,k) \le -\rho \quad \forall (x,\tau,k) \in D \cap \Upsilon_{\mathcal{A}}(\delta,\Delta). \tag{12}$$

Using this lemma, we take $T > 2\eta/\rho$. Now, suppose there exists a solution x to $\widetilde{\mathcal{H}}_{tv}$ such that $\Theta_T(x)$ is nonempty, i.e., there exists $(t,j) \in \operatorname{dom} x$ such that $t+j \ge T$ and $x(s,i) \in \Omega_{\mathcal{A}}(\delta,\Delta) \cap \Pi(\overline{C} \cup D)$ (the latter set since all solutions x to $\widetilde{\mathcal{H}}_{tv}$ take values in this set) for all $(s,i) \in \operatorname{dom} x$ with $s+i \le t+j$. It follows using (10) for $x(0,0)$ and $x(t,j)$, denoting the initial value of (τ,k) by (τ_0,k_0), integrating and summing (11)-(12), and using that $x(s,i) \in \Pi(C)$ for almost all times s, that

$$-\eta \le \widetilde{V}(t,j) \le \widetilde{V}(0,0) - (t+j)\rho \le \eta - T\rho < -\eta,$$

where $\widetilde{V}(t,j) := V(x(t,j), t+\tau_0, j+k_0)$. This being impossible, we conclude that $\Theta_T(x)$ is empty for each solution x and $\mathcal{A}$ is UGA for $\widetilde{\mathcal{H}}_{tv}$. Now, we show that this implies that $\mathcal{A}$ is UGA for $\mathcal{H}_{tv}$. For arbitrary $\varepsilon, r > 0$, let $\alpha \in \mathcal{K}_\infty$ come from the UGS property of $\mathcal{H}_{tv}$ and define $\delta = \alpha^{-1}(\varepsilon)$. Let T come from the UGA property of $\widetilde{\mathcal{H}}_{tv}$ with parameters δ and r (δ plays the role of ε in Definition 2.3). For solutions x to $\mathcal{H}_{tv}$ with $|x(0,0)| \le r$ that are also solutions to $\widetilde{\mathcal{H}}_{tv}$ there is nothing to check. Let x be a solution to $\mathcal{H}_{tv}$ with $|x(0,0)| \le r$ that is not a solution to $\widetilde{\mathcal{H}}_{tv}$. Then, there exists $(t^*,j^*) \in \operatorname{dom} x$ such that $x(t^*,j^*) \notin \overline{C} \cup D$. It follows that the solution resulting from truncating x up to (t^*, j^*-1) is a solution to $\widetilde{\mathcal{H}}_{tv}$. Then, using the UGA property of $\widetilde{\mathcal{H}}_{tv}$, when $t^*+j^*-1 \ge T$ we have that $|x(t,j)|_{\mathcal{A}} \le \delta$, $t+j \ge t^*+j^*-1$, $(t,j) \in \operatorname{dom} x$. Using the UGS property of $\mathcal{H}_{tv}$, UGA of $\mathcal{A}$ for $\mathcal{H}_{tv}$ follows with parameters ε, δ, and $T+1$ (in the order introduced in Definition 2.3). ■

B. Proof of Lemma 5.1

We use the following auxiliary lemmas, which are adaptations of Claim 1 and Claim 2 in [8], culminating in a theorem that generalizes the nonlinear version of Finsler's lemma given in [1, Theorem A.1].

Lemma 5.2: Let $\Psi \subset \mathbb{R}^p$ be a nonempty compact set, $Y_i : \Psi \to \mathbb{R}$, $i \in \{1,2,\ldots,m\}$, be continuous functions and $Y_0 : \Psi \to \{0\}, Y_{m+1} : \Psi \to \{1\}$ be constant functions such that

$$\text{for each } j \in \{m-1, m\}, \text{ if } Y_i(z) = 0 \text{ for all } i \in \{0,1,\ldots,j\} \text{ then } Y_{j+1}(z) \le 0. \tag{13}$$

Then, there exists $\varepsilon > 0$ such that:

$$Y_i(z) = 0 \text{ for all } i \in \{1,2,\ldots,m-1\} \text{ imply } Y_m(z) \le -\varepsilon. \tag{14}$$

Proof: By contradiction, suppose that for each $n \in \mathbb{Z}_{\ge 1}$ there exist $z_n \in \Psi$ such that $Y_i(z_n) = 0$ for all $i \in \{0,1,\ldots,m-1\}$ and $Y_m(z_n) > -\frac{1}{n}$. By compactness of Ψ, the continuity of Y_m, and the property (13) with $j = m-1$, the sequence $\{z_n\}_{i=1}^\infty$ has an accumulation point $z^* \in \Psi$ such that $Y_i(z^*) = 0$ for all $i \in \{0,1,\ldots,m\}$. Then, using the property (13) with $j = m$, this implies that $Y_{m+1}(z^*) \le 0$. This is a contradiction since, by definition, $Y_{m+1}(z^*) = 1$. ■

Lemma 5.3: Let $\Psi \subset \mathbb{R}^p$ be a nonempty compact set, $Y_i : \Psi \to \mathbb{R}$, $i \in \{1,2,\ldots,m\}$, $m \ge 2$, be continuous functions and $Y_0 : \Psi \to \{0\}, Y_{m+1} : \Psi \to \{1\}$ be constant functions such that

$$\text{for each } j \in \{0,1,\ldots,m\}, \text{ if } Y_i(z) = 0 \text{ for all } i \in \{0,1,\ldots,j\} \text{ then } Y_{j+1}(z) \le 0. \tag{15}$$

Let $\ell \in \{2,3,\ldots,m\}$, $\widetilde{\varepsilon} > 0$, and a continuous function $\widetilde{Y}_\ell : \Psi \to \mathbb{R}$ be given. Then, Property 1 implies Property 2.

Property 1: A) $Y_i(z) = 0$ *for all* $i \in \{1, 2, \ldots, \ell - 1\}$ implies B) $\widetilde{Y}_\ell(z) \leq -\widetilde{\varepsilon}$.

Property 2: there exists $K^*_{\ell-1} > 0$ *such that:* A) $Y_i(z) = 0$ *for all* $i \in \{1, 2, \ldots, \ell-2\}$ implies B) $K_{\ell-1}Y_{\ell-1}(z) + \widetilde{Y}_\ell(z) \leq -\widetilde{\varepsilon}/2$ *for all* $K_{\ell-1} \geq K^*_{\ell-1}$.

Proof: By property (15), Property 2A implies $Y_{\ell-1}(z) \leq 0$. Therefore, Property 2A implies

$$K_{\ell-1}Y_{\ell-1}(z) + \widetilde{Y}_\ell(z) \leq \widetilde{Y}_\ell(z) \qquad \forall K_{\ell-1} \geq 0.$$

If $Y_{\ell-1}(z) = 0$ then, due to Property 1, Property 2B holds for all $K_{\ell-1} \geq 0$ whenever Property 2A holds. We claim further that there exists $\tau > 0$ such that Property 2B holds whenever Property 2A holds and $Y_{\ell-1}(z) > -\tau$. Suppose not, that is, for each integer n there exists $z_n \in \Psi$ such that $Y_{\ell-1}(z_n) > -\frac{1}{n}$ and

$$\widetilde{Y}_\ell(z_n) > -\frac{\widetilde{\varepsilon}}{2}. \tag{16}$$

Then, by compactness of Ψ, continuity of $Y_{\ell-1}$, and the nested property (15), the sequence $\{z_n\}_{n=1}^\infty$ has an accumulation point $z^* \in \Psi$ such that $Y_{\ell-1}(z^*) = 0$. Then, there exists a subsequence of $\{z_n\}_{n=1}^\infty$, which we will not relabel, converging to z^*. Then, Property 1 implies $\widetilde{Y}_\ell(z^*) \leq -\widetilde{\varepsilon}$. By continuity of $\widetilde{Y}$ this contradicts (16) for large enough n.

It follows from the continuity of $\widetilde{Y}_\ell$ and compactness of Ψ that we can pick $K^*_{\ell-1} > 0$ large enough to satisfy $\max_{z\in\Psi} \widetilde{Y}_\ell(z) \leq \tau K^*_{\ell-1} - \widetilde{\varepsilon}/2$. Hence, Property 2A implies Property 2B. ■

Theorem 5.4: Let Ψ_c, Ψ_d *be compact subsets of* $\mathbb{R}^p$, $Y_{c,i} : \Psi_c \to \mathbb{R}$, $Y_{d,i} : \Psi_d \to \mathbb{R}$, $i \in \{1, 2, \ldots, m\}$, *be continuous functions and* $Y_{c,0}, Y_{d,0} : \mathbb{R}^p \to \{0\}$, $Y_{c,m+1}, Y_{d,m+1} : \mathbb{R}^p \to \{1\}$ *be constant functions such that, for each* $j \in \{0, 1, \ldots, m\}$,

1) *if* $z \in \Psi_c$ *and* $Y_{c,i}(z) = 0$ *for all* $i \in \{0, 1, \ldots, j\}$ *then* $Y_{c,j+1}(z) \leq 0$,
2) *if* $z \in \Psi_d$ *and* $Y_{d,i}(z) = 0$ *for all* $i \in \{0, 1, \ldots, j\}$ *then* $Y_{d,j+1}(z) \leq 0$.

Then there exist $K_i > 0$, $i \in \{1, 2, \ldots, m-1\}$, *and* $\rho > 0$ *such that*

$$\begin{array}{lcll} \sum_{i=1}^{m-1} K_iY_{c,i}(z) + Y_{c,m}(z) & \leq & -\rho & \forall z \in \Psi_c, \\ \sum_{i=1}^{m-1} K_iY_{d,i}(z) + Y_{d,m}(z) & \leq & -\rho & \forall z \in \Psi_d. \end{array} \tag{17}$$

Proof: If either Ψ_c, Ψ_d, or both sets are empty, then there is nothing to check for the corresponding inequalities in (17). By assumption, (13) in Lemma 5.2 holds for both sets of functions $Y_{c,i}$ and $Y_{d,i}$. Apply Lemma 5.2 to each set of functions to generate $\varepsilon_c > 0$, $\varepsilon_d > 0$ and define $\varepsilon := \min\{\varepsilon_c, \varepsilon_d\}$. Lemma 5.2 implies that Property 1 of Lemma 5.3 holds, for both sets of functions, with $\ell = m$, $\widetilde{\varepsilon} = \varepsilon$, and $\widetilde{Y}_{c,\ell} = Y_{c,m}$ and $\widetilde{Y}_{d,\ell} = Y_{d,m}$. Since Property 1A holds, Property 2A holds. Then, from Property 2 of Lemma 5.3, there exists $K_{m-1} > 0$ such that

$$\begin{array}{l} z \in \Psi_c \ , \ Y_{c,i}(z) = 0 \ , \ i \in \{1, 2, \ldots, m-2\} \ \text{implies} \\ \qquad K_{m-1}Y_{c,m-1}(z) + Y_{c,m}(z) \leq -\frac{\varepsilon}{2} \\ z \in \Psi_d \ , \ Y_{d,i}(z) = 0 \ , \ i \in \{1, 2, \ldots, m-2\} \ \text{implies} \\ \qquad K_{m-1}Y_{d,m-1}(z) + Y_{d,m}(z) \leq -\frac{\varepsilon}{2} \end{array}$$

Then, Lemma 5.3 can be applied again with $\ell = m-1$, $\widetilde{\varepsilon} = \varepsilon/2$, and $\widetilde{Y}_{c,\ell} = K_{m-1}Y_{c,m-1} + Y_{c,m}$ and $\widetilde{Y}_{d,\ell} = K_{m-1}Y_{d,m-1} + Y_{d,m}$. Proceeding in this way, the result holds with $\rho := \varepsilon/2^{m-1}$. ■

Lemma 5.1 now follows from Theorem 5.4 by setting $\Psi_c := \left(\overline{\Pi(C)} \cap \Omega_{\mathcal{A}}(\delta, \Delta)\right) \times \mu\overline{\mathbb{B}}$, $\Psi_d := \left(\overline{\Pi(D)} \cap \Omega_{\mathcal{A}}(\delta, \Delta)\right) \times \mu\overline{\mathbb{B}}$, $Y_{c,i} := u_{c,i}$, $Y_{d,i} := u_{d,i}$, $i \in \{1, 2, \ldots, m\}$, and then taking $V := \sum_{i=1}^{m-1} K_iV_i + V_m$ and $\eta := \mu + \sum_{i=1}^{m-1} K_i\mu$.

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