Inverse-Optimal Input-to-State Stabilizing Control for Hybrid Systems

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Abstract— **This paper studies the problem of control design methods to stabilize a hybrid system under disturbances as an inverse-optimality problem. Sufficient conditions are given to guarantee input-to-state stability of a hybrid system with disturbances only. Next, it is shown that a hybrid system, with inputs and disturbances, can be rendered input-to-state controlled stable under the existence of a control Lyapunov function (CLF) using pointwise minimum norm (min-norm) stabilizing feedback laws. Finally, it is proven that every CLF is a meaningful value function for a two-player zero-sum hybrid game in the context of stability, and that every pointwise min-norm stabilizing control feedback law is optimal for such a game with meaningful cost functionals. The main results are illustrated in a one-degree-of-freedom juggling system.**

I. INTRODUCTION

Studying decision-making processes for dynamical systems that exhibit both continuous-time and discrete-time behavior under the presence of adversarial actions has gained attention in recent years. Notions that relate inputs and the state of the system conveniently allow the design of control strategies to certify a desired property, such as the stability of a set of interest, and simultaneously reduce the effect of a disturbance. Applying only continuous-time results fails when seeking to measure and optimize the performance of the system at discrete-time events. Systems that interconnect physical and computational components or include continuous dynamics plus timers that expire, communication switches, impacts, or resets require a specialized control design formulation to address the effect of adversarial actions.

These observations led us to study the design of control strategies to stabilize *hybrid systems* under disturbances as a two-player zero-sum game. Following the framework in [1], a player P_1 selects the control input to minimize a cost functional, while a player P_2 designs a disturbance to maximize it. Given that the design of the optimal strategy (known as the *saddle-point equilibrium*) and the computation of the optimal cost via solving the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations represents a nontrivial task, we propose an inverse optimality approach. By *inverse* (instead of *direct*), we mean that given a feedback law, we are searching for stage costs, which must be meaningful in a well-defined sense.

Input-to-state stability notions trace back to [2], [3] for continuous-time systems, [4] for discrete-time systems, and [5], [6] for hybrid systems modeled as in [7], for impulsive systems in [8], and [9] for switched systems. Inverse optimal design of stabilizing controllers for nonlinear continuous-time systems with disturbances was studied in [10], [11] with a two-player zero-sum formulation of a differential game.

Motivated by the computational complexity of tools designing stabilizing controllers for hybrid systems as in [7] under disturbances, extending the work in [10], we present an inverse-optimal approach to formulate such problem as a two-player zero-sum game as in [1], [12], [13]. A key contribution of this paper is the formulation of two-player zero-sum games with hybrid constraints as in [7] encoding the inverse-optimal design of stabilizing controllers in the presence of disturbances. In particular, in Section III, we introduce the notion of pre-input-to-state stability for hybrid systems with disturbance inputs only, and the corresponding Lyapunov functions. Theorem 5.3 shows that a hybrid system, with inputs and disturbances, can be rendered pre-input-to-state stable, under the existence of a control Lyapunov function (CLF), with a pointwise minimum norm (min-norm) continuous control feedback law. Finally, in Theorem 6.3 we show that every CLF is a meaningful value function for the two-player zero-sum hybrid game in the context of stability and that every pointwise min-norm control feedback law is optimal for such a game.

Notation. Let $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{R}_{>0} := (0, \infty)$. For a vector $x \in \mathbb{R}^n$, we denote by $x^\top$ its transpose and by $|x|$ its Euclidean norm. Given two vectors x and y, we write $(x, y) = [x^\top y^\top]^\top$, and $\langle x, y\rangle$ denotes the Euclidean inner product. For a set $\mathcal{C} \subset \mathbb{R}^n$, we denote by $\text{int}\,\mathcal{C}$ its interior and by $\overline{\mathcal{C}}$ its closure. If $\mathcal{C}$ is nonempty, then the distance from $x \in \mathbb{R}^n$ to $\mathcal{C}$ is $|x|_{\mathcal{C}} := \inf_{y\in\mathcal{C}} |y - x|$. A function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function ($\alpha \in \mathcal{K}$) if α is zero at zero, continuous, and strictly increasing. It is a class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is class-$\mathcal{K}$, $a = \infty$, and $\lim_{r\to\infty} \alpha(r) = \infty$. A continuous function $\beta : [0, a] \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function ($\beta \in \mathcal{KL}$) if, for each $s \in \mathbb{R}_{\geq 0}$, we have that $r \mapsto \beta(r, s) \in \mathcal{K}$, and if, for each $r \in \mathbb{R}_{\geq 0}$, $s \mapsto \beta(r, s)$ is decreasing and $\lim_{s\to\infty} \beta(r, s) = 0$. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, we denote the Lie derivative of h along f as $L_f h(x) := \langle \nabla h(x), f(x)\rangle$ for all $x \in \mathbb{R}^n$.

II. PRELIMINARIES

A. Hybrid Dynamical Systems

This paper considers the hybrid systems modeling framework in [7]. Therein, the continuous dynamics of the system are modeled by differential equations, while the discrete dynamics are modeled by difference equations. Based on

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this, a hybrid dynamical system $\mathcal{H}$ on $\mathbb{R}^n$ affine in the input $u = (u_C, u_D) = ((u_{C1}, u_{C2}), (u_{D1}, u_{D2})) \in \mathbb{R}^{m_C+m_D}$, where $u_1 := (u_{C1}, u_{D1}) \in \mathbb{R}^{m_{C1}+m_{D1}}$ is a control input and $u_2 := (u_{C2}, u_{D2}) \in \mathbb{R}^{m_{C2}+m_{D2}}$ is a disturbance, is defined as

$$\mathcal{H} \; : \; \begin{cases} \dot{x} = F(x, u_C) := f(x) + f_{u1}(x)u_{C1} + f_{u2}(x)u_{C2} \\ \hfill (x, u_C) \in C, \\ x^+ = G(x, u_D) := g(x) + g_{u1}(x)u_{D1} + g_{u2}(x)u_{D2} \\ \hfill (x, u_D) \in D \end{cases} \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, and, for each $i \in \{1, 2\}$, $f_{ui} : \mathbb{R}^n \to \mathbb{R}^{n\times m_{Ci}}$, $g_{ui} : \mathbb{R}^n \to \mathbb{R}^{n\times m_{Di}}$. The *flow map* $F : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}^n$ captures the continuous evolution of the system, when the state and continuous input are in the *flow set* C. The *jump map* $G : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}^n$ describes the discrete evolution of the system when the state and discrete input are in the *jump set* D.

Since solutions to the dynamical system $\mathcal{H}$ as in (1) can exhibit both continuous and discrete behavior, we use ordinary time $t \in \mathbb{R}_{\geq 0}$ to determine the amount of flow elapsed and a counter $j \in \mathbb{N}$ that keeps track of the number of jumps that have occurred. Based on this parametrization of time, the concept of hybrid time domain is proposed as follows.

Definition 2.1: (Hybrid time domain) *A set $\tilde{E} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if there exists $J \in \mathbb{N}$ such that*

$$\tilde{E} = \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}) \tag{2}$$

for some finite sequence of times $\{t_j\}_{j=0}^{J+1}$ satisfying $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if it is the union of a nondecreasing sequence $E_1 \subset E_2 \subset E_3 \subset \ldots$ of compact hybrid time domains.

A hybrid signal is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I_\phi^j := \{t : (t, j) \in \operatorname{dom} \phi\}$.

Definition 2.2: (Hybrid arc) *A hybrid signal $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is called a hybrid arc if, for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I_ϕ^j.*

Definition 2.3: (Hybrid input) *A hybrid signal u is a hybrid input if, for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I_u^j.*

A solution to the hybrid system $\mathcal{H}$ with input is defined as follows.

Definition 2.4: (Solution to $\mathcal{H}$) *A pair (ϕ, u) defines a solution to (1) if $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is a hybrid arc, $u = (u_C, u_D) : \operatorname{dom} u \to \mathbb{R}^{m_C+m_D}$ is a hybrid input with $\operatorname{dom} \phi = \operatorname{dom} u$, and*

- $(\phi(0, 0), u_C(0, 0)) \in \overline{C}$ *or* $(\phi(0, 0), u_D(0, 0)) \in D$,
- *For each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior $\operatorname{int} I_\phi^j$, we have, for all $t \in \operatorname{int} I_\phi^j$,*
$$(\phi(t, j), u_C(t, j)) \in C$$
and, for almost all $t \in I_\phi^j$,
$$\frac{d\phi}{dt}(t, j) = F(\phi(t, j), u_C(t, j)),$$
- *For each $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$,*
$$\begin{aligned} &(\phi(t, j), u_D(t, j)) \in D \\ &\phi(t, j + 1) = G(\phi(t, j), u_D(t, j)). \end{aligned}$$

The $\mathcal{L}^\infty$ norm of a hybrid signal $r = (r_C, r_D)$ is given by $\|r\|_{(t,j)} := \max\left\{\|r_C\|_{(t,j)}, \|r_D\|_{(t,j)}\right\}$ where

$$\|r_C\|_{(t,j)} := \max_{j'\leq j} \operatorname*{ess\,sup}_{t'\in I_r^j} |r(t', j')|, \tag{3a}$$

$$\|r_D\|_{(t,j)} := \sup_{(t',j')\in\Gamma(r), t'+j'\leq t+j} |r(t', j')|, \tag{3b}$$

where $\Gamma(r) := \{(t, j) \in \operatorname{dom} r : (t, j + 1) \in \operatorname{dom} r\}$. For notational convenience, $\|r\|_\#$ denotes $\lim_{t+j\to N} \|r\|_{(t,j)}$, where $N = \sup_{(t,j)\in\operatorname{dom} r} t + j \in [0, +\infty]$. We say a solution pair (ϕ, u) to $\mathcal{H}$ is maximal if it cannot be extended, and we say that it is complete if $\operatorname{dom} \phi$ is unbounded. Given a nonempty set $M \subset \mathbb{R}^n$, we denote by $\widehat{\mathcal{S}}_{\mathcal{H}}(M)$ the set of solution pairs (ϕ, u) to $\mathcal{H}$ as in (1) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_{\mathcal{H}}(M) \subset \widehat{\mathcal{S}}_{\mathcal{H}}(M)$ denotes all maximal solutions from M. We define $\operatorname{dom}_t \phi := \{t \in \mathbb{R}_{\geq 0} : \exists j \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$, $\operatorname{dom}_j \phi := \{j \in \mathbb{N}_{\geq 0} : \exists t \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$, $\sup_t \operatorname{dom} \phi := \sup \operatorname{dom}_t \phi$ and $\sup_j \operatorname{dom} \phi := \sup \operatorname{dom}_j \phi$. We also define the projection of $\star \in \{C, D\}$ onto $\mathbb{R}^n$ as $\Pi(\star) := \{\xi \in \mathbb{R}^n : \exists u_\star \text{ s.t. } (\xi, u_\star) \in \star\}$.

B. Hybrid Systems with Disturbance Inputs

Consider $\mathcal{H}$ as in (1) with the control input u_1 assigned by the feedback law $\kappa_1 := (\kappa_{C1}, \kappa_{D1}) : \Pi(C) \cup \Pi(D) \to \mathbb{R}^{m_{C1}+m_{D1}}$, and with disturbance u_2. The resulting hybrid system is

$$\mathcal{H}_{\kappa_1} : \begin{cases} \dot{x} \;= F(x, (\kappa_{C1}(x), u_{C2})) =: F_{\kappa_1}(x, u_{C2}) \\ \hfill (x, u_{C2}) \in C_{\kappa_1}, \\ x^+ = G(x, (\kappa_{D1}(x), u_{D2})) =: G_{\kappa_1}(x, u_{D2}) \\ \hfill (x, u_{D2}) \in D_{\kappa_1} \end{cases} \tag{4}$$

where $\star_{\kappa_1} := \{(x, u_{\star 2}) : (x, (\kappa_{\star 1}(x), u_{\star 2})) \in \star\}$, for each $\star \in \{C, D\}$. The pair (ϕ, u_2) is a solution to $\mathcal{H}_{\kappa_1}$ if the conditions in Definition 2.4 are satisfied.

III. INPUT-TO-STATE STABILITY FOR HYBRID SYSTEMS

Given a hybrid system $\mathcal{H}$ as in (1) and a control feedback law κ_1, we are interested in studying conditions to guarantee stability of a compact set $\mathcal{A} \subset \mathbb{R}^n$ with respect to $\mathcal{H}_{\kappa_1}$ as in (4) under the presence of a disturbance $u_2 = (u_{C2}, u_{D2})$, in the following sense.

Definition 3.1: (Pre-input-to-state stability) *Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, the system $\mathcal{H}_{\kappa_1}$ in (4) is pre-input-to-state* *stable (pre-ISS) with respect to the disturbance u_2 and the set $\mathcal{A}$ if there exist $\beta \in \mathcal{KL}$ and $\sigma \in \mathcal{K}_\infty$ such that, for each $\xi \in \mathbb{R}^n$, each $(\phi, u_2) \in \mathcal{S}_{\mathcal{H}_{\kappa_1}}(\xi)$ satisfies, for each $(t,j) \in \operatorname{dom}\phi$,*

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\left\{\beta(|\xi|_{\mathcal{A}}, t+j), \sigma\left(\|u_2\|_{(t,j)}\right)\right\}. \tag{5}$$

Definition 3.1 is essentially the definition of ISS in [6] for the case of disturbances where $x \mapsto |x|_{\mathcal{A}}$ is employed as the proper indicator for $\mathcal{A}$ and maximal solutions are not required to be complete. It subsumes the standard notion of ISS for continuous-time and discrete-time systems.

The conditions guaranteeing that a hybrid system is pre-ISS without computing solutions rely on Lyapunov functions, as defined next.

Definition 3.2: (Pre-ISS Lyapunov function candidate) *Given a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (1) and a compact set $\mathcal{A} \subset \mathbb{R}^n$, the function $V : \operatorname{dom} V \to \mathbb{R}$ defines a pre-ISS-Lyapunov function (pre-ISS-LF) candidate for $\mathcal{H}$ with respect to $\mathcal{A}$ if the following conditions hold:*

1) $\overline{\Pi(C)} \cup \Pi(D) \cup G(D) \subset \operatorname{dom} V$;
2) *V is continuously differentiable on an open set containing $\overline{\Pi(C)}$;*
3) *there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \tag{6}$$

for all $x \in \Pi(C) \cup \Pi(D) \cup G(D)$.

IV. PROBLEM STATEMENT

Given a compact set $\mathcal{A}$ and a control feedback law κ_1, we say $\mathcal{H}$ is *pre-input-to-state stabilizable* when the corresponding hybrid closed-loop system $\mathcal{H}_{\kappa_1}$ as in (4) is pre-ISS with respect to the disturbance u_2 and $\mathcal{A}$. In this paper, we aim to design a control feedback law κ_1 such that $\mathcal{H}_{\kappa_1}$ is pre-ISS with respect to u_2 and $\mathcal{A}$, but also that solves a zero-sum hybrid game.

Problem 4.1: (Inverse-optimal pre-ISS) *Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, design a control feedback law κ_1 such that the hybrid closed-loop system $\mathcal{H}_{\kappa_1}$ is pre-ISS with respect to the disturbance u_2 and $\mathcal{A}$. In addition, determine the cost functional that κ_1 minimizes under the worst-case disturbance u_2.*

Remark 4.2: (Relation to the literature) *This problem was solved in [10] for continuous-time systems without constraints, i.e., the case in which $\mathcal{H} = (\mathbb{R}^n \times \mathbb{R}^m, F, \emptyset, \star)$ and $\mathcal{A} = \{0\}$, where $\star$ denotes an arbitrary jump map.*

To motivate our inverse-optimality approach, we present an example for which the hybrid HJBI equations [1, Theorem 3.8] do not have a solution, but the approach in this paper leads to an inverse-optimal construction. Consider a hybrid system with state $x \in \mathbb{R}$, inputs $u_C = (u_{C1}, u_{C2}) \in \mathbb{R}^2$ and $u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} &= u_{C1} + x u_{C2} & \quad x \in [\mu, \delta] \\ x^+ &= \sigma + P^\top u_D & \quad x = \mu \end{aligned} \tag{7}$$

where $P = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^\top \in \mathbb{R}^2$ and $\delta > \mu > \sigma > 0$. Given the initial condition $\xi \in [\mu, \delta]$, consider the following zero-sum hybrid game

$$\underset{u=(u_1,u_2)\in\mathcal{U}}{\underset{u_1}{\text{minimize}}\ \underset{u_2}{\text{maximize}}} \quad \mathcal{J}(\xi, u) \tag{8}$$

where $\mathcal{J}$ is the cost associated to the unique solution (ϕ, u) to (7) with $\phi(0,0) = \xi$, and it is given by

$$\begin{aligned} \mathcal{J}(\xi, u) := &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} \Big(\phi(t,j)^2 \\ &\qquad + u_C^\top(t,j) \begin{bmatrix} 1 & 0 \\ 0 & -\zeta^2 \end{bmatrix} u_C(t,j) \Big)\, dt \\ &+ \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \left(\phi(t,j)^2 - \sigma^2\right) + \limsup_{\substack{(t,j)\in\sup\operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} q(\phi(t,j)) \end{aligned}$$

with parameter $\zeta > 0$ and some appropriate[1] $q : \mathbb{R}^n \to \mathbb{R}$. For optimality [1, Theorem 3.8], we are looking for a function $V : \operatorname{dom} V \to \mathbb{R}_{\geq 0}$ that is continuously differentiable and that satisfies

- for each $x \in [\mu, \delta]$

$$\begin{aligned} 0 = \underset{u_C=(u_{C1},u_{C2})\in\mathbb{R}^2}{\min_{u_{C1}} \max_{u_{C2}}} &\Big\{ x^2 + u_C^\top \begin{bmatrix} 1 & 0 \\ 0 & -\zeta^2 \end{bmatrix} u_C \\ &+ \frac{dV}{dx}(x) \begin{bmatrix} 1 & x \end{bmatrix} u_C \Big\}. \end{aligned} \tag{9}$$

Solving for u_C in (9), we have

$$\frac{dV}{dx}(x) = 2x\sqrt{\frac{\zeta^2}{\zeta^2 - x^2}}. \tag{10}$$

- for $x = \mu$

$$V(x) = \underset{(u_{D1},u_{D2})\in\mathbb{R}^2}{\min_{u_{D1}} \max_{u_{D2}}} \left\{ x^2 - \sigma^2 + V(\sigma + P u_D) \right\} \tag{11}$$

Solving for u_D in (11), we have

$$u_D^\star \in \left\{ u_D \in \mathbb{R}^2 : \frac{dV}{dx}(\sigma + P u_D) = 0 \right\}.$$

Equations (9) and (11) are referred to as the Hamilton-Jacobi-Bellman-Isaacs (HJBI) hybrid equations for the zero-sum hybrid game in (8). Notice that (10) cannot be solved when $x > \zeta$ and if, in addition, we have that $\zeta < \mu$, then $\mu \notin \operatorname{dom} V$ and, as a result, (11) cannot be solved either.

V. INPUT-TO-STATE STABILIZABILITY

In this section, we use pre-ISS control Lyapunov functions (pre-ISS-CLFs) as a synthesis tool to stabilize a hybrid system. First, we introduce definitions and preliminary results on pre-ISS-CLFs for hybrid systems with disturbances.

[1]We refer the reader to [1, Theorem 3.8] for general sufficient conditions on the terminal cost q and the value function V for optimality.

A. Input-to-State Stability Control Lyapunov Functions

Given a hybrid system $\mathcal{H} = (C, F, D, G)$ we define the projection of $\star \in \{C, D\}$ onto $\mathbb{R}^n \times \mathbb{R}^{m_{\star 2}}$ as

$$\Pi_{u_{\star 1}}(\star) = \{(x, u_{\star 2}) : \exists u_{\star 1} \text{ s.t. } (x, (u_{\star 1}, u_{\star 2})) \in \star\}.$$

In addition, we define, for each $\star \in \{C, D\}$, the set of admissible control inputs during flows ($\star = C$) and during jumps ($\star = D$) at each state and disturbance as

$$\Psi_\star(x, u_{\star 2}) := \{u_{\star 1} \in \mathbb{R}^{m_{\star 1}} : (x, (u_{\star 1}, u_{\star 2})) \in \star\}.$$

Definition 5.1: (pre-ISS control Lyapunov functions) *Given a hybrid system $\mathcal{H} = (C, F, D, G)$ as in* (1) *and a compact set $\mathcal{A} \subset \mathbb{R}^n$, a pre-ISS-LF candidate V is a pre-ISS control Lyapunov function (pre-ISS-CLF) for $\mathcal{H}$ with respect to u_2 and $\mathcal{A}$ if there exist $\alpha_C, \alpha_D, \rho \in \mathcal{K}_\infty$ such that*

$$\begin{aligned} & |x|_\mathcal{A} \geq \rho(|u_{C2}|), (x, u_{C2}) \in \Pi_{u_{C1}}(C) \\ & \Rightarrow \inf_{u_{C1} \in \Psi_C(x, u_{C2})} \langle \nabla V(x), F(x, u_C) \rangle \leq -\alpha_C(|x|_\mathcal{A}) \end{aligned} \quad (12a)$$

and

$$\begin{aligned} & |x|_\mathcal{A} \geq \rho(|u_{D2}|), (x, u_{D2}) \in \Pi_{u_{D1}}(D) \\ & \Rightarrow \inf_{u_{D1} \in \Psi_D(x, u_{D2})} V(G(x, u_D)) - V(x) \leq -\alpha_D(|x|_\mathcal{A}). \end{aligned} \quad (12b)$$

To characterize the effect of inputs on the stability conditions at jumps, we restrict our analysis to the family of systems and Lyapunov functions that satisfy the next assumption.

Assumption 5.2: (Control-affine upper bound on the change of V after a jump) *Given a hybrid system $\mathcal{H} = (C, F, D, G)$ as in* (1) *and a function $V : \operatorname{dom} V \to \mathbb{R}$, suppose that there exist functions $\widehat{V}_{L1} : \mathbb{R}^n \to \mathbb{R}^{m_{D1}}$ and $\widehat{V}_{L2} : \mathbb{R}^n \to \mathbb{R}^{m_{D2}}$ such that, for all $(x, u_D) \in D$,*

$$\begin{aligned} V(G(x, u_D)) &= V\Big(g(x) + g_{u1}(x) u_{D1} + g_{u2}(x) u_{D2}\Big) \\ &\leq V(g(x)) + \widehat{V}_{L1}(x) u_{D1} + \widehat{V}_{L2}(x) u_{D2}. \end{aligned} \quad (13)$$

Leveraging the previous assumption, the results in the following section are used to establish a connection between the existence of a pre-ISS-CLF and a control feedback law κ_1 that pre-input-to-state stabilizes the hybrid closed-loop system $\mathcal{H}_{\kappa_1}$.

B. Input-to-State Stabilizing Minimum Norm Control

We can endow a hybrid system $\mathcal{H}$ with an input-to-state-stability property by solving a quadratic program (QP) in terms of a pre-ISS-CLF V. Namely, given $\alpha_C, \rho \in \mathcal{K}_\infty$, we define, for all $x \in \Pi(C)$,

$$\omega_C(x) := L_f V(x) + \alpha_C(|x|_\mathcal{A}) + |L_{f_{u_2}} V(x)| \rho^{-1}(|x|_\mathcal{A}) \quad (14)$$

and introduce the following QP:

$$\begin{aligned} \kappa_{C1_{QP}}(x) := \arg\min_{v \in \mathbb{R}^{m_{C1}}} \quad & |v|^2 \\ \text{subject to} \quad & L_{f_{u1}} V(x) v \leq -\omega_C(x) \end{aligned} \quad (15)$$

whose closed-form solution is given by

$$\kappa_{C1_{QP}}(x) = \begin{cases} -\dfrac{\max\{0, \omega_C(x)\}}{|L_{f_{u1}} V(x)|^2} L_{f_{u1}} V(x) & \\ & \text{if } L_{f_{u1}} V(x) \neq 0, \\ 0 & \text{if } L_{f_{u1}} V(x) = 0. \end{cases} \quad (16)$$

Similarly, given a function $\alpha_D \in \mathcal{K}$, under Assumption 5.2, we define, for all $x \in \Pi(D)$,

$$\begin{aligned} \omega_D(x) := V\big(g(x)\big) - V(x) + \alpha_D(|x|_\mathcal{A}) \\ + |\widehat{V}_{L2}(x)| \rho^{-1}(|x|_\mathcal{A}) \end{aligned} \quad (17)$$

and introduce the following QP:

$$\begin{aligned} \kappa_{D1_{QP}}(x) := \arg\min_{v \in \mathbb{R}^{m_{D1}}} \quad & |v|^2 \\ \text{subject to} \quad & \widehat{V}_{L1}(x) v \leq -\omega_D(x) \end{aligned} \quad (18)$$

whose closed-form solution is expressed as

$$\kappa_{D1_{QP}}(x) = \begin{cases} -\dfrac{\max\{0, \omega_D(x)\}}{|\widehat{V}_{L1}(x)|^2} \widehat{V}_{L1}(x) & \\ & \text{if } \widehat{V}_{L1}(x) \neq 0, \\ 0 & \text{if } \widehat{V}_{L1}(x) = 0. \end{cases} \quad (19)$$

With (16) and (19) we establish the following result.

Theorem 5.3: (QP control via pre-ISS-CLFs) *Given a hybrid system $\mathcal{H}$ as in* (1) *and a compact set $\mathcal{A} \subset \mathbb{R}^n$, suppose that there exists a pre-ISS-CLF V for $\mathcal{H}$ with respect to the disturbance u_2 and $\mathcal{A}$ such that Assumption 5.2 holds. Then, the feedback law $\kappa_1 = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$, with $\kappa_{C1_{QP}}$ as in* (16) *and $\kappa_{D1_{QP}}$ as in* (19)*, renders the resulting hybrid closed-loop system $\mathcal{H}_{\kappa_1}$ pre-ISS with respect to u_2 and $\mathcal{A}$.*

Remark 5.4: (Noncompleteness of solutions under QP control) *Notice that the optimization in* (15) *and* (18) *is carried over $\mathbb{R}^{m_{C1}}$ and $\mathbb{R}^{m_{D1}}$, respectively, instead of over the constraint sets $\Psi_\star$, $\star \in \{C, D\}$, as in Definition 5.1. This allows to compute the closed-form control feedback law $\kappa_1 = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$ which may potentially lead to maximal noncomplete solutions to $\mathcal{H}_{\kappa_1}$. The "pre" term in the results accounts for this trade-off. We refer the reader to [7, Proposition 2.34] for sufficient conditions for completeness of solutions for the hybrid closed-loop system $\mathcal{H}_{\kappa_1}$.*

VI. INVERSE-OPTIMAL STABILIZING CONTROL

Since the control input u_1, assigned by a control feedback law, aims to stabilize $\mathcal{H}$ to a compact set $\mathcal{A}$ but the disturbance u_2 seeks to prevent it, we formulate a zero-sum hybrid game that captures such a setting following [1]. Specifically, in this section, given a control feedback law κ_1 such that $\mathcal{H}_{\kappa_1}$ is pre-ISS with respect to the disturbance u_2 and $\mathcal{A}$, we are interested in determining the cost functional that κ_1 minimizes.

In particular, consider $\xi \in \overline{\Pi(C)} \cup \Pi(D)$, an input action $u := (u_1, u_2) = ((u_{C1}, u_{D1}), (u_{C2}, u_{D2}))$, the stage cost for flows $\mathcal{L}_C : C \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $\mathcal{L}_D : D \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \overline{\Pi(C)} \cup \Pi(D) \to \mathbb{R}$. Under the assumption of uniqueness of solutions[2], we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ as

$$\begin{aligned}\mathcal{J}(\xi, u) := &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} \mathcal{L}_C(\phi(t,j), u_C(t,j))dt \\ &+ \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \mathcal{L}_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) \\ &+ \limsup_{\substack{(t,j)\to \sup \operatorname{dom}\phi \\ (t,j)\in \operatorname{dom}\phi}} q(\phi(t,j))\end{aligned} \tag{20}$$

where $t_{\sup_j \operatorname{dom}\phi + 1} := \sup_t \operatorname{dom}\phi$ defines the upper limit of the last integral, and $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to $\operatorname{dom}\phi$. The two-player zero-sum hybrid game is formulated as

$$\underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}(\xi,\mathcal{A})}{\underset{u_1}{\text{minimize}}\ \underset{u_2}{\text{maximize}}} \quad \mathcal{J}(\xi, u) \tag{21}$$

where $\mathcal{U}_{\mathcal{H}}(\xi, \mathcal{A})$ is the set of joint input actions $u = (u_1, u_2)$, where u_1 is assigned by a control feedback law κ_1, yielding solutions from ξ such that $\mathcal{H}_{\kappa_1}$ is pre-ISS with respect to u_2 and $\mathcal{A}$.

The next definition introduces the notion of a value function for the hybrid game in (21).

Definition 6.1: (Value function) *Given a hybrid system $\mathcal{H}$ as in* (1)*, a compact set $\mathcal{A} \subset \mathbb{R}^n$, and $\xi \in \mathbb{R}^n$, the value function at ξ, when it exists, is given by*

$$\mathcal{J}^*(\xi) := \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}(\xi,\mathcal{A})}{\min_{u_1}\max_{u_2}} \quad \mathcal{J}(\xi, u). \tag{22}$$

A. Inverse-Optimal Min-Norm Control

In this section, we provide sufficient conditions to solve (21) when u_1 is assigned by the control feedback law κ_1 expressed as the pointwise optimal result of a QP, as stated in Section V-B.

For a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (1) and a compact set $\mathcal{A}$, let V be a pre-ISS-CLF for $\mathcal{H}$ with respect to the disturbance u_2 and $\mathcal{A}$ such that Assumption 5.2 holds. Given $\gamma \in \mathcal{K}_\infty$, we define the stage cost for flows $\mathcal{L}_C$, the stage cost for jumps $\mathcal{L}_D$, and the terminal cost q as

$$\begin{aligned}\mathcal{L}_C(x, u_C) :=& \mathcal{L}_{1C}(x) + \frac{1}{2}\frac{|L_{f_{u_1}}V(x)|^2}{\max\{0, \omega_C(x)\}}|u_{C1}|^2 \\ &- \lambda\gamma\left(\frac{|u_{C2}|}{\lambda}\right) \qquad \forall (x, u_C) \in C\end{aligned} \tag{23a}$$

$$\begin{aligned}\mathcal{L}_D(x, u_D) :=& \mathcal{L}_{1D}(x) + \frac{1}{2}\frac{|\widehat{V}_{L1}(x)|^2}{\max\{0, \omega_D(x)\}}|u_{D1}|^2 \\ &- \lambda\gamma\left(\frac{|u_{D2}|}{\lambda}\right) \qquad \forall (x, u_D) \in D\end{aligned} \tag{23b}$$

$$q(x) := V(x) \qquad \forall x \in \overline{\Pi(C)} \cup \Pi(D) \tag{23c}$$

where ω_C is defined in (14) and ω_D in (17). We remind the reader that we approach the optimal control problem in (21) as an inverse problem: we design the optimal control feedback law κ_1 and then the stage costs $\mathcal{L}_{1C}$ and $\mathcal{L}_{1D}$ in (23).

A key tool in our analysis to solve the hybrid game in (21) is the Legendre–Fenchel transform, which is defined as follows.

Definition 6.2: (Legendre-Fenchel transform of a class-$\mathcal{K}_\infty$ function [10, Lemma A.1]) *For a class-$\mathcal{K}_\infty$ function γ whose derivative exists and is also a class-$\mathcal{K}_\infty$ function, the Legendre–Fenchel transform of γ is defined, for all $r \geq 0$, as*

$$\bar{\gamma}(r) := \int_0^r (\gamma')^{-1}(s)ds = r(\gamma')^{-1}(r) - \gamma((\gamma')^{-1}(r))$$

where $(\gamma')^{-1}$ stands for the inverse function of $\gamma' := \dfrac{d\gamma}{dr}$.

The next result shows that every pointwise minimum norm control feedback law $\kappa_1 := (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$, as in (16) and (19), is optimal for a meaningful hybrid game.

Theorem 6.3: (Inverse-optimal QP-based control) *Given a hybrid system $\mathcal{H} = (C, F, D, G)$ as in* (1) *and a compact set $\mathcal{A} \subset \mathbb{R}^n$, suppose that there exists a pre-ISS-CLF V for $\mathcal{H}$ with respect to the disturbance u_2 and $\mathcal{A}$. If there exist functions $\widehat{V}_{L1} : \Pi(D) \to \mathbb{R}^{m_{D1}}$ and $\widehat{V}_{L2} : \Pi(D) \to \mathbb{R}^{m_{D2}}$ such that Assumption 5.2 holds for all $(x, u_D) \in D$, then, $\kappa_1 := (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$, with values as in* (16) *and* (19)*, is such that the hybrid closed-loop system $\mathcal{H}_{\kappa_1}$ is pre-ISS with respect to u_2 and $\mathcal{A}$, and minimizes, for all $\xi \in \overline{\Pi(C)} \cup \Pi(D)$, all $\lambda \in (0, 1]$, and all $\gamma \in \mathcal{K}_\infty$ with $\gamma' \in \mathcal{K}_\infty$, the cost $\mathcal{J}$ in* (20) *with*

$$\begin{aligned}\mathcal{L}_{1C}(x) := -\Big(&L_f V(x) - \frac{1}{2}\max\{0, \omega_C(x)\} \\ &+ \lambda\bar{\gamma}(|L_{f_{u_2}}V(x)|)\Big)\end{aligned} \tag{24a}$$

and

$$\begin{aligned}\mathcal{L}_{1D}(x) := -\Big(&V(g(x)) - V(x) - \frac{1}{2}\max\{0, \omega_D(x)\} \\ &+ \lambda\bar{\gamma}(|\widehat{V}_{L2}(x)|)\Big)\end{aligned} \tag{25a}$$

where ω_C and ω_D defined in (14) *and* (17)*, respectively.*

We illustrate the inverse-optimal approach in the following example.

Example 1 (One-degree-of-freedom juggling system): In the following, we illustrate our min-norm-based inverse-optimal approach for a one-degree-of-freedom juggling system on a moving platform under disturbances. In particular, consider the following hybrid system:

$$\mathcal{H} : \begin{cases} \dot{x} = \begin{bmatrix} x_2 \\ -\mu_C \end{bmatrix} & x \in C \\ x^+ = \begin{bmatrix} u_{D1} + u_{D2} \\ -\mu_D x_2 \end{bmatrix} & (x, u_D) \in D \end{cases} \tag{26}$$

[2]Sufficient conditions to establish uniqueness of solutions to $\mathcal{H}$ from $\xi \in \overline{\Pi(C)} \cup \Pi(D)$ can be found in [14, Proposition 2.11].

where $x := (x_1, x_2) \in \mathbb{R}^2$, $\mu_C > 0$, $\mu_D \in [0, 1)$, and

$$C := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$$
$$D := \{(x, u_D) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1 = u_{D1} + u_{D2} > 0, x_2 \leq 0\}.$$

Let $\mathcal{A} = \{(0, 0)\}$ and consider the *modified* energy function given by

$$V(x) := \left(1 + \frac{1-\mu_D^2}{\pi(1+\mu_D^2)} \arctan x_2\right)\left(\mu_C x_1 + \frac{1}{2}x_2^2\right)$$
$$= (1 + \theta \arctan x_2)\left(\mu_C x_1 + \frac{1}{2}x_2^2\right).$$

To show that V is a pre-ISS-LF candidate as in Definition 3.2, notice that: i) $\operatorname{dom} V = \mathbb{R}^2$, ii) V is continuously differentiable, and iii) we have that, for all $x \in C$,

$$\frac{1+3\mu_D^2}{2(1+\mu_D^2)}\alpha_1(|x|) \leq V(x) \leq \frac{3+\mu_D^2}{2(1+\mu_D^2)}\alpha_2(|x|)$$

where $\mathbb{R}_{\geq 0} \ni r \mapsto \alpha_1(r) := \min\left\{\frac{r^2}{4}, \mu_C \frac{r}{\sqrt{2}}\right\}$ and $\mathbb{R}_{\geq 0} \ni r \mapsto \alpha_2(r) := \frac{1}{2}r^2 + \mu_C r$. Then, V is a pre-ISS-LF candidate for $\mathcal{H}$ with respect to $\mathcal{A}$. In addition, observe that V satisfies Assumption 5.2 with

$$\widehat{V}_{L1}(x) = \widehat{V}_{L2}(x) = \mu_C\,(1 + \theta \arctan(-\mu_D x_2)),$$

and it can also be shown that V is a pre-ISS-CLF for $\mathcal{H}$ with respect to u_2 and $\mathcal{A}$. Then, by Theorem 5.3, $\mathcal{H}$ can be rendered pre-ISS with respect to u_2 and $\mathcal{A}$, and Theorem 6.3 shows that the control feedback law $\kappa_1 = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$, with values as in (16) and (19), minimizes the cost $\mathcal{J}$ in (20) under the maximizing disturbance u_2, where the value function is $\mathcal{J}^*(\xi) = V(\xi)$ for all $\xi \in \{\eta \in \mathbb{R}^2 : \eta_1 \geq 0\}$. This is confirmed by Fig. 1.

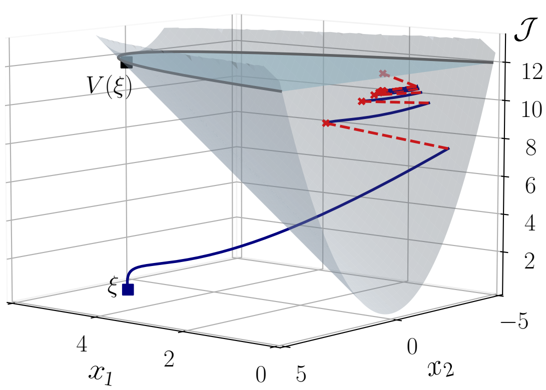

Fig. 1 – Phase portrait of a solution to $\mathcal{H}$ as in (26) from ξ. The evolution of the cost of the solution (blue-red), rendered by the saddle-point equilibrium strategy, is shown. Notice that $\mathcal{J}^*(\xi) = V(\xi)$ as guaranteed by Theorem 6.3.

VII. Conclusion and Future Work

In this paper, we study the problem of designing a stabilizing controller for a hybrid system under disturbances as an inverse optimal problem. We characterize the control feedback law as the pointwise solution of a quadratic program (QP) and formulate the problem as a two-player zero-sum hybrid game against the worst-case disturbance. Our approach is inverse: we design the optimal control feedback law κ_1 and then the cost functional for which κ_1 is optimal. In particular, we show that every control Lyapunov function (CLF) is a meaningful value function for the game, and that every pointwise min-norm control feedback law is optimal—even though its construction is independent of any explicit cost functional or the HJBI hybrid equations. Future work includes designing projection tools to deal with feedback laws that render maximal solutions complete and relaxing Assumption 5.2.

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