Pointwise Optimal Feedback Laws for Hybrid Inclusions using Multiple Control Barrier Functions

Carlos A. Montenegro G., Hannah M. Sweatland, Keith A. Currier, Warren E. Dixon, and Ricardo G. Sanfelice

***Abstract*— This paper studies the problem of designing optimization-based controllers, with desired regularity properties, encoding the satisfaction of multiple state constraints via barrier functions for hybrid systems, modeled as hybrid inclusions. Sufficient conditions are given to guarantee forward invariance or asymptotic stability of a closed set K for a hybrid closed-loop system, even with discontinuous feedback laws. However, robustness of such properties is not necessarily guaranteed. Thus, we present sufficient conditions for the continuity of the optimization-based feedback laws that render the hybrid closed-loop system well-posed. A numerical simulation of a one-degree-of-freedom juggling system illustrating the main result of the paper is presented.**

I. INTRODUCTION

The notion of control barrier functions (CBFs) is a particularly common and useful tool to design feedback laws that render a set K forward invariant for a control system, which can span multiple domains, such as power systems, robotics, aerospace, and biology, to name a few [1], [2]. An important aspect to consider, when CBFs are used for design, is ensuring that resulting feedback laws enjoy appropriate regularity properties (local boundedness, Lipschitz continuity, smoothness, etc.) to guarantee that the closed-loop system is well-posed. Motivated by this, in this paper, we study the continuity properties of pointwise optimal CBF-based feedback laws for hybrid systems, modeled as hybrid inclusions, following the framework in [3]. In addition, we also analyze the induced stability properties of K using CBFs and the nominal robustness of the hybrid closed-loop system.

Because *safety*-critical systems often require the satisfaction of multiple state constraints, it can be especially useful to define the set K with multiple CBFs as in [4], [5]. Although it is possible to combine multiple state constraints into a single scalar barrier function [6], [7], the resulting barrier function is generally nonsmooth, leading to discontinuous feedback laws. To overcome this issue, in [4] the authors present a framework to study forward invariance using multiple barrier functions for hybrid inclusions without inputs, and in [5] a framework is presented for constrained differential inclusions with inputs. The majority of the CBF literature studies pointwise optimal feedback laws for nonlinear continuous- or discrete-time systems, encoded as parametric optimization problems with no more than one or two constraints, for which results guaranteeing local Lipschitz continuity [8] and regularity properties [9] are well known. In [10], the authors characterize the set of points where a minimum norm feedback law for a nonlinear continuous-time dynamical system is discontinuous. They also provide sufficient conditions for the controller to grow unbounded or not as near a point of discontinuity. In [5], the authors provide sufficient conditions for the continuity of optimization-based feedback laws using multiple control barrier functions for constrained differential inclusions. To the best of our knowledge, there are no such results for hybrid inclusions.

In this paper, we present an approach to design continuous optimization-based controllers that encode multiple state constraints via barrier functions for hybrid dynamical systems, modeled as in [3]. One of the main contributions of this paper is given by sufficient conditions for forward pre-invariance (Theorem 3.5) or pre-asymptotic stability (Theorem 3.6) of a closed set K for a hybrid closed-loop system. Unlike [11] for nonlinear systems, the results herein do not require Lipschitz continuity, or even continuity, of the feedback laws. However, the latter implies that invariance or stability of K for the closed-loop system may not hold in the presence of arbitrarily small disturbances. To address this issue, extending the work in [5], we present sufficient conditions for the continuity of pointwise optimal feedback laws (Theorem 4.3), encoded as a parametric optimization problem with an arbitrary number of constraints, such that the closed-loop system is well-posed (Corollary 4.4). Finally, in Section V, we provide a numerical example to demonstrate our theoretical results.

Notation. Let $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{R}_{>0} := (0, \infty)$. For a given $d \in \mathbb{N} \setminus \{0\}$, the shorthand $[d] := \{1, 2, \dots, d\}$ is used. For a vector $x \in \mathbb{R}^n$, we denote by $x^\top$ its transpose and by $|x|$ its Euclidean norm. Given two vectors, x and y, we write $(x, y) = [x^\top y^\top]^\top$, and $\langle x, y\rangle$ denotes the Euclidean inner product. Given a vector $x \in \mathbb{R}^n$ and a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, the distance from x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. We represent by $\mathbb{B}$ the closed Euclidean unit ball, by $x + \varepsilon\mathbb{B}$ the closed ball of radius ε centered at x,

C. A. Montenegro and R. G. Sanfelice are with the Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064. Email: {camonten, ricardo}@ucsc.edu. H. M. Sweatland, K. Currier, and W. E. Dixon are with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville FL 32611- 6250, USA. Email: {hsweatland, k.currier, wdixon}@ufl.edu.

Research partially supported by NSF Grants no. CNS-2039054 and CNS-2111688, by AFOSR Grants nos. FA9550-23-1-0145, FA9550-23-1-0313, and FA9550-23-1-0678, by AFRL Grant nos. FA8651-22-1-0017 and FA8651-23-1-0004, by ARO Grant no. W911NF-20-1-0253, and by DoD Grant no. W911NF-23-1-0158.

and by $\mathcal{A} + \varepsilon\mathbb{B} := \{a + b : a \in \mathcal{A}, b \in \varepsilon\mathbb{B}\}$ the Minkowski sum of $\mathcal{A}$ and $\varepsilon\mathbb{B}$. We denote as $\operatorname{int}\mathcal{A}$ the interior of $\mathcal{A}$, as $\overline{\mathcal{A}}$ its closure, and as $\partial\mathcal{A}$ its boundary. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s)$ for each $r \in \mathbb{R}_{\geq 0}$. Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The domain of S is $\operatorname{dom} S := \{x \in \mathbb{R}^n : S(x) \neq \emptyset\}$. We say S is outer semicontinuous (osc) at $\bar{x}$ if $\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x})$ and inner semicontinuous (isc) at $\bar{x}$ if $\liminf_{x\to\bar{x}} S(x) \supset S(\bar{x})$. The map S is called continuous at $\bar{x}$ if both conditions hold, i.e., $S(x) \to S(\bar{x})$ as $x \to \bar{x}$.

II. PRELIMINARIES

A. Hybrid Dynamical Systems

This paper considers hybrid systems modeled based on the framework in [3], where the continuous dynamics of the system are modeled by differential inclusions and the discrete dynamics are modeled by difference inclusions. Formally, a hybrid system $\mathcal{H}_P$ on $\mathbb{R}^n$ with input $u := (u_C, u_D) \in \mathcal{U}_C \times \mathcal{U}_D \subset \mathbb{R}^{m_C + m_D} = \mathbb{R}^m$ is defined as

$$\mathcal{H}_P \ : \begin{cases} \dot{x} \in F_P(x, u_C) & (x, u_C) \in C_P \\ x^+ \in G_P(x, u_D) & (x, u_D) \in D_P. \end{cases} \quad (1)$$

The *flow map* $F_P : \mathbb{R}^n \times \mathbb{R}^{m_C} \rightrightarrows \mathbb{R}^n$ defines the continuous evolution of the system when the state and continuous input are in the *flow set* $C_P \subset \mathbb{R}^n \times \mathcal{U}_C$. The *jump map* $G_P : \mathbb{R}^n \times \mathbb{R}^{m_D} \rightrightarrows \mathbb{R}^n$ defines the discrete evolution of the system when the state and discrete input are in the *jump set* $D_P \subset \mathbb{R}^n \times \mathcal{U}_D$. We assume that the sets $\mathcal{U}_C \subset \mathbb{R}^{m_C}$ and $\mathcal{U}_D \subset \mathbb{R}^{m_D}$ are bounded[1].

Since solutions to a hybrid system $\mathcal{H}$ as in (1) can exhibit both continuous and discrete behavior, we use ordinary time $t \in \mathbb{R}_{\geq 0}$ to track the amount of flow elapsed and a counter $j \in \mathbb{N}$ to track of the number of jumps that have occurred. Based on this hybrid parametrization of time, the concept of a hybrid time domain is proposed as follows.

Definition 2.1: (Hybrid time domain) *A set $\tilde{E} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if*

$$\tilde{E} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\}) \quad (2)$$

where $J \in \mathbb{N}$ and $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J \leq t_{J+1}$. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if it is the union of a nondecreasing sequence $E_1 \subset E_2 \subset E_3 \subset \ldots$ of compact hybrid time domains.

A hybrid signal is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I_\phi^j := \{t : (t, j) \in \operatorname{dom}\phi\}$ and $\operatorname{rge}\phi := \{\phi(t, j) : (t, j) \in \operatorname{dom}\phi\}$.

Definition 2.2: (Hybrid arc) *A hybrid signal $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is called a hybrid arc if, for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I_ϕ^j. A hybrid arc ϕ is said to be compact if $\operatorname{dom}\phi$ is compact.*

Definition 2.3: (Hybrid input) *A hybrid signal u is a hybrid input if, for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I_u^j.*

A solution to a hybrid system is defined as follows.

Definition 2.4: (Solution to $\mathcal{H}_P$) *A pair (ϕ, u) defines a solution to (1) if $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is a hybrid arc, $u = (u_C, u_D) : \operatorname{dom} u \to \mathcal{U}_C \times \mathcal{U}_D$ is a hybrid input with $\operatorname{dom}\phi = \operatorname{dom} u$, and*

(S0) $(\phi(0, 0), u_C(0, 0)) \in \overline{C_P}$ *or* $(\phi(0, 0), u_D(0, 0)) \in D_P$,

(S1) *For each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior $\operatorname{int} I_\phi^j$, we have that, for all $t \in \operatorname{int} I_\phi^j$,*

$$(\phi(t, j), u_C(t, j)) \in C_P$$

and, for almost all $t \in I_\phi^j$,

$$\frac{d\phi}{dt}(t, j) \in F_P\Big(\phi(t, j), u_C(t, j)\Big),$$

(S2) *For all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j + 1) \in \operatorname{dom}\phi$,*

$$(\phi(t, j), u_D(t, j)) \in D_P$$
$$\phi(t, j + 1) \in G_P\Big(\phi(t, j), u_D(t, j)\Big).$$

We say a solution pair (ϕ, u) to $\mathcal{H}_P$ is maximal if it cannot be extended, and we say it is complete if $\operatorname{dom}\phi$ is unbounded. Given a nonempty set $M \subset \mathbb{R}^n$, we denote by $\widehat{\mathcal{S}}_{\mathcal{H}_P}(M)$ the set of solution pairs (ϕ, u) to $\mathcal{H}_P$ as in (1) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_{\mathcal{H}_P}(M) \subset \widehat{\mathcal{S}}_{\mathcal{H}_P}(M)$ denotes all maximal solutions from M. For each $\star \in \{C, D\}$, we define the projections of $\star_P$ onto $\mathbb{R}^n$ as

$$\Pi(\star_P) := \{\xi \in \mathbb{R}^n : \exists u_\star \text{ s.t. } (\xi, u\star) \in \star_P\}.$$

B. Hybrid Closed-Loop Systems

Consider the hybrid system resulting from $\mathcal{H}_P$ with the input $u := (u_C, u_D)$ assigned by a feedback law $\kappa := (\kappa_C, \kappa_D) : \Pi(C_P) \cup \Pi(D_P) \to \mathcal{U}_C \times \mathcal{U}_D$. The resulting hybrid system is

$$\mathcal{H}_\kappa \ : \begin{cases} \dot{x} \in F_P(x, \kappa_C(x)) =: F_\kappa(x) & x \in C_\kappa \\ x^+ \in G_P(x, \kappa_D(x)) =: G_\kappa(x) & x \in D_\kappa \end{cases} \quad (3)$$

where $\star_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_\star(x)) \in \star_P\}$, for each $\star \in \{C, D\}$. A hybrid arc $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is a solution to $\mathcal{H}_\kappa$ if ϕ satisfies Definition 2.4 for the case of no input u.

Well-posed dynamical systems refer to a class of dynamical systems where the solutions enjoy very useful structural properties [3, Lemma 2.21]. A hybrid closed-loop system $\mathcal{H}_\kappa$ as in (3) is well-posed if, in particular, for every (graphically) convergent sequence of solutions to $\mathcal{H}_\kappa$ that is locally bounded, its (graphical) limit is also a solution.

[1]This is a mild assumption since control inputs are almost always limited in real-world applications due to physical, safety, and practical constraints.

Finally, we introduce the notions of forward invariance and asymptotic stability of a set for a hybrid closed-loop system.

Definition 2.5: (Forward invariance) *Given a hybrid closed-loop system $\mathcal{H}_\kappa$ as in (3), a nonempty set $K \subset \mathbb{R}^n$ is said to be forward pre-invariant for $\mathcal{H}_\kappa$ if for every $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(K)$, we have that* $\operatorname{rge}\phi \subset K$.

Definition 2.6: (Local pre-asymptotic stability) *Consider a hybrid closed-loop system $\mathcal{H}_\kappa$ as in (3). A closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be locally pre-asymptotically stable (LpAS) for $\mathcal{H}_\kappa$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that every $\phi \in \widehat{\mathcal{S}}_{\mathcal{H}_\kappa}(\mathcal{A} + \delta\mathbb{B})$ satisfies $\operatorname{rge}\phi \subset \mathcal{A} + \varepsilon\mathbb{B}$, and moreover, each maximal solution ϕ is bounded and, if complete, satisfies*

$$\lim_{\substack{(t,j)\in\operatorname{dom}\phi \\ t+j\to\infty}} |\phi(t,j)|_{\mathcal{A}} = 0.$$

In Definition 2.5 and Definition 2.6, if maximal solutions to $\mathcal{H}_\kappa$ are complete, then we can drop the prefix "pre."

III. CONTROL BARRIER FUNCTIONS AND SAFEGUARDING FEEDBACK LAWS

In this paper, given a closed set K and a hybrid system $\mathcal{H}_P$ as in (1), we address the problem of designing a continuous feedback law $\kappa := (\kappa_C, \kappa_D)$ that renders K forward pre-invariant or LpAS for $\mathcal{H}_\kappa$. To this end, barrier functions (BFs) serve as a synthesis tool to guarantee invariance of a set of interest [2], [12]. In particular, given $K \subset \mathbb{R}^n$, suppose that there exists a continuous function $B : \operatorname{dom} B \to \mathbb{R}$ such that

$$K := \{x \in \Pi(C_P) \cup \Pi(D_P) : B(x) \leq 0\} \tag{4}$$

which is closed when $\Pi(C_P) \cup \Pi(D_P)$ is closed. In the following, we will extend this notion where the set K is given by the intersection of multiple sets, in turn, defined by multiple functions B_i, $i \in [d]$ for some $d \in \mathbb{N} \setminus \{0\}$.

Definition 3.1: (Multiple barrier function candidate) *Given the hybrid system $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ and a set $K \subset \Pi(C_P) \cup \Pi(D_P)$, the function $B : \operatorname{dom} B \to \mathbb{R}^d$ is a multiple barrier function (mBF) candidate for $\mathcal{H}_P$ with respect to K if the following conditions hold:*

1) $\overline{\Pi(C_P)} \cup \Pi(D_P) \cup G_P(D_P) \subset \operatorname{dom} B$;
2) *For each $i \in [d]$, B_i is continuously differentiable on $\overline{\Pi(C_P)}$;*
3) *For each $i \in [d]$,*
$$B_i(x) > 0 \qquad \forall x \in \left(\overline{\Pi(C_P)} \cup \Pi(D_P)\right) \setminus K_i,$$
where
$$K_i := \{x \in \Pi(C_P) \cup \Pi(D_P) : B_i(x) \leq 0\} \tag{5}$$
4) $K = \bigcap_{i\in[d]} K_i$.

We also introduce the primary design parameter in the form of a performance function $\gamma := (\gamma_C, \gamma_D)$, which will be used to define a set of control inputs that constrain the worst-case growth of an mBF candidate during flows and at jumps.

Assumption 3.2: (Performance functions) *Given a hybrid system $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ and a set K, let B be an mBF candidate for $\mathcal{H}_P$ with respect to K. Suppose that there exist $\gamma_C : \Pi(C_P) \to \mathbb{R}^d$, $\gamma_D : \Pi(D_P) \to \mathbb{R}^d$, and, for each $i \in [d]$, an open neighborhood $\mathcal{N}_i$ of*

$$M_i := \{x \in \partial K : B_i(x) = 0\} \tag{6}$$

such that

1) *for each $i \in [d]$,*
$$\gamma_C^{(i)}(x) \geq 0 \qquad \forall x \in (\mathcal{N}_i \setminus K_i) \cap \Pi(C_P); \tag{7}$$
2) *for each $i \in [d]$,*
$$\gamma_D^{(i)}(x) \geq 0 \qquad \forall x \in K_i \cap \Pi(D_P). \tag{8}$$

Given an mBF candidate B as in Definition 3.1 and $\gamma := (\gamma_C, \gamma_D) : \mathbb{R}^n \to \mathbb{R}^{2d}$, define the functions $\Gamma_C : \Pi(C_P) \to \mathbb{R}^d$ and $\Gamma_D : \Pi(D_P) \to \mathbb{R}^d$ such that their i-th component is given by

$$\Gamma_C^{(i)}(x, u_C) := \sup_{f\in F_P(x,u_C)} \langle \nabla B_i(x), f\rangle + \gamma_C^{(i)}(x), \tag{9a}$$

$$\Gamma_D^{(i)}(x, u_D) := \sup_{g\in G_P(x,u_D)} B_i(g) + \gamma_D^{(i)}(x). \tag{9b}$$

Observe that, for each $i \in [d]$, the value of $(x, u_C) \mapsto \Gamma_C^{(i)}(x, u_C)$ and $(x, u_D) \mapsto \Gamma_D^{(i)}(x, u_D)$ contains the worst-case growth of each $x \mapsto B_i(x)$ for any possible direction of flow in F_P or jump in G_P, respectively. If F_P and G_P are nonempty and locally bounded for each $(x, u_C) \in C_P$ or $(x, u_D) \in D_P$, respectively, then it follows that $\Gamma_C^{(i)}(x, u_C) < \infty$ and $\Gamma_D^{(i)}(x, u_D) < \infty$ for every $i \in [d]$. Allowing F_P or G_P to be unbounded is physically unrealistic as it would lead to solutions that flow arbitrarily fast or jump to arbitrarily large values. This motivates the following assumption.

Assumption 3.3: (Data of $\mathcal{H}_P$) *For a hybrid system $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ as in (1), suppose that i) $F_P : \mathbb{R}^n \times \mathbb{R}^{m_C} \rightrightarrows \mathbb{R}^n$ is locally bounded relative*[2] *to C_P and $C_P \subset \operatorname{dom} F_P$, and ii) $G_P : \mathbb{R}^n \times \mathbb{R}^{m_D} \rightrightarrows \mathbb{R}^n$ is locally bounded relative to D_P and $D_P \subset \operatorname{dom} G_P$.*

Next, we define, for each $\star \in \{C, D\}$, the set of admissible control inputs during flows ($\star = C$) and during jumps ($\star = D$) at each state as

$$\Psi_\star(x) := \{u_\star \in \mathcal{U}_\star : (x, u_\star) \in \star_P\}, \tag{10}$$

and introduce the definition of (multiple) control barrier functions used throughout this paper (cf. [5]).

Definition 3.4: (Multiple control barrier functions) *Given a hybrid system $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ as in (1) satisfying Assumption 3.3 and a closed set K, let B be an mBF candidate for $\mathcal{H}_P$ with respect to K, and $\gamma_C : \Pi(C_P) \to \mathbb{R}^d$ and $\gamma_D : \Pi(D_P) \to \mathbb{R}^d$ be performance functions satisfying Assumption 3.2. Suppose that*

[2]See [3, Definition A.32].

1) *for every* $x \in \Pi(C_P)$ *there exists an admissible input* $u_C \in \Psi_C(x)$ *such that* $\Gamma_C^{(i)}(x, u_C) \leq 0$ *for all* $i \in [d]$*;*
2) *for every* $x \in \Pi(D_P)$ *there exists an admissible input* $u_D \in \Psi_D(x)$ *such that* $\Gamma_D^{(i)}(x, u_D) \leq 0$ *for all* $i \in [d]$*.*

Then, B *is said to be a multiple control barrier function (mCBF) for* $\mathcal{H}_P$ *with respect to* (K, γ)*, where* $\gamma := (\gamma_C, \gamma_D)$*.*

Given an mCBF for $\mathcal{H}_P$ with respect to (K, γ), we define, for each $\star \in \{C, D\}$, the set-valued mapping $\mathbb{U}_\star : \Pi(\star_P) \rightrightarrows \mathcal{U}_\star$ as

$$\mathbb{U}_\star(x) := \left\{ u_\star \in \Psi_\star(x) : \max_{i \in [d]} \Gamma_\star^{(i)}(x, u_\star) \leq 0 \right\}. \quad (11)$$

Notice that the performance functions γ_C and γ_D shape $\mathbb{U}_C$ and $\mathbb{U}_D$, acting as tuning parameters that constrain the worst-case growth of the change of an mCBF along flows and jumps to guarantee forward pre-invariance or pre-asymptotic stability of a set of interest. When, for each $\star \in \{C, D\}$, a feedback law $x \mapsto \kappa_\star(x)$ is a selection from (11); that is, $\kappa_\star(x) \in \mathbb{U}_\star(x)$ for each $x \in \Pi(\star_P)$, we say that $\kappa := (\kappa_C, \kappa_D)$ is a safeguarding feedback law.

The following theorem is one of the main results of this section. It states that mCBFs provide invariance properties of a closed set K for the hybrid closed-loop system $\mathcal{H}_\kappa$, with $\kappa_\star(x) \in \mathbb{U}_\star(x)$ for all $x \in \Pi(\star_P)$ and each $\star \in \{C, D\}$.

Theorem 3.5: (Forward pre-invariance) *Given a hybrid system* $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ *as in (1), a closed set* K*, and a performance function* $\gamma := (\gamma_C, \gamma_D) : \Pi(C_P) \cup \Pi(D_P) \to \mathbb{R}^{2d}$ *satisfying Assumption 3.2, suppose that* B *is an mCBF for* $\mathcal{H}_P$ *with respect to* (K, γ)*. Let* $\kappa := (\kappa_C, \kappa_D)$ *be such that, for each* $\star \in \{C, D\}$*, we have* $\kappa_\star(x) \in \mathbb{U}_\star(x)$ *for all* $x \in \Pi(\star_P)$*. If* $G_P(D_P) \subset \Pi(C_P) \cup \Pi(D_P)$*, then* K *is forward pre-invariant for* $\mathcal{H}_\kappa$ *as in (3).*

Related notions of CBFs in the literature for continuous-time nonlinear systems, such as zeroing CBFs, are defined so that a set K is not just forward invariant, but also asymptotically stable [13, Proposition 2]. The following result provides sufficient conditions for pre-asymptotic stability of a set K for the closed-loop system using multiple CBFs.

Theorem 3.6: (Pre-asymptotic stability) *Given a hybrid system* $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ *as in (1), a closed set* K*, and a performance function* $\gamma := (\gamma_C, \gamma_D) : \Pi(C_P) \cup \Pi(D_P) \to \mathbb{R}^{2d}$*, suppose that* B *is an mCBF for* $\mathcal{H}_P$ *with respect to* (K, γ)*. Let* $\kappa := (\kappa_C, \kappa_D)$ *be such that, for each* $\star \in \{C, D\}$*, we have* $\kappa_\star(x) \in \mathbb{U}_\star(x)$ *for all* $x \in \Pi(\star_P)$*. If, for each* $i \in [d]$*, there exists an open neighborhood* $\mathcal{N}_i$ *of* M_i*, defined as in (6), such that*

$$\gamma_C^{(i)}(x) > 0 \qquad \forall x \in \left(\bigcup_{i \in [d]} \mathcal{N}_i \setminus K \right) \cap \Pi(C_P) \quad (12a)$$

$$\gamma_D^{(i)}(x) \geq 0 \qquad \forall x \in \left(\bigcup_{i \in [d]} \mathcal{N}_i \setminus K \right) \cap \Pi(D_P), \quad (12b)$$

then K *is LpAS for* $\mathcal{H}_\kappa$ *as in (3).*

IV. CONTINUITY OF POINTWISE OPTIMAL SAFEGUARDING FEEDBACK LAWS

In this section, given a hybrid system $\mathcal{H}_P$ and a closed set K, we study the continuity of a class of optimization-based feedback laws, with a general class of objective functions and arbitrary number of constraints, to guarantee forward pre-invariance or pre-asymptotic stability of K for the hybrid closed-loop system $\mathcal{H}_\kappa$. To this end, given a performance function $\gamma := (\gamma_C, \gamma_D)$, suppose that B is an mCBF for $\mathcal{H}_P$ with respect to (K, γ) and let us introduce, for each $\star \in \{C, D\}$, the optimal solution map $\kappa_\star : \Pi(\star_P) \rightrightarrows \mathcal{U}_\star$ as

$$\kappa_\star(x) := \underset{u_\star \in \mathbb{U}_\star(x)}{\arg\min}\, Q_\star(x, u_\star) \quad (13)$$

where $Q_\star : \star_P \to \mathbb{R}$ is referred to as the *objective function*. Notice that, since B is an mCBF for $\mathcal{H}_P$ with respect to (K, γ), we see that (13) is feasible. However, the optimal solution maps κ_C and κ_D may not necessarily be single valued nor continuous. To address these issues, we first present the following lemmas.

Lemma 4.1: (Continuity of set-valued mappings defined by inequalities) *Consider the sets* $\mathcal{X} \subset \mathbb{R}^n$*,* $\mathcal{U} \subset \mathbb{R}^m$*, and a set-valued mapping* $S : \mathcal{X} \rightrightarrows \mathcal{U}$ *defined by*

$$S(x) := \{u \in \mathcal{U} : \eta_i(x, u) \leq 0 \quad \forall i \in [d]\} \quad \forall x \in \mathcal{X} \quad (14)$$

where, for each $i \in [d]$*, the function* $\eta_i : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ *is continuous. If* $\mathcal{U}$ *is closed, then* S *is osc at every* $x \in \mathcal{X}$*. Additionally, suppose that* $\mathcal{U}$ *is convex. If, for each* $x \in \mathcal{X}$ *and for each* $i \in [d]$*, the map* $u \mapsto \eta_i(x, u)$ *is convex and there exists* $\bar{u} \in \mathcal{U}$ *such that* $\eta_i(x, \bar{u}) < 0$*, then* S *is isc, and thus continuous, at* x*.*

Lemma 4.1 is helpful in showing that $\mathbb{U}_C$ and $\mathbb{U}_D$ are continuous as set-valued mappings; namely, isc and osc.

The next result leverages Lemma 4.1, together with other mild assumptions, to prove the continuity of optimal solution maps of the form (13).

Lemma 4.2: (Continuity of optimal solution maps) *Consider the sets* $\mathcal{X} \subset \mathbb{R}^n$ *and* $\mathcal{U} \subset \mathbb{R}^m$*. Given a function* $Q : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ *and a set-valued mapping* $\mathbb{U} : \mathcal{X} \rightrightarrows \mathcal{U}$*, let* $\kappa : \mathcal{X} \rightrightarrows \mathcal{U}$ *be defined as*

$$\kappa(x) := \underset{u \in \mathbb{U}(x)}{\arg\min}\, Q(x, u) \qquad \forall x \in \mathcal{X}. \quad (15)$$

Suppose that Q *and* $\mathbb{U}$ *are continuous and that* $\mathbb{U}$ *is also locally bounded relative to* $\mathcal{X}$*. Then,* κ *is osc at every* $x \in \mathcal{X}$*. If, additionally, for each* $x \in \mathcal{X}$*, the map* $u \mapsto Q(x, u)$ *is strictly convex on* $\mathbb{U}(x)$*, then* κ *is single valued and continuous.*

In [5, Lemma 3], the authors provide sufficient conditions for continuity of κ as in (15), where it is assumed that the cost function Q is level-bounded in u, locally uniformly in x. This property is equivalent to local boundedness of the level set $x \mapsto \ell_Q(x, \lambda) := \{u \in \mathbb{R}^m : Q(x, u) \leq \lambda\}$ for every $\lambda \in \mathbb{R}$. Local boundedness holds if Q is continuous and, for each x, the map $u \mapsto Q(x, u)$ is convex and inf-compact (cf. [14, Lemma 5.7]). In contrast, Lemma 4.2 only requires the continuity of Q at the expense of the constraint set $\mathbb{U}$ being locally bounded.

The main result of this section provides sufficient conditions for the continuity of κ_C and κ_D defined in (13).

Theorem 4.3: (Continuity of pointwise optimal safeguarding feedback laws) *Given a hybrid system $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ as in (1), a closed set K, and a performance function $\gamma := (\gamma_C, \gamma_D) : \Pi(C_P) \cup \Pi(D_P) \to \mathbb{R}^{2d}$, suppose that B is an mCBF for $\mathcal{H}_P$ with respect to (K, γ). Let $\mathbb{U}_C$ and $\mathbb{U}_D$ be defined as in (11) and consider, for each $\star \in \{C, D\}$, the objective function $Q_\star : \star_P \to \mathbb{R}$. Furthermore, suppose that $\mathcal{U}_C$ and $\mathcal{U}_D$ are bounded and that, for each $i \in [d]$ and each $\star \in \{C, D\}$, the following conditions hold:*

(A1) *$\star_P$ is closed;*
(A2) *$\Gamma_\star^{(i)}(x, u_\star)$ is continuous on $\star_P$;*
(A3) *for every $x \in \Pi(\star_P)$ the map $u_\star \mapsto \Gamma_\star^{(i)}(x, u_\star)$ is convex on $\mathbb{U}_\star(x)$ and $\Psi_\star(x)$ is convex on $\mathcal{U}_\star$;*
(A4) *there exists $u_\star \in \Psi_\star(x)$ such that $\Gamma_\star^{(i)}(x, u_\star) < 0$ for every $x \in \Pi(\star_P)$;*
(A5) *the objective function $Q_\star$ is continuous and, for every $x \in \Pi(\star_P)$, the map $u_\star \mapsto Q_\star(x, u_\star)$ is strictly convex on $\mathbb{U}_\star(x)$.*

Then, κ_C and κ_D, defined in (13), are single valued and continuous.

The next result shows that if, for each $\star \in \{C, D\}$, $x \mapsto \kappa_\star(x)$ is the pointwise optimal selection from (11), and $\kappa_\star$ is continuous, then the resulting hybrid closed-looped system is well-posed.

Corollary 4.4: (Well-posedness of $\mathcal{H}_\kappa$) *Given a hybrid system $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ as in (1) satisfying Assumption 3.3, a closed set $K \subset \mathbb{R}^n$, and a performance function $\gamma := (\gamma_C, \gamma_D) : \Pi(C_P) \cup \Pi(D_P) \to \mathbb{R}^{2d}$, suppose that B is an mCBF for $\mathcal{H}_P$ with respect to (K, γ). Let $\mathbb{U}_C$ and $\mathbb{U}_D$ be defined as in (11), and consider the feedback law $\kappa := (\kappa_C, \kappa_D)$ in (13). If the sets $\mathcal{U}_C$ and $\mathcal{U}_D$ are bounded and the following conditions hold:*

(B1) *(A1)-(A5) in Theorem 4.3;*
(B2) *F_P is osc relative to C_P and $F_P(x, u_C)$ is convex for each $(x, u_C) \in C_P$;*
(B3) *G_P is osc relative to D_P.*

Then, the hybrid closed-loop system $\mathcal{H}_\kappa$ is well-posed.

Several conclusions arise from the well-posedness of $\mathcal{H}_\kappa$. In particular, the basin of pre-attraction[3] of K for $\mathcal{H}_\kappa$, denoted by $\mathcal{B}_K$, is an open set containing K [15, Proposition 7.4] and K is uniformly pAS for $\mathcal{H}_\kappa$ on $\mathcal{B}_K$; that is, there exists $\beta \in \mathcal{KL}$ such that, for every $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(\mathcal{B}_K)$, we have, by [15, Theorem 7.12], that

$$|\phi(t,j)|_K \leq \beta\left(|\phi(0,0)|_K, t+j\right) \quad \forall (t,j) \in \operatorname{dom}\phi. \quad (16)$$

In addition, K is robustly pre-asymptotically stable on B_K in the sense of [15, Definition 7.18].

[3]Given a hybrid system $\mathcal{H}$, the basin of pre-attraction of a pre-asymptotically stable set $\mathcal{A}$ for $\mathcal{H}$, denoted $\mathcal{B}_\mathcal{A}$, is given by the set of points $\xi \in \mathbb{R}^n$ such that every $\phi \in \mathcal{S}_\mathcal{H}(\xi)$ is bounded and, if ϕ is complete, then $\lim_{t+j\to\infty} |\phi(t,j)|_\mathcal{A} = 0$.

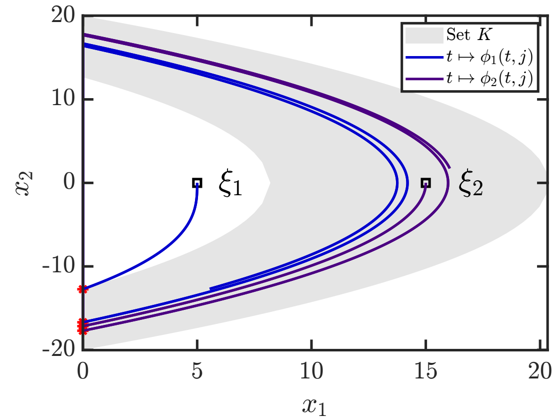

Fig. 1 – The phase portrait of the state trajectory ϕ to $\mathcal{H}$ from $\xi \in K$, under the minimum norm selection $x \mapsto \kappa_\star(x)$ from $x \mapsto \mathbb{U}_\star(x)$, for each $\star \in \{C, D\}$, with values as in (13).

V. SIMULATION RESULTS

To illustrate our results, consider the one-degree-of-freedom juggling system described in [16, Example 3.4], where a ball with height $x_1 \geq 0$ and vertical velocity x_2 is attached to a nonelastic string of length $h_{\max} > 0$ and with a known coefficient of restitution $\lambda_p \in [0, 1]$. Suppose that the surface where the ball bounces is actuated and that the string breaks when the ball moves with a velocity larger than $x_2 > v_{\max} \geq \sqrt{\frac{2\theta h_{\max}}{1-\lambda_p^2}}$ when it is at $h_{\max}$. At contacts with the surface, the coefficient of restitution is unknown though constrained to the interval $[0, 1]$. Consider the function $E : \mathbb{R}^2 \to \mathbb{R}$ describing the total energy of the system as

$$E(x) := \theta x_1 + \frac{1}{2} x_2^2 \qquad \forall x \in \mathbb{R}^2$$

for some $\theta > 0$. Define $\overline{v} := \sqrt{2E(h_{\max}, \lambda_p v_{\max})}$ and, for some $u_{\max} \in [0, v_{\max} - \overline{v}]$, suppose that the input when the ball is in the air is restricted by $u_C \in [-u_{\max}, u_{\max}]$ and when the ball impacts the actuated surface by $u_D \in [0, u_{\max}]$. This system —including parametric uncertainties—is modeled as a hybrid system $\mathcal{H}_P = (C_P, F_P, D_P, G_P)$ as in (1) with

$$F_P(x, u_C) := \begin{bmatrix} x_2 \\ -\theta + u_C \end{bmatrix} \qquad \forall (x, u_C) \in C_P$$

$$G_P(x, u_D) := \begin{cases} \begin{bmatrix} x_1 \\ [0, -x_2] + u_D \end{bmatrix} & \text{if} \quad (x, u_D) \in D_P^{(1)} \\ \begin{bmatrix} x_1 \\ \min\{-\lambda_p x_2, -\delta_p\} \end{bmatrix} & \text{if} \quad (x, u_D) \in D_P^{(2)} \end{cases}$$

where $0 < \delta_p < v_{\max}$ and $D_P := D_P^{(1)} \cup D_P^{(2)}$ with

$$C_P := [0, h_{\max}] \times (-\infty, v_{\max}] \times [-u_{\max}, u_{\max}]$$
$$D_P^{(1)} := \{0\} \times [-\overline{v}, 0] \times [0, u_{\max}]$$

$$D_P^{(2)} := \{h_{\max}\} \times [0, v_{\max}] \times [0, u_{\max}],$$

and it can be shown that $G_P \subset \Pi(C_P) \cup \Pi(D_P)$. The control objective is to constrain the total energy of the system between the interval $[E_1, E_2]$, where $0 < E_1 < 2\theta h_{\max} < 2\theta h_{\max} + \delta_p^2 < E_2 < 2E(h_{\max}, v_{\max})$. Thus, we consider $K := K_1 \cap K_2$, where, for each $i \in \{1, 2\}$,

$$K_i := \left\{x \in \Pi(C_P) : (-1)^i(2\theta x_1 + x_2^2 - E_i) \leq 0\right\}.$$

An mBF candidate B for $\mathcal{H}_P$ with respect to K has components given by

$$B_i(x) := (-1)^i(2\theta x_1 + x_2^2 - E_i) \qquad i \in \{1, 2\}.$$

In fact, all the items in Definition 3.1 hold by construction of B. Next, consider the following performance function $\gamma = (\gamma_C, \gamma_D)$, where, for each $i \in \{1, 2\}$,

$$\gamma_C^{(i)}(x) := \alpha B_i(x) \text{ and } \gamma_D^{(i)}(x) := \beta \max\{0, B_i(x)\}$$

for some $\alpha, \beta > 0$, and observe that this choice satisfies Assumption 3.2. Furthermore, (9a) gives, for all $(x, u_C) \in C_P$ and for each $i \in \{1, 2\}$,

$$\Gamma_C^{(i)}(x, u_C) = (-1)^i\,(2x_2u_C) + \gamma_C^{(i)}(x). \tag{17a}$$

With $\Psi_C(x) = [-u_{\max}, u_{\max}]$, it can be shown that $\Pi(C_P) \ni x \mapsto \mathbb{U}_C(x) \neq \emptyset$. Similarly, for all $(x, u_D) \in D_P$, we see that

- For all $(x, u_D) \in D_P^{(1)}$ and for each $i \in \{1, 2\}$,

$$\Gamma_D^{(i)}(x, u_D) = (-1)^i\left((u_D - (i-1)x_2)^2 - E_i\right) + \gamma_D^{(i)}(x). \tag{17b}$$

- For all $(x, u_D) \in D_P^{(2)}$ and for each $i \in \{1, 2\}$, we have that

$$\Gamma_D^{(i)}(x, u_D) = B_i(h_{\max}, \min\{-\lambda_p x_2, -\delta_p\}) + \gamma_D^{(i)}(x). \tag{17c}$$

Moreover, with $\Psi_D(x) = [0, u_{\max}]$, it can also be shown that $\Pi(D_P) \ni x \mapsto \mathbb{U}_D(x) \neq \emptyset$. Thus, B is an mCBF for $\mathcal{H}_P$ with respect to (K, γ). This implies that K is forward pre-invariant (LpAS) for $\mathcal{H}_\kappa$, for any selection $\kappa_\star(x) \in \mathbb{U}_\star(x)$ for all $x \in \Pi(\star_P)$ and each $\star \in \{C, D\}$, by Theorem 3.5 (Theorem 3.6).

Figure 1 illustrates that the set K is pre-asymptotically stable and forward pre-invariant for $\mathcal{H}_\kappa$, despite the uncertainty in the coefficient of restitution at impacts with the controlled surface. Namely, when $\xi \notin K$, we have that the state trajectories to $\mathcal{H}_\kappa$ are bounded, converge, and stay in K. Whereas, if $\xi \in K$, the state trajectory to $\mathcal{H}_\kappa$ does not leave K.

VI. CONCLUSION & FUTURE WORK

Given a hybrid system $\mathcal{H}_P$ and a closed set K, we introduce sufficient conditions to render K forward pre-invariant or LpAS for the hybrid closed-loop system $\mathcal{H}_\kappa$ using optimization-based controllers. To provide robustness guarantees for the properties of K for $\mathcal{H}_\kappa$, we also present sufficient conditions for the continuity of the feedback law which, in turn, renders $\mathcal{H}_\kappa$ well-posed. Future work includes studying approaches to systematically certify that, given a multiple barrier function candidate B, the set-valued mappings $\mathbb{U}_C$ and $\mathbb{U}_D$ are not empty at each $x \in \Pi(C_P)$ and $\Pi(D_P)$, respectively.

REFERENCES

[1] J. Zeng, B. Zhang, and K. Sreenath, "Safety-Critical Model Predictive Control with Discrete-Time Control Barrier Function," in 2021 American Control Conference (ACC). IEEE, 2021, pp. 3882–3889.

[2] A. D. Ames, J. W. Grizzle, and P. Tabuada, "Control Barrier Function based Quadratic Programs with Application to Adaptive Cruise Control," in 53rd IEEE Conference on Decision and Control, 2014, pp. 6271–6278.

[3] R. G. Sanfelice, Hybrid Feedback Control. Princeton University Press, 2021.

[4] M. Maghenem and R. Sanfelice, "Sufficient Conditions for Forward Invariance and Contractivity in Hybrid Inclusions using Barrier Functions," Automatica, vol. 124, p. 109328, 11 2020.

[5] A. Isaly, M. Ghanbarpour, R. G. Sanfelice, and W. E. Dixon, "On the Feasibility and Continuity of Feedback Controllers Defined by Multiple Control Barrier Functions," IEEE Transactions on Automatic Control, vol. 69, pp. 7326–7339, 2024.

[6] P. Glotfelter, J. Cortés, and M. Egerstedt, "Nonsmooth Barrier Functions with Applications to Multi-Robot Systems," IEEE Control Systems Letters, vol. 1, no. 2, pp. 310–315, 2017.

[7] ——, "A Nonsmooth Approach to Controller Synthesis for Boolean Specifications," IEEE Transactions on Automatic Control, vol. 66, no. 11, pp. 5160–5174, 2021.

[8] M. Jankovic, "Robust Control Barrier Functions for Constrained Stabilization of Nonlinear Systems," Automatica, vol. 96, pp. 359–367, 2018.

[9] P. Mestres, A. Allibhoy, and J. Cortés, "Regularity Properties of Optimization-Based Controllers," European Journal of Control, vol. 81, p. 101098, 2025.

[10] M. Alyaseen, N. Atanasov, and J. Cortés, "Continuity and Boundedness of Minimum-Norm CBF-Safe Controllers," IEEE Transactions on Automatic Control, pp. 1–8, 2025.

[11] X. Tan, W. S. Cortez, and D. V. Dimarogonas, "High-Order Barrier Functions: Robustness, Safety, and Performance-Critical Control," IEEE Transactions on Automatic Control, vol. 67, no. 6, pp. 3021–3028, 2022.

[12] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Robustness of Control Barrier Functions for Safety Critical Control," IFAC-PapersOnLine, vol. 48, no. 27, pp. 54–61, 2015.

[13] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control Barrier Function Based Quadratic Programs for Safety Critical Systems," IEEE Transactions on Automatic Control, vol. 62, no. 8, pp. 3861–3876, 2017.

[14] G. Still, "Lectures on Parametric Optimization: An Introduction," Preprint, Optimization Online, 2018.

[15] R. Goebel, R. G. Sanfelice, and A. R. Teel, Hybrid Dynamical Systems: Modeling, Stability, and Robustness. New Jersey: Princeton University Press, 2012.

[16] J. Chai and R. G. Sanfelice, "Forward Invariance of Sets for Hybrid Dynamical Systems (Part II)," IEEE Transactions on Automatic Control, vol. 66, no. 1, pp. 89–104, 2021.